2000 Canadian Mathematics Olympiad Solutions
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The Year 2000 Canadian Mathematics Olympiad was written on Wednesday April 2, by 98 high school students across Canada. A correct and well presented solution to any of the five questions was awarded seven points. This year's exam was a somewhat harder than usual, with the mean score being 8.37 out of 35. The top few scores were: 30, 28, 27, 22, 20, 20, 20. The first, second and third prizes are awarded to: Daniel Brox (Sentinel Secondary BC), David Arthur (Upper Canada College ON), and David Pritchard (Woburn Collegiate Institute ON).

1. At 12:00 noon, Anne, Beth and Carmen begin running laps around a circular track of length three hundred meters, all starting from the same point on the track. Each jogger maintains a constant speed in one of the two possible directions for an indefinite period of time. Show that if Anne's speed is different from the other two speeds, then at some later time Anne will be at least one hundred meters from each of the other runners. (Here, distance is measured along the shorter of the two arcs separating two runners.)

   Comment: We were surprised by the difficulty of this question, having awarded an average grade of 1.43 out of 7. We present two solutions; only the first appeared among the graded papers.

   Solution 1: By rotating the frame of reference we may assume that Anne has speed zero, that Beth runs at least as fast as Carmen, and that Carmen's speed is positive. If Beth is no more than twice as fast as Carmen, then both are at least 100 meters from Anne when Carmen has run 100 meters. If Beth runs more that twice as fast as Carmen, then Beth runs a stretch of more than 200 meters during the time Carmen runs between 100 and 200 meters. Some part of this stretch lies more than 100 meters from Anne, at which time both Beth and Carmen are at least (in fact, more than) 100 meters away from Anne.

   Solution 2: By rotating the frame of reference we may assume Anne's speed to equal zero, and that the other two runners have non-zero speed. We may assume that Beth is running at least as fast as Carmen. Suppose that it takes t seconds for Beth to run 200 meters. Then at all times in the infinite set $T = \{t, 2t, 4t, 8t, \ldots\}$, Beth is exactly 100 meters from Anne. At time t, Carmen has traveled exactly d meters where $0 < d \le 200$. Let k be the least integer such that $2^k d \ge 100$. Then $k \ge 0$ and $100 \le 2^k d \le 200$, so at time $2^k t \in T$ both Beth and Carmen are at least 100 meters from Anne.

2. A *permutation* of the integers $1901, 1902, \ldots, 2000$ is a sequence $a_1, a_2, \ldots, a_{100}$ in which each of those integers appears exactly once. Given such a permutation, we form the sequence of partial sums

   $$s_1 = a_1, \ \ s_2 = a_1 + a_2, \ \ s_3 = a_1 + a_2 + a_3, \ \ldots \ , \ s_{100} = a_1 + a_2 + \cdots + a_{100}.$$

   How many of these permutations will have no terms of the sequence $s_1, \ldots, s_{100}$ divisible by three?

   Comment: This question was the easiest and most straight forward, with an average grade of 3.07.

   Solution: Let $\{1901, 1902, \ldots, 2000\} = R_0 \cup R_1 \cup R_2$ where each integer in R_i is congruent to i modulo 3. We note that $|R_0| = |R_1| = 33$ and $|R_2| = 34$. Each permutation $S = (a_1, a_2, \ldots, a_{100})$ can be uniquely specified by describing a sequence $S' = (a'_1, a'_2, \ldots, a'_{100})$ of

residues modulo 3 (containing exactly 33 zeros, 33 ones and 34 twos), and three permutations (one each of R_0, R_1, and R_2). Note that the number of permutations of R_i is exactly $|R_i|! = 1 \cdot 2 \cdots |R_i|$.

The condition on the partial sums of S depends only on the sequence of residues S'. In order to avoid a partial sum divisible by three, the subsequence formed by the 67 ones and twos in S' must equal either $1, 1, 2, 1, 2, \ldots, 1, 2$ or $2, 2, 1, 2, 1, \ldots, 2, 1$. Since $|R_2| = |R_1| + 1$, only the second pattern is possible. The 33 zero entries in S' may appear anywhere among $a'_1, a'_2, \ldots, a'_{100}$ provided that $a'_1 \neq 0$. There are $\binom{99}{33} = \frac{99!}{33!\,66!}$ ways to choose which entries in S' equal zero. Thus there are exactly $\binom{99}{33}$ sequences S' whose partial sums are not divisible by three. Therefore the total number of permutations S satisfying this requirement is exactly

$$\binom{99}{33} \cdot 33! \cdot 33! \cdot 34! = \frac{99! \cdot 33! \cdot 34!}{66!}.$$

Incidently, this number equals approximately $4.4 \cdot 10^{138}$.

3. Let $A = (a_1, a_2, \ldots, a_{2000})$ be a sequence of integers each lying in the interval $[-1000, 1000]$. Suppose that the entries in A sum to 1. Show that some nonempty subsequence of A sums to zero.

Comment: This students found this question to be the most difficult, with an average grade of 0.51, and only one perfect solution among 100 papers.

Solution: We may assume no entry of A is zero, for otherwise we are done. We sort A into a new list $B = (b_1, \ldots, b_{2000})$ by selecting elements from A one at a time in such a way that $b_1 > 0$, $b_2 < 0$ and, for each $i = 2, 3, \ldots, 2000$, the sign of b_i is opposite to that of the partial sum

$$s_{i-1} = b_1 + b_2 + \cdots + b_{i-1}.$$

(We can assume that each $s_{i-i} \neq 0$ for otherwise we are done.) At each step of the selection process a candidate for b_i is guaranteed to exist, since the condition $a_1 + a_2 + \cdots + a_{2000} = 1$ implies that the sum of unselected entries in A is either zero or has sign opposite to s_{i-1}.

From the way they were defined, each of $s_1, s_2, \ldots, s_{2000}$ is one of the 1999 nonzero integers in the interval $[-999, 1000]$. By the Pigeon Hole Principle, $s_j = s_k$ for some j, k satisfying $1 \leq j < k \leq 2000$. Thus $b_{j+1} + b_{j+2} + \cdots + b_k = 0$ and we are done.

4. Let $ABCD$ be a convex quadrilateral with

$$\angle CBD = 2\angle ADB,$$
$$\angle ABD = 2\angle CDB$$
$$\text{and} \qquad AB = CB.$$

Prove that $AD = CD$.

Comment: There are several different solutions to this, including some using purely trigonometric arguments (involving the law of sines and standard angle sum formulas). We present here two prettier geometric arguments (with diagrams). The first solution is perhaps the more attractive of the two. Average grade: 1.84 out of 7.

Solution 1 (from contestant Keon Choi): Extend DB to a point P on the circle through A and C centered at B. Then $\angle CPD = \frac{1}{2}\angle CBD = \angle ADB$ and $\angle APD = \frac{1}{2}\angle ABD = \angle CDB$,

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so $APCD$ is a parallelogram. Now PD bisects AC so BD is an angle bisector of isosceles triangle ABC. We have

$$\angle ADB = \frac{1}{2}\angle CBD = \frac{1}{2}\angle ABD = \angle CDB$$

so DB is the angle bisector of $\angle ADC$. As DB bisects the base of triangle ADC, this triangle must be isosceles and $AD = CD$.

Solution 2: Let the bisector of $\angle ABD$ meet AD at E. Let the bisector of $\angle CBD$ meet CD at F. Then $\angle FBD = \angle BDE$ and $\angle EBD = \angle BDF$, which imply $BE \parallel FD$ and $BF \parallel ED$. Thus $BEDF$ is a parallelogram whence

$$BD \text{ intersects } EF \text{ at its midpoint } M. \qquad (1)$$

On the other hand since BE is an angle bisector, we have $\frac{AB}{BD} = \frac{AE}{ED}$. Similarly we have $\frac{CB}{BD} = \frac{CF}{FD}$. By assumption $AB = CB$ so $\frac{AE}{ED} = \frac{CF}{FD}$ which implies $EF \parallel AC$. Thus $\triangle DEF$ and $\triangle DAC$ are similar, which implies by (1) that BD intersects AC at its midpoint N. Since $\triangle ABC$ is isosceles, this implies $AC \perp BD$. Thus $\triangle NAD$ and $\triangle NCD$ are right triangles with equal legs and hence are congruent. Thus $AD = CD$.
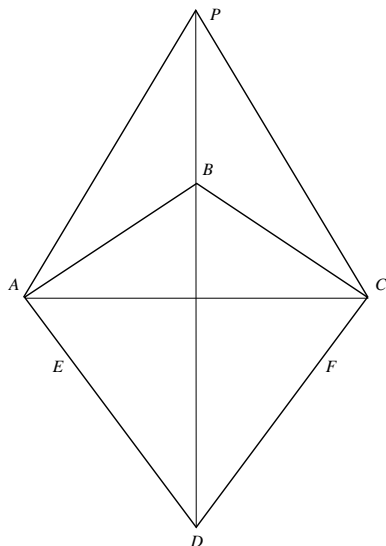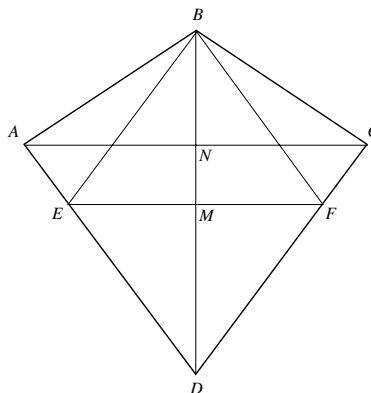

Diagram for Solution 1   Diagram for Solution 2

5. Suppose that the real numbers $a_1, a_2, \ldots, a_{100}$ satisfy

$$a_1 \geq a_2 \geq \cdots \geq a_{100} \geq 0$$

$$a_1 + a_2 \leq 100$$

$$a_3 + a_4 + \cdots + a_{100} \leq 100.$$

Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all possible sequences $a_1, a_2, \ldots, a_{100}$ which achieve this maximum.

Comment: All of the correct solutions involved a sequence of adjustments to the variables, each of which increase $a_1^2 + a_2^2 + \cdots + a_{100}^2$ while satisfying the constraints, eventually arriving at the two optimal sequences: $100, 0, 0, \ldots, 0$ and $50, 50, 50, 50, 0, 0, \ldots, 0$. We present here a sharper proof, which might be arrived at after guessing that the optimal value is 100^2. Average grade: 1.52 out of 7.

Solution: We have $a_1 + a_2 + \cdots + a_{100} \leq 200$, *so*

$$
\begin{aligned}
a_1^2 + a_2^2 + \cdots + a_{100}^2 \quad &\leq \quad (100 - a_2)^2 + a_2^2 + a_3^2 + \cdots + a_{100}^2 \\
&= \quad 100^2 - 200a_2 + 2a_2^2 + a_3^2 + \cdots + a_{100}^2 \\
&\leq \quad 100^2 - (a_1 + a_2 + \cdots + a_{100})a_2 + 2a_2^2 + a_3^2 + \cdots + a_{100}^2 \\
&= \quad 100^2 + (a_2^2 - a_1 a_2) + (a_3^2 - a_3 a_2) + (a_4^2 - a_4 a_2) + \cdots + (a_{100}^2 - a_{100} a_2) \\
&= \quad 100^2 + (a_2 - a_1)a_2 + (a_3 - a_2)a_3 + (a_4 - a_2)a_4 + \cdots + (a_{100} - a_2)a_{100}
\end{aligned}
$$

Since $a_1 \geq a_2 \geq \cdots \geq a_{100} \geq 0$, *none of the terms* $(a_i - a_j)a_i$ *is positive. Thus* $a_1^2 + a_2^2 + \cdots + a_{100}^2 \leq 10,000$ *with equality holding if and only if*

$$
a_1 = 100 - a_2 \qquad and \qquad a_1 + a_2 + \cdots + a_{100} = 200
$$

and each of the products

$$
(a_2 - a_1)a_2, \quad (a_3 - a_2)a_3, \quad (a_4 - a_2)a_4, \quad \cdots \quad , \quad (a_{100} - a_2)a_{100}
$$

equals zero. Since $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_{100} \geq 0$, *the last condition holds if and only if for some* $i \geq 1$ *we have* $a_1 = a_2 = \cdots = a_i$ *and* $a_{i+1} = \cdots = a_{100} = 0$. *If* $i = 1$, *then we get the solution* $100, 0, 0, \ldots, 0$. *If* $i \geq 2$, *then from* $a_1 + a_2 = 100$, *we get that* $i = 4$ *and the second optimal solution* $50, 50, 50, 50, 0, 0, \ldots, 0$.