1. Let x, y and z be positive real numbers. Show that $x^2 + xy^2 + xyz^2 \geq 4xyz - 4$.

Solution. Note that

$$x^2 \geq 4x - 4, \quad y^2 \geq 4y - 4, \quad \text{and} \quad z^2 \geq 4z - 4,$$

and therefore

$$x^2 + xy^2 + xyz^2 \geq (4x - 4) + x(4y - 4) + xy(4z - 4) = 4xyz - 4.$$

2. For any positive integers n and k, let $L(n, k)$ be the least common multiple of the k consecutive integers n, $n + 1$, $\ldots$, $n + k - 1$. Show that for any integer b, there exist integers n and k such that $L(n, k) > b\,L(n + 1, k)$.

Solution. I. Let $p > b$ be prime, let $n = p^3$ and $k = p^2$. If $p^3 < i < p^3 + p^2$, then no power of p greater than 1 divides i, while p divides $p^3 + p$. It follows that $L(p^3, p^2) = p^2 L(p^3 + 1, p^2 - 1)$. A similar calculation shows that $L(p^3 + 1, p^2) = pL(p^3 + 1, p^2 - 1)$. Thus $L(p^3, p^2) = pL(p^3 + 1, p^2) > bL(p^3 + 1, p^2)$.

II. Let $m > 1$. Then $L(m!-1, m+1)$ is the least common multiple of the integers from $m!-1$ to $m!+m-1$. But $m!-1$ is relatively prime to all of $m!, m!+1, \ldots, m!+m-1$. It follows that $L(m!-1, m+1) = (m!-1)M$, where $M = \text{lcm}(m!, m!+1, \ldots, m!+m-1)$.

Now consider $L(m!, m+1)$. This is $\text{lcm}(M, m!+m)$. But $m!+m = m((m-1)!+1)$, and m divides M. Thus $\text{lcm}(M, m! + m) \leq M((m - 1)! + 1)$, and

$$\frac{L(m! - 1, m + 1)}{L(m!, m + 1)} \geq \frac{m! - 1}{(m - 1)! + 1}.$$

Since m can be arbitrarily large, so can $L(m! - 1, m + 1)/L(m!, m + 1)$. Therefore taking $n = m! - 1$ for sufficiently large m, and $k = m + 1$, works.

3. Let $ABCD$ be a convex quadrilateral and let P be the point of intersection of AC and BD. Suppose that $AC + AD = BC + BD$. Prove that the internal angle bisectors of $\angle ACB$, $\angle ADB$, and $\angle APB$ meet at a common point.

Solution. I. Construct A' on CA so that $AA' = AD$ and B' on CB such that $BB' = BD$. Then we have three angle bisectors that correspond to the perpendicular bisectors of $A'B'$, $A'D$, and $B'D$. These perpendicular bisectors are concurrent, so the angle bisectors are also concurrent. This tells us that the external angle bisectors at A and B meet at the excentre of PDB. A symmetric argument for C finishes the problem.

II. Note that the angle bisectors $\angle ACB$ and $\angle APB$ intersect at the excentres of $\triangle PBC$ opposite C and the angle bisectors of $\angle ADB$ and $\angle APB$ intersect at the excentres of $\triangle PAD$ opposite D. Hence, it suffices to prove that these two excentres coincide.

Let the excircle of $\triangle PBC$ opposite C touch side PB at a point X, line CP at a point Y and line CB at a point Z. Hence, $CY = CZ$, $PX = PY$ and $BX = BZ$. Therefore, $CP + PX = CB + BX$. Since $CP + PX + CB + BX$ is the perimeter of $\triangle CBP$, $CP + PX = CB + BX = s$, where s is the semi-perimeter of $\triangle CBP$. Therefore,

$$PX = CB + BX - CP = \frac{s}{2} - CP = \frac{CB + BP + PC}{2} - CP = \frac{CB + BP - PC}{2}.$$

Similarly, if we let the excircle of $\triangle PAD$ opposite D touch side PA at a point X', then

$$PX' = \frac{DA + AP - PD}{2}.$$

Since both excircles are tangent to AC and BD, if we show that $PX = PX'$, then we would show that the two excircles are tangent to AC and BD at the same points, i.e. the two excircles are identical. Hence, the two excentres coincide.

We will use the fact that $AC + AD = BC + BD$ to prove that $PX = PX'$. Since $AC + AD = BC + BD$, $AP + PC + AD = BC + BP + PD$. Hence, $AP + AD - PD = BC + BP - PC$. Therefore, $PX = PX'$, as desired.

4. A number of robots are placed on the squares of a finite, rectangular grid of squares. A square can hold any number of robots. Every edge of each square of the grid is classified as either *passable* or *impassable*. All edges on the boundary of the grid are impassable.

You can give any of the commands *up*, *down*, *left*, or *right*. All of the robots then simultaneously try to move in the specified direction. If the edge adjacent to a robot in that direction is passable, the robot moves across the edge and into the next square. Otherwise, the robot remains on its current square. You can then give another command of *up*, *down*, *left*, or *right*, then another, for as long as you want.

Suppose that for any individual robot, and any square on the grid, there is a finite sequence of commands that will move that robot to that square. Prove that you can also give a finite sequence of commands such that *all* of the robots end up on the same square at the same time.

Solution. We will prove any two robots can be moved to the same square. From that point on, they will always be on the same square. We can then similarly move a third robot onto the same square as these two, and then a fourth, and so on, until all robots are on the same square.

Towards that end, consider two robots A and B. Let $d(A, B)$ denote the minimum number of commands that need to be given in order to move A to the square on which B is *currently* standing. We will give a procedure that is guaranteed to decrease $d(A, B)$. Since $d(A, B)$ is a non-negative integer, this procedure will eventually decrease n to 0, which finishes the proof.

Let $n = d(A, B)$, and let $S = \{s_1, s_2, \ldots, s_n\}$ be a minimum sequence of moves that takes A to the square where B is currently standing. Certainly A will not run into an impassable edge during this sequence, or we could get a shorter sequence by removing that command. Now suppose B runs into an impassable edge after some command s_i. From that point, we can get A to the square on which B started with the commands $s_{i+1}, s_{i+2}, \ldots, s_n$ and then to the square where B is currently with the commands $s_1, s_2, \ldots, s_{i-1}$. But this was only $n - 1$ commands in total, and so we have decreased $d(A, B)$ as required.

Otherwise, we have given a sequence of n commands to A and B, and neither ran into an impassable edge during the execution of these commands. In particular, the vector v connecting A to B on the grid must have never changed. We moved A to the position $B = A + v$, and therefore we must have also moved B to $B + v$. Repeating this process k times, we will move A to $A + kv$ and B to $B + kv$. But if $v \neq (0, 0)$, this will eventually force B off the edge of the grid, giving a contradiction.

5. A bookshelf contains n volumes, labelled 1 to n, in some order. The librarian wishes to put them in the correct order as follows. The librarian selects a volume that is too far to the right, say the volume with label k, takes it out, and inserts it in the k-th position. For example, if the bookshelf contains the volumes 1, 3, 2, 4 in that order, the librarian could take out volume 2 and place it in the second position. The books will then be in the correct order 1, 2, 3, 4.

(a) Show that if this process is repeated, then, however the librarian makes the selections, all the volumes will eventually be in the correct order.
(b) What is the largest number of steps that this process can take?

Solution. (a) If t_k is the number of times that volume k is selected, then we have $t_k \leq 1 + (t_1 + t_2 + \cdots + t_{k-1})$. This is because volume k must move to the right between selections, which means some volume was placed to its left. The only way that can happen is if a lower-numbered volume was selected. This leads to the bound $t_k \leq 2^{k-1}$. Furthermore, $t_n = 0$ since the nth volume will never be too far to the right. Therefore if N is the total number of moves then

$$N = t_1 + t_2 + \cdots + t_{n-1} \leq 1 + 2 + \cdots + 2^{n-2} = 2^{n-1} - 1,$$

and in particular the process terminates.

(b) Conversely, $2^{n-1}-1$ moves are required for the configuration $(n, 1, 2, 3, \ldots, n-1)$ if the librarian picks the rightmost eligible volume each time.

This can be proved by induction: if at a certain stage we are at $(x, n-k, n-k+1, \ldots, n-1)$, then after $2^k - 1$ moves, we will have moved to $(n-k, n-k+1, \ldots, n-1, x)$ without touching any of the volumes further to the left. Indeed, after $2^{k-1} - 1$ moves, we get to $(x, n-k+1, n-k+2, \ldots, n-1, n-k)$, which becomes $(n-k, x, n-k+1, n-k+2, \ldots, n-1)$ after 1 more move, and then $(n-k, n-k+1, \ldots, n-1, x)$ after another $2^{k-1} - 1$ moves. The result follows by taking $k = n-1$.