Lossless estimates for asymptotic methods with applications to propagation features for dispersive equations

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Motivation

- A model: the free Schrödinger equation on the line:

$$(S) \qquad \begin{cases} \left[i\partial_t + \partial_{xx} \right] u(t) = 0 & \forall\, t \geqslant 0 \\ u(0) = u_0 \end{cases} .$$

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- Existing result: If $u_0 \in L^1(\mathbb{R})$ then

$$\left\| u(t,.) \right\|_{L^\infty(\mathbb{R})} \leqslant \frac{\left\| u_0 \right\|_{L^1}}{2\sqrt{\pi}}\, t^{-\frac{1}{2}} .$$

Notion of frequency band

Definition 1

Let $p_1 < p_2$ be two finite real numbers. An initial datum u_0 is in the frequency band $[p_1, p_2]$ if and only if $\mathcal{F}u_0$ is a function which satisfies

$$supp\,\mathcal{F}u_0 \subseteq [p_1, p_2] \,.$$

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- In terms of quantum mechanics, an initial datum $u_0 \in L^2(\mathbb{R})$ in the frequency band $[p_1, p_2]$ means that its initial momentum is localized in this interval ;
- Localization of the initial momentum $\rightsquigarrow$ Description of propagation features.

If u_0 is in a frequency band $[p_1, p_2]$ then the solution formula of the free Schrödinger equation is given by

$$u(t,x) = \frac{1}{2\pi}\int_{p_1}^{p_2} \mathcal{F}u_0(p)\, e^{-itp^2+ixp}\, dp\ .$$

By defining new functions :

$$U(p) := \frac{1}{2\pi}\mathcal{F}u_0(p) \qquad , \qquad \Psi(p,t,x) := -p^2 + \frac{x}{t}\,p\ ,$$

we can rewrite the solution as

$$u(t,x) = \int_{p_1}^{p_2} U(p)\, e^{it\Psi(p,t,x)}\, dp\ . \tag{1}$$

The rewriting (1) is of the form

$$I(\omega) = \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp$$

$\rightsquigarrow$ Oscillatory integral with respect to a large parameter ω, where

- U is the amplitude ;
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Idea: Study the asymptotic behaviour of $I(\omega)$ to derive information on the time-asymptotic behaviour of the solution of the free Schrödinger equation (and other dispersive equations).

Phase depending on parameters

Oscillatory integral :

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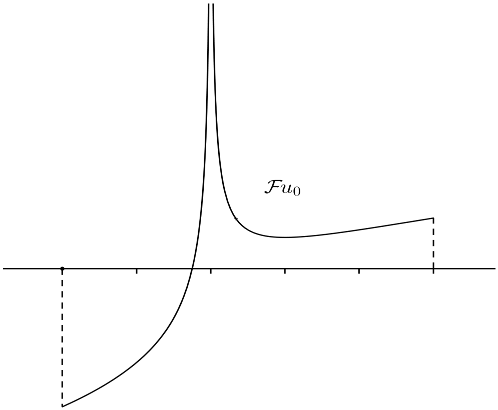
The abstract results on oscillatory integrals remain valid in the case of phases depending on parameters, including the large parameter: the proofs are word-for-word the same.

Singular frequency

Definition 2

A singular frequency is an integrable singular point of $\mathcal{F}u_0$.



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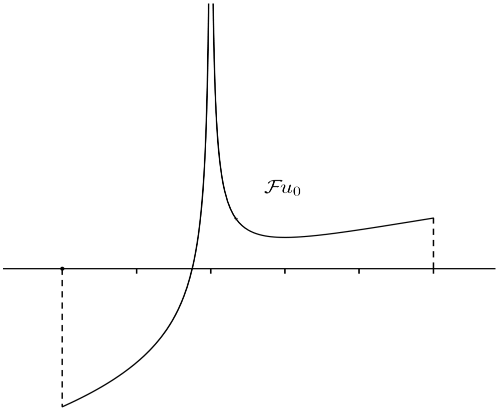

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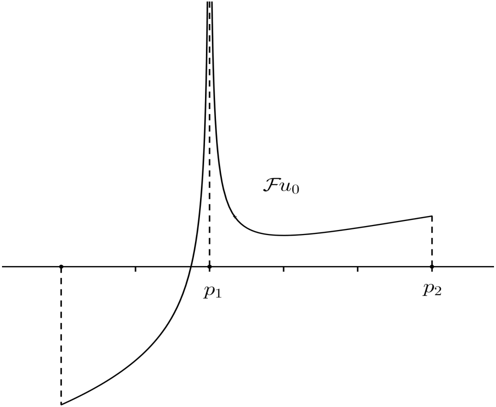

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- The expansion fails when the frequency band approaches the critical frequencies.

$\rightsquigarrow$ These critical values play a similar role as the singular frequencies !

Plan of the thesis defence

1. Explicit error estimates for the stationary phase method in one variable
2. Lossless error estimates and applications to the free Schrödinger equation
3. Optimal van der Corput estimates describing the interaction between a stationary point of the phase and a singularity of the amplitude
4. Applications to evolution equations given by Fourier multipliers covering Schrödinger-type and hyperbolic examples
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Theorem (Hörmander, 1984)

Let $p_1 < p_2$ be two finite real numbers and let X be an open neighborhood of $[p_1, p_2]$. Let $U \in \mathcal{C}_0^2\big([p_1, p_2], \mathbb{C}\big)$ and $\psi \in \mathcal{C}^4(X, \mathbb{C})$ such that $Im\, \psi \geqslant 0$ on X. If there exists $p_0 \in X$ such that $\psi'(p_0) = 0$, $\psi''(p_0) \neq 0$ and $Im\, \psi(p_0) = 0$, $\psi'(p) \neq 0$ on $[p_1, p_2] \backslash \{p_0\}$, then

$$\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp \; - \; \sqrt{2\pi}\, e^{-i\frac{\pi}{4}}\, e^{i\omega\psi(p_0)}\, \frac{U(p_0)}{\sqrt{\psi''(p_0)}}\, \omega^{-\frac{1}{2}} \right| \leqslant C(\psi)\, \|U\|_{\mathcal{C}^2([p_1,p_2])}\, \omega^{-1},$$

for all $\omega > 0$. Moreover $C(\psi) \geqslant 0$ is bounded when ψ stays in a bounded set in $\mathcal{C}^4(X)$ and $|p - p_0| / |\psi'(p)|$ has a uniform bound.

1. The stationary phase method of A. Erdélyi

Choice: Stationary phase method of A. Erdélyi, *Asymptotic expansions*, 1956.

The approach is specific to dimension 1.

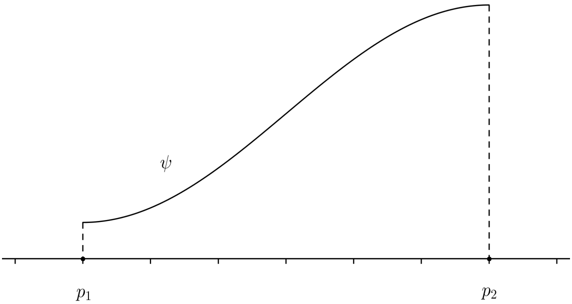
- Precise remainder estimates ;
- Singular amplitudes are allowed ;
- Stationary points of non-integer order can be studied.

We formulate the result of A. Erdélyi in a modern way and we provide a detailed proof.

Assumption $(\mathbf{P1}_{\rho_1,\rho_2,N})$. For $\rho_1, \rho_2 \geqslant 1$ and $N \in \mathbb{N}\backslash\{0\}$, let $\psi \in \mathcal{C}^1\big([p_1, p_2], \mathbb{R}\big)$ be a function satisfying

$$\psi'(p) = (p - p_1)^{\rho_1 - 1}(p_2 - p)^{\rho_2 - 1}\tilde{\psi}(p) \ ,$$

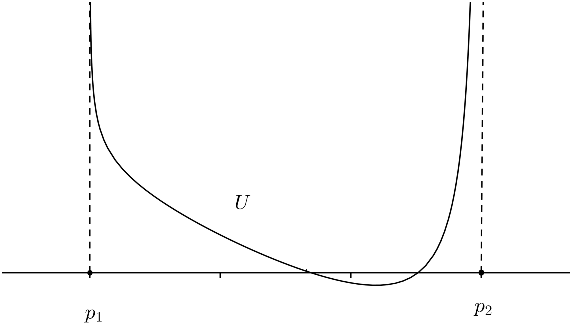
where $\tilde{\psi} \in \mathcal{C}^N\big([p_1, p_2], \mathbb{R}\big)$ is positive. The points p_j $(j = 1, 2)$ are called *stationary points* of ψ of order $\rho_j - 1$.



Assumption (A1$_{\mu_1,\mu_2,N}$). For $\mu_1, \mu_2 \in (0,1]$ and $N \in \mathbb{N}\backslash\{0\}$, let $U : (p_1, p_2) \longrightarrow \mathbb{C}$ be a function defined by

$$U(p) = (p - p_1)^{\mu_1 - 1}(p_2 - p)^{\mu_2 - 1}\tilde{u}(p) \ ,$$

where $\tilde{u} \in \mathcal{C}^N\big([p_1, p_2], \mathbb{C}\big)$ and $\tilde{u}(p_j) \neq 0$ if $\mu_j \neq 1$ $(j = 1, 2)$. The points p_j are called *singular points* of U.



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Let $N \in \mathbb{N}\backslash\{0\}$, let $\rho_1, \rho_2 \geqslant 1$ and $\mu_1, \mu_2 \in (0,1)$. Suppose that the functions $\psi : [p_1, p_2] \longrightarrow \mathbb{R}$ and $U : (p_1, p_2) \longrightarrow \mathbb{C}$ satisfy Assumption $(P1_{\rho_1,\rho_2,N})$ and Assumption $(A1_{\mu_1,\mu_2,N})$, respectively. Then we have

$$\begin{cases} \displaystyle\int_{p_1}^{p_2} U(p) e^{i\omega\psi(p)}\, dp = \sum_{j=1,2} \Big(A_N^{(j)}(\omega) + R_N^{(j)}(\omega) \Big) , \\ \displaystyle\Big| R_N^{(j)}(\omega) \Big| \leqslant \frac{1}{(N-1)!} \frac{1}{\rho_j} \Gamma\left(\frac{N}{\rho_j}\right) \int_0^{s_j} s^{\mu_j - 1} \left| \frac{d^N}{ds^N}[\nu_j k_j](s) \right| ds \, \omega^{-\frac{N}{\rho_j}} , \end{cases}$$

for all $\omega > 0$, where

- $\displaystyle A_N^{(j)}(\omega) := e^{i\omega\psi(p_j)} \sum_{n=0}^{N-1} \Theta_{n+1}^{(j)}(\rho_j, \mu_j) \frac{d^n}{ds^n}[k_j](0)\, \omega^{-\frac{n+\mu_j}{\rho_j}} ,$
- $\displaystyle R_N^{(j)}(\omega) := (-1)^{N+1+j} e^{i\omega\psi(p_j)} \int_0^{s_j} \phi_N^{(j)}(s, \omega, \rho_j, \mu_j) \frac{d^N}{ds^N}[\nu_j k_j](s)\, ds .$

The undefined items are explained in the following.

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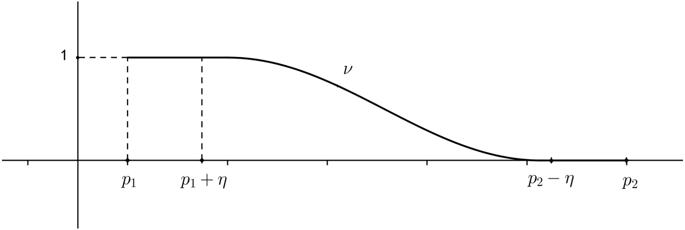
The undefined items are explained in the following.

1. Splitting of the integral

For this purpose, we use a smooth cut-off function $\nu : [p_1, p_2] \longrightarrow \mathbb{R}$ satisfying

$$\begin{cases} \nu = 1 & \text{on} \quad [p_1, p_1 + \eta] \,, \\ \nu = 0 & \text{on} \quad [p_2 - \eta, p_2] \,, \\ 0 \leqslant \nu \leqslant 1 \,, \end{cases}$$

where $\eta \in \left(0, \frac{p_2 - p_1}{2}\right)$ is fixed.



2. Substitution

We carry out different substitutions in each resulting integral to simplify the phase. To do so, we employ the explicit $\mathcal{C}^{N+1}$-diffeomorphisms *(not proved in the source)* :

- $$\varphi_j : p \in I_j \longmapsto \Big((-1)^j \big(\psi(p_j) - \psi(p) \big) \Big)^{\frac{1}{\rho_j}} \in [0, s_j]$$

with $I_1 := [p_1, p_2 - \eta]$, $I_2 := [p_1 + \eta, p_2]$, $s_1 := \varphi_1(p_2 - \eta)$ and $s_2 := \varphi_2(p_1 + \eta)$.

This creates regular functions related to the amplitude :

- $$k_j : s \in [0, s_j] \longmapsto U\big(\varphi_j^{-1}(s)\big)\, s^{1-\mu_j} \left(\varphi_j^{-1}\right)'(s) \in \mathbb{C}$$

and the cut-off functions become

- $$\nu_1 : s \in [0, s_1] \longmapsto \nu \circ \varphi_1^{-1}(s) \in \mathbb{R} \; ;$$
- $$\nu_2 : s \in [0, s_2] \longmapsto (1 - \nu) \circ \varphi_2^{-1}(s) \in \mathbb{R} \; .$$

3. Integration by parts

We integrate by parts to create the expansion.

The n-th primitive on $[0, s_j]$ of $s \longmapsto s^{\mu_j - 1} e^{(-1)^{j+1} i \omega s^{\rho_j}}$ is given by

- $$\phi_n^{(j)}(s, \omega, \rho_j, \mu_j) := \frac{(-1)^n}{(n-1)!} \int_{\Lambda^{(j)}(s)} (z-s)^{n-1}\, z^{\mu_j - 1}\, e^{(-1)^{j+1} i \omega z^{\rho_j}}\, dz \, .$$

(formula without proof in the source)

The integration path

- $$\Lambda^{(j)}(s) := \left\{ s + t e^{(-1)^{j+1} i \frac{\pi}{2\rho_j}} \in \mathbb{C} \,\middle|\, t \geqslant 0 \right\} \subset \mathbb{C}$$

has been chosen to be contained in a region of controllable oscillations.

4. Remainder estimate

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5. Calculation of the coefficients of the expansion.

- $$\Theta_{n+1}^{(j)}(\rho_j, \mu_j) := \frac{(-1)^{j+1}}{n!\,\rho_j}\,\Gamma\Big(\frac{n+\mu_j}{\rho_j}\Big)\,e^{(-1)^{j+1} i \frac{\pi}{2}\,\frac{n+\mu_j}{\rho_j}}\,.$$

$\square$

Suppose $\mu_1 = 1$, *i.e* p_1 is not a singular point of the amplitude U. Then the highest term of $A_N^{(1)}(\omega)$ behaves like

$$e^{i\omega\psi(p_1)}\,\Theta_N^{(1)}(\rho_1, 1)\,\frac{d^{N-1}}{ds^{N-1}}[k_1](0)\,\omega^{-\frac{N}{\rho_1}} \quad \sim \quad \omega^{-\frac{N}{\rho_1}} \quad , \quad \omega \to +\infty \,.$$

Moreover let us recall that the remainder is estimated as follows :

$$\left| R_N^{(1)}(\omega) \right| \lesssim \omega^{-\frac{N}{\rho_1}} \,.$$

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1. Absence of singularities: lack of precision

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The remedy proposed by Erdélyi leads to complicated formulas when written down and does not seem possible in the case of stationary points of non integer order.

1. Improved remainder estimate in the case of regular amplitude

Theorem 2

Let $N \in \mathbb{N} \backslash \{0\}$ and assume $\mu_j = 1$ and $\rho_j \geqslant 2$ for a certain $j \in \{1, 2\}$. Suppose that the functions $\psi : [p_1, p_2] \longrightarrow \mathbb{R}$ and $U : (p_1, p_2) \longrightarrow \mathbb{C}$ satisfy Assumption $(P1_{\rho_1,\rho_2,N})$ and Assumption $(A1_{\mu_1,\mu_2,N})$, respectively. Then the statement of Theorem 1 is still true and, for $\gamma \in (0, 1)$ and

$$\delta := \frac{\gamma + N}{\rho_j} \in \left(\frac{N}{\rho_j}, \frac{N+1}{\rho_j} \right) ,$$

we have

$$\left| R_N^{(j)}(\omega) \right| \leqslant L_{\gamma,\rho_j,N} \int_0^{s_j} s^{-\gamma} \left| \frac{d^N}{ds^N} [\nu_j k_j](s) \right| ds \; \omega^{-\delta} ,$$

for all $\omega > 0$ and for a certain constant $L_{\gamma,\rho_j,N} > 0$.

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By applying the previous theorems to

$$\int_{p_1}^{p_0} (p - p_1)^{-\frac{1}{4}} \, e^{-i\omega(p-p_0)^2} \, dp \ ,$$

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$$\int_{p_1}^{p_0} (p-p_1)^{-\frac{1}{4}}\, e^{-i\omega(p-p_0)^2}\, dp = \frac{\sqrt{\pi}}{2}\, e^{-i\frac{\pi}{4}}\, (p_0-p_1)^{-\frac{1}{4}}\, \omega^{-\frac{1}{2}}$$

$$+ \frac{\Gamma\left(\frac{3}{4}\right)}{2^{\frac{1}{4}}}\, e^{i\frac{3\pi}{8}}\, e^{-i\omega(p_0-p_1)^2}\, (p_0-p_1)^{-\frac{1}{4}}\, \omega^{-\frac{3}{4}} + R_1^{(1)}(\omega,p_0) + R_1^{(2)}(\omega,p_0)$$

where $\omega > 0$, and we have for $\delta \in \left(\frac{3}{4},1\right)$,

- $$\left|R_1^{(1)}(\omega,p_0)\right| \leqslant \int_0^{\frac{8}{9}(p_0-p_1)^2} s^{-\frac{1}{4}} \left|\left(\nu_1 k_1\right)'(s)\right| ds\, \omega^{-1};$$
- $$\left|R_1^{(2)}(\omega,p_0)\right| \leqslant L_{\gamma,2,1} \int_0^{\frac{2}{3}(p_0-p_1)} s^{-\gamma} \left|\left(\nu_2 k_2\right)'(s)\right| ds\, \omega^{-\delta}.$$

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where $p_0 > p_1$, we obtain

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$$+ \frac{\Gamma\left(\frac{3}{4}\right)}{2^{\frac{1}{4}}} \, e^{i\frac{3\pi}{8}} \, e^{-i\omega(p_0-p_1)^2} \, (\mathbf{p_0} - \mathbf{p_1})^{-\frac{1}{4}} \, \omega^{-\frac{3}{4}} + R_1^{(1)}(\omega, p_0) + R_1^{(2)}(\omega, p_0)$$

where $\omega > 0$, and we have for $\delta \in \left(\frac{3}{4}, 1\right)$,

- $$\left| R_1^{(1)}(\omega, p_0) \right| \leqslant \int_0^{\frac{8}{9}(p_0-p_1)^2} s^{-\frac{1}{4}} \left| \left(\nu_1 k_1\right)'(s) \right| ds \, \omega^{-1};$$
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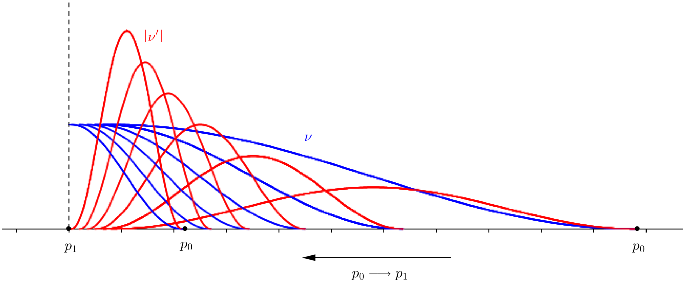
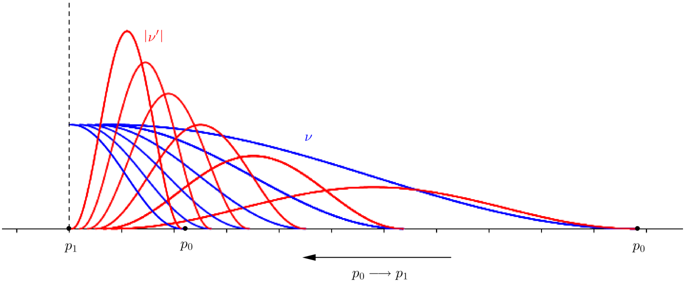
1. Blow-up of the smooth cut-off function





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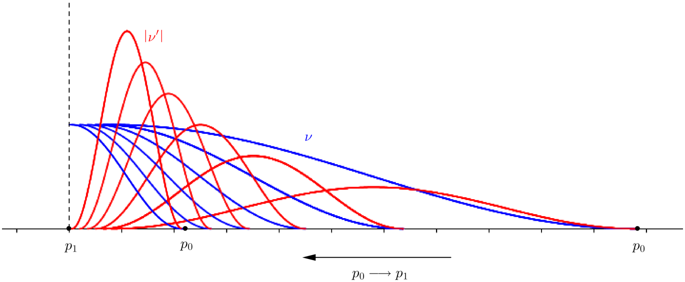

$\rightsquigarrow$ The smooth cut-off function contributes **artificially** to the blow-up of the remainder.

<u>Idea:</u> Improve the result of Erdélyi by replacing the smooth cut-off function with a characteristic function to obtain **lossless** error estimates.

Plan of the thesis defence

1. Explicit error estimates for the stationary phase method in one variable
2. Lossless error estimates and applications to the free Schrödinger equation
3. Optimal van der Corput estimates describing the interaction between a stationary point of the phase and a singularity of the amplitude
4. Applications to evolution equations given by Fourier multipliers covering Schrödinger-type and hyperbolic examples
5. Time asymptotic behaviour of approximate solutions of Schrödinger equations with both potential and initial condition in frequency bands

Theorem 3

Let $\rho_1, \rho_2 \geqslant 1$ and $\mu_1, \mu_2 \in (0,1)$. Suppose that $\psi : [p_1, p_2] \longrightarrow \mathbb{R}$ and $U : (p_1, p_2) \longrightarrow \mathbb{C}$ satisfy Assumption ($P1_{\rho_1,\rho_2,1}$) and Assumption ($A1_{\mu_1,\mu_2,1}$), respectively. Then we have

$$\begin{cases} \displaystyle\int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp = \sum_{j=1,2} \Big(A^{(j)}(\omega) + R_1^{(j)}(\omega, q) + R_2^{(j)}(\omega, q) \Big)\ , \\ \displaystyle\Big| R_1^{(j)}(\omega, q) \Big| \leqslant \frac{1}{\rho_j}\, \Gamma\Big(\frac{1}{\rho_j}\Big) \int_0^{s_j} s^{\mu_j - 1} \big| (k_j)'(s) \big|\, ds\ \omega^{-\frac{1}{\rho_j}}\ , \\ \displaystyle\Big| R_2^{(j)}(\omega, q) \Big| \leqslant \frac{\rho_j - \mu_j}{\rho_j}\, \Gamma\Big(\frac{1}{\rho_j}\Big) \Big| U(q)\, (\varphi_j)'(q)^{-1} \Big| \varphi_j(q)^{-\rho_j}\ \omega^{-\left(1 + \frac{1}{\rho_j}\right)}\ , \end{cases}$$

for all $\omega > 0$ and for a fixed $q \in (p_1, p_2)$, where

- $A^{(j)}(\omega) := e^{i\omega\psi(p_j)}\, k_j(0)\, \Theta^{(j)}(\rho_j, \mu_j)\, \omega^{-\frac{\mu_j}{\rho_j}}$;
- $R_1^{(j)}(\omega, q) := (-1)^j\, e^{i\omega\psi(p_j)} \displaystyle\int_0^{s_j} \phi^{(j)}(s, \omega, \rho_j, \mu_j)\, (k_j)'(s)\, ds$;
- $R_2^{(j)}(\omega, q) := (-1)^j\, i\, \dfrac{\mu_j - \rho_j}{\rho_j}\, e^{i\omega\psi(p_j)}\, k_j(s_j) \displaystyle\int_{\Lambda^{(j)}(s_j)} z^{\mu_j - \rho_j - 1} e^{(-1)^{j+1} i\omega z^{\rho_j}}\, dz\ \omega^{-1}$.

2. Stationary phase method without smooth cut-off function

Remarks:

- The notations are the same as those of Theorem 1 ;

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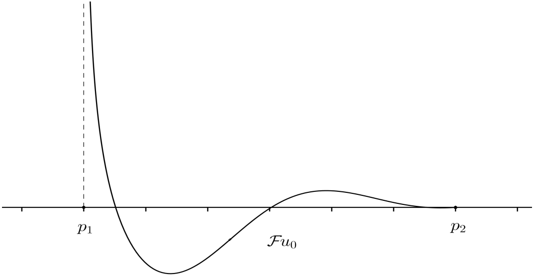
- This theorem permits in the applications to obtain the explicit dependence of the remainder on the distance between the stationary point of the phase and the singular point of the amplitude.

2. Free Schrödinger equation and hypotheses on the initial data

We consider now the free Schrödinger equation on the line:

$$(S) \qquad \begin{cases} \left[i\partial_t + \partial_{xx}\right] u(t) = 0 & \forall\, t \geqslant 0 \\ u(0) = u_0 \end{cases} .$$

Condition ($\mathbf{C1}_{[p_1,p_2],\mu}$). Fix $\mu \in (0,1)$ and let $p_1 < p_2$ be two finite real numbers. A tempered distribution u_0 on $\mathbb{R}$ satisfies Condition ($\mathrm{C1}_{[p_1,p_2],\mu}$) if and only if $\mathcal{F}u_0 \equiv 0$ on $\mathbb{R}\backslash[p_1,p_2]$ and $\mathcal{F}u_0$ verifies Assumption ($\mathrm{A1}_{\mu,1,1}$) on $[p_1,p_2]$, with $\mathcal{F}u_0(p_2) = 0$.



2. Exploit the potential of refinement

We recall that the solution formula of (S) is given by

$$u(t,x) = \frac{1}{2\pi} \int_{p_1}^{p_2} \mathcal{F}u_0(p)\, e^{-itp^2+ixp}\, dp \ ,$$

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Possibilities:

- Provide uniform remainder estimates in parameter regions where the stationary point p_0 is far from the singularity p_1, e.g

$$p_1 + \varepsilon < p_0 < p_2 \ .$$

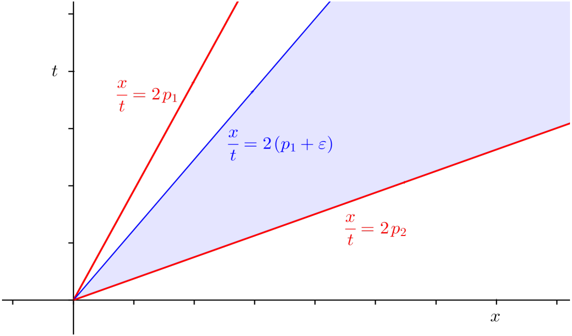
2. The condition $p_1 + \varepsilon < p_0 < p_2$ in terms of t and x

Let $p_1 < p_2$ be two finite real numbers and fix $\varepsilon > 0$ such that

$$p_1 + \varepsilon < p_2 \, .$$

Then we define the cone $\mathfrak{C}_S\big(p_1 + \varepsilon, p_2\big)$ as follows :

$$\mathfrak{C}_S\big(p_1 + \varepsilon, p_2\big) := \left\{ (t, x) \in \mathbb{R}_+^* \times \mathbb{R} \;\middle|\; 2(p_1 + \varepsilon) < \frac{x}{t} < 2p_2 \right\} \, .$$



Theorem 4

Suppose that u_0 *satisfies Condition* $(C1_{[p_1,p_2],\mu})$ *and fix* $\varepsilon > 0$ *such that* $p_1 + \varepsilon < p_2$.

Then for all $(t,x) \in \mathfrak{C}_S\big(p_1+\varepsilon, p_2\big)$, *there exist two complex numbers* $H(t,x,u_0)$ *and* $K_\mu(t,x,u_0)$ *satisfying*

$$\left| u(t,x) - H(t,x,u_0)\, t^{-\frac{1}{2}} - K_\mu(t,x,u_0)\, t^{-\mu} \right| \leqslant \sum_{k=1}^{8} R_k(u_0,\varepsilon)\, t^{-\alpha_k(\mu)} \,,$$

where $R_k(u_0,\varepsilon) \geqslant 0$ $(k = 1, \dots 8)$ *are constants independent from* t *and* x, *and* $\alpha_k(\mu) > \max\left\{\mu, \dfrac{1}{2}\right\}$.

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where $\tilde{R}_k(u_0) \geqslant 0$ $(k = 1, \ldots 8)$ are constants independent from t and x, $\alpha_k(\mu) > \max\left\{\mu, \frac{1}{2}\right\}$ and $\beta_k(\mu) \geqslant 0$.

Remarks:

- $H(t,x,u_0) = \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, \tilde{u}\left(\frac{x}{2t}\right) \left(\frac{x}{2t} - p_1\right)^{\mu-1}$;
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- F. Ali Mehmeti, R. Haller-Dintelmann, V. Régnier, *The Influence of the Tunnel Effect on the L^∞-time Decay*. Oper. Theory Adv. Appl. **221** (2012), 11-24 ;

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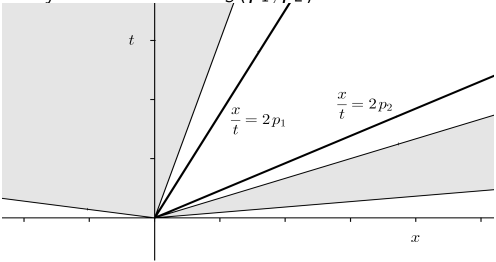
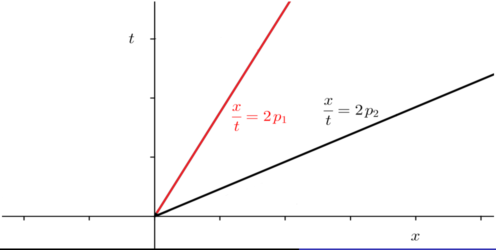
By a similar work, we obtain asymptotic expansions with uniform remainder estimates in arbitrary cones outside $\mathfrak{C}_S(p_1, p_2)$:



Moreover, we expand the solution on the direction $\frac{x}{2t} = p_1$:



2. Outside $\mathfrak{C}_S(p_1, p_2)$

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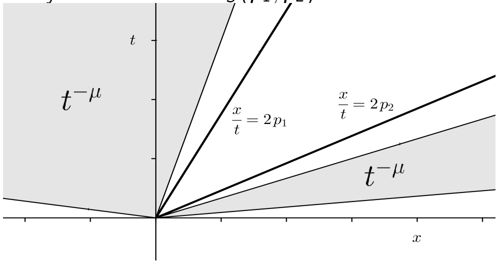

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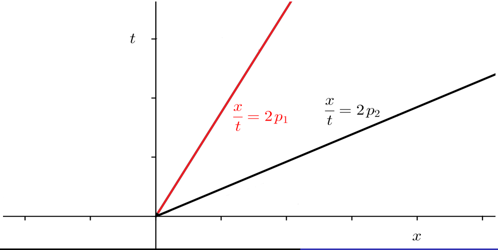

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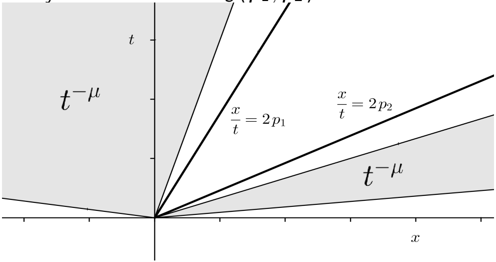

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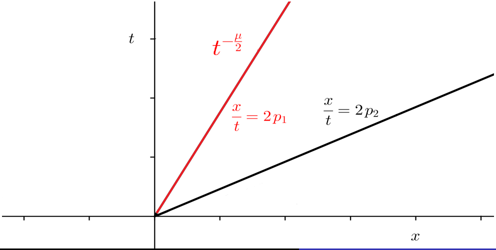
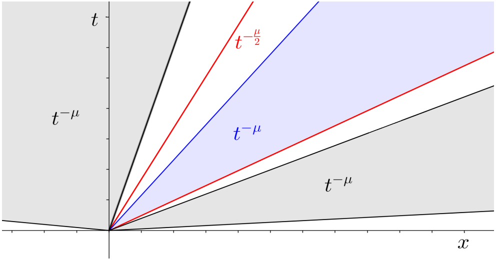
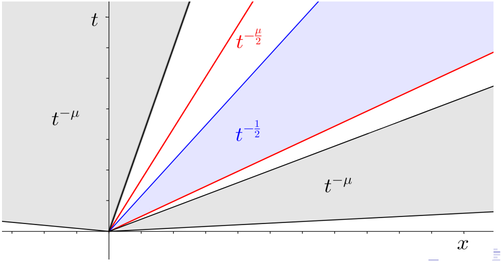


2. Summary

- Case $\mu \in \left(0, \frac{1}{2}\right]$:



- Case $\mu \in \left(\frac{1}{2}, 1\right)$: *(physical case)*



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which is of the form

$$\int_{p_1}^{p_2} U(p)\, e^{i\omega\left(-(p-p_0)^2+p_0^2\right)}\, dp\ .$$

Possibilities:

- Provide uniform remainder estimates in parameter regions where the stationary point p_0 is far from the singularity p_1, e.g

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2. Exploit the potential of refinement

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<u>Possibilities:</u>

- Provide uniform remainder estimates in parameter regions where the stationary point p_0 is far from the singularity p_1, e.g

$$p_1+\varepsilon < p_0 < p_2\ .$$

- Establish uniform estimates in parameter regions approaching p_1 along curves parametrized by ω, e.g

$$p_1+\omega^{-\vartheta} \leqslant p_0 < p_2\ .$$

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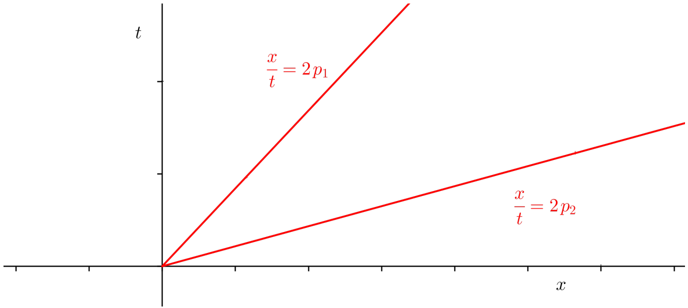
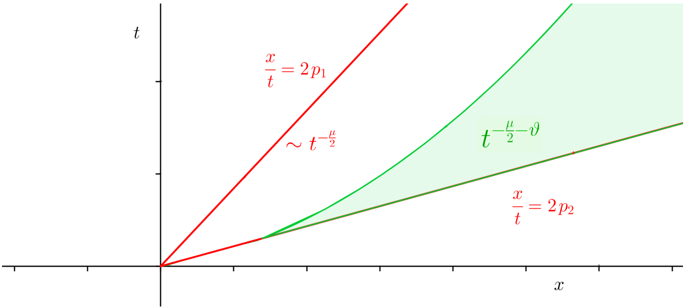
- Show the optimality of the previous estimates by expanding the integral on the curves given by

$$p_0 = p_1+\omega^{-\vartheta} \ .$$



t
$\frac{x}{t} = 2p_1$
$\frac{x}{t} = 2p_2$
x


t
$\frac{x}{t} = 2p_1$
$\sim t^{-\frac{\mu}{2}}$
$t^{-\frac{\mu}{2}-\vartheta}$
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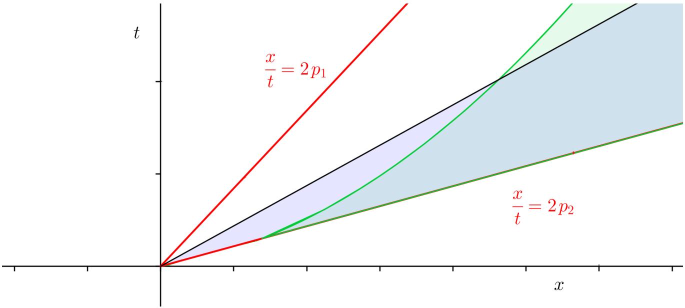
x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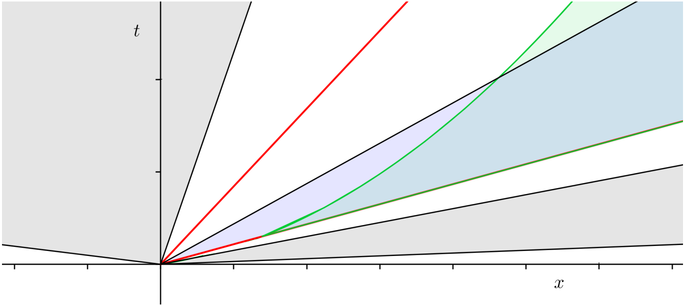
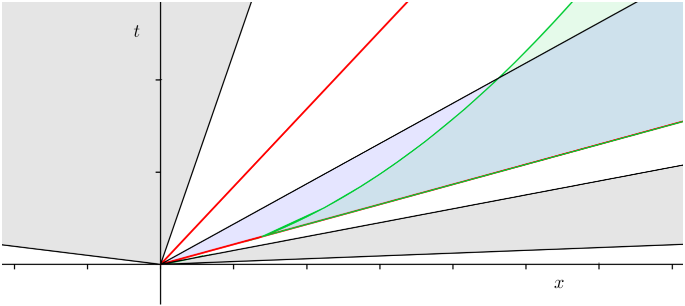
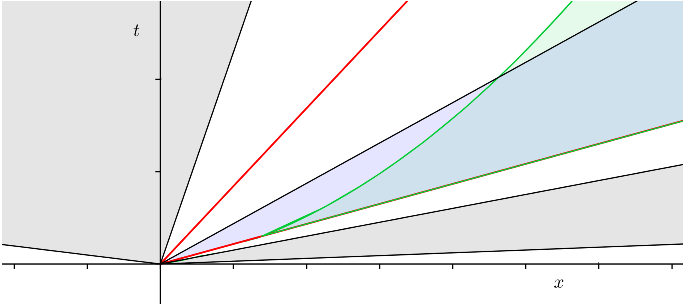
x

2. Summary and observation





$\rightsquigarrow$ Existence of forbidden regions (in white here) due to the **inherent** blow-up of the asymptotic expansions.



⇝ Existence of forbidden regions (in white here) due to the **inherent** blow-up of the asymptotic expansions.

Strategy: Give up the idea to control the forbidden regions by expansions to one term in favour of uniform estimates with respect to the position of the stationary point.

⇝ L^∞-norm estimates of the solution which permit to control the decay of the solution, even in the forbidden regions.

Plan of the thesis defence

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3. Van der Corput lemma

Theorem (Stein, 1993)

Let $\psi : [p_1, p_2] \longrightarrow \mathbb{R}$ *and* $U : [p_1, p_2] \longrightarrow \mathbb{C}$ *be two smooth functions. Suppose that* $\left|\psi^{(k)}\right| \geqslant 1$ *on* $[p_1, p_2]$*. Then*

$$\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp \right| \leqslant (5 \times 2^{k-1} - 2) \Big(\|U\|_{L^\infty(p_1,p_2)} + \|U'\|_{L^1(p_1,p_2)} \Big)\, \omega^{-\frac{1}{k}} \ ,$$

holds when

1. $k \geqslant 2$, *or*
2. $k = 1$ *and* ψ' *is monotonic.*

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- T. Alazard, N. Burq, C. Zuily, *A stationary phase type method. To appear in Proc. Amer. Math. Soc.* arXiv:1511.01439v1 [math.AP] (2015) ;
- M. Ruzhansky, *Multidimensional decay in van der Corput lemma*. Studia Math. **208** (2012), 1-10.

Let $p_1 < p_2$ be two finite real numbers let I be an open interval such that $[p_1, p_2] \subset I$.

Assumption $(\mathrm{P2}_{p_0,\rho})$. Let $p_0 \in I$ and $\rho > 1$. A function $\psi : I \longrightarrow \mathbb{R}$ satisfies Assumption $(\mathrm{P2}_{p_0,\rho})$ if and only if $\psi \in \mathcal{C}^1(I) \cap \mathcal{C}^2(I \backslash \{p_0\})$ and there exists a function $\tilde{\psi} : I \longrightarrow \mathbb{R}$ such that

$$\forall\, p \in I \qquad \psi'(p) = |p - p_0|^{\rho - 1}\, \tilde{\psi}(p)\ ,$$

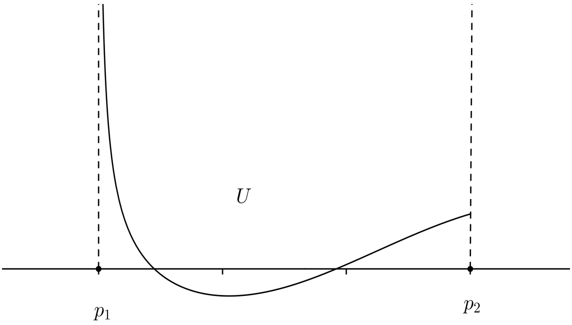
where $\left|\tilde{\psi}\right| : I \longrightarrow \mathbb{R}$ is assumed continuous and does not vanish on I.

Example: Let $\psi \in \mathcal{C}^N(I)$ for a certain $N \geqslant 2$, and let $p_0 \in I$. Suppose that $\psi^{(k)}(p_0) = 0$ for $k = 1, \ldots, N-1$, and $\left|\psi^{(N)}\right| > 0$ on I. Then ψ satisfies Assumption $(\mathrm{P2}_{p_0,N})$.

Assumption ($\mathbf{A2}_{p_1,\mu}$). Let $\mu \in (0,1]$. A function $U : (p_1, p_2] \longrightarrow \mathbb{C}$ satisfies Assumption ($\mathrm{A2}_{p_1,\mu}$) if and only if there exists a function $\tilde{u} : [p_1, p_2] \longrightarrow \mathbb{C}$ such that

$$\forall\, p \in (p_1, p_2] \qquad U(p) = (p - p_1)^{\mu-1}\, \tilde{u}(p)\ ,$$

where $\tilde{u} \in \mathcal{C}^1\big([p_1, p_2], \mathbb{C}\big)$ and $\tilde{u}(p_1) \neq 0$ if $\mu \neq 1$.



3. Van der Corput lemma with singular amplitude and stationary point of real order

Theorem 5

Let $\rho > 1$, $\mu \in (0,1]$ and choose $p_0 \in I$. Suppose that the functions $\psi : I \longrightarrow \mathbb{R}$ and $U : (p_1, p_2] \longrightarrow \mathbb{C}$ satisfy Assumption $(P2_{p_0,\rho})$ and Assumption $(A2_{p_1,\mu})$, respectively. Moreover suppose that ψ' is monotone on the intervals $\{p \in I \,|\, p < p_0\}$ and $\{p \in I \,|\, p > p_0\}$. Then

$$\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp \right| \leqslant C(U,\psi)\, \omega^{-\frac{\mu}{\rho}} \ ,$$

for all $\omega > 0$, where the constant $C(U,\psi) > 0$ is given by

$$C(U,\psi) := \frac{3}{\mu}\, \|\tilde{u}\|_{L^\infty(p_1,p_2)} + \left(8\, \|\tilde{u}\|_{L^\infty(p_1,p_2)} + 2\, \left\|\tilde{u}'\right\|_{L^1(p_1,p_2)} \right) \left(\min_{p \in [p_1,p_2]} \left|\tilde{\psi}(p)\right| \right)^{-1} .$$

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Proof: classical methods (Stein,Zygmund) + factorization hypotheses (Erdélyi).

3. Absence of a stationary point: uniformity versus optimal decay rate

Theorem 6

Let $\mu \in (0,1]$. Suppose that the function $U : (p_1, p_2] \longrightarrow \mathbb{C}$ satisfies Assumption $(A2_{p_1,\mu})$. Moreover suppose that $\psi : I \longrightarrow \mathbb{R}$ belongs to $\mathcal{C}^2([p_1, p_2])$, and that ψ' does not vanish and is monotone on $[p_1, p_2]$. Then

$$\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp \right| \leqslant C_c(U,\psi)\, \omega^{-\mu} ,$$

for all $\omega > 0$, where the constant $C_c(U,\psi) > 0$ is given by

$$C_c(U,\psi) := \frac{1}{\mu} \left\| \tilde{u} \right\|_{L^\infty(p_1,p_2)} + \Big(4 \left\| \tilde{u} \right\|_{L^\infty(p_1,p_2)} + \left\| \tilde{u}' \right\|_{L^1(p_1,p_2)} \Big) \left(\min_{p \in [p_1,p_2]} \left| \psi'(p) \right| \right)^{-1} .$$

The decay rate $\omega^{-\mu}$ is attained for a suitable choice of ψ and U.

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Now we consider the following class of dispersive equations on the line:

$$(S_f) \qquad \left\{ \begin{array}{ll} \Big[i\,\partial_t - f\big(D\big) \Big] u(t) = 0 & \forall\, t \geqslant 0 \\ u(0) = u_0 & \end{array} \right. ,$$

where the symbol f satisfies $f'' > 0$.

In particular, the free Schrödinger equation belongs to the above class because its symbol f_S is given by $f_S(p) = p^2$.

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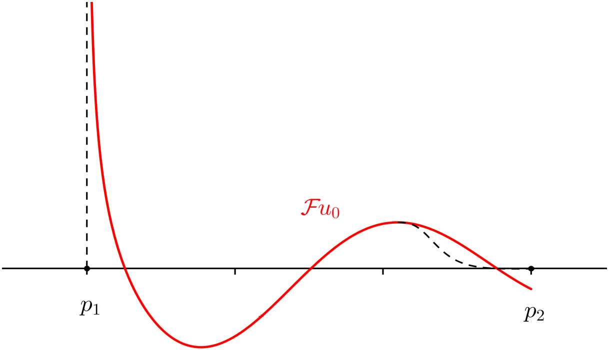
- M. Ben Artzi, F. Treves, *Uniform Estimates for a Class of Dispersive Equations*. J. Funct. Anal. **120** (1994), 264-299.

Condition ($\mathbf{C2}_{[p_1,p_2],\mu}$). Fix $\mu \in (0,1]$ and let $p_1 < p_2$ be two finite real numbers. A tempered distribution u_0 on $\mathbb{R}$ satisfies Condition ($C2_{[p_1,p_2],\mu}$) if and only if $\mathcal{F}u_0 \equiv 0$ on $\mathbb{R}\backslash[p_1,p_2]$ and $\mathcal{F}u_0$ verifies Assumption ($A2_{p_1,\mu}$) on $[p_1,p_2]$.



Theorem 7

Suppose that u_0 satisfies Condition $(C2_{[p_1,p_2],\mu})$ and choose two finite real numbers $\tilde{p}_1 < \tilde{p}_2$ such that $[p_1, p_2] \subset (\tilde{p}_1, \tilde{p}_2) =: \tilde{I}$. Then we have

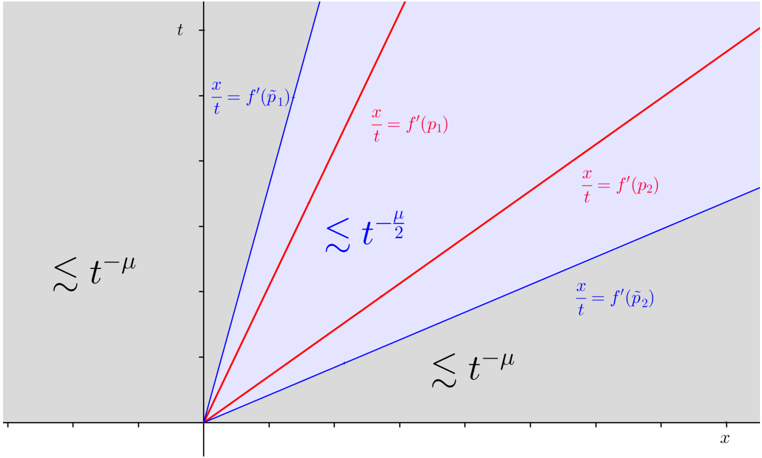
- $\forall\,(t,x) \in \mathfrak{C}\big(f'(\tilde{p}_1), f'(\tilde{p}_2)\big) \qquad \big|u(t,x)\big| \leqslant c(u_0, f)\, t^{-\frac{\mu}{2}}$,
- $\forall\,(t,x) \in \mathfrak{C}\big(f'(\tilde{p}_1), f'(\tilde{p}_2)\big)^c \qquad \big|u(t,x)\big| \leqslant c_{\tilde{I}}(u_0, f)\, t^{-\mu}$.

The constants $c(u_0, f), c_{\tilde{I}}(u_0, f) \geqslant 0$ are independent from t and x, and the two decay rates are optimal.

Remark:

$$\mathfrak{C}(\alpha, \beta) := \left\{ (t,x) \in \mathbb{R}_+^* \times \mathbb{R} \;\middle|\; \alpha < \frac{x}{t} < \beta \right\} .$$

Illustration:



4. Particular case: the free Schrödinger equation with singular frequency

Theorem 8

Suppose that u_0 satisfies Condition $(C2_{[p_1,p_2],\mu})$. Then the solution $u_S : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{C}$ of the free Schrödinger equation (S) on $\mathbb{R}$ satisfies

$$\forall\, t \geqslant 1 \qquad \big\| u_S(t,.) \big\|_{L^\infty(\mathbb{R})} \leqslant c_S(u_0)\, t^{-\frac{\mu}{2}} \ .$$

The constant $c_S(u_0) \geqslant 0$ is independent from t and x, and the decay rate is optimal.

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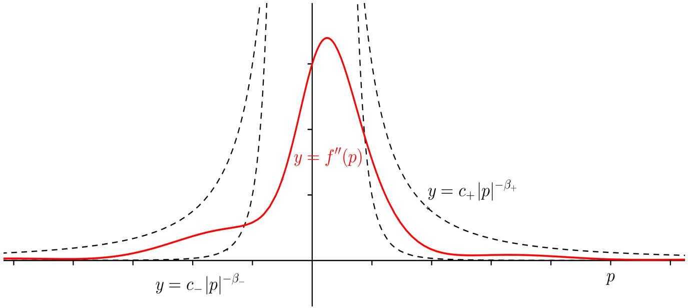
- T. Cazenave, F.B. Weissler , *Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations*. Math. Z. **228** (1998), 83-120.
- T. Cazenave, J. Xie, L. Zhang, *A note on decay rates for Schrödinger's equation*. Proc. Amer. Math. Soc. **138** (2010) no. 1, 199-207.

Condition ($\mathbf{S}_{\beta_+,\beta_-,R}$). Fix $\beta_- \geqslant \beta_+ > 1$ and $R \geqslant 1$.
A $\mathcal{C}^\infty$-function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies Condition ($\mathrm{S}_{\beta_+,\beta_-,R}$) if and only if the second derivative of f verifies

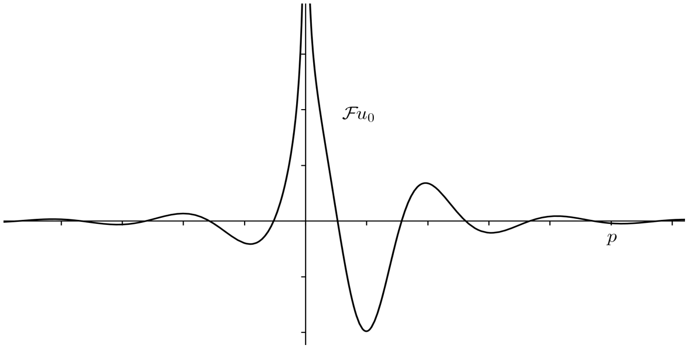
$$\exists\, c_+ \geqslant c_- > 0 \qquad \forall\, |p| \geqslant R \qquad c_- \, |p|^{-\beta_-} \leqslant f''(p) \leqslant c_+ \, |p|^{-\beta_+} \ .$$



Condition (C3$_\mu$). Fix $\mu \in (0, 1]$. A tempered distribution u_0 on $\mathbb{R}$ satisfies Condition (C3$_\mu$) if and only if there exists a $\mathcal{C}^1$-function $\tilde{u} : \mathbb{R} \longrightarrow \mathbb{C}$ such that

$$\forall\, p \in \mathbb{R} \setminus \{0\} \qquad \mathcal{F}u_0(p) = |p|^{\mu-1}\, \tilde{u}(p) \ ,$$

where $\tilde{u}$ tends *sufficiently fast* to 0 at $\pm\infty$, and $\tilde{u}(0) \neq 0$ if $\mu \neq 1$.



Theorem 9

Suppose that the symbol f satisfies Condition ($S_{\beta_+,\beta_-,R}$) and that the initial datum u_0 satisfies Condition ($C3_\mu$), where $\mu \in (0,1]$. Then we have

- $\forall\,(t,x) \in \mathfrak{C}(a,b) \qquad \big|u(t,x)\big| \leqslant c^{(1)}(u_0,f)\, t^{-\frac{\mu}{2}} + c^{(2)}(u_0,f)\, t^{-\frac{1}{2}}$,
- $\forall\,(t,x) \in \mathfrak{C}(a,b)^c \qquad \big|u(t,x)\big| \leqslant c_c^{(1)}(u_0,f)\, t^{-\mu} + c_c^{(2)}(u_0,f)\, t^{-1}$,

All the constants are independent from t and x, and the two finite real numbers $a < b$ are defined by

$$a := \lim_{p\to-\infty} f'(p) \qquad , \qquad b := \lim_{p\to+\infty} f'(p) \,.$$

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- $\forall\,(t,x) \in \mathfrak{C}(a,b)^c \qquad \big|u(t,x)\big| \leqslant c_c^{(1)}(u_0,f)\,t^{-\mu} + c_c^{(2)}(u_0,f)\,t^{-1}$,

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- Application to the Klein-Gordon equation on the line, leading a time-asymptotic concentration of the wave packets in the light-cone issued by the origin.

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5. Schrödinger equation with potential

Finally we consider the Schrödinger equation with potential on the line,

$$(SP) \qquad \begin{cases} i\,\partial_t u(t) = -\partial_{xx} u(t) + V(x)u(t) \\ u(0) = u_0 \end{cases},$$

for $t \geqslant 0$, where $V \in W^{1,\infty}(\mathbb{R})$.

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for $t \geqslant 0$, where $V \in W^{1,\infty}(\mathbb{R})$.

For $u_0 \in H^2(\mathbb{R})$, this equation has a unique solution $u \in \mathcal{C}^1\big(\mathbb{R}_+, H^1(\mathbb{R})\big)$ which can be represented as a series, namely,

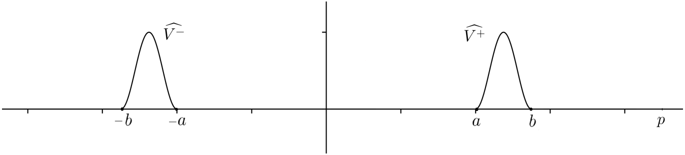
$$\forall\, t \geqslant 0 \qquad \lim_{N \to +\infty} \left\| u(t) - \sum_{n=0}^{N} S_n(t) u_0 \right\|_{H^1(\mathbb{R})} = 0 ,$$

called Dyson-Philips series, where

$$\begin{cases} S_0(t)u_0 := \mathcal{F}^{-1}\Big(e^{-it\,\cdot^2}\,\widehat{u_0}\Big) , \\ S_{n+1}(t)u_0 := -i\displaystyle\int_0^t S_n(t-s)\,V(x)\,S_0(s)u_0\,ds , \quad \forall\, n \in \mathbb{N} . \end{cases}$$

Condition ($\mathcal{P}_{a,b}$). Let $a < b$ be two finite positive real numbers. An element V of $L^2(\mathbb{R})$ satisfies Condition ($\mathcal{P}_{a,b}$) if and only if $\widehat{V}$ is an even real-valued $\mathcal{C}^1$-function on $\mathbb{R}$ which verifies

$$supp\ \widehat{V} \subseteq [-b,-a] \cup [a,b]\ .$$



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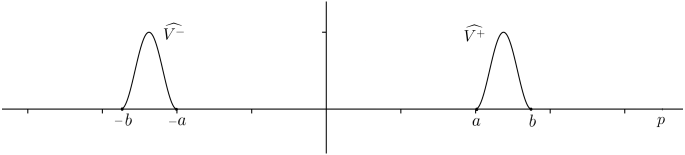


$$V = \mathcal{F}^{-1}\left(\chi_{[-b,-a]}\widehat{V}\right) + \mathcal{F}^{-1}\left(\chi_{[a,b]}\widehat{V}\right) =: V^- + V^+$$

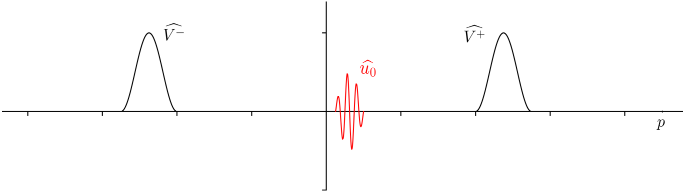
$$\Longrightarrow \qquad S_1(t)u_0 = S_1^-(t)u_0 + S_1^+(t)u_0 \, .$$

Condition ($C4_{[p_1,p_2]}$). Let $p_1 < p_2$ be two finite real numbers. An element u_0 of $H^2(\mathbb{R})$ satisfies Condition ($C4_{[p_1,p_2]}$) if and only if $\widehat{u_0}$ is a $\mathcal{C}^1$-function on $\mathbb{R}$ which verifies

$$supp\, \widehat{u_0} \subseteq [p_1, p_2]\ .$$

Hypothesis ($\mathcal{H}1$). The initial datum u_0 and the potential V verify Condition ($C4_{[p_1,p_2]}$) and Condition ($\mathcal{P}_{a,b}$) respectively, with

$$a - \frac{b}{2} > 0 \quad \text{and} \quad \frac{b}{2} - a < p_1 < p_2 < a - \frac{b}{2} \quad \text{and} \quad 0 \notin [p_1, p_2]\ .$$

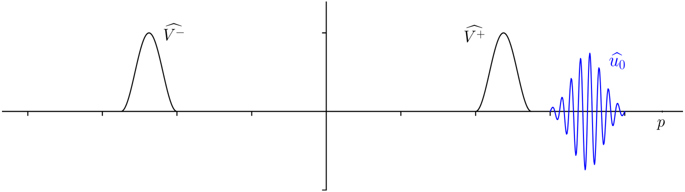


Condition ($C4_{[p_1,p_2]}$). Let $p_1 < p_2$ be two finite real numbers. An element u_0 of $H^2(\mathbb{R})$ satisfies Condition ($C4_{[p_1,p_2]}$) if and only if $\widehat{u_0}$ is a $\mathcal{C}^1$-function on $\mathbb{R}$ which verifies

$$supp\ \widehat{u_0} \subseteq [p_1, p_2] \ .$$

Hypothesis ($\mathcal{H}2$). The initial datum u_0 and the potential V verify Condition ($C4_{[p_1,p_2]}$) and Condition ($\mathcal{P}_{a,b}$) respectively, with

$$b < p_1 \ .$$

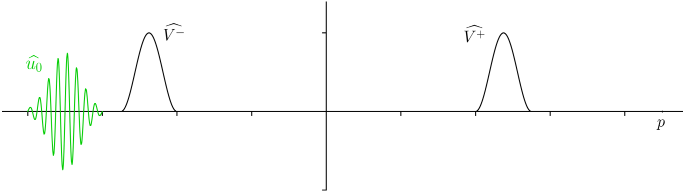


Condition $(\mathrm{C4}_{[p_1,p_2]})$. Let $p_1 < p_2$ be two finite real numbers. An element u_0 of $H^2(\mathbb{R})$ satisfies Condition $(\mathrm{C4}_{[p_1,p_2]})$ if and only if $\widehat{u_0}$ is a $\mathcal{C}^1$-function on $\mathbb{R}$ which verifies

$$supp\ \widehat{u_0} \subseteq [p_1, p_2] \ .$$

Hypothesis $(\mathcal{H}3)$. The initial datum u_0 and the potential V verify Condition $(\mathrm{C4}_{[p_1,p_2]})$ and Condition $(\mathcal{P}_{a,b})$ respectively, with

$$p_2 < -b \ .$$



5. Asymptotic behaviour of the non-perturbed term

Theorem 10

Suppose that u_0 satisfies Condition $(C4_{[p_1,p_2]})$. Then we have for all $(t,x) \in \mathfrak{C}_S\big(p_1,p_2\big)$,

$$\Big| \big(S_0(t)u_0\big)(x) - H_0(t,x,u_0)\, t^{-\frac{1}{2}} \Big| \leqslant C_0(u_0)\, t^{-\delta} \,,$$

where

$$H_0(t,x,u_0) := \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, \widehat{u_0}\Big(\frac{x}{2t}\Big) \,.$$

Moreover, if we choose two finite real numbers $\tilde{p}_1 < \tilde{p}_2$ such that $[p_1,p_2] \subset (\tilde{p}_1,\tilde{p}_2) =: \tilde{I}$, then we have for all $(t,x) \in \mathfrak{C}_S\big(\tilde{p}_1,\tilde{p}_2\big)^c$,

$$\Big| \big(S_0(t)u_0\big)(x) \Big| \leqslant C_{0,\tilde{I}}(u_0)\, t^{-1} \,.$$

Theorem 11 (Part 1)

Suppose that one of the three hypotheses ($\mathcal{H}1$), ($\mathcal{H}2$) or ($\mathcal{H}3$) is verified. Then we have for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

- $$\big(S_1^-(t)u_0\big)(x) = \frac{1}{2\pi}\int_{p_1-b}^{p_2-a} W_1^-(p)\, e^{-itp^2+ixp}\, dp + \frac{1}{2\pi}\int_{p_1-b}^{p_2-a} W_2^-(t,p)\, e^{-itp^2+ixp}\, dp\, t^{-1}\,,$$

- $$\big(S_1^+(t)u_0\big)(x) = \frac{1}{2\pi}\int_{p_1+a}^{p_2+b} W_1^+(p)\, e^{-itp^2+ixp}\, dp + \frac{1}{2\pi}\int_{p_1+a}^{p_2+b} W_2^+(t,p)\, e^{-itp^2+ixp}\, dp\, t^{-1}\,.$$

Theorem 11 (Part 2)

For $t \geqslant 0$, the function $W_1^{\pm}, W_2^{\pm}(t,.) : \mathbb{R} \longrightarrow \mathbb{C}$ are defined by

- $$W_1^{\pm}(p) := \mp \int_{\pm a}^{\pm b} \frac{\widehat{V}(y)\, \widehat{u_0}(p-y)}{q(y,p)}\, dy \ ,$$
- $$W_2^{\pm}(t,p) := \mp\, i \int_{\pm a}^{\pm b} \partial_y \left[\frac{\widehat{V}(y)\, \widehat{u_0}(p-y)}{q(y,p)\, \partial_y q(y,p)} \right] e^{-it\, q(y,p)}\, dy \ ,$$

for all $p \in \mathbb{R}$, where $q(y,p) = (p-y)^2 - p^2$. Note that for fixed $t \geqslant 0$, we have

- $$\mathrm{supp}\, W_1^{-} = \mathrm{supp}\, W_2^{-}(t,.) \subseteq [p_1 - b, p_2 - a] \ ,$$
- $$\mathrm{supp}\, W_1^{+} = \mathrm{supp}\, W_2^{+}(t,.) \subseteq [p_1 + a, p_2 + b] \ .$$

Theorem 12

Suppose that the hypotheses of Theorem 11 are satisfied. Then we have for all $(t,x) \in \mathfrak{C}_S\big(p_1+a, p_2+b\big)$,

$$\Big|\big(S_1^+(t)u_0\big)(x) - H_1^+(t,x,u_0)\, t^{-\frac{1}{2}}\Big| \leqslant C_1^+(u_0,V,\delta)\, t^{-\delta} + C_2^+(u_0,V)\, t^{-1} \ ,$$

where $\delta \in \left(\frac{1}{2}, 1\right)$ *and*

$$H_1^+(t,x,u_0) := \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, W_1^+\Big(\frac{x}{2t}\Big) \ .$$

Moreover, if we choose two finite real numbers $\tilde{p}_1 < \tilde{p}_2$ *such that* $[p_1,p_2] \subset (\tilde{p}_1,\tilde{p}_2) =: \tilde{I}$, *then we have for all* $(t,x) \in \mathfrak{C}_S\big(\tilde{p}_1+a, \tilde{p}_2+b\big)^c$,

$$\Big|\big(S_1^+(t)u_0\big)(x)\Big| \leqslant C_{1,\tilde{I}}^+(u_0,V)\, t^{-1} \ .$$

Theorem 13

Suppose that the hypotheses of Theorem 11 are satisfied. Then we have for all $(t,x) \in \mathfrak{C}_S\big(p_1 - b, p_2 - a\big)$,

$$\Big| \big(S_1^-(t)u_0\big)(x) - H_1^-(t,x,u_0)\, t^{-\frac{1}{2}} \Big| \leqslant C_1^-(u_0,V,\delta)\, t^{-\delta} + C_2^-(u_0,V)\, t^{-1} \ ,$$

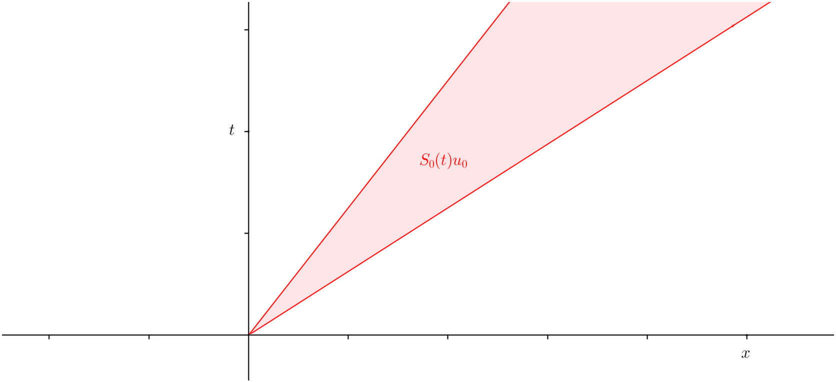
where $\delta \in \left(\frac{1}{2}, 1\right)$ *and*

$$H_1^-(t,x,u_0) := \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, W_1^-\Big(\frac{x}{2t}\Big) \ .$$

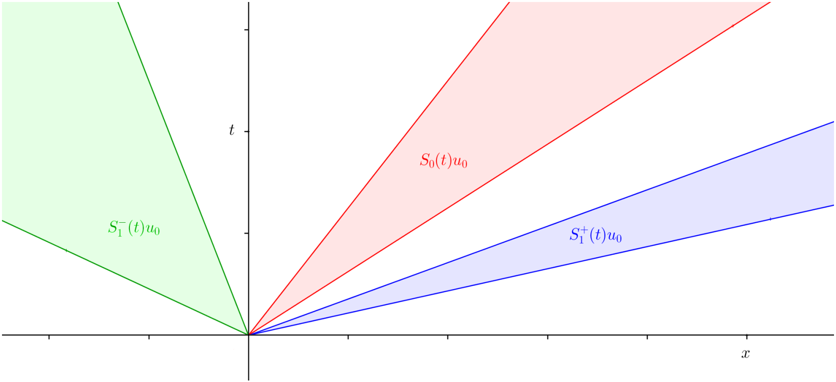
Moreover, if we choose two finite real numbers $\tilde{p}_1 < \tilde{p}_2$ *such that* $[p_1,p_2] \subset (\tilde{p}_1,\tilde{p}_2) =: \tilde{I}$, *then we have for all* $(t,x) \in \mathfrak{C}_S\big(\tilde{p}_1 - b, \tilde{p}_2 - a\big)^c$,

$$\Big| \big(S_1^-(t)u_0\big)(x) \Big| \leqslant C_{1,\tilde{I}}^-(u_0,V)\, t^{-1} \ .$$

Case of small and positive initial momentum:



Case of small and positive initial momentum:



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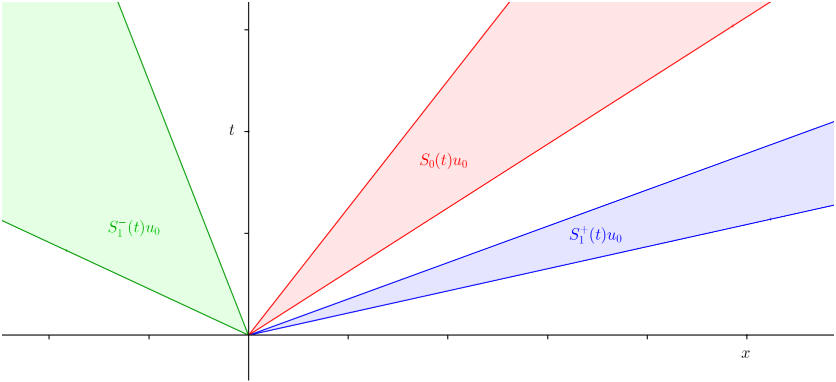
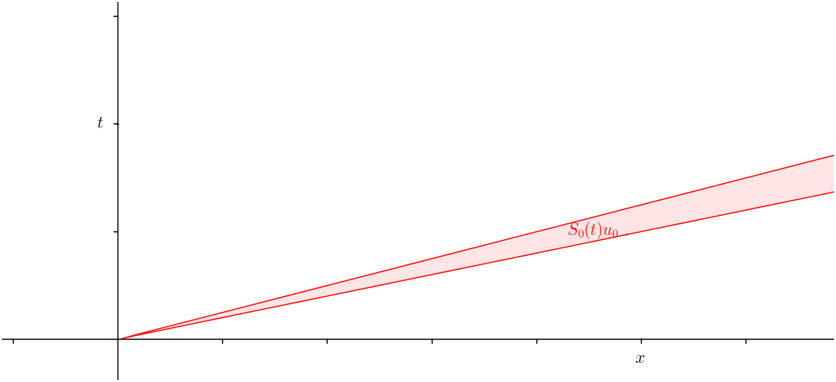
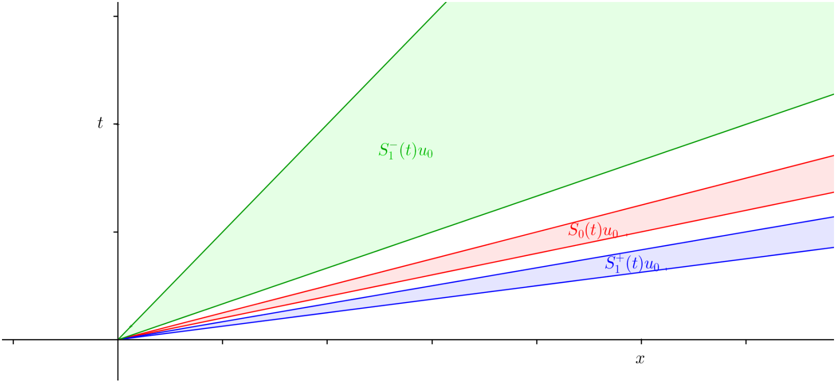


$\rightsquigarrow$ $S_1^-(t)u_0$: reflected part , $S_1^+(t)u_0$: transmitted part

Case of high and positive initial momentum:



Case of high and positive initial momentum:



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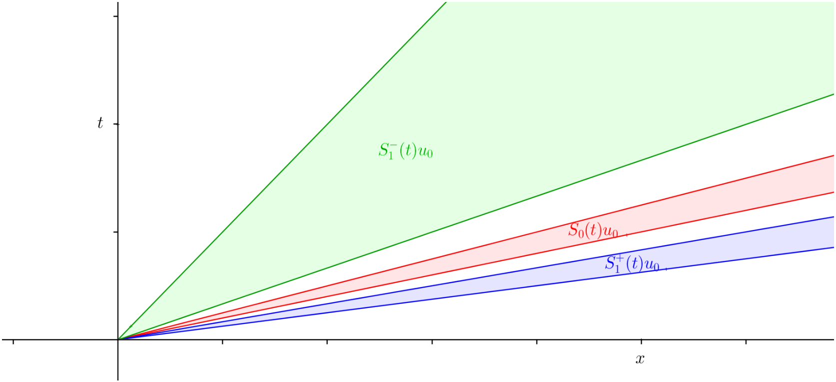

$\rightsquigarrow$ $S_1^-(t)u_0$: retarded transmission , $S_1^+(t)u_0$: advanced transmission

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