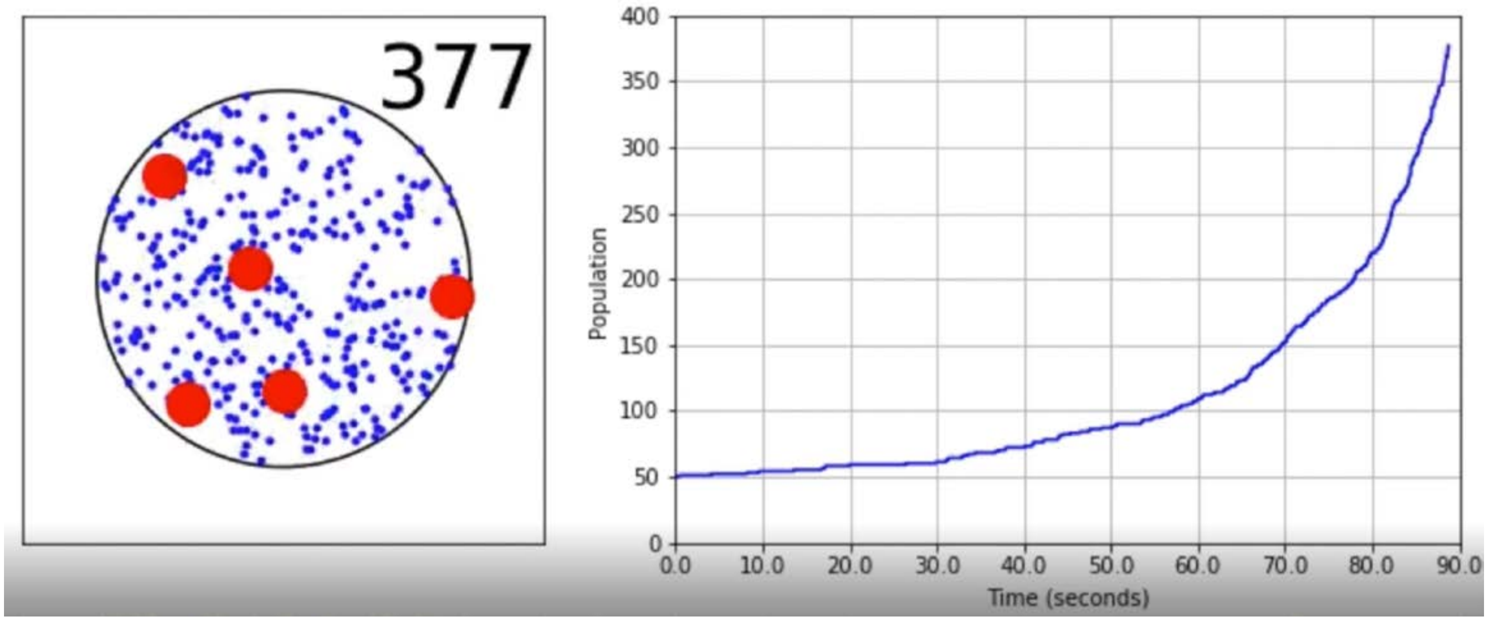
The collision model

RabbitMath.ca

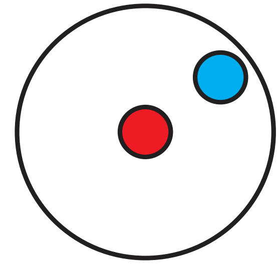
The Animation



radius of the circle: $r = 5$
Particle speed: $v = 4$
collision proximity: $\delta = 0.02$
frame freq: 25 frames/s
initial pop $N = 50$

The dynamic equation.

Expected # offspring of a focal particle in one time unit (1/25th of a sec)



$$\varepsilon = (N-1)\frac{\pi\delta^2}{\pi r^2} = (N-1)\frac{\pi(0.02)^2}{\pi(5)^2} = (N-1)\frac{0.0004}{25} = \frac{N-1}{62500}$$

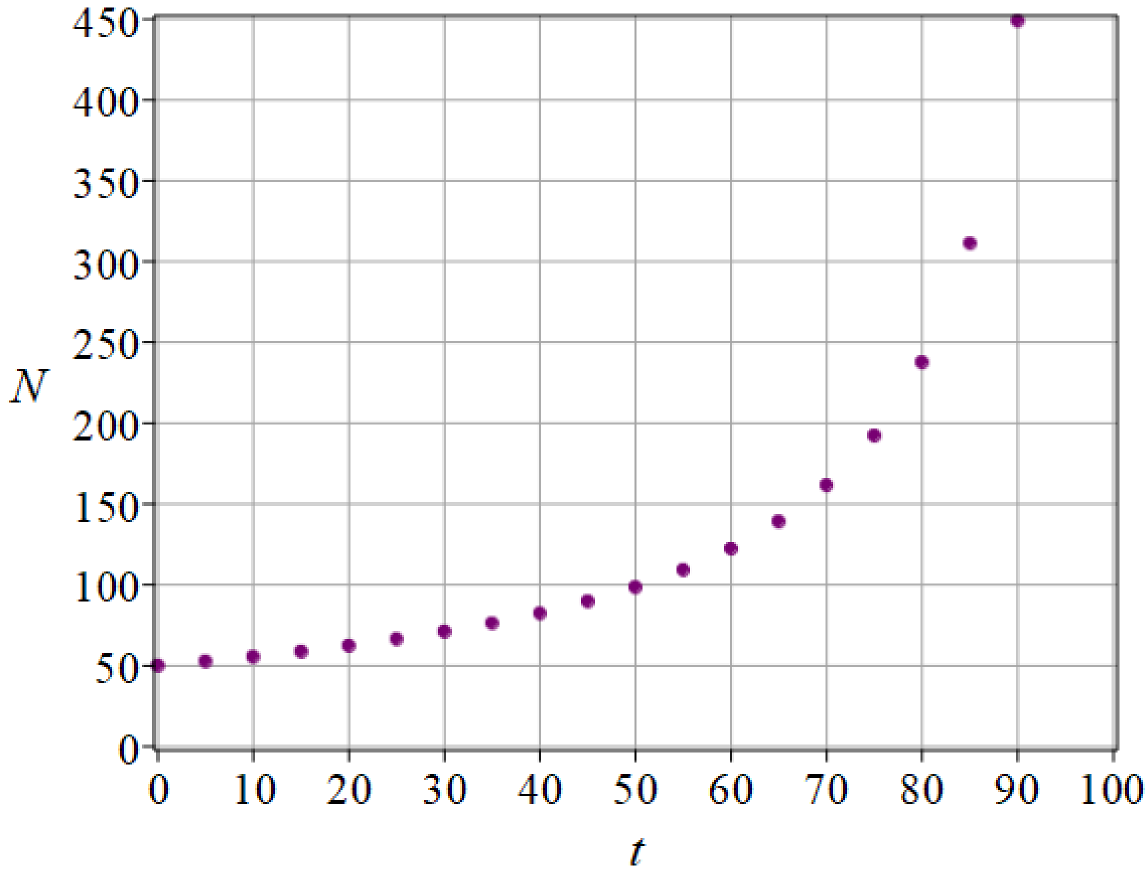
Number of new particles in one time unit

$$\Delta N = \frac{\varepsilon N}{2} = \frac{N}{2}\left(\frac{N-1}{62500}\right) = \frac{N(N-1)}{125000}$$

Written as a recursive equation:

$$N_{\tau+1} = N_\tau + \frac{N_\tau(N_\tau - 1)}{125000} \qquad N_0 = 50$$

Note: τ is time measured in 1/25th of a second. We reserve t for time in seconds.



Time t	Size N
0	50
5	52.6
10	55.4
15	58.6
20	62.2
25	66.3
30	70.9
35	76.2
40	82.4
45	89.8
50	98.5
55	109.1
60	122.3
65	139.2
70	161.5
75	192.3
80	237.7
85	311.3
90	450.9

$$N_{\tau+1} = N_\tau + \frac{N_\tau(N_\tau - 1)}{125000} \qquad N_0 = 50$$

Approximating the discrete difference equation with a differential equation.

The difference equation

$$\Delta N = \frac{N(N-1)}{125000}$$

A differential equation:

$$\frac{\mathrm{d}N}{\mathrm{d}\tau} = \frac{N(N-1)}{125000}$$

Change time to seconds:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N(N-1)}{5000}$$

Separate variables:

$$\frac{\mathrm{d}N}{N(N-1)} = \frac{\mathrm{d}t}{5000}$$

Partial fractions:

$$\left(\frac{1}{N-1} - \frac{1}{N}\right)\mathrm{d}N = \frac{\mathrm{d}t}{5000}$$

Now we can integrate:

$$\left(\frac{1}{N-1} - \frac{1}{N}\right) \mathrm{d}N = \frac{\mathrm{d}t}{5000}$$

Integrate:

$$\ln\left(\frac{N-1}{N}\right) = \frac{t}{5000} + c$$

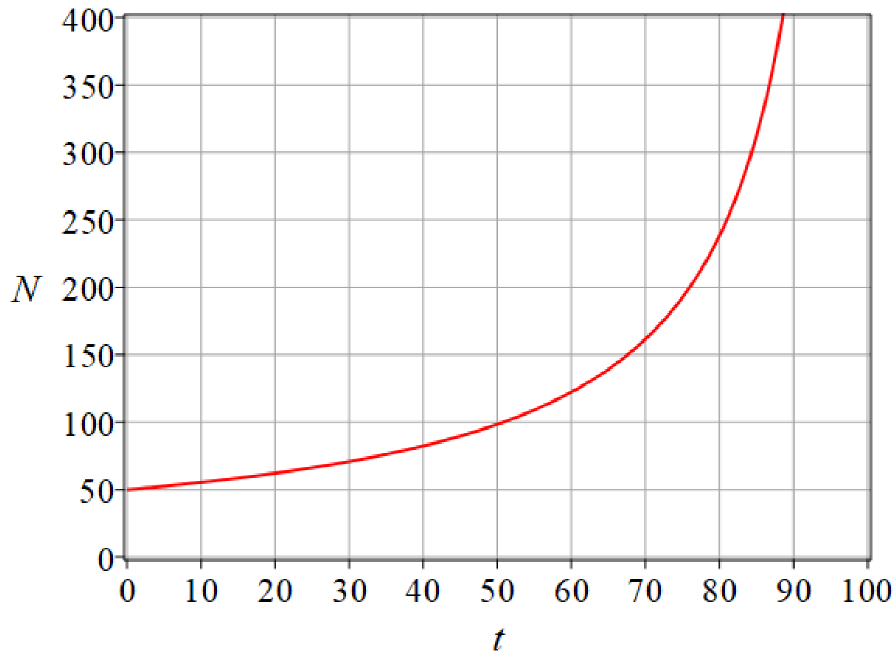
Exponentiate:

$$\frac{N-1}{N} = ke^{\frac{t}{5000}}$$

Solve for k: $\quad N(0) = 50 \quad \Rightarrow \quad k = 0.98$

Solve for N:

$$N = \frac{1}{1 - 0.98e^{\left(\frac{t}{5000}\right)}}$$



$$N_{\tau+1} = N_\tau + \frac{N_\tau(N_\tau - 1)}{125000}$$

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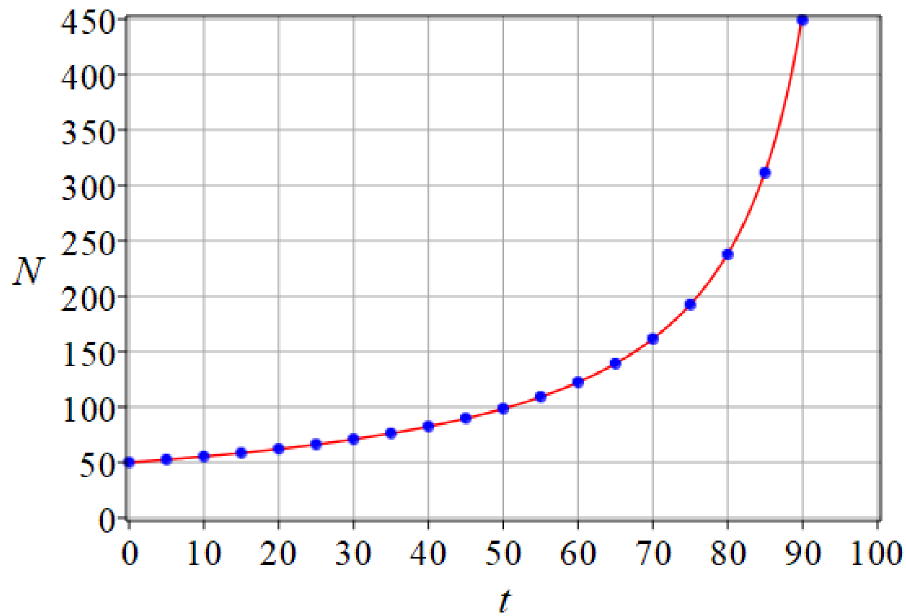
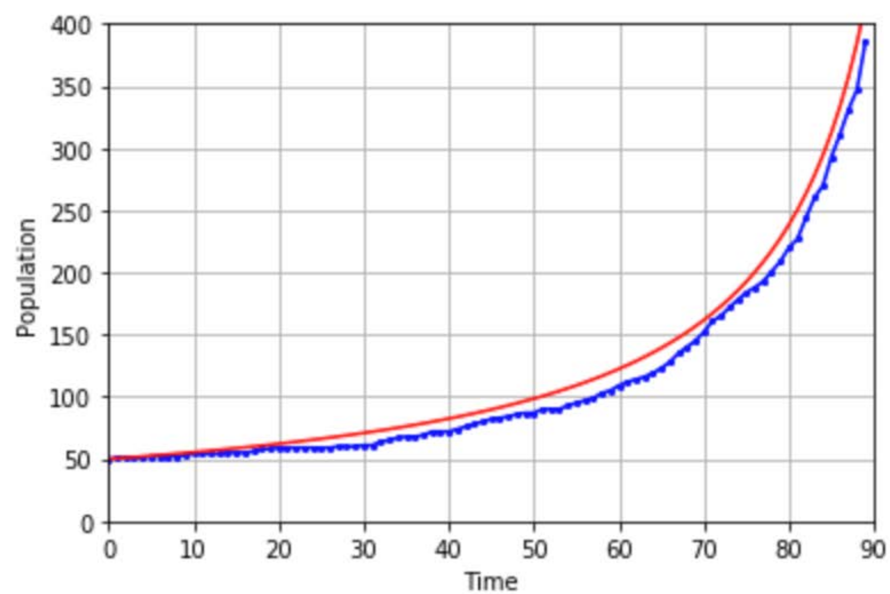


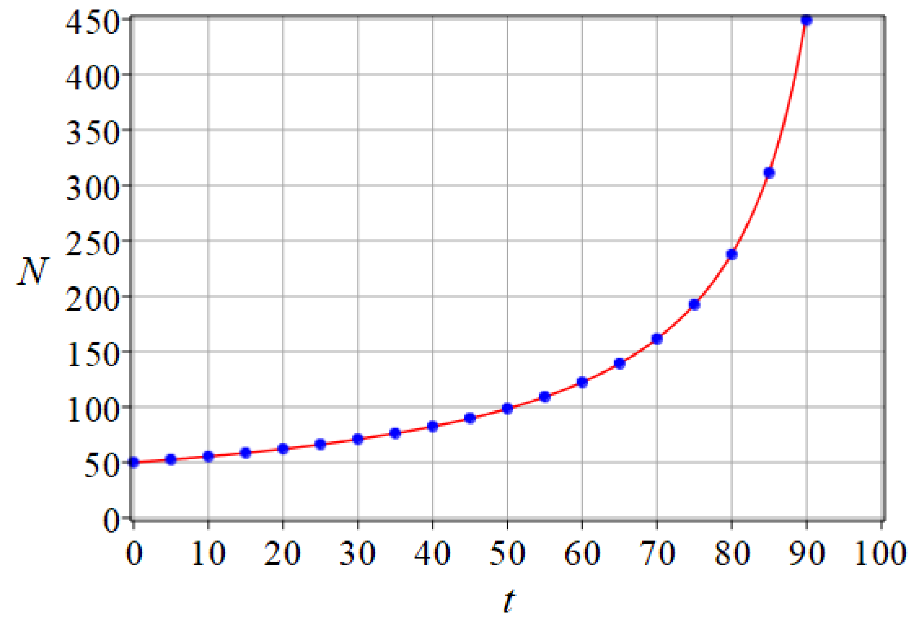
The original data and

the solution to the DE



$$N_{\tau+1} = N_\tau + \frac{N_\tau(N_\tau - 1)}{125000}$$

$$N = \frac{1}{1 - 0.98e^{\left(\frac{t}{5000}\right)}}$$



Asymptote?

$$1 - 0.98e^{\left(\frac{t}{5000}\right)} = 0$$

Solve for t:

$$t = 101$$

Red curve asymptotes at 101, blue dots last forever.