

Thinking about Thinking: Visual Patterns

By Michael Pershan

0.

This piece is all about visual patterns. In particular, it's about how people think about visual patterns.

I think that thinking about how kids might think about a type of problem is one of the most productive planning a teacher can do. (Lots of thinking in that last sentence!) What it comes down to is using kids' thinking to plan for our mathematical goals, questions, hints and feedback. If we know the different ways students can think about a problem, then we can more clearly articulate our mathematical goals for students. The goal is, presumably, to help kids think in one of the ways that they can't yet. Once we have clear goals we can develop a framework in which to think about the questions we ask, how to give hints during the activity, and what feedback to give after the lesson.

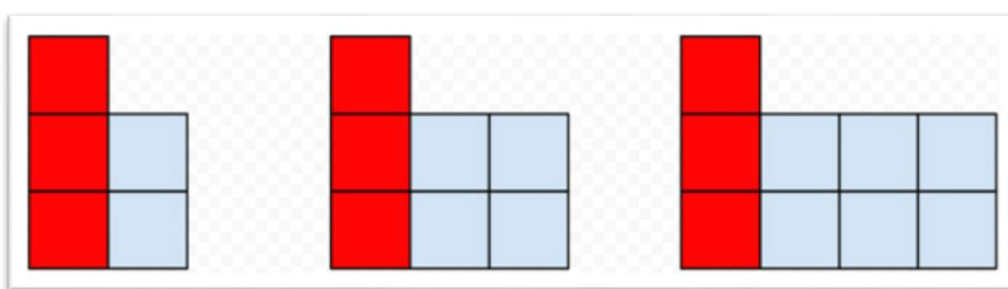
What follows is my take on how my students think about visual patterns. After describing my way of seeing this thinking, I try to draw a clean line between this scheme and the mathematics that visual patterns can help my students learn. This makes my options for instructional goals much clearer. Then, I use the thinking and instructional goals as a framework for planning the questions, hints and feedback that could be helpful to students for these types of problems.

I don't believe that there is one true way to describe all the possible ways to approach some problem. Instead, there are lots of true and useful ways to describe student thinking, but some are more or less useful than others. The key, I think, is to be explicit and systematic, so that classroom experience helps us improve our descriptions and make them more useful. This may seem like a philosophical point (it is) but I mention it at the outset to invite you to disagree and describe the landscape of thinking about visual patterns in a way completely different from my own, if it serves your needs better.

Besides for all this thinking about thinking, there are a lot of math problems interspersed throughout the text. Some of them I made up, others I've ripped off from others (with attribution). I've done this because I find it helpful to think about the math myself when I'm trying to think about how students think about it. I've left out the answers, because the thinking matters more than the answers. I hope you enjoy them!

1.

Let's start with a math question:



Source: visualpatterns.org

2.

Gotcha! I didn't ask a question at all. This is just an image. If you found yourself asking a question, well, isn't that interesting?

What question did you ask, and what does that question say about how you were looking at this picture?

Personally, I can't avoid thinking about the fourth, next figure when I look at this image. Just as I can't help but understand what you mean when you speak a sentence, I can't help but think about what the fourth image would look like. This says something about me and how I have learned to think about these sorts of things.

You can find kids who aren't like me at all. They can look at the above and just see a *picture*, not a pattern. They can see it as a bunch of squares of different colors all cobbled together into an assortment of shapes and arrangements. They don't feel compelled to see it as growing – they're free of this burden.

It makes me wonder if there are any visual patterns I can see without this burden of wondering what comes next, that I would see just as *pictures* and as nothing more. If I could, it wouldn't just be hard for me when you ask, "What's the next step?" – it would be surprising.



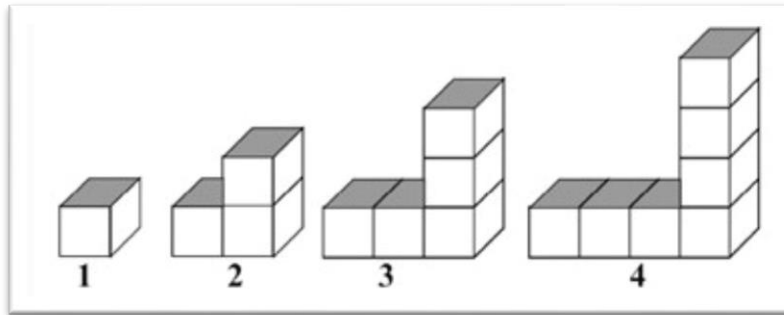
28 14 24

Did you turn this string of numbers into a question? Or were you able to just see it as a bunch of numbers?

All of this goes to show that seeing patterns is something we learn how to do. There are different ways to see patterns, some of which are more helpful for our needs than others. That's what happens when we spend lots of time working with a certain type of pattern – we learn how to see it.

3.

Every student in my classes has always been able to see these visual patterns as growing things, and so they've never been shocked by a request to draw the next step. (Though, they might be outraged if I asked them to find the next number in "28, 14, 24.") Though they are always able to see that the pattern is growing, they aren't always able to find a useful way of seeing that growth.

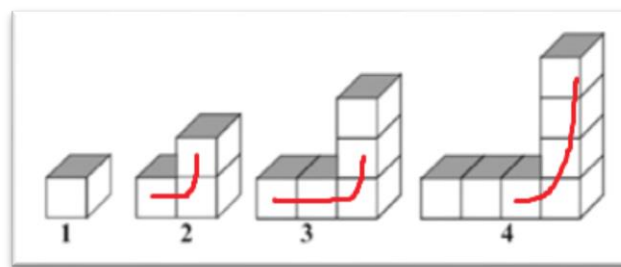


Above is a visual pattern problem – the “L-Shape pattern” – I gave to my class of third grade math students, including Toni. Toni was working alone at the pattern, and she called me over after several minutes of fruitless confusion. She was trying, but failing to make sense of this pattern, to see it in some recognizably useful way.

“There is no pattern,” she tells me.

“Interesting. Tell me more.”

She made clear that she saw it as growing, but she also knew perfectly well that this wasn’t enough. She couldn’t grasp it, see it, taste it in any particularly helpful way. She drew lines in weird ways that didn’t reveal an underlying structure of the pattern.



My head started spinning as I tried to come up with a way to help Toni. Should I ask a question? Point something out to her? Should I just listen? Experience told me to resist the temptation to try to direct her thinking without trying to fully understand it. I asked her to explain herself, but she just repeated what she had already told me.

I tried to ask Toni some general, non-invasive questions – “What else did you try? Did that work? Why didn’t that work? What do you want to try next?” – but Toni was getting more than a bit impatient, and by the way she looked around I figured she was (understandably) worried about falling behind her classmates.

I decided to shift tactics. I knew what I wanted to help Toni see. She was looking for a pattern in the growth, but she was having trouble getting specific about it. I wanted to ask a question that would draw Toni’s attention to helpful features of the pattern’s growth and help her get specific about precisely how this shape is changing.

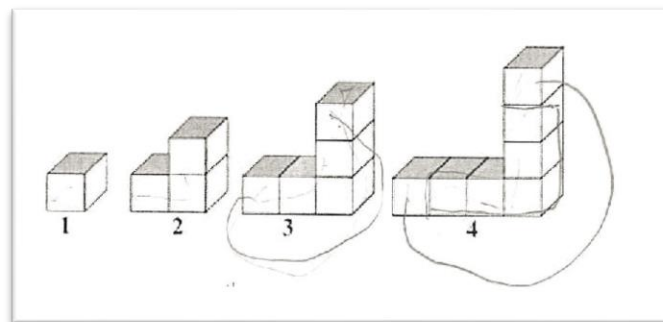
This would involve a bit of guessing on my part, though, since I didn’t really know what question would work!

My first question was a promising dud: “Can you see the previous step in the following step?”

To which Toni responded, “no.”

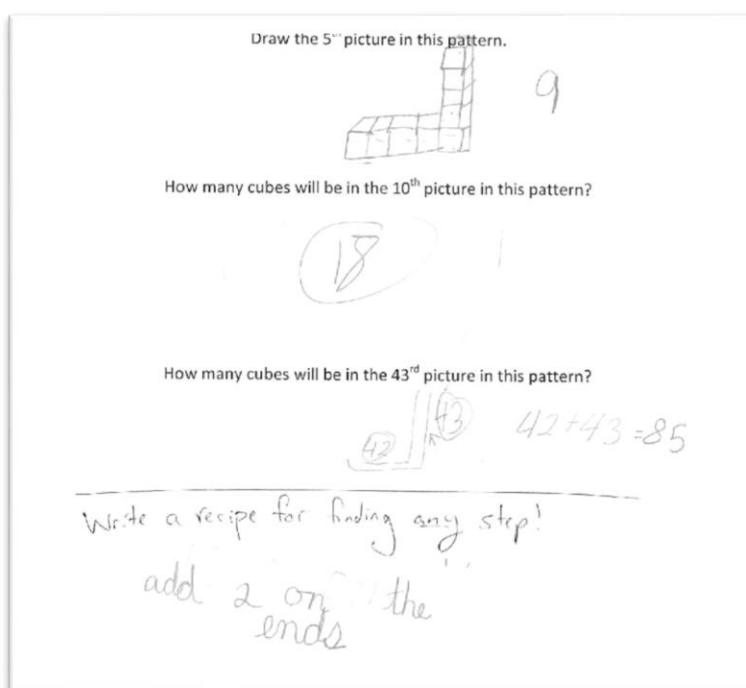
I tried again, this time directing her attention more directly: “Do you see the *second* picture in the *third*? Imagine that you were building the third picture from the second. Where would you put the extra bricks?”

Bingo. She grabbed her pencil and started sketching.



Why did that question work? I think it's because it encouraged Toni to see the static picture on the page as a changing thing. Toni had lots of experience playing with blocks and adding on parts to existing doodles. By asking her to think of one picture in the next, I helped direct her thinking to this analogy, and she was able to see the pattern's growth in a useful way that related to things she had lots of experience with.

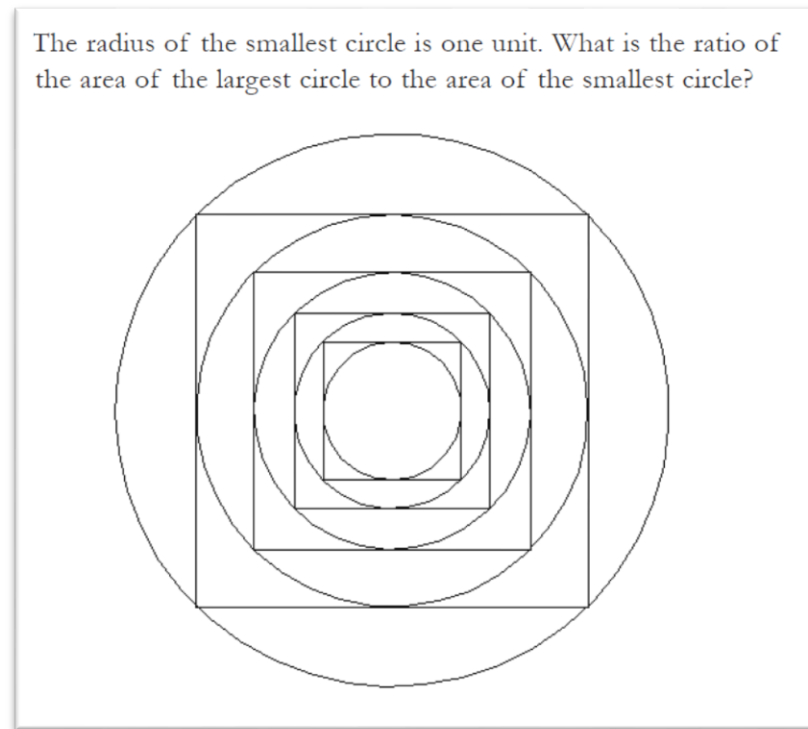
The rest of her work is interesting too. She did it all on her own. I like it because in just one page of work she sees this pattern in at least three or four different ways, some that we'd recognize as very sophisticated.



Notice how she finds the 5th picture by adding on another square to each end, but she finds the 10th step without drawing a picture. Then she finds the 42nd picture by relating the dimensions of the picture to the pattern stage. In some ways, she's getting started, but in others she's ready to talk algebra.

Toni made the move from thinking of this pattern statically to seeing it as a thing in motion, changing throughout time.

Incidentally, this way of seeing static things dynamically is helpful for lots of mathematical problems, though I doubt most students would see the connection between all these different problems on their own.



Source: Park Math Curriculum

6.

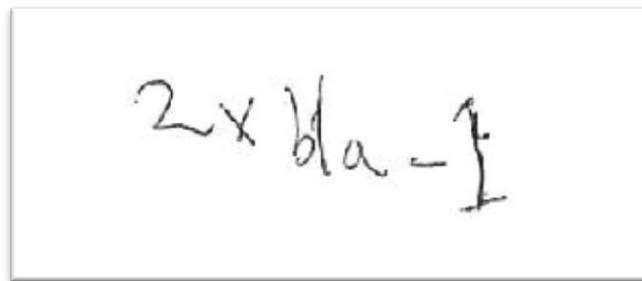
Here are the major ways that I see people thinking about visual pattern problems, like the constantly growing L-Shape pattern before.

- A. **Recursively** – By noticing a pattern in the growth rate and using that to calculate a given step. In the L-Shape pattern, this sort of thinking might lead you to draw a given step picture by repeatedly adding on bricks to both ends of the L. Or you might count by 2s until arriving at some step. Even if you take a few shortcuts (“Well, five 2s makes 10”) I think that’s still recursive thinking about the pattern.

B. **Relationally** – By relating the step number of the pattern to some aspects of each picture. This is what Toni did with the 42nd step. She said, “Well, there will be a tower of cubes going straight up that will have 43 cubes in it, and then the bottom part of the L will contain 42 cubes.”

C. **Functionally** – By fitting the pattern to the equation of a function. This sort of thinking is how I approach most of the constant growth patterns. I say something along the lines of, “Well, this is growing constantly. So it’s going to be the growth rate times the step number plus some constants. The growth rate is two, and n stands for the step number, and it looks like if my equation will work it has to be 2 times n minus 1.” Then I just use that for the 42nd step.

I saw recursive and relational thinking in Toni’s work, above, but I didn’t see functional thinking. If I had seen functional thinking, I wouldn’t have expected it to take a fully algebraic form. More likely, given Toni’s age, she would have formulated a functional view in some other, less formal way.

A photograph of a piece of paper with the handwritten equation $2 \times 42a - 1$ in black ink. The handwriting is somewhat informal and slightly blurry.

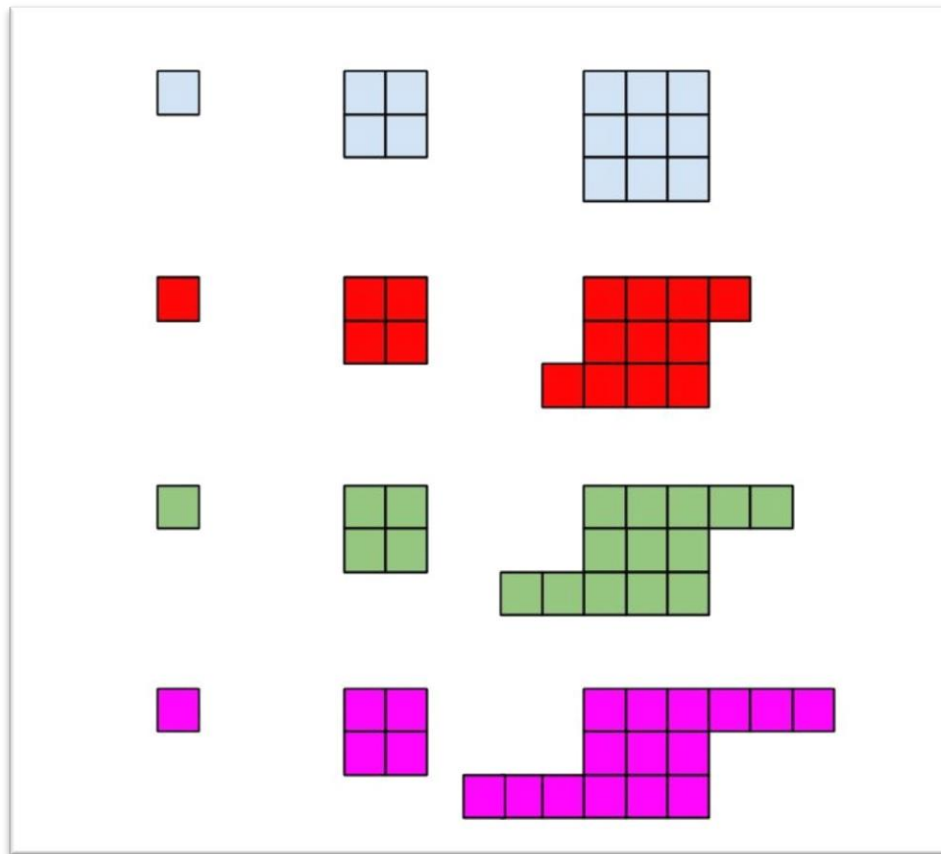
At the beginning of my conversation with Toni, she was stuck. In my scheme, this is because she couldn’t see the pattern recursively or relationally, at first. When she called me over, I made a decision to help direct her attention towards a recursive way of seeing the pattern. I could have decided to direct her attention towards the connections between the step number and the pictures, but I didn’t. (Should I have?)

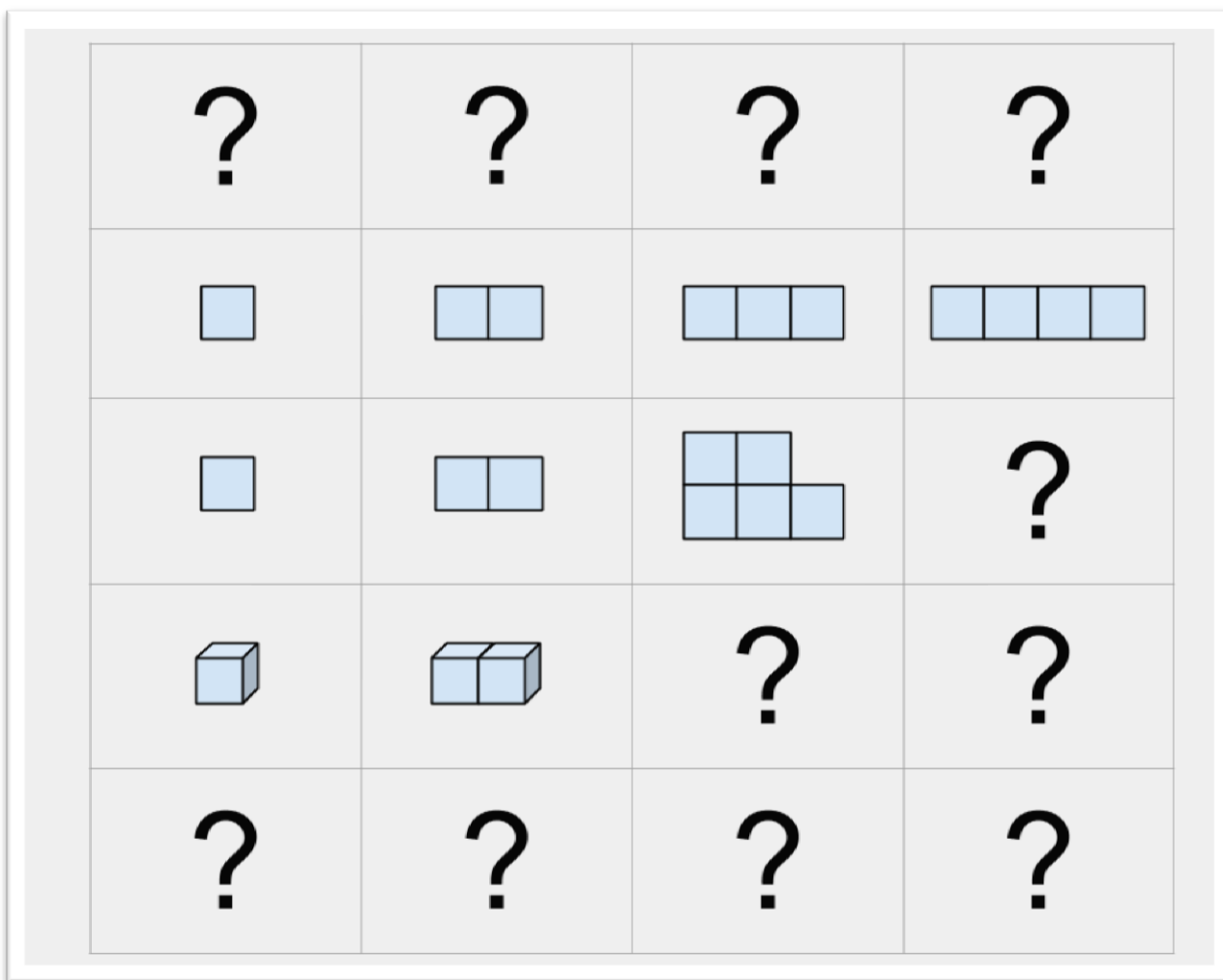
I’m not surprised that Toni didn’t think about the pattern functionally. Functional thinking is often hard, but it’s so, so important.

7.

The L-Shape pattern changes by the same number of cubes each time – its growth is constant. Of course, there are lots of non-constantly growing visual patterns.

My experience is that there are ways of seeing these sorts of non-constant growth patterns that are immensely helpful in helping me quickly see the structure of the pattern. When those structural cues aren't there, though, it becomes essentially the same as a numerical pattern, and inevitably becomes a little bit more challenging for me.





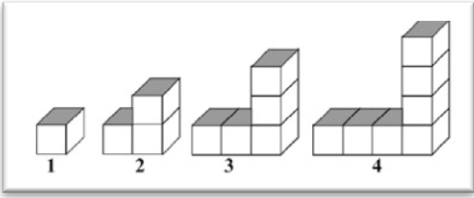
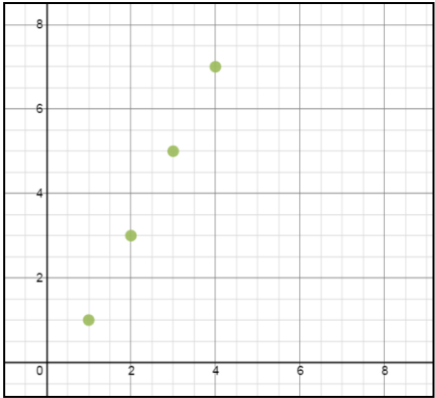
How many patterns can you find in this square?

8.

Visual patterns – who needs them? After all, very little in the world comes in the form of a neat little sequence of growing Tetris pieces. (A growing doodle, perhaps. Windows of a rising building. Towers of children's blocks. Apples, being laid out for display.)

Far more common in school than visual patterns are patterns that show themselves through numbers, graphs, or tables. The L-Shape pattern that appears above could easily been presented in any of these

three other forms. These other forms are more common, flexible and useful. Why bother with all this picture-pattern stuff?

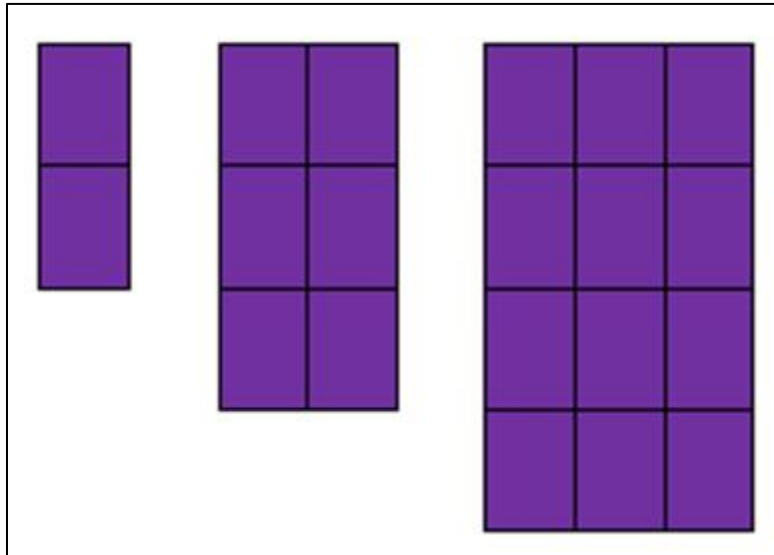
	$1, 3, 5, 7, _, _, _$																
<table border="1" data-bbox="344 630 604 1108"> <thead> <tr> <th>x</th><th>y</th></tr> </thead> <tbody> <tr> <td>-1</td><td>?</td></tr> <tr> <td>0</td><td>?</td></tr> <tr> <td>1</td><td>1</td></tr> <tr> <td>2</td><td>3</td></tr> <tr> <td>3</td><td>5</td></tr> <tr> <td>4</td><td>7</td></tr> <tr> <td>5</td><td>?</td></tr> </tbody> </table>	x	y	-1	?	0	?	1	1	2	3	3	5	4	7	5	?	 $(5, _) ?$
x	y																
-1	?																
0	?																
1	1																
2	3																
3	5																
4	7																
5	?																

I see three types of thinking about visual patterns: recursive, relational and functional thinking. Relational thinking – that connecting of the step and a dimension of the picture – is not available when the pattern is presented numerically, or in a table or a graph. Relational thinking is this perspective that is only useful for visual patterns. It's what makes visual patterns different.

(Don't graphs allow for special, graphical ways of finding a step in a pattern? Graphical patterns are different, too.)

In a sense, visual patterns are easier for students than other representations of patterns. I see this most often when my students work with non-linear visual patterns. Recursive and functional thinking often

doesn't occur to them. Relational thinking, on the other hand, eventually occurs to many of my young students, and they'll use this to make sense of patterns that would otherwise be inaccessible.



Relational thinking is great, but it's not broadly useful. The most powerful perspective on a pattern is functional thinking, the holy grail of many a high school course. It's the sort of thinking that helps an expert quickly look at a pattern and make careful predictions about any step in the sequence. Many students don't get there, though. The journey from recursive to functional thinking can be rocky. It's hard for a lot of kids.

Relational thinking can only really be applied when the pattern is presented in a visual form. It's certainly beautiful, but it's not broadly useful all on its own. To the extent that relational thinking isn't just beautiful, but also useful, it's because relational thinking can help students gain this hard-to-obtain functional perspective. The important question, then, is how do students develop a functional perspective out of a relational one?

I am a high school math teacher, and I am proud to announce that I have a functional way of seeing this table:

x	y
1	12
2	20
3	30
4	42

That means that I can tell you that the 10th row will have a y-value of 110, the 1000th row has a y of 1,001,000, and that I can say with confidence the values of any row in this table that you'd like to know.

The goal, for many students, is for them to be able to do likewise. How do we help them get there? How can visual patterns help out in this learning?

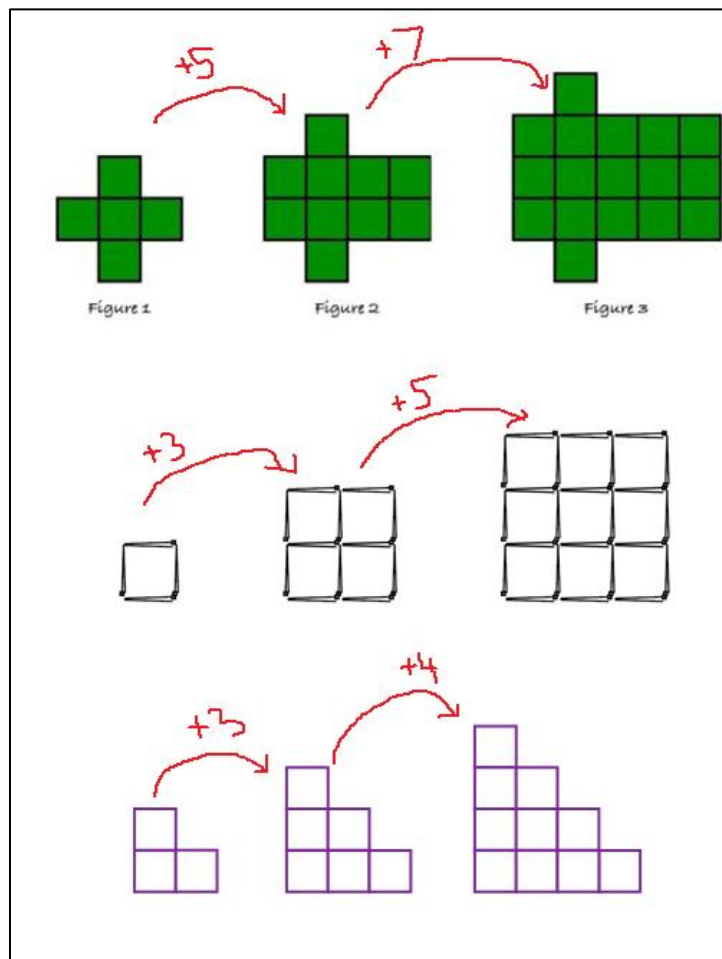
The story starts with recursive thinking, something that a lot of students will attempt. While these students will be able to see a pattern in the growth, that pattern is unlikely to help make big leaps ahead. To make progress, they'll still have to go through each and every row.

x	y
1	12
2	20
3	30
4	42

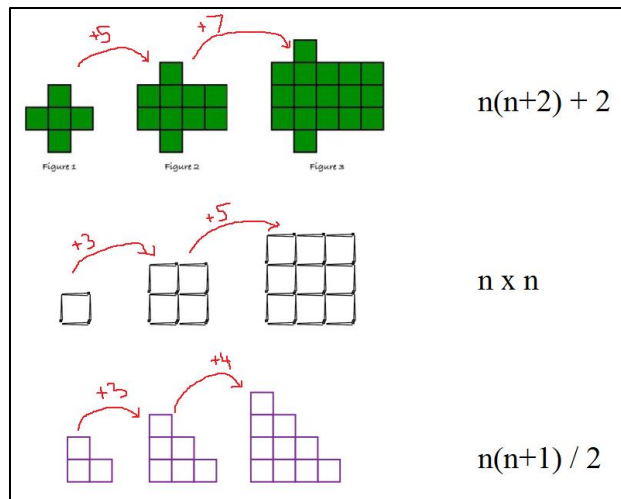


(Though, there are some clever shortcuts a student might take! See, Gauss.)

Imagine a kid who had spent a lot of time working with visual patterns. She might remember that there is a group of visual patterns whose recursive growth feels awfully similar. She associates this constantly growing growth with the rectangle family of patterns.



But there are more things that she knows about this family of patterns. After all, when she studied these patterns her teacher pointed out that the equations that the class was deriving – using relational thinking – were all similar in some important ways.



In short, our imaginary student associates a certain kind of recursive pattern (constantly growing growth) with a certain type of function (something-plus-n times something-plus-n), via experience deriving these equations relationally.

Going back to the original table, our student might associate constantly increasing growth with this “n-plus-something times n-plus-something” function. She would probably have to make a few guesses of how exactly to see the y-column as multiplication, but I can imagine her figuring it out before too long.

x	y	
1	12	$3 \times 4 = 12$
2	20	$4 \times 5 = 20$
3	30	$5 \times 6 = 30$
4	42	$6 \times 7 = 42$
		so... $n \times (n+1)$

Red arrows indicate growth in the y-column: +8 from 12 to 20, and +10 from 20 to 30.

In sum, the progression of learning might go something like this:

- **Visual Patterns** – “I can see these recursively and relationally!”

- **Patterns in Visual Patterns** – “I noticed patterns in the equations that I get relationally for this type of pattern, so now I see what sorts of functions are created by this sort of recursive growth.”
- **Non-Visual Patterns** – “I can see these recursively, and I remember from visual patterns that this sort of growth relates to a certain type of function. Now I can find a function that fits this growth.”
- **Eventually** – “I notice patterns in the equations that I get from these non-visual patterns, and now I know what sort of functions to associate with this sort of growth.”

Phew!

10.

In short, the role of relational thinking can be to facilitate an association between certain “families” of growth and certain “families” of functional equations. That’s knowledge that students can use when they see recursive growth in tables, graphs or numbers.

All of that learning, though, was imagined. There are many ways for this whole sequence of learning to fall apart in real life. Here are two big ones that I worry about:

- The plan is for relational thinking to be the handmaiden between recursive and functional thinking. But what if kids stop noticing recursive thinking once they start seeing these patterns relationally? The whole project would collapse.
- If students prefer relational thinking in their work with visual patterns, they might never see the patterns in the equations they are producing. If students don’t think functionally about visual patterns, they surely won’t be able to use that knowledge to develop functional knowledge with graphs, tables or numbers. Once again, the whole thing collapses.

These worries are not insurmountable, but they do take care and subtlety to avoid. To avoid the first, I would need to design activities and conversations that help draw students' attention to recursive thinking, so that I can be sure that they see it. To avoid the second, we will need to likewise make noticing the patterns within these equations a major focus for an activity and conversation.

This sort of curricular work is not easy, but it is important. We teachers often cheat when we're planning a sequence of lessons by rooting the exercise in something beyond mathematical thinking. We'll talk about building a narrative, or we'll talk about how the lessons string together in some intellectually pleasing way (but without any reason to think that students will follow this string!).

The alternative is what I did above. I started by thinking through a particular kid's thinking about visual patterns, but then I made my thinking more systematic – an attempt to cover every important possibility and growth I'd expect to see in student work. That helped me identify what my goals are for these activities, and it helped me understand why these visual patterns might be valuable. Above, it helped me lay out a fairly detailed sequence for how learning might go.

With all this in mind, I'm ready to think about what feedback or hints might be most helpful to help a kid with these sorts of problems.

11.

When we see a student's work, we sometimes decide to give hints (during the activity) or feedback (afterwards). The feedback shouldn't always be the same for every student, though. It should depend on the sort of thinking they have already done, and should help them do important thinking that they haven't yet done. Because our feedback ought to be conditional on thinking – both the thinking we see in kids and the thinking we'd like to help them see – our *thinking about feedback* likewise ought to be conditional on our *thinking about thinking*.

In other words, everything above needed to come before everything that follows. Understanding student thinking comes first.

What feedback is most helpful for students who are working on a visual pattern problem? It depends entirely on where she is and where she's going. When we arrive at her desk she might be completely stuck, without any way of making sense of the problem. Alternatively, she might have a productive perspective on the pattern, seeing it either recursively, relationally or functionally.

One reason to give feedback¹ or a hint² is because we're aiming to help students adopt a perspective on visual patterns they aren't currently using. Maybe a student is seeing a pattern recursively, but we want to help them be able to also see it functionally. Maybe a student is completely stuck, and we'd like her to be able to see a pattern relationally. Those two cases hardly exhaust the possibilities, but this table does.

	Aiming for: recursive	Aiming for: relational	Aiming for: functional
Currently: no way of thinking			
Currently: recursive thinking			
Currently: relational thinking			
Currently: functional thinking			

This is my feedback table, a planner. You and I might fill out this table differently depending on our kids, our curriculum and the specifics of our classes. That's fine, but if we both find the recursive/relational/functional scheme true and useful for our students, we'll be able to have a healthy

¹ Are there other ways to do this? Sure! Is feedback one way? Yep.

² Are hints always a good idea? Nope. Are they sometimes a good idea? Of course. Imagine a student who adopts a recursive perspective and knows that there are quicker ways to find the 43rd step, and is frustrated with his recursive approach. He calls you over for help. I think that's a good time for a hint. Imagine a student who is the only kid in class still using a recursive perspective after a few days in class. Everyone is ready to move on. Tomorrow you have to do work that depends on having a relational perspective. Imagine a kid who has a hard time paying attention during class discussions. Some teachers talk about never ever giving their students hints, but I think that's a mistake. Hints are helpful.

conversation about our ideas for feedback and hints. All is good, as long as our talk of feedback is grounded in a shared perspective on student thinking.

Below are my ideas for filling out this table.³

Hints/Feedback for Visual Patterns

	Aiming for: recursive thinking	Aiming for: relational thinking	Aiming for: functional thinking
Shows no way of making sense of the problem	“Can you see the picture in the 2 nd step in the 3 rd ?”	“Can you see the step number in each picture?” Alternatively, start with recursive thinking.	Start with recursive or relational thinking.
Shows recursive thinking	Nudge towards either relational or functional perspectives.	“If you’re adding 2 each time, then how many squares will be in this row in the 99 th step?” “Can you write or draw a description of the 99 th picture?”	“You’re adding 2 all these times. Can you experiment with shortcuts to quickly do all that addition?” “Can you calculate how your answers would be different if the pattern grew by 10 each time?”
Shows relational thinking	“Can you come up with a rule for how many more bricks get added each time?”	Nudge towards either recursive or functional perspectives.	“Try drawing a pattern that fits $3n+1$ and $3n+2$.” “What if the pattern had another row? Try writing a rule for how the equation changes depending on the rows.”
Shows functional thinking	“Try to write a rule that explains how the pattern grows from step to step.” “What would the growth be like if your equation was $-2n + 4$?”	“Try to draw a pattern that would fit the equation $5n+1$.”	Extension: “What if this pattern grew by 3 instead of by 2 each time? What would your equation look like then?”

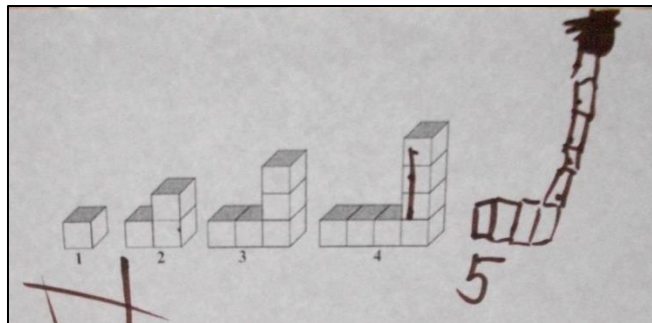
³ I doubt that we’d really have much of a need to show students who are thinking relationally about the L-Shaped pattern how to think recursively about it. I think it could be more likely to be useful for the non-constantly growing rectangle pattern. The dimensions of the rectangle are so easy to see, and the changing growth rate is often unfamiliar to students, that I can see this being hard for some kids. Moreover, I argued above that if visual patterns are useful, it’s because students see connections between recursive and functional perspectives. It might be fun and useful to make a table like this for a certain “family” of visual pattern, like the family constantly growing patterns or the family of quadratic patterns.

My point in this section isn't to sell you on my feedback questions. (Though, now that we mention it, do you like them? Which would you change?) Instead, I'm advocating for an approach to feedback and hints.

This approach is both optimistic and pessimistic. Some people would like to talk about the "best" feedback to offer students on a given problem. I'm pessimistic that anything like that could work. Feedback is conditional on student thinking and our goals, or else it's a waste of everyone's time and we might as well go back to circling every wrong move with red pen. Still, I'm optimistic that we can productively talk to each other about feedback, as long as we ground the discussion in our shared vision of how kids think.

12.

Once, I was working with a young student on the L-Shape pattern. I asked him (Shaun) what he saw ("it's becoming an L") and if he could draw the next shape in the pattern ("that's easy!"). Here is what he drew:



I asked Shaun to elaborate, and he was happy to. He explained that the pattern is growing, straight up. He then counted off 5 squares rising up from the bottom ("oops, that's six," and he crosses out the one on top).

How should we categorize this work? If it were correct, it would be as clear an example of recursive thinking as we might hope for. Should we *still* consider this recursive thinking, or is it more like Toni's pre-recursive, totally-stuck thinking before?

I see Shaun's thinking as definitely recursive, but he has growth to do within the recursive perspective.

What I love about Shaun's thinking (along with Toni's before) is that there is a lot of development within a recursive perspective. It doesn't happen all at once. In this case, Shaun's way of seeing looks for recursive growth in *one direction*, and he hasn't yet adopted a perspective that allows him to see *multi-dimensional* growth.

It also seems to me that Shaun reveals the beginnings of a relational perspective in his analysis. He said, as a matter of course, that the fifth step in the pattern will have five squares rising up from the base of the L. This is a clear attempt to relate the step number of the pattern to an element of the picture. In his analysis, though, he is off by one, since the "tower" height is only equal to the step number if you include a square from the base. In this way, Shaun is at a stage in his development where he is able to relate the *step number* to part of the picture, but he isn't yet relating *plus-or-minus the step number* to part of the picture. (Or, alternatively, he is seeing the step number in the pattern but is not yet *compensating for overlaps*.) Eventually, we would like Shaun to be able to see the pattern in more sensitive ways.

Shaun doesn't fit well into the recursive/relational/functional scheme. To be clear, I think that scheme is still true of his work (it accurately describes his work) but it's not useful. Shaun's work, at this stage, isn't about moving from recursive to functional perspectives. Instead, he has work to do within recursive and relational perspectives.

Shaun is in second grade. If I taught a class of second graders who were a lot like Shaun, I would have written a very different analysis of student thinking with visual patterns. Maybe it would look like this:

- A. One-directional Perspective** – Sees constant growth that is happening in one part of the diagram and can use it to calculate given steps, but struggles to make sense of growth that is happening in several parts of the diagram at once.

B. Multi-directional Perspective – Sees constant growth that is happening in several parts of the diagram and can use it to calculate given steps, but struggles to see this growth as coordinated, i.e. says “it’s growing by 1 here, here and here” but doesn’t understand when someone says “it’s growing by 3 each time.”

C. Recursive Perspective – Sees constant growth quickly, fluently and accurately.

Maybe it would look entirely different. Maybe this isn’t the most useful thinking to describe for our students. Maybe it is. I don’t know!

Supposing that this scheme is true and useful, though, it might be fun to think about questions that would help Shaun adopt a multi-directional perspective. I bet the question that helped Toni would help Shaun – can you see the second picture in the third? I also bet that if I drew his attention to the multi-dimensional growth (“I notice that the base is getting longer too!”) that he could accommodate this, and his thinking would evolve.

I love a lot of things about teaching – the kids, the colleagues, the math, summer – but this sort of thinking about thinking is maybe my favorite thing about the work itself.

13.

Math is beautiful, powerful and surprising. I don’t always feel this way, but the last time I did was when I didn’t solve a geometry problem. The problem featured a big triangle that had been split into two smaller ones, and I was tasked with finding a missing angle. As days passed I gathered a flock of insights, but I couldn’t arrange them into an answer. This was killing me, so I flipped on my computer and looked up a solution. A half hour into *that* effort (in my defense the solution was in Dutch), I took a step back from the screen. I took out a piece of paper, tried to recreate the argument I had just read. Alright, alright, but there was a missing step in my argument...ah! The missing idea slipped into place and there it was, a proof.

I love how transportable math is, how this problem gave me something to think about in the shower and on walks. I love how good it felt to finally understand that proof, and it hardly mattered that I couldn't solve it on my own. (Do we ever?) I told myself the argument while giving the baby a bath, that's how nice it was to think about.

If this is what makes math great, what is it that makes *teaching* math great? I would say teaching math is a lot like math: it is beautiful, powerful and surprising, and in roughly the same sense. What I mean is, teaching is a lot of fun, intellectually speaking.

I think this often gets lost when we talk about our work. We teachers have an easy time expressing the emotional or motivational aspects of our work, and we mathies would be *very* happy to talk and talk and talk about our favorite topics or problems for as long as you'll listen. What's much harder to talk about is how classroom work is interesting to think about. That's what I set out to write about in this piece.

This piece has a perspective on what exactly the intellectual work of teaching consists of. It's important to make that perspective explicit, because it's sort of the point of the whole thing. The perspective goes like this: what matters, most of all, are the ways our students can think about a mathematical situation, and the work of a teacher is to listen, describe and systematize these ways of thinking.

"But wait," you say, "what matters most is engagement! If kids aren't engaged then no thinking happens."

Well, I certainly agree with that, and I'd happily throw in other equally important aspects of classroom math: self-concept, mindset, metacognition, having fun, content knowledge, procedural knowledge, interpersonal status, problem solving ability, feedback, effort... We can keep going and going!

Teaching is complex, and there are a lot of things that matter the most. The predominant way of talking about teaching is to pick a favorite thing (feedback!) and show how that thing applies in different teaching situations (feedback on quizzes! feedback after classwork! feedback during classwork!). Let's call this a top-down approach, in that we start with big, big abstract concepts and then apply those concepts to myriad classroom situations.

My approach in this piece is a bottom-up one, because I'm not sure that we can really talk about any of these big-picture aspects of teaching at such a high level of generality. Such talk leads to massive confusion over terms, jargon, our constant need to define and clarify our meaning, and a need to create strong oppositions between GOODWORDS and BADWORDS in education. ("That's not feedback, that's a grade. That's not a problem, it's a drill. That's not understanding, that's memorizing," and so on.) It's frustrating, and we never seem to get anywhere or learning anything from our failures to communicate.

I feel a need to ground our talk of teaching in specifics, and I think the most promising way to do this is through student thinking. That's why, in this piece, I focused on visual patterns and tried to articulate the way I see student thinking on these types of mathematical situations. I included mathematical work itself, to draw attention to our own thinking about visual patterns. It's why I held off on talking about hints or feedback until I'd said my piece about student thinking.

There are a lot of ways of thinking that have never been described in a true or useful way. This is the work of classroom teaching and teachers – no one else can really do it well, and almost no one is doing it at all. (Too busy making worksheets and activities, I suppose.)

Someone needs to do this work. Won't you give it a shot?