Supplementary Appendix for "Nonparametric Tests for Treatment Effect Heterogeneity with Duration Outcomes" by Pedro H. C. Sant'Anna

This supplementary appendix contains details of how to construct tests for the null hypothesis of zero conditional (restricted) local average treatment effect (Section S.1.1), and for the null hypothesis of homogeneous conditional local average treatment effect (Section S.1.2), and all the mathematical proofs of the theoretical results presented in the main text (Section S.2).

S.1 Additional Tests under Endogenous Treatment Allocations

S.1.1 Testing for Zero Conditional Local Average Treatment Effect

The goal of this subsection is to show how one can adapt the test of zero conditional average treatment effect described in Section 4.1 to the local treatment effect setup described in Section 4.3.

Denote the restricted conditional local average treatment effects by

$$\begin{aligned} \Upsilon_{\bar{\tau}}^{clate}(\mathbf{X}) \equiv\ & \mathbb{E}\left[Y(1)\,1\{Y(1)\leq\bar{\tau}\}\,|\mathbf{X}, pop=comp\right] \\ & -\mathbb{E}\left[Y(0)\,1\{Y(0)\leq\bar{\tau}\}\,|\mathbf{X}, pop=comp\right], \end{aligned}$$

Our goal is to test the null hypothesis

$$H_0^{late}: \Upsilon_{\bar{\tau}}^{clate}(\mathbf{X}) = 0\ a.s., \tag{S.1}$$

against H_1^{late}, which is simply the negation of (S.1). The null (S.1) is analogous to (4.1) within the LTE setup.

From Theorem 3.1 of Abadie (2003), we have that under Assumption 4.1,

$$\begin{aligned} \Upsilon_{\bar{\tau}}^{1,clate}(\mathbf{X}) &\equiv \mathbb{E}\left[Y(1)\,1\{Y(1)\leq\bar{\tau}\}\,|\mathbf{X}, pop=comp\right] \\ &= \frac{1}{\Gamma(\mathbf{X})}\left(\mathbb{E}\left[\left.\frac{TZY1\{Y\leq\bar{\tau}\}}{q_0(\mathbf{X})}\right|\mathbf{X}\right]\right. \\ &\quad \left.-\mathbb{E}\left[\left.\frac{T(1-Z)Y1\{Y\leq\bar{\tau}\}}{1-q_0(\mathbf{X})}\right|\mathbf{X}\right]\right), \end{aligned} \tag{S.2}$$

and

$$\begin{aligned} \Upsilon_{\bar{\tau}}^{0,clate}(\mathbf{X}) &\equiv \mathbb{E}\left[Y(0)\,1\{Y(0)\leq\bar{\tau}\}\,|\mathbf{X}, pop=comp\right] \\ &= \frac{1}{\Gamma(\mathbf{X})}\left(\mathbb{E}\left[\left.\frac{(1-T)(1-Z)Y1\{Y\leq\bar{\tau}\}}{1-q_0(\mathbf{X})}\right|\mathbf{X}\right]\right. \\ &\quad \left.-\mathbb{E}\left[\left.\frac{(1-T)ZY1\{Y\leq\bar{\tau}\}}{q_0(\mathbf{X})}\right|\mathbf{X}\right]\right) \end{aligned} \tag{S.3}$$

where

$$\Gamma(\mathbf{X})=\mathbb{E}\left[\left(\frac{TZ}{q_0(\mathbf{X})}-\frac{T(1-Z)}{1-q_0(\mathbf{X})}\right)\Bigg|\mathbf{X}\right],$$

and $\Gamma(\mathbf{X})=P(T(1)>T(0)|\mathbf{X})>0$ $a.s.$ see Lemma 2.1 of Abadie (2003).

Combining these results with those analogous to Lemma 1 and Lemma 2 (and Theorem 7), we have that H_0^{late} is true if and only if

$$I_{\bar{\tau}}^{late}(\mathbf{x})=0 \ a.e. \text{ in } \mathcal{X}_X,$$

where $I_{\bar{\tau}}^{late}(\mathbf{x})=I_{\bar{\tau}}^{1,late}(\mathbf{x})-I_{\bar{\tau}}^{0,late}(\mathbf{x})$, and for $t\in\{0,1\}$,

$$\begin{aligned}I^{t,ldte}(\mathbf{x})\equiv(2t-1)\Bigg\{&\mathbb{E}^{km}\left[\frac{Q1\{Q\leq\bar{\tau}\}}{q_0(\mathbf{X})}1\{X\leq\mathbf{x}\}|T=t,Z=1\right]\mathbb{P}(T=t,Z=1)\\&-\mathbb{E}^{km}\left[\frac{Q1\{Q\leq\bar{\tau}\}}{1-q_0(\mathbf{X})}1\{X\leq\mathbf{x}\}|T=t,Z=0\right]\mathbb{P}(T=t,Z=0)\Bigg\}\end{aligned}$$

Then, as discussed in Section 2.3, our KS type test statistic for hypothesis (S.1) is

$$KS_{\bar{\tau},n}^{late}=\sqrt{n}\sup_{\mathbf{x}\in\mathcal{X}_X}\left|\hat{I}_{\bar{\tau},n}^{late}(\mathbf{x})\right|,$$

where $\hat{I}_{\bar{\tau},n}^{late}(\mathbf{x})=\hat{I}_{\bar{\tau},n}^{1,late}(\mathbf{x})-\hat{I}_{\bar{\tau},n}^{0,late}(\mathbf{x})$,

$$\begin{aligned}\hat{I}_{\bar{\tau},n}^{1,late}(\mathbf{x}) &= \frac{n_{11}}{n}\sum_{i=1}^{n_{11}}W_{in_{11}}\frac{Q_{1:n_{11}}1\{Q_{1:n_{11}}\leq\bar{\tau}\}1\left\{\mathbf{X}_{[i:n_{11}]}\leq\mathbf{x}\right\}}{\hat{q}_n\left(X_{[i:n_{11}]}\right)}\\&\quad-\frac{n_{10}}{n}\sum_{i=1}^{n_{10}}W_{in_{10}}\frac{Q_{1:n_{10}}1\{Q_{1:n_{10}}\leq\bar{\tau}\}1\left\{\mathbf{X}_{[i:n_{10}]}\leq\mathbf{x}\right\}}{1-\hat{q}_n\left(\mathbf{X}_{[i:n_{10}]}\right)},\\\hat{I}_{\bar{\tau},n}^{0,late}(\mathbf{x}) &= \frac{n_{00}}{n}\sum_{j=1}^{n_{00}}W_{jn_{00}}\frac{Q_{j:n_{00}}1\{Q_{j:n_{00}}\leq\bar{\tau}\}1\left\{\mathbf{X}_{[j:n_{00}]}\leq\mathbf{x}\right\}}{1-\hat{q}_n\left(\mathbf{X}_{[j:n_{00}]}\right)}\\&\quad-\frac{n_{01}}{n}\sum_{j=1}^{n_{01}}W_{jn_{01}}\frac{Q_{j:n_{01}}1\{Q_{j:n_{01}}\leq\bar{\tau}\}1\left\{\mathbf{X}_{[j:n_{01}]}\leq\mathbf{x}\right\}}{\hat{q}_n\left(\mathbf{X}_{[j:n_{01}]}\right)},\end{aligned}$$

with $W_{in_{tz}}$ as defined in (B.4). The discussion for the Cramér-von Mises test is the same and is therefore omitted.

In the next theorem, we state the asymptotic properties of $KS_{\bar{\tau},n}^{late}$. Using an analogous procedure to the one described in Section 3.4, let $c_{\bar{\tau},\alpha,n}^{late,*}$ denote the bootstrap critical value of the $KS_{\bar{\tau},n}^{late}$.

Theorem S.1 *Suppose Assumptions 4.1-4.2 are satisfied. Further, suppose that for the sub-population of compliers, Assumption A.2, A.7, and A.8 are satisfied, and that q_0 and its SLE $\hat{q}_n$ satisfy the analogous of Assumptions A.4 and A.5. Then, for a fixed $\bar{\tau}\leq\tau$,*

1. *Under H_0^{late}, $\lim_{n\to\infty}\mathbb{P}_n\left\{KS_{\bar{\tau},n}^{late}>c_{\bar{\tau},\alpha,n}^{late,*}\right\}=\alpha$.*

2. *Under* H_1^{late}, $\lim_{n\to\infty}\mathbb{P}_n\left\{KS_{\bar{\tau},n}^{late} > c_{\bar{\tau},\alpha,n}^{late,*}\right\} = 1$.

3. *Under* $H_{1,n}^{late}: \Upsilon_{\bar{\tau}}^{late}(\mathbf{X}) = \frac{1}{\sqrt{n}} h_{\bar{\tau}}^{late}(\mathbf{X})$ *a.s., if* $h_{\bar{\tau}}^{late}(\cdot)$ *is an integrable function, and the set* $h_{\bar{\tau},n}^{late} \equiv \left\{\mathbf{x}\in\mathcal{X}_X : n^{-1/2}h_{\bar{\tau}}^{late}(\mathbf{x}) \neq 0\right\}$ *has positive Lebesgue measure,* $\lim_{n\to\infty}\mathbb{P}_n\left\{KS_{\bar{\tau},n}^{late} > c_{\bar{\tau},\alpha,n}^{late,*}\right\} > \alpha$.

The proof of Theorem S.1 follows from the same steps as the proofs of Theorems 5 and 7, and therefore is omitted.

S.1.2 Testing for Homogeneous Conditional Local Average Treatment Effect

The goal of this subsection is to show how one can adapt the test of homogenous conditional average treatment effect described in Section 4.2 to the local treatment effect setup described in Section 4.3. More precisely, the goal is to test the null hypothesis

$$H_0^{l\hom} : \exists\ \Upsilon_{\bar{\tau}}^{l} \in \mathbb{R} : \Upsilon_{\bar{\tau}}^{clate}(\mathbf{X}) = \Upsilon_{\bar{\tau}}^{l}\ a.s., \tag{S.4}$$

against $H_1^{l\hom}$, which is simply the negation of (S.1). The null (S.1) is analogous to (4.2) within the LTE setup.

From Theorem 3.1 of Abadie (2003) and the results discussed in Section S.1.1, $H_0^{l\hom}$ is true if and only if

$$\Upsilon_{\bar{\tau}}^{1,clate}(\mathbf{X}) - \Upsilon_{\bar{\tau}}^{0,clate}(\mathbf{X}) = \Upsilon_{\bar{\tau}}^{late}\ a.s., \tag{S.5}$$

where $\Upsilon_{\bar{\tau}}^{1,clate}(\mathbf{X})$ and $\Upsilon_{\bar{\tau}}^{0,clate}(\mathbf{X})$ are defined as in (S.2) and (S.3), respectively, and $\Upsilon_{\bar{\tau}}^{late} = \Upsilon_{\bar{\tau}}^{1,late} - \Upsilon_{\bar{\tau}}^{0,late}$ with

$$\begin{aligned}
\Upsilon_{\bar{\tau}}^{1,late} &= \left(\mathbb{E}\left[\frac{TZY1\{Y\leq\bar{\tau}\}}{q_0(\mathbf{X})}\right] - \mathbb{E}\left[\frac{T(1-Z)Y1\{Y\leq\bar{\tau}\}}{1-q_0(\mathbf{X})}\right]\right),\\
\Upsilon_{\bar{\tau}}^{0,late} &= \frac{1}{\Gamma}\left(\mathbb{E}\left[\frac{(1-T)(1-Z)Y1\{Y\leq\bar{\tau}\}}{1-q_0(\mathbf{X})}\right] - \mathbb{E}\left[\frac{(1-T)ZY1\{Y\leq\bar{\tau}\}}{q_0(\mathbf{X})}\right]\right),
\end{aligned}$$

and

$$\Gamma = \mathbb{E}\left[\left(\frac{TZ}{q_0(\mathbf{X})} - \frac{T(1-Z)}{1-q_0(\mathbf{X})}\right)\right] > 0.$$

Multiplying both sides of (S.5) by $\Gamma(\mathbf{X}) > 0$ $a.s.$, and rearranging the terms, we have that (S.5) is equivalent to

$$\begin{aligned}
&\mathbb{E}\left[T\left(\frac{Z}{q_0(\mathbf{X})} - \frac{(1-Z)}{1-q_0(\mathbf{X})}\right)\left(Y1\{Y\leq\bar{\tau}\} - \Upsilon_{\bar{\tau}}^{late}\right)\middle|\mathbf{X}\right]\\
&-\mathbb{E}\left[(1-T)\left(\frac{(1-Z)}{1-q_0(\mathbf{X})} - \frac{Z}{q_0(\mathbf{X})}\right)Y1\{Y\leq\bar{\tau}\}\middle|\mathbf{X}\right]\\
=&\ 0\ a.s..
\end{aligned}$$

Combining the aforementioned results with those analogous to Lemma 1 and Lemma 2 (and Theorem 7), we have that $H_0^{l\,\text{hom}}$ is true if and only if

$$I_{\bar{\tau}}^{l\,\text{hom}}(\mathbf{x}) = 0 \; a.e. \text{ in } \mathcal{X}_X,$$

where $I_{\bar{\tau}}^{l\,\text{hom}}(\mathbf{x}) = I_{\bar{\tau}}^{1,l\,\text{hom}}(\mathbf{x}) - I_{\bar{\tau}}^{0,l\,\text{hom}}(\mathbf{x})$,

$$\begin{aligned} & I^{t,l\,\text{hom}}(\mathbf{x}) \\ & \equiv (2t-1)\left\{ \mathbb{E}^{km}\left[\frac{\left(Q1\{Q \le \bar{\tau}\} - T\Upsilon_{\bar{\tau}}^{late}\right)1\{\mathbf{X} \le \mathbf{x}\}}{q_0(\mathbf{X})} \middle| T=t, Z=1 \right] \mathbb{P}(T=t, Z=1) \right. \\ & \quad \left. - \mathbb{E}^{km}\left[\frac{\left(Q1\{Q \le \bar{\tau}\} - T\Upsilon_{\bar{\tau}}^{late}\right)1\{\mathbf{X} \le \mathbf{x}\}}{1-q_0(\mathbf{X})} \middle| T=t, Z=0 \right] \mathbb{P}(T=t, Z=1) \right\} \end{aligned}$$

Then, as discussed in Section 2.3, our KS type test statistic for hypothesis (S.4) is

$$KS_{\bar{\tau},n}^{l\,\text{hom}} = \sqrt{n} \sup_{\mathbf{x} \in \mathcal{X}_X} \left| I_{\bar{\tau}}^{l\,\text{hom}}(\mathbf{x}) \right|,$$

where $\hat{I}_{\bar{\tau},n}^{l,\text{hom}}(\mathbf{x}) = \hat{I}_{\bar{\tau},n}^{1,l\,\text{hom}}(\mathbf{x}) - \hat{I}_{\bar{\tau},n}^{0,l\,\text{hom}}(\mathbf{x})$,

$$\begin{aligned} \hat{I}_{\bar{\tau},n}^{1,l\,\text{hom}}(x) &= \frac{n_{11}}{n}\sum_{i=1}^{n_{11}} W_{in_{11}} \frac{\left(Q_{1:n_{11}}1\{Q_{1:n_{11}} \le \bar{\tau}\} - \hat{\Upsilon}_{\bar{\tau},n}^{late}\right)1\left\{\mathbf{X}_{[i:n_{11}]} \le \mathbf{x}\right\}}{\hat{q}_n\left(\mathbf{X}_{[i:n_{11}]}\right)} \\ & \quad - \frac{n_{10}}{n}\sum_{i=1}^{n_{10}} W_{in_{10}} \frac{\left(Q_{1:n_{10}}1\{Q_{1:n_{10}} \le \bar{\tau}\} - \hat{\Upsilon}_{\bar{\tau},n}^{late}\right)1\left\{\mathbf{X}_{[i:n_{10}]} \le \mathbf{x}\right\}}{1-\hat{q}_n\left(\mathbf{X}_{[i:n_{10}]}\right)}, \\ \hat{I}_{\bar{\tau},n}^{0,l\,\text{hom}}(x) &= \frac{n_{00}}{n}\sum_{j=1}^{n_{00}} W_{jn_{00}} \frac{Q_{j:n_{00}}1\{Q_{j:n_{00}} \le \bar{\tau}\}1\left\{\mathbf{X}_{[j:n_{00}]} \le \mathbf{x}\right\}}{1-\hat{q}_n\left(\mathbf{X}_{[j:n_{00}]}\right)} \\ & \quad - \frac{n_{01}}{n}\sum_{j=1}^{n_{01}} W_{jn_{01}} \frac{Q_{j:n_{01}}1\{Q_{j:n_{01}} \le \bar{\tau}\}1\left\{\mathbf{X}_{[j:n_{01}]} \le \mathbf{x}\right\}}{\hat{q}_n\left(\mathbf{X}_{[j:n_{01}]}\right)}, \end{aligned}$$

with $W_{in_{tz}}$ as defined in (B.4). The discussion for the Cramér-von Mises test is the same and is therefore omitted.

In the next theorem, we state the asymptotic properties of $KS_{\bar{\tau},n}^{l\,\text{hom}}$. Using an analogous procedure to the one described in Section 3.4, let $c_{\bar{\tau},\alpha,n}^{l\,\text{hom},*}$ denote the bootstrap critical value of the $KS_{\bar{\tau},n}^{l\,\text{hom}}$.

Theorem S.2 *Suppose Assumptions 4.1-4.2 are satisfied. Further, suppose that for the sub-population of compliers, Assumption A.2, A.7, and A.8 are satisfied, and that q_0 and its SLE $\hat{q}_n$ satisfy the analogous of Assumptions A.4 and A.5. Then, for a fixed $\bar{\tau} \le \tau$,*

1. *Under $H_0^{l\,\text{hom}}$, $\lim_{n\to\infty} \mathbb{P}_n\left\{KS_{\bar{\tau},n}^{l\,\text{hom}} > c_{\bar{\tau},\alpha,n}^{l\,\text{hom},*}\right\} = \alpha$.*

2. *Under* $H_1^{l\,\hom}$, $\lim_{n\to\infty}\mathbb{P}_n\left\{KS_{\bar{\tau},n}^{l\,\hom} > c_{\bar{\tau},\alpha,n}^{l\,\hom,*}\right\} = 1$.

3. *Under* $H_{1,n}^{l\,\hom}: \Upsilon_{\bar{\tau}}^{clate}(\mathbf{X}) - \Upsilon_{\bar{\tau}}^{l} = n^{-1/2}h_{\bar{\tau}}^{l\,\hom}(\mathbf{X})$ *a.s., if* $h_{\bar{\tau}}^{l\,\hom}(\cdot)$ *is an integrable function, and the set* $h_{\bar{\tau},n}^{l\,\hom} \equiv \left\{\mathbf{x}\in\mathcal{X}_X : n^{-1/2}h_{\bar{\tau}}^{l\,\hom}(\mathbf{x}) \neq 0\right\}$ *has positive Lebesgue measure, then* $\lim_{n\to\infty}\mathbb{P}_n\left\{KS_{\bar{\tau},n}^{l\,\hom} > c_{\bar{\tau},\alpha,n}^{l\,\hom,*}\right\} > \alpha$.

The proof of Theorem S.2 follows from the same steps as the proofs of Theorems 7 and 7, and therefore is omitted.

S.2 Mathematical Proofs

Before proving the main results of the article, we first introduce some notation. For a generic set $\mathcal{G}$, let $l^\infty(\mathcal{G})$ be the Banach space of all uniformly bounded real functions on $\mathcal{G}$ equipped with the uniform metric $\|f\|_{\mathcal{G}} \equiv \sup_{z\in\mathcal{G}}|f(z)|$. We consider convergence in distribution of empirical processes in the metric space $\left(l^\infty(\mathcal{G}), \|f\|_{\mathcal{G}}\right)$ in the sense of J. Hoffman-Jørgensen (see, e.g. , van der Vaart and Wellner (1996)). For any generic Euclidean random vector ξ on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, $\mathcal{X}_X$ denotes its state space and $P_{\boldsymbol{\xi}}$ denotes its induced probability measure with corresponding distribution function $F_\xi(\cdot) = P_\xi(-\infty,\cdot]$. Define $\mathcal{W} \equiv [-\infty,\tau]\times\mathcal{X}_X$. Throughout the appendix, denote $\mathcal{C}$ as a generic finite constant that may change from expression to expression. Finally, all random variables are defined on a common probability space $(\Omega,\mathcal{A},\mathbb{P})$.

S.2.1 Proofs of Lemmas 1 and 2

First, we present the proofs of the identification result in Lemma 1, and the characterization of the null hypothesis in Lemma 2.

Proof of Lemma 1: In the absence of censoring, we have that, for $t\in\{0,1\}$, by Assumption 2.1 and the law of total probability,

$$\begin{aligned}\mathbb{E}\left[h\left(Y(t),\mathbf{X},t\right)\right] &= \mathbb{E}\left[\frac{1\{T=t\}\,h\left(Y,\mathbf{X},T\right)}{\mathbb{P}\left(T=t|\mathbf{X}\right)}\right]\\ &= \int h\left(\bar{y},\bar{\mathbf{x}},t\right)F_{Y,\mathbf{X}|T}\left(d\bar{y},d\bar{\mathbf{x}}|t\right)\mathbb{P}\left(T=t\right),\end{aligned}$$

where the first equality is due to Rosenbaum and Rubin (1983). Thus, our result follows from noticing that, as shown by (Stute, 1993, Remark 1.4) and (Sant'Anna, 2016, Proposition 1),

$$F_{Q,\mathbf{X}|T}^{km}(y,\mathbf{x}|t) = \begin{cases} F_{Y,\mathbf{X}|T}(y,\mathbf{x}|t), & if\ y<\tau,\\ F_{Y,\mathbf{X}|T}(y-,\mathbf{x}|t) + 1\{\tau(t)\in A(t)\}\,F_{Y,\mathbf{X}|T}(\{\tau\},\mathbf{x}|t) & if\ y\geq\tau,\end{cases}$$

whenever censoring is present and Assumption 2.2 is satisfied. Conditional results follows from taking the appropriate Radon-Nikodym derivative of $F_{Q,\mathbf{X}|T}^{km}(y,\mathbf{x}|t)$. ■

Proof of Lemma 2: From Lemma 1, we have that, $\forall y \in [-\infty, \tau]$,

$$\Upsilon(y|\mathbf{X}) = \mathbb{E}^{km}\left[\frac{T1\{Q \le y\}}{p_0(\mathbf{X})} - \frac{(1-T)1\{Q \le y\}}{1-p_0(\mathbf{X})}\middle|\mathbf{X}\right] \text{ a.s.}$$

Define the finite Borel measure

$$v(B) = \int_B \Upsilon(y|\mathbf{x})\,\mathbb{P}_{\mathbf{X}}(d\mathbf{x}),$$

where $\mathbb{P}_X$ is the probability measure associated with $\mathbf{X}$ and $B \in \mathcal{B}$, with $\mathcal{B}$ the Borel σ-field of $\mathbb{R}^k$ in the product topology. Then, the proof follows directly from Lemma 1 of Escanciano (2006). ■

S.2.2 Proofs: Testing for Zero Conditional Distributional Treatment Effect

Before we proceed to the proof Lemma 3, we introduce the following auxiliary lemma.

Lemma S.1 *Under Assumptions 2.1 (ii), A.2, A.4 and A.5,* $|\hat{p}_n(\mathbf{x}) - p_0(\mathbf{x})|_\infty = o_{\mathbb{P}}\left(n^{-1/4}\right)$.

Proof of Lemma S.1: Define the pseudo true propensity score $p^{pseudo}(\mathbf{x}) = L\left(\mathbf{R}^L(\mathbf{x})'\boldsymbol{\pi}_{0,L}\right)$, where

$$\boldsymbol{\pi}_{0,L} = \arg\max_{\boldsymbol{\pi}} \mathbb{E}\left[p_0(\mathbf{X})\log\left(\mathcal{L}\left(\mathbf{R}^L(\mathbf{X})'\boldsymbol{\pi}\right)\right) + (1-p_0(\mathbf{X}))\log\left(1-\mathcal{L}\left(\mathbf{R}^L(\mathbf{X})'\boldsymbol{\pi}\right)\right)\right].$$

Let $\boldsymbol{\zeta}(L) = \sup_{\mathbf{x}\in\mathcal{X}}\left|\mathbf{R}^L(\mathbf{x})\right|$. Then, it follows from the triangle inequality, the mean value theorem, and Lemmas 1 and 2 of Hirano et al. (2003),

$$\begin{aligned}
|\hat{p}_n(\mathbf{x}) - p_0(\mathbf{x})|_\infty &= \left|\hat{p}_n(\mathbf{x}) - p^{pseudo}(\mathbf{x}) + p^{pseudo}(\mathbf{x}) - p_0(\mathbf{x})\right|_\infty \\
&\le \left|\hat{p}_n(\mathbf{x}) - p^{pseudo}(\mathbf{x})\right|_\infty + \left|p^{pseudo}(\mathbf{x}) - p_0(\mathbf{x})\right|_\infty \\
&\le \mathcal{C}\boldsymbol{\zeta}(L)\left\|\boldsymbol{\pi}_{n,L} - \boldsymbol{\pi}_{0,L}\right\| + \left|p^{pseudo}(\mathbf{x}) - p_0(\mathbf{x})\right|_\infty \\
&= O_{\mathbb{P}}\left(\boldsymbol{\zeta}(L)\sqrt{\frac{L}{n}}\right) + O_{\mathbb{P}}\left(\boldsymbol{\zeta}(L)L^{-s/2k}\right) \\
&= o_{\mathbb{P}}\left(n^{-1/4}\right)
\end{aligned}$$

where the last equality follow from Assumption A.5. ■

In the following we prove Lemma 3. With some abuse of notation, let

$$\begin{aligned}
I_1^0(y,\mathbf{x},p) &= \mathbb{E}\left[\frac{1-T}{1-p(\mathbf{X})}1\{Y \le y\}1\{\mathbf{X} \le \mathbf{x}\}\right], \\
I_1^1(y,\mathbf{x},p) &= \mathbb{E}\left[\frac{T}{p(\mathbf{X})}1\{Y \le y\}1\{\mathbf{X} \le \mathbf{x}\}\right], \\
I_{1n}^0(y,\mathbf{x},p) &= \mathbb{E}_n^{km}\left[\frac{1-T}{1-p(\mathbf{X})}1\{Q \le y\}1\{\mathbf{X} \le \mathbf{x}\}\right]
\end{aligned}$$

$$I_1^1(y,\mathbf{x},p) \;=\; \mathbb{E}_n^{km}\left[\frac{T}{p(\mathbf{X})}1\{Q\leq y\}1\{\mathbf{X}\leq\mathbf{x}\}\right].$$

Proof of Lemma 3: Notice that

$$\begin{aligned}\sqrt{n}\left(\hat{I}_{1,n}(y,\mathbf{x})-I_1(y,\mathbf{x})\right) &= \sqrt{n}\left(I_{1n}^1(y,\mathbf{x},\hat{p}_n)-I_1^1(y,\mathbf{x},p_0)\right)\\ &\quad -\sqrt{n}\left(I_{1n}^0(y,\mathbf{x},\hat{p}_n)-I_1^0(y,\mathbf{x},p_0)\right).\end{aligned}$$

Therefore, we can work with $\sqrt{n}\left(I_{1n}^1(y,\mathbf{x},\hat{p}_n)-I_1^1(y,\mathbf{x},p)\right)$ and $\sqrt{n}\left(I_{1n}^0(y,\mathbf{x},\hat{p}_n)-I_1^0(y,\mathbf{x},p)\right)$ separately. Given that these two functionals have symmetric construction, we only provide detailed arguments for the linear representation of $\sqrt{n}\left(I_{1n}^1(y,\mathbf{x},\hat{p}_n)-I_1^1(y,\mathbf{x},p)\right)$.

First, write

$$\begin{aligned}&\sqrt{n}\left(I_{1n}^1(y,\mathbf{x},\hat{p}_n)-I_1^1(y,\mathbf{x},p_0)\right)\\ =\;& \sqrt{n}\int\varphi_{y,x,p_0}^1(\bar{y},\bar{\mathbf{x}},\bar{t})\left[F_n^{km}(d\bar{y},d\bar{\mathbf{x}},d\bar{t})-F(d\bar{y},d\bar{\mathbf{x}},d\bar{t})\right]\\ &+\sqrt{n}\int\left[\varphi_{y,x,\hat{p}_n}^1(\bar{y},\bar{\mathbf{x}},\bar{t})-\varphi_{y,x,p_0}^1(\bar{y},\bar{\mathbf{x}},\bar{t})\right]\left(F_n^{km}(d\bar{y},d\bar{\mathbf{x}},d\bar{t})-F(d\bar{y},d\bar{\mathbf{x}},d\bar{t})\right)\\ &+\sqrt{n}\int\left[\varphi_{y,x,\hat{p}_n}^1(\bar{y},\bar{\mathbf{x}},\bar{t})-\varphi_{y,x,p_0}^1(\bar{y},\bar{\mathbf{x}},\bar{t})\right]F(d\bar{y},d\bar{\mathbf{x}},d\bar{t})\\ =\;& \mathbb{A}_{1n}+\mathbb{A}_{2n}+\mathbb{A}_{3n},\end{aligned}$$

where, for a generic $p(\cdot)$, $\varphi_{y,\mathbf{x},p}^1(\bar{y},\bar{\mathbf{x}},\bar{t})=\bar{t}1\{\bar{y}\leq y\}1\{\bar{\mathbf{x}}\leq\mathbf{x}\}/p(\bar{x})$.

From Theorem 1.1 of Stute (1996), we have that, under Assumptions 2.1, 2.2, and A.6,

$$\mathbb{A}_{1n}=\frac{1}{\sqrt{n}}\sum_{i=1}^n\left(\eta_{1,i}(y,\mathbf{x})-I_1^1(y,\mathbf{x})\right), \tag{S.1}$$

where $\eta_{1,i}(y,\mathbf{x})$ is defined as in (3.6).

Next, by a Taylor expansion argument, we have that

$$\begin{aligned}A_{2n} &= \sqrt{n}\int\left(\frac{\bar{t}1\{\bar{y}\leq y\}1\{\bar{\mathbf{x}}\leq\mathbf{x}\}(\bar{p}(\bar{\mathbf{x}})-p_0(\bar{\mathbf{x}}))}{p_0^2(\bar{\mathbf{x}})}\right)\left(F_n^{km}(d\bar{y},d\bar{\mathbf{x}},d\bar{t})-F(d\bar{y},d\bar{\mathbf{x}},d\bar{t})\right)\\ &\leq \mathcal{C}\left|(\hat{p}_n(\bar{\mathbf{x}})-p_0(\bar{\mathbf{x}}))\right|_\infty\sqrt{n}\int_{\mathcal{W}}\left(F_n^{km}(d\bar{y},d\bar{\mathbf{x}},d\bar{t})-F(d\bar{y},d\bar{\mathbf{x}},d\bar{t})\right)\\ &\leq \mathcal{C}\left|\hat{p}_n(\bar{\mathbf{x}})-p_0(\bar{\mathbf{x}})\right|_\infty O_{\mathbb{P}}(1)\\ &= o_{\mathbb{P}}\left(n^{-1/4}\right)O_{\mathbb{P}}(1)\\ &= o_{\mathbb{P}}(1),\end{aligned}\tag{S.2}$$

where the first inequality follows from $|\bar{p}(\mathbf{X})-p_0(\mathbf{X})|\leq|\hat{p}_n(\mathbf{X})-p_0(\mathbf{X})|$, $p_0>\varepsilon$ *a.s.*, and $\bar{t}1\{\cdot\leq y\}1\{\cdot\leq x\}\leq 1$ *a.s.*, the second inequality follows from Theorem 1.1 of Stute (1996), and the last part follows from Lemma S.1.

Finally, we analyze the $\mathbb{A}_{3n}$ term. Note that we can rewrite $\mathbb{A}_{3n}$ as

$$\begin{aligned}\mathbb{A}_{3n} &= \sqrt{n}\left(I_1^1(y,\mathbf{x},\hat{p}_n) - I_1^1(y,\mathbf{x},p_0) - \Gamma_1^1(y,\mathbf{x},p_0)[\hat{p}_n - p_0]\right)\\ &\quad + \sqrt{n}\left(\Gamma_1^1(y,\mathbf{x},p_0)[\hat{p}_n - p_0]\right)\\ &= \mathbb{A}_{3n}^1 + \mathbb{A}_{3n}^2,\end{aligned}$$

where

$$\Gamma_1^1(y,\mathbf{x},p)[p-p_0] = -\mathbb{E}\left[\frac{T1\{Y\le y\}1\{\mathbf{X}\le\mathbf{x}\}}{p_0^2(\mathbf{X})}(p(\mathbf{X}) - p_0(\mathbf{X}))\right]$$

is the pathwise derivative of $I_1^1(y,\mathbf{x},p_0)$ with respect to p. But notice that

$$\begin{aligned}&\left|I_1^1(y,\mathbf{x},\hat{p}_n) - I_1^1(y,\mathbf{x},p_0) - \Gamma_1^1(y,\mathbf{x},p_0)[\hat{p}_n - p_0]\right|_\infty\\ \le\ & 2\left|\mathbb{E}\left[\frac{T1\{Y\le y\}1\{\mathbf{X}\le\mathbf{x}\}}{p_0^3(\mathbf{X})}(\bar{p}(\mathbf{X}) - p_0(\mathbf{X}))^2\right]\right|_\infty\\ \le\ & 2\left|\mathbb{E}\left[\frac{T1\{Y\le y\}1\{\mathbf{X}\le\mathbf{x}\}}{p_0^3(\mathbf{X})}\right]\right|_\infty |p(\mathbf{x}) - p_0(\mathbf{x})|_\infty^2\\ \le\ & \mathcal{C}\,|p(\mathbf{x}) - p_0(\mathbf{x})|_\infty^2\\ =\ & o_{\mathbb{P}}\left(n^{-1/2}\right)\end{aligned}$$

where the first inequality follows from a Taylor expansion argument, the second inequality follows from $|\bar{p}(\mathbf{X}) - p_0(\mathbf{X})| \le |p(\mathbf{X}) - p_0(\mathbf{X})|$, and the last one from $p_0 > \varepsilon$ *a.s.*, and $T1\{Y\le y\}1\{\mathbf{X}\le\mathbf{x}\}\le 1$ *a.s.*, and Lemma S.1. Thus, we have that

$$\mathbb{A}_{3n}^1 = o_{\mathbb{P}}(1), \tag{S.3}$$

uniformly in $(y,\mathbf{x})\in\mathcal{W}$.

To complete the proof of the linear representation of $\sqrt{n}(I_{1n}(y,\mathbf{x}) - I_1(y,\mathbf{x}))$, we have to show that

$$\mathbb{A}_{3n}^2 = -\frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{F_{Y(1)|X}(y|\mathbf{X}_i)1\{\mathbf{X}_i\le\mathbf{x}\}}{p_0(\mathbf{X}_i)}(T_i - p_0(\mathbf{X}_i)) + o_{\mathbb{P}}(1). \tag{S.4}$$

To this end, notice that by the Law of Iterated Expectations, we have that

$$\Gamma_1^1(y,\mathbf{x},p_0)[\hat{p}_n - p] = \mathbb{E}\left[-\frac{F_{Y(1)|X}(y|\mathbf{X})1\{\mathbf{X}\le\mathbf{x}\}}{p_0(\mathbf{X})}(\hat{p}_n(\mathbf{X}) - p_0(\mathbf{X}))\right].$$

Then, by using the same arguments as in Hirano et al. (2003)'s Addendum, we have that

$$\begin{aligned}&\sqrt{n}\mathbb{E}\left[-\frac{F_{Y(1)|\mathbf{X}}(y|\mathbf{X})1\{\mathbf{X}\le\mathbf{x}\}(\hat{p}_n(\mathbf{X}) - p_0(\mathbf{X}))}{p_0(\mathbf{X})}\right]\\ =\ & \frac{-1}{\sqrt{n}}\sum_{i=1}^n\frac{F_{Y(1)|\mathbf{X}}(y|\mathbf{X}_i)1\{\mathbf{X}_i\le\mathbf{x}\}}{p_0(\mathbf{X}_i)}(T_i - p_0(\mathbf{X}_i))\end{aligned}$$

$$
\begin{aligned}
& +O_{\mathbb{P}}\left(\boldsymbol{\zeta}(L) L^{-\frac{s}{2k}}\right)+O_{\mathbb{P}}\left(\frac{\boldsymbol{\zeta}(L)^2}{\sqrt{n}}\right)+O_{\mathbb{P}}\left(\sqrt{n}\boldsymbol{\zeta}(L) L_n^{-\frac{s}{2k}}\right) \\
& +O_{\mathbb{P}}\left(\frac{\boldsymbol{\zeta}(L)^{\frac{11}{2}}}{\sqrt{n}}\right)+O_{\mathbb{P}}\left(\max\left(L^{-\frac{1}{2k}}, \boldsymbol{\zeta}(L_n) L^{-\frac{s}{2k}}\right)\right) \\
= & \frac{-1}{\sqrt{n}} \sum_{i=1}^{n} \frac{F_{Y(1)|\mathbf{X}}(y|\mathbf{X}_i) 1\{\mathbf{X}_i \leq \mathbf{x}\}}{p_0(\mathbf{X}_i)}(T_i-p_0(\mathbf{X}_i)) \\
& +O_{\mathbb{P}}\left(n^{-\left(\frac{s}{2k}-1\right)v}\right)+O_{\mathbb{P}}\left(n^{2v-\frac{1}{2}}\right)+O_{\mathbb{P}}\left(n^{-\left(\frac{s}{2k}-1\right)v+\frac{1}{2}}\right) \\
& +O_{\mathbb{P}}\left(n^{\frac{11}{2}v-1/2}\right)+O_{\mathbb{P}}\left(\max\left(L^{1-\frac{s}{2k}}, L^{-\frac{1}{2k}}\right)\right) \\
= & \frac{-1}{\sqrt{n}} \sum_{i=1}^{n} \frac{F_{Y(1)|\mathbf{X}}(y|\mathbf{X}_i) 1\{\mathbf{X}_i \leq \mathbf{x}\}}{p_0(\mathbf{X}_i)}(T_i-p_0(\mathbf{X}_i))+o_{\mathbb{P}}(1),
\end{aligned}
$$

where the second equality follows from using power series and setting $L = a \cdot N^v$ as in Assumption A.5, and the last equality follows from the conditions we have imposed on v. Then, (S.4) follows.

By combining (S.1)-(S.4), we have that

$$
\begin{aligned}
& \sqrt{n}\left(I_{1n}^1(y, \mathbf{x}, \hat{p}_n)-I_1^1(y, \mathbf{x}, p)\right)= \\
& \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n}\left(\eta_{1,i}(y, \mathbf{x})-I_1^1(y, \mathbf{x})-\frac{F_{Y(1)|\mathbf{X}}(y|\mathbf{X}_i) 1\{\mathbf{X}_i \leq \mathbf{x}\}}{p_0(\mathbf{X}_i)}(T_i-p_0(\mathbf{X}_i))\right)+o_{\mathbb{P}}(1), \quad \text{(S.5)}
\end{aligned}
$$

completing the proof of the asymptotic linear representation of $\sqrt{n}\left(I_{1n}^1(y, \mathbf{x}, \hat{p}_n)-I_1^1(y, \mathbf{x}, p)\right)$.

Following the same steps as above, we can show that

$$
\begin{aligned}
& \sqrt{n}\left(I_{1n}^0(y, \mathbf{x}, \hat{p}_n)-I_1^0(y, \mathbf{x}, p)\right)= \\
& \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n}\left(\eta_{0,i}(y, \mathbf{x})-I_1^0(y, \mathbf{x})+\frac{F_{Y(0)|\mathbf{X}}(y|\mathbf{X}_i) 1\{\mathbf{X}_i \leq \mathbf{x}\}}{1-p_0(\mathbf{X}_i)}(T_i-p_0(\mathbf{X}_i))\right)+o_{\mathbb{P}}(1). \quad \text{(S.6)}
\end{aligned}
$$

Then, the proof of Lemma 3 follows from (S.5) and (S.6). ■

Next, we prove our main theorems.

Proof of Theorem 1: From the asymptotic representation in Lemma 3, it suffices to prove the convergence of the dominant term. To this end, define the class of real-valued measurable functions

$$
\begin{aligned}
\mathcal{F}=\{ & \left(\bar{y}, \bar{\mathbf{x}}, \bar{t}, \bar{\delta}\right) \rightarrow \varphi_{(y,\mathbf{x})} \equiv \eta(y, \mathbf{x}) \\
& -\left(\frac{F_{Y(1)|\mathbf{X}}(y|\bar{\mathbf{x}})}{p_0(\bar{\mathbf{x}})}+\frac{F_{Y(0)|\mathbf{X}}(y|\bar{\mathbf{x}})}{1-p_0(\bar{\mathbf{x}})}\right) 1\{\bar{\mathbf{x}} \leq \mathbf{x}\}(\bar{t}-p(\bar{\mathbf{x}})):(y, \mathbf{x}) \in \mathcal{W}\}
\end{aligned}
$$

where $\eta(y, \mathbf{x})$ is defined as in (3.7)

Our goal is to show that class of functions $\mathcal{F}$ is Donsker. By Theorem 2.10.6 in van der Vaart and Wellner (1996) it suffices to show that, for $t = 0, 1$, the classes of functions

$$\begin{aligned}
\mathcal{G}_{1,t} &\equiv \left\{\left(\bar{y}, \bar{\mathbf{x}}, \bar{t}, \bar{\delta}\right) \to \xi_t\left(\bar{y}, \bar{x}, \bar{t}; y, \mathbf{x}\right) : (y, \mathbf{x}) \in \mathcal{W}\right\}, && \text{(S.7)}\\
\mathcal{G}_{2} &\equiv \Bigg\{ \left(\bar{y}, \bar{\mathbf{x}}, \bar{t}, \bar{\delta}\right) \to \left(\frac{F_{Y(1)|\mathbf{X}}\left(y|\bar{\mathbf{x}}\right)}{p_0\left(\bar{\mathbf{x}}\right)} + \frac{F_{Y(0)|\mathbf{X}}\left(y|\bar{\mathbf{x}}\right)}{1 - p_0\left(\bar{\mathbf{x}}\right)}\right) \times \\
&\qquad \left\{1\left\{\bar{\mathbf{x}} \le \mathbf{x}\right\}\right\}\left(\bar{t} - p_0\left(\bar{\mathbf{x}}\right)\right) : (y, \mathbf{x}) \in \mathcal{W}\Bigg\}\\
\mathcal{G}_{3,t} &\equiv \left\{\left(\bar{y}, \bar{\mathbf{x}}, \bar{t}, \bar{\delta}\right) \to \gamma_{t,1}\left(\bar{y}; y, \mathbf{x}\right) : (y, \mathbf{x}) \in \mathcal{W}\right\}, && \text{(S.8)}\\
\mathcal{G}_{4,t} &\equiv \left\{\left(\bar{y}, \bar{\mathbf{x}}, \bar{t}, \bar{\delta}\right) \to \gamma_{t,2}\left(\bar{y}; y, \mathbf{x}\right) : (y, \mathbf{x}) \in \mathcal{W}\right\}, && \text{(S.9)}\\
\mathcal{G}_{5,t} &= \left\{\left(\bar{y}, \bar{\mathbf{x}}, \bar{t}, \bar{\delta}\right) \to \gamma_{t,0}\left(\bar{y}\right)\right\}, && \text{(S.10)}\\
\mathcal{G}_{6} &= \left\{\left(\bar{y}, \bar{\mathbf{x}}, \bar{t}, \bar{\delta}\right) \to \delta\right\} && \text{(S.11)}
\end{aligned}$$

are Donsker. Here, recall that ξ_1 and ξ_0 are as defined in(3.4) and (3.5), and $\gamma_{t,0}(\cdot)$, $\gamma_{t,1}(\cdot)$ and $\gamma_{t,2}(\cdot)$ are defined as in Section 3.1. Now, notice that both $\mathcal{G}_{1,t}$ and $\mathcal{G}_2$ are VC-class with square integrable envelope functions. Therefore, by Theorem 2.6.8 in van der Vaart and Wellner (1996), these classes of functions are Donsker. Both $\mathcal{G}_{5,t}$ and $\mathcal{G}_6$ are not indexed by y nor by x, and so they are clearly Donsker. Next consider $\mathcal{G}_{3,t}$ and $\mathcal{G}_{4,t}$. In order to prove that these classes of functions are Donsker, by Theorem 2.5.6 of van der Vaart and Wellner (1996), it suffices to show that, for $i = 3, 4$,

$$\int_0^\infty \sqrt{\ln N_{[\cdot]}\left(\varepsilon, \mathcal{G}_{i,t}, L_2(P)\right)} d\varepsilon < \infty \tag{S.12}$$

where P is the probability measure corresponding to the joint distribution of $(Q, \delta, T, \mathbf{X})$, and $L_2(P)$ is the $L_2 - norm$. Notice that

$$\gamma_{t,2}(\omega) = \int\int \frac{1\left\{\bar{v} < \omega, \bar{v} < \bar{\omega}\right\}\xi_t\left(\bar{\omega}, \bar{\mathbf{x}}, \bar{t}; y, \mathbf{x}\right)}{\left[1 - H_t\left(\bar{v}\right)\right]^2}\gamma_{t,0}\left(\bar{\omega}\right) H_{t,0}\left(d\bar{v}\right) H_{t,11}\left(d\bar{\omega}, d\bar{\mathbf{x}}\right),$$

and $\gamma_{t,1}(\omega) = -\gamma_{t,1}^{-}(\omega)$, where

$$\gamma_{t,1}^{-}(\omega) = \frac{-1}{1 - H_t(\omega)}\int 1\left\{\bar{\omega} > \omega\right\}\xi_t\left(\bar{\omega}, \bar{\mathbf{x}}, \bar{t}; y, \mathbf{x}\right)\gamma_{t,0}\left(\bar{w}\right) H_{t,11}\left(d\bar{\omega}, d\bar{\mathbf{x}}\right)$$

are non-decreasing, bounded functions. Therefore, by Theorem 2.7.5 in van der Vaart and Wellner (1996), we have that, for a fixed $\varepsilon > 0$ and $i = 3, 4$, $\ln N_{[\cdot]}\left(\varepsilon, \mathcal{G}_{i,t}, L_2(P)\right) \le K\varepsilon^{-1}$, where K is an arbitrary constant. Hence the integral in (S.12) is finite, and the classes of functions $\mathcal{G}_{3,t}$ and $\mathcal{G}_{4,t}$, $t = 0, 1$, are Donsker.

We have just shown that $\mathcal{F}$ is Donsker, that is, we have proved that

$$\sqrt{n}\left(\hat{I}_{1,n} - I_1\right)(y, \mathbf{x}) \Rightarrow C_\infty(y, \mathbf{x})$$

where C_∞ is a tight Gaussian process in $l^\infty(\mathcal{W})$ with zero mean and covariance function given by (3.11). Since under H_0, $I_1(y, \mathbf{x}) = 0$ $a.e.$ in $\mathcal{W}$, the proof is completed.■

Proof of Theorem 2: Notice that we can always write

$$\begin{aligned}\sqrt{n}\hat{I}_{1,n}\left(y,\mathbf{x}\right) &= \sqrt{n}\left(\hat{I}_{1,n}-I_1\right)\left(y,\mathbf{x}\right)+\sqrt{n}I\left(y,\mathbf{x}\right)\\ &= D_{1,n}\left(y,\mathbf{x}\right)+D_{2,n}\left(y,\mathbf{x}\right).\end{aligned}$$

From the proof of Theorem 1, we have that

$$\sqrt{n}\left(\hat{I}_{1,n}-I_{1,n}\right)\Rightarrow C_\infty,$$

and therefore $D_{1,n}\left(y,\mathbf{x}\right)=O_{\mathbb{P}}\left(1\right)$. On the other hand, under the alternative $I_1\left(y,\mathbf{x}\right)\neq 0$ for some $\left(y,\mathbf{x}\right)$. Therefore $D_{2,n}\left(y,\mathbf{x}\right)=O_{\mathbb{P}}\left(n^{1/2}\right)$. Hence, under H_1,

$$\sqrt{n}\sup_{\left(y,\mathbf{x}\right)\in\mathcal{W}}\left|\hat{I}_{1,n}\left(y,\mathbf{x}\right)\right|\xrightarrow{p}\infty.$$

Since under H_0, $I_1\left(y,\mathbf{x}\right)=0$ for all $\left(y,\mathbf{x}\right)$, $KS_n=O_{\mathbb{P}}\left(1\right)$, and therefore $c_\alpha^{KS}=O\left(1\right)$ almost surely, we conclude that

$$\lim_{n\to\infty}P\left\{KS_n>c_\alpha^{KS}\right\}=1.$$

Analogously, we have that

$$\lim_{n\to\infty}P\left\{CvM_n>c_\alpha^{CvM}\right\}=1.$$

■

Proof of Theorem 3: As in the proof of Theorem 2, we can always write

$$\begin{aligned}\sqrt{n}\hat{I}_{1,n}\left(y,\mathbf{x}\right) &= \sqrt{n}\left(\hat{I}_{1,n}-I_1\right)\left(y,\mathbf{x}\right)+\sqrt{n}I\left(y,\mathbf{x}\right)\\ &= D_{1,n}\left(y,\mathbf{x}\right)+D_{2,n}\left(y,\mathbf{x}\right)\end{aligned}$$

From the proof of Theorem 1, we have that

$$\sqrt{n}\left(\hat{I}_{1,n}-I_1\right)\left(y,\mathbf{x}\right)\Rightarrow C_\infty,$$

and therefore $D_{1,n}\left(y,\mathbf{x}\right)=O_{\mathbb{P}}\left(1\right)$. On the other hand, under the local alternatives of the type $H_{1,n}$, $\sqrt{n}I_1\left(y,\mathbf{x}\right)=\mathbb{E}\left[h\left(y,\mathbf{x}\right)1\left\{\mathbf{X}\leq\mathbf{x}\right\}\right]=O_{\mathbb{P}}\left(1\right)$. Hence, under $H_{1,n}$,

$$\sqrt{n}\hat{I}_{1,n}\left(y,\mathbf{x}\right)\Rightarrow C_\infty+R\left(y,\mathbf{x}\right)$$

in $l^\infty\left(\mathcal{W}\right)$.■

Before we proceed with the proof of Theorem 4, we provide the following auxiliary results.

Lemma S.2 *Suppose Assumptions 2.1, A.2-A.5 hold. Additionally, assume that, for each y, $F_{Y(t)|\mathbf{X}}\left(y|\mathbf{x}\right)$ is continuously differentiable of order $m>k$, where k is the dimension of $\mathbf{X}$.*

Then,

$$\sup_{(y,\mathbf{x})\in\mathcal{W}}\left|\hat{F}^{km}_{Y(t)|\mathbf{X},n}(y|\mathbf{x})-F_{Y(t)|\mathbf{X}}(y|\mathbf{x})\right|=o_{\mathbb{P}}(1)$$

Proof of Lemma S.2: For a matrix $\mathbf{A}$, let $\|\mathbf{A}\|$ denote the matrix norm of A such that $\|\mathbf{A}\|=\sqrt{tr(\mathbf{A}'\mathbf{A})}$. Due to the symmetry of arguments, we only prove that

$$\sup_{(y,\mathbf{x})\in\mathcal{W}}\left|\hat{F}^{km}_{Y(1)|\mathbf{X},n}(y|\mathbf{x})-F_{Y(1)|\mathbf{X}}(y|\mathbf{x})\right|=o_{\mathbb{P}}(1).$$

Define

$$\begin{aligned}
\Phi_L(y) &= \frac{1}{n}\sum_{i=1}^{n}\frac{\delta_i T_i 1\{Q_i\le y\}\gamma_{1,0}(Q_i)\mathbf{R}^L(\mathbf{X}_i)}{\hat{p}_n(\mathbf{X}_i)},\\
\Phi_L^{KM}(y) &= \frac{n_1}{n}\sum_{i=1}^{n_1}W_{in_1}\frac{1\{Q_{i:n_1}\le y\}\mathbf{R}^L\left(\mathbf{X}_{[i:n_1]}\right)}{\hat{p}_n\left(\mathbf{X}_{[i:n_1]}\right)},\\
\zeta_L &= \frac{1}{n}\sum_{i=1}^{n\mathbf{R}L}(\mathbf{X}_i)\mathbf{R}^L(\mathbf{X}_i)'.
\end{aligned}$$

Notice that from Lemma S.1,

$$\begin{aligned}
\Phi_L(y) &= \frac{1}{n}\sum_{i=1}^{n}\frac{\delta_i T_i 1\{Q_i\le y\}\gamma_{1,0}(Q_i)\mathbf{R}^L(\mathbf{X}_i)}{p_0(\mathbf{X}_i)}\\
&\quad -\frac{1}{n}\sum_{i=1}^{n}\frac{\delta_i T_i 1\{Q_i\le y\}\gamma_{1,0}(Q_i)\mathbf{R}^L(\mathbf{X}_i)(\hat{p}_n(\mathbf{X}_i)-p_0(\mathbf{X}_i))}{\hat{p}_n(\mathbf{X}_i)p_0(\mathbf{X}_i)}\\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{\delta_i T_i 1\{Q_i\le y\}\gamma_{1,0}(Q_i)\mathbf{R}^L(\mathbf{X}_i)}{p_0(\mathbf{X}_i)}+o_p\left(n^{-1/4}\right),
\end{aligned}$$

uniformly in $(y,\mathbf{x})\in\mathcal{W}$. Analogously,

$$\Phi_L^{KM}(y)=\frac{n_1}{n}\sum_{i=1}^{n_1}W_{in_1}\frac{1\{Q_{i:n_1}\le y\}\mathbf{R}^L\left(X_{[i:n_1]}\right)}{p_0\left(\mathbf{X}_{[i:n_1]}\right)}+o_p\left(n^{-1/4}\right).$$

Thus, we have that

$$\begin{aligned}
&\Phi_L^{KM}(y)-\Phi_L(y)\\
&= \frac{n_1}{n}\sum_{i=1}^{n_1}W_{in_1}\frac{1\{Q_{i:n_1}\le y\}\mathbf{R}^L\left(\mathbf{X}_{[i:n_1]}\right)}{p_0\left(\mathbf{X}_{[i:n_1]}\right)}\\
&\quad -\frac{1}{n}\sum_{i=1}^{n}\frac{\delta_i T_i 1\{Q_i\le y\}\gamma_{1,0}(Q_i)R^L(\mathbf{X}_i)}{p_0(\mathbf{X}_i)}\\
&\quad +o_{\mathbb{P}}\left(n^{-1/4}\right)\\
&= o_P(1)
\end{aligned}$$

where the last equality follows from Theorem 1 of Stute (1993).

From the above results, we have that

$$\hat{F}^{km}_{Y(1)|\mathbf{X},n}\left(y|\mathbf{x}\right)=\Phi_{L}\left(y\right)^{'}\zeta_{L}^{-1}\mathbf{R}^{L}\left(x\right)+o_{\mathbb{P}}\left(1\right)$$

uniformly in $(y,\mathbf{x})\in\mathcal{W}$. Next, by noticing that the variance of $\delta T1\left\{Q\leq\cdot\right\}\gamma_{0}\left(Q\right)/p_{0}\left(\mathbf{X}\right)$ conditional on $\mathbf{X}$ is bounded for all $(y,\mathbf{x})\in\mathcal{W}$, the uniform bound in Newey (1997) for power series estimators applies:

$$\begin{aligned}\sup_{(y,\mathbf{x})\in\mathcal{W}}\left|\Phi_{L}\left(y\right)^{'}\zeta_{L}^{-1}\mathbf{R}^{L}\left(\mathbf{x}\right)-F_{Y(1)|\mathbf{X}}\left(y|\mathbf{x}\right)\right| &\leq \mathcal{C}\left(L^{\frac{3}{2}}n^{-\frac{1}{2}}+L^{1-\frac{m}{k}}\right)\\ &= o_{\mathbb{P}}\left(1\right)\end{aligned}$$

where the last step follows from Assumption A.5 and that $m>k$. ■

Lemma S.3 *Let $\mathcal{H}$ be the collection of all distribution functions bounded away from 0 and 1 that satisfy Assumption A.4. For a given $(y,\mathbf{x})\in\mathcal{W}$, and $p\in\mathcal{H}$, let*

$$\begin{aligned}\xi_{0}\left(\bar{y},\bar{\mathbf{x}},\bar{t};y,\mathbf{x},p\right) &= \left(\frac{1-\bar{t}}{1-p\left(\bar{\mathbf{x}}\right)}\right)1\left\{\bar{y}\leq y\right\}1\left\{\bar{\mathbf{x}}\leq\mathbf{x}\right\},\\ \xi_{1}\left(\bar{y},\bar{\mathbf{x}},\bar{t};y,\mathbf{x},p\right) &= \left(\frac{\bar{t}}{p\left(\bar{\mathbf{x}}\right)}\right)1\left\{\bar{y}\leq y\right\}1\left\{\bar{\mathbf{x}}\leq\mathbf{x}\right\}.\end{aligned}$$

Under the same Assumptions as Lemma S.2, the classes of real-valued measurable functions

$$\begin{aligned}\mathcal{G}_{0} &\equiv \left\{\left(\bar{y},\bar{\mathbf{x}},\bar{t}\right)\rightarrow\xi_{0}\left(\bar{y},\bar{\mathbf{x}},\bar{t};y,\mathbf{x},p\right):(y,\mathbf{x})\in\mathcal{W},p\in\mathcal{H}\right\},\\ \mathcal{G}_{1} &\equiv \left\{\left(\bar{y},\bar{\mathbf{x}},\bar{t}\right)\rightarrow\xi_{1}\left(\bar{y},\bar{\mathbf{x}},\bar{t};y,\mathbf{x},p\right):(y,\mathbf{x})\in\mathcal{W},p\in\mathcal{H}\right\},\end{aligned}$$

are Donsker.

Proof of Lemma S.3: From the fact that for some $\varepsilon>0$, $\varepsilon<p\left(\cdot\right)<1-\varepsilon$ *a.s.*, the integrability conditions in Assumptions 2.1, and Theorem 2.10.6 of van der Vaart and Wellner (1996), it suffices to show that the classes of functions $\left\{1\left\{\cdot\leq y\right\}1\left\{\cdot\leq\mathbf{x}\right\},(y,\mathbf{x})\in\mathcal{W}\right\}$, $\{p,p\in\mathcal{H}\}$, $\{y\}$, and $\{t\}$ are Donsker. To this end, notice that under the smoothness conditions imposed by Assumptions A.2, A.4 and A.5 it follows that the class of functions $\{p,p\in\mathcal{H}\}$ is Donsker, cf. Theorem 2.7.1 of van der Vaart and Wellner (1996). The functions $\{y\}$ and $\{t\}$ are clearly Donsker because they are not indexed by anything. The class of functions $\left\{1\left\{\cdot\leq y\right\}1\left\{\cdot\leq\mathbf{x}\right\},(y,\mathbf{x})\in\mathcal{W}\right\}$ is a VC-class, and hence it is Donsker, cf. Theorem 2.6.4 of van der Vaart and Wellner (1996). Thus, the proof is completed.■

Lemma S.4 *Let $\mathcal{H}_{F}$ be the collection of all m-times continuously differentiable conditional CDF, where $m>k$, where k is the dimension of the conditioning set. For a given $(y,\mathbf{x})\in\mathcal{W}$, $p\in\mathcal{H}$, $F_{Y(t)|\mathbf{X}}\in\mathcal{H}_{F}$, $t\in\{0,1\}$, let*

$$\alpha_{1}\left(\bar{\mathbf{x}},\bar{t};y,\mathbf{x},p,F_{Y(1)|\mathbf{X}}\right) = -\frac{F_{Y(1)|\mathbf{X}}\left(y|\bar{\mathbf{x}}\right)1\left\{\bar{\mathbf{x}}\leq\mathbf{x}\right\}}{p\left(\bar{\mathbf{x}}\right)}\left(\bar{t}-p\left(\bar{\mathbf{x}}\right)\right),$$

$$\alpha_0\left(\bar{\mathbf{x}},\bar{t};y,\mathbf{x},p,F_{Y(0)|\mathbf{X}}\right) = \frac{F_{Y(0)|\mathbf{X}}\left(y|\bar{\mathbf{x}}\right)1\left\{\bar{\mathbf{x}}\leq\mathbf{x}\right\}}{1-p\left(\bar{x}\right)}\left(\bar{t}-p\left(\bar{\mathbf{x}}\right)\right)$$

Under the same Assumptions as Lemma S.2, the classes of real-valued measurable functions

$$\begin{aligned}\mathcal{G}_{0,0} &\equiv \left\{(\bar{\mathbf{x}},\bar{t})\to\alpha_0\left(\bar{\mathbf{x}};y,\mathbf{x},p,F_{Y(0)|\mathbf{X}}\right):(y,\mathbf{x})\in\mathcal{W},p\in\mathcal{H},F_{Y(0)|\mathbf{X}}\in\mathcal{H}_F\right\},\\ \mathcal{G}_{1,1} &\equiv \left\{(\bar{\mathbf{x}},\bar{t})\to\alpha_1\left(\bar{\mathbf{x}};y,\mathbf{x},p,F_{Y(1)|\mathbf{X}}\right):(y,\mathbf{x})\in\mathcal{W},p\in\mathcal{H},F_{Y(1)|\mathbf{X}}\in\mathcal{H}_F\right\},\end{aligned}$$

are Donsker.

Proof of Lemma S.3: From the fact that for some $\varepsilon>0$, $\varepsilon<p(\cdot)<1-\varepsilon$ *a.s.*, the integrability conditions in Assumptions 2.1, and Theorem 2.10.6 of van der Vaart and Wellner (1996), it suffices to show that the classes of functions $\{1\{\cdot\leq\mathbf{x}\},(y,\mathbf{x})\in\mathcal{W}\}$, $\{p,p\in\mathcal{H}\}$, $\left\{F_{Y(t)|\mathbf{X}}\left(y|\bar{\mathbf{x}}\right),(y,\mathbf{x})\in\mathcal{W},F_{Y(t)|\mathbf{X}}\in\mathcal{H}_F\right\}$, $t=\{0,1\}$, and $\{t\}$ are Donsker. Given that in Lemma S.3 we already have shown all these but $\left\{F_{Y(t)|\mathbf{X}}\left(y|\bar{\mathbf{x}}\right),(y,\mathbf{x})\in\mathcal{W},F_{Y(t)|\mathbf{X}}\in\mathcal{H}_F\right\}$, it suffices to show that the later is Donsker. To this end, notice that underthe smoothness conditions imposed, it follows that the class of functions $\left\{F_{Y(t)|\mathbf{X}}\left(y|\bar{\mathbf{x}}\right),(y,\mathbf{x})\in\mathcal{W},F_{Y(t)|\mathbf{X}}\in\mathcal{H}_F\right\}$ is Donsker, cf. Theorem 2.7.1 of van der Vaart and Wellner (1996). Thus, the proof is completed.■

Next, we proceed with the proof of Theorem 4.

Proof of Theorem 4: For $t\in\{0,1\}$, denote

$$\hat{\eta}_{t,i}\left(y,\mathbf{x},\hat{p}_n\right)=\hat{\xi}_t\left(Q_i,\mathbf{X}_i,T_i;y,\mathbf{x},\hat{p}_n\right)\hat{\gamma}_{t,0}\left(Q_i\right)\delta_{t,i}+\hat{\gamma}_{t,1}\left(Q_i\right)\left(1-\delta_i\right)-\hat{\gamma}_{t,2}\left(Q_i\right)$$

and

$$\hat{\eta}_i\left(y,\mathbf{x},\hat{p}_n\right)=\hat{\eta}_{1,i}\left(y,\mathbf{x},\hat{p}_n\right)-\hat{\eta}_{0,i}\left(y,\mathbf{x},\hat{p}_n\right)$$

and $\hat{\gamma}_{t,0}$, $\hat{\gamma}_{t,1}$ and $\hat{\gamma}_{t,2}$ are the empirical analogous of $\gamma_{t,0}$, $\gamma_{t,1}$ and $\gamma_{t,2}$ as defined in (3.7), with the true propensity score p_0 replaced by the SLE $\hat{p}_n$.

The proof follows two steps. In the first step in this proof is to show that

$$\begin{aligned}&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\hat{\eta}_i(y,\mathbf{x},\hat{p}_n)+\hat{\alpha}^{KM}\left(\mathbf{X}_i;y,\mathbf{x},\hat{p}_n,\hat{F}^{km}_{Y(1)|\mathbf{X},n},\hat{F}^{km}_{Y(0)|\mathbf{X},n}\right)\left(T_i-\hat{p}_n\left(\mathbf{X}_i\right)\right)\right)\\ &=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\eta_i(y,\mathbf{x})+\alpha\left(\mathbf{X}_i;y,\mathbf{x}\right)\left(T_i-p_0\left(\mathbf{X}_i\right)\right)\right)+o_{\mathbb{P}}\left(1\right)\end{aligned}\tag{S.13}$$

uniformly in $(y,\mathbf{x})\in\mathcal{W}$, that is, there is no estimation effect coming from replacing the true $\eta(y,\mathbf{x})$, $\alpha\left(\mathbf{X};y,\mathbf{x}\right)$ and $p_0\left(\mathbf{X}\right)$ by their nonparametric estimators.

In the second step, we prove that, under H_0, H_1 or $H_{1,n}$,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\eta_i(y,\mathbf{x},p)-\alpha\left(\mathbf{X}_i;y,\mathbf{x}\right)\left(T_i-p\left(\mathbf{X}_i\right)\right)\right)V_i\tag{S.14}$$

converges weakly to the same limit process as in Theorem 1.

We proceed with the proof of the first step. To show this, it suffices to show that the classes of functions

$$\left\{ \begin{array}{c} \eta_i(y, \mathbf{x}, p_n) - \alpha\left(\mathbf{X}_i; y, \mathbf{x}, p, F_{Y(0)|\mathbf{X}}, F_{Y(1)|\mathbf{X}}\right)\left(T_i - p\left(\mathbf{X}\right)\right) : \\ (y, \mathbf{x}) \in \mathcal{W}, p \in \mathcal{H}, \left(F_{Y(0)|\mathbf{X}}, F_{Y(1)|\mathbf{X}}\right) \in \mathcal{H}_F \times \mathcal{H}_F \end{array} \right\}$$

is Donsker. However, this follows from Lemmas S.3 and S.4. Therefore, by a stochastic equicontinuity argument, the Glivenko-Cantelli Theorem and the triangle inequality,

$$\sup_{(y,\mathbf{x})\in\mathcal{W}} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\hat{\eta}_i(y, \mathbf{x}, \hat{p}_n) - \eta_i(y, \mathbf{x})\right) \right| = o_{\mathbb{P}}\left(1\right). \tag{S.15}$$

$$\sup_{(y,\mathbf{x})\in\mathcal{W}} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \begin{array}{c} \hat{\alpha}^{KM}\left(\mathbf{X}_i; y, \mathbf{x}, \hat{F}^{km}_{Y(1)|\mathbf{X},n}, \hat{F}^{km}_{Y(0)|\mathbf{X},n}\right)\left(T_i - \hat{p}\left(\mathbf{X}_i\right)\right) \\ -\alpha\left(\mathbf{X}_i; y, \mathbf{x}\right)\left(T_i - p\left(\mathbf{X}_i\right)\right) \end{array} \right| = o_{\mathbb{P}}\left(1\right). \tag{S.16}$$

Combining (S.15) and (S.16), we have established (S.13), finishing the proof of the first step.

Next, let's consider (S.14). Define the classes of real measurable functions

$$\begin{aligned} \mathcal{G}_* \equiv &\{\left(\bar{w}, \bar{\mathbf{x}}, \bar{t}, \bar{\delta}, \bar{v}\right) \in \mathcal{X}_Y \times \mathcal{X}_{\mathbf{X}} \times \{0,1\} \times \{0,1\} \times \mathcal{X}_v \to \\ &\left(\eta(y, \mathbf{x}) + \alpha\left(\bar{\mathbf{x}}; y, \mathbf{x}\right)\left(\bar{t} - p_0\left(\bar{\mathbf{x}}\right)\right)\right)\bar{v} : (y, \mathbf{x}) \in \mathcal{W}\}, \end{aligned}$$

The class $\mathcal{G}_*$ is Donsker, since

$$\mathcal{G} \equiv \left\{\left(\bar{w}, \bar{\mathbf{x}}, \bar{t}, \bar{\delta}\right) \to \eta(y, \mathbf{x}) + \alpha\left(\bar{\mathbf{x}}; y, \mathbf{x}\right)\left(\bar{t} - p_0\left(\bar{\mathbf{x}}\right)\right) : (y, \mathbf{x}) \in \mathcal{W}\right\},$$

is Donsker, see Theorem 2.9.6 in van der Vaart and Wellner (1996). Then, since $\mathbb{P}^*_n g = 0$ for all $g \in \mathcal{G}_*$,

$$I^*_{1,n}\left(y, \mathbf{x}\right) = \frac{1}{n} \sum_{i=1}^{n} \left(\eta_i\left(y, \mathbf{x}\right) + \alpha\left(\mathbf{X}_i; y, \mathbf{x}\right)\left(T_i - p\left(\mathbf{X}_i\right)\right)\right) V_i + o_{\mathbb{P}^*_n}\left(n^{-1/2}\right), \tag{S.17}$$

uniformly in $(y, \mathbf{x}) \in \mathcal{W}$.

The expansion (S.17), and the multiplier functional central limit theorem, see Theorem 2.9.6 in van der Vaart and Wellner (1996), imply that $\sqrt{n} I^*_{1,n}\left(y, \mathbf{x}\right)$ converges weakly (in probability) to the same weak limit as $\sqrt{n}\hat{I}_{1,n}\left(y, \mathbf{x}\right)$ in $l^\infty\left(\mathcal{W}\right)$ under H_0, H_1 or H_{1n}.

This completes the proof of Theorem 4.■

S.2.3 Proofs: Testing for Zero Conditional Average Treatment Effects

In this subsection we prove Theorem 5. To this end, we first present two auxiliary lemmas. The first one, we establish the integrated moment representation of the null H_0^{cate} in (4.1). The second Lemma establishes the asymptotic linear representation of the two-step Kaplan-Meier integral.

Lemma S.5 *Suppose Assumptions 2.1-2.2 hold. Assume that the parametric family $w(\mathbf{X},\mathbf{x})$ satisfy Assumption A.1, and let $\bar{\tau} \leq \tau$. Then*

$$\Upsilon_{\bar{\tau}}^{cate}(\mathbf{X}) = 0 \ a.s. \Leftrightarrow I_{\bar{\tau}}^{cate}(\mathbf{x}) = 0 \ a.e \ in \ \mathcal{X}_X,$$

where $I_{\bar{\tau}}^{cate}(\mathbf{x}) = I_{\bar{\tau}}^{1,cate}(\mathbf{x}) - I_{\bar{\tau}}^{0}(\mathbf{x})$, with

$$I_{\bar{\tau}}^{t,cate}(\mathbf{x}) \equiv \mathbb{E}^{km}\left[\frac{1\{T=t\}\, Q1\{Q\leq\bar{\tau}\}}{\mathbb{P}(T=t|\mathbf{X})}1\{\mathbf{X}\leq\mathbf{x}\}\right],\ t\in\{0,1\}.$$

Proof of Lemma S.5: From Lemma 1, we have that for some $\bar{\tau} \leq \tau$

$$\Upsilon_{\bar{\tau}}^{cate}(\mathbf{X}) = \mathbb{E}^{km}\left[\frac{TQ1\{Q\leq\bar{\tau}\}}{p_0(\mathbf{X})} - \frac{(1-T)\,Q1\{Q\leq\bar{\tau}\}}{1-p_0(\mathbf{X})}\middle|\mathbf{X}\right] \text{ a.s.},$$

provided (2.6) is ruled out (as we do). The rest of the proof follows from the exact same arguments as the proof of Lemma 2 as is therefore omitted. ■

Before stating the next Lemma, we need to introduce some notation. Let

$$\eta_{\bar{\tau}}^{cate}(\mathbf{x}) = \eta_{\bar{\tau},1}^{cate}(\mathbf{x}) - \eta_{\bar{\tau},0}^{cate}(\mathbf{x}),$$

with

$$\eta_{\bar{\tau},t}^{cate}(\mathbf{x}) = \xi_{\bar{\tau},t}^{cate}(Q,\mathbf{X},T;\mathbf{x})\,\gamma_{t,0}(Q)\,\delta + \gamma_{t,1,\bar{\tau}}^{cate}(Q;\mathbf{x})(1-\delta) - \gamma_{t,2,\bar{\tau}}^{cate}(Q;\mathbf{x}),$$

$t\in\{0,1\}$, where

$$\begin{aligned} \xi_{\bar{\tau},1}^{cate}(Q,\mathbf{X},T;\mathbf{x}) &= \frac{TQ1\{Q\leq\bar{\tau}\}\,1\{\mathbf{X}\leq\mathbf{x}\}}{p_0(\mathbf{X})}, \\ \xi_{\bar{\tau},0}^{cate}(Q,\mathbf{X},T;\mathbf{x}) &= \frac{(1-T)\,Q1\{Q\leq\bar{\tau}\}\,1\{\mathbf{X}\leq\mathbf{x}\}}{1-p_0(\mathbf{X})}, \end{aligned}$$

$\gamma_{t,0}$ is defined in (3.1), and $\gamma_{t,1,\bar{\tau}}^{cate}(Q;\mathbf{x})$ and $\gamma_{t,2,\bar{\tau}}^{cate}(Q;\mathbf{x})$ are defined as (3.2) and (3.3), respectively, but with ξ_t replaced by $\xi_{\bar{\tau},t}^{cate}$. Furthermore, denote $\alpha_{\bar{\tau}}^{cate}(\mathbf{X};\mathbf{x}) = \alpha_{\bar{\tau},1}^{cate}(\mathbf{X};\mathbf{x}) - \alpha_{\bar{\tau},0}^{cate}(\mathbf{X};\mathbf{x})$, with

$$\begin{aligned} \alpha_{\bar{\tau},1}^{cate}(\mathbf{X};\mathbf{x}) &= -\frac{\mathbb{E}(Y(1)1\{Y(1)\leq\bar{\tau}\}|\mathbf{X})\,1\{\mathbf{X}\leq\mathbf{x}\}}{p_0(\mathbf{X})}, \\ \alpha_{\tau,0}^{cate}(\mathbf{X};\mathbf{x}) &= \frac{\mathbb{E}(Y(0)1\{Y(0)\leq\bar{\tau}\}|\mathbf{X})\,1\{\mathbf{X}\leq\mathbf{x}\}}{1-p_0(\mathbf{X})}. \end{aligned}$$

Lemma S.6 *Under the same assumptions as in Theorem 5, we have that uniformly in $\mathbf{x}\in\mathcal{X}_X$,*

$$\sqrt{n}\left(\hat{I}_{\bar{\tau},n}^{cate}(\mathbf{x}) - I_{\bar{\tau}}^{cate}(\mathbf{x})\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\left[\eta_{\bar{\tau},i}^{cate}(\mathbf{x}) - I_{\bar{\tau}}^{cate}(\mathbf{x})\right] + \alpha_{\bar{\tau}}^{cate}(\mathbf{X}_i;\mathbf{x})\,(T_i - p_0(\mathbf{X}_i))\right\} + o_{\mathbb{P}}(1), \tag{S.18}$$

Proof of Lemma S.6: With some abuse of notation, let

$$\begin{aligned}
I_{\bar{\tau}}^{0,cate}(\mathbf{x},p) &= \mathbb{E}\left[\frac{1-T}{1-p(\mathbf{X})}Y1\{Y\leq\bar{\tau}\}1\{\mathbf{X}\leq\mathbf{x}\}\right], \\
I_{\bar{\tau}}^{1,cate}(\mathbf{x},p) &= \mathbb{E}\left[\frac{T}{p(\mathbf{X})}Y1\{Y\leq\bar{\tau}\}1\{\mathbf{X}\leq\mathbf{x}\}\right], \\
I_{\bar{\tau},n}^{0,cate}(\mathbf{x},p) &= \mathbb{E}_n^{km}\left[\frac{1-T}{1-p(\mathbf{X})}Q1\{Q\leq\bar{\tau}\}1\{\mathbf{X}\leq\mathbf{x}\}\right] \\
I_{\bar{\tau},n}^{1,cate}(\mathbf{x},p) &= \mathbb{E}_n^{km}\left[\frac{T}{p(\mathbf{X})}Q1\{Q\leq\bar{\tau}\}1\{\mathbf{X}\leq\mathbf{x}\}\right].
\end{aligned}$$

Then, notice that

$$\begin{aligned}
\sqrt{n}\left(\hat{I}_{\bar{\tau},n}^{cate}(\mathbf{x})-I_{\bar{\tau}}^{cate}(\mathbf{x})\right) &= \sqrt{n}\left(I_{\bar{\tau},n}^{1,cate}(\mathbf{x},\hat{p}_n)-I_{\bar{\tau}}^{1,cate}(\mathbf{x},p_0)\right) \\
&\quad -\sqrt{n}\left(I_{\bar{\tau},n}^{0}(\mathbf{x},\hat{p}_n)-I_{\bar{\tau}}^{0,cate}(\mathbf{x},p_0)\right).
\end{aligned}$$

Therefore, we can work with $\sqrt{n}\left(I_{\bar{\tau},n}^{t,cate}(\mathbf{x},\hat{p}_n)-I_{\bar{\tau}}^{t,cate}(\mathbf{x},p_0)\right)$, $t\in\{0,1\}$, separately. Given that these two functionals have symmetric construction, we only provide detailed arguments for the linear representation of

$$\sqrt{n}\left(I_{\bar{\tau},n}^{1,cate}(\mathbf{x},\hat{p}_n)-I_{\bar{\tau}}^{1,cate}(\mathbf{x},p_0)\right).$$

To consider the most general case, we set $\bar{\tau}=\tau$, implying that $1\{Y\leq\bar{\tau}\}=1$ *a.s.* and $1\{Q\leq\bar{\tau}\}=1$ *a.s.*. This way, to avoid (even more) cumbersome notation, we drop the index $\bar{\tau}$.

As in the proof of Lemma 3, we

$$\begin{aligned}
&\sqrt{n}\left(I_n^{1,cate}(y,\mathbf{x},\hat{p}_n)-I^{1,cate}(y,\mathbf{x},p_0)\right) \\
=\ & \sqrt{n}\int\varphi_{\mathbf{x},p_0}^{1,cate}(\bar{y},\bar{\mathbf{x}},\bar{t})\left[F_n^{km}(d\bar{y},d\bar{\mathbf{x}},d\bar{t})-F(d\bar{y},d\bar{\mathbf{x}},d\bar{t})\right] \\
&+\sqrt{n}\int\left[\varphi_{\mathbf{x},\hat{p}_n}^{1,cate}(\bar{y},\bar{\mathbf{x}},\bar{t})-\varphi_{x,p_0}^{1,cate}(\bar{y},\bar{\mathbf{x}},\bar{t})\right]\left(F_n^{km}(d\bar{y},d\bar{\mathbf{x}},d\bar{t})-F(d\bar{y},d\bar{\mathbf{x}},d\bar{t})\right) \\
&+\sqrt{n}\int\left[\varphi_{\mathbf{x},\hat{p}_n}^{1,cate}(\bar{y},\bar{\mathbf{x}},\bar{t})-\varphi_{x,p_0}^{1,cate}(\bar{y},\bar{\mathbf{x}},\bar{t})\right]F(d\bar{y},d\bar{\mathbf{x}},d\bar{t}) \\
=\ & \mathbb{A}_{1n}^{cate}+\mathbb{A}_{2n}^{cate}+\mathbb{A}_{3n}^{cate},
\end{aligned}$$

where, for a generic $p(\cdot)$, $\varphi_{\mathbf{x},p}^{1,cate}(\bar{y},\bar{\mathbf{x}},\bar{t})=\bar{t}\bar{y}1\{\bar{\mathbf{x}}\leq\mathbf{x}\}/p(\bar{\mathbf{x}})$.

Then, following the same type of arguments as in the proof of Lemma 3, we have that, uniformly in $x\in\mathcal{X}_X$,

$$\begin{aligned}
\mathbb{A}_{1n}^{cate} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\eta_i^{1,cate}(\mathbf{x})-I^{1,cate}(\mathbf{x})\right)+o_{\mathbb{P}}(1), \\
A_{2n}^{cate} &= o_{\mathbb{P}}(1),
\end{aligned}$$

$$\mathbb{A}_{3n}^{cate} = \frac{-1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbb{E}\left(Y\left(1\right)|\mathbf{X}_i\right)1\left\{\mathbf{X}_i \leq \mathbf{x}\right\}}{p_0\left(\mathbf{X}_i\right)}\left(T_i - p_0\left(\mathbf{X}_i\right)\right) + o_{\mathbb{P}}\left(1\right)$$

concluding the proof. ■

Proof of Theorem 5: Once we have proved the validity of the linear representation (S.18), the proof of the weak converge of the process $\sqrt{n}\left(\hat{I}_{\bar{\tau},n}^{cate} - I_{\bar{\tau}}^{cate}\right)(\mathbf{x})$ under H_0^{cate}, H_1^{cate} and $H_{1,n}^{cate}$ follows the same steps of Theorems 1, 2 and 3**,** and the validity of the bootstrap follows the reasoning of Theorem 4 in a routine fashion. Details are omitted.■

S.2.4 Proofs: Testing for Homogeneous Average Treatment Effects

In this subsection we prove Theorem 6. To this end, we first present two auxiliary lemmas. The first one, we establish the integrated moment representation of the null H_0^{hom} in (4.2). The second Lemma establishes the asymptotic linear representation of the two-step Kaplan-Meier integral.

Lemma S.7 *Suppose Assumptions 2.1-2.2 hold. Assume that the parametric family $w(\mathbf{X}, \mathbf{x})$ satisfy Assumption A.1, and let $\bar{\tau} \leq \tau$. Then, we have that*

$$H_0^{\text{hom}} : \exists\ \Upsilon_{\bar{\tau}} \in \mathbb{R} : \Upsilon_{\bar{\tau}}^{cate}\left(\mathbf{X}\right) = \Upsilon_{\bar{\tau}}\ a.s.$$

is true if and only if

$$I_{\bar{\tau}}^{\text{hom}}(\mathbf{x}) = 0 \ a.e. \ in \ \mathcal{X}_X,$$

where $I_{\bar{\tau}}^{\text{hom}}(\mathbf{x}) = I_{\bar{\tau}}^{1,\text{hom}}(\mathbf{x}) - I_{\bar{\tau}}^{0,\text{hom}}(\mathbf{x})$, with, for $t \in \{0,1\}$,

$$I_{\bar{\tau}}^{t,\text{hom}}\left(\mathbf{x}\right) \equiv \mathbb{E}^{km}\left[1\left\{T=t\right\}\left(\frac{Q1\left\{Q \leq \bar{\tau}\right\}}{\mathbb{P}\left(T=t|\mathbf{X}\right)} - \left(2T-1\right)I_{\bar{\tau}}^{ate}\right)1\left\{\mathbf{X} \leq \mathbf{x}\right\}\right],\ t \in \left\{0,1\right\},$$

and

$$I_{\bar{\tau}}^{ate} \equiv \mathbb{E}^{km}\left[\frac{TQ1\left\{Q \leq \bar{\tau}\right\}}{p_0\left(\mathbf{X}\right)}\right] - \mathbb{E}^{km}\left[\frac{\left(1-T\right)Q1\left\{Q \leq \bar{\tau}\right\}}{1-p_0\left(\mathbf{X}\right)}\right].$$

Proof of Lemma S.7: From Lemma 1, we have that for some $\bar{\tau} \leq \tau$

$$\Upsilon_{\bar{\tau}}^{cate}\left(\mathbf{x}\right) = \mathbb{E}^{km}\left[\frac{TQ1\left\{Q \leq \bar{\tau}\right\}}{p_0\left(\mathbf{X}\right)} - \frac{\left(1-T\right)Q1\left\{Q \leq \bar{\tau}\right\}}{1-p_0\left(\mathbf{X}\right)}\middle|\mathbf{X}\right] \text{ a.s.},$$

and that

$$\mathbb{E}^{km}\left[Y\left(1\right)1\left\{Y\left(1\right) \leq \bar{\tau}\right\} - Y\left(0\right)1\left\{Y\left(0\right) \leq \bar{\tau}\right\}\right] = I_{\bar{\tau}}^{ate},$$

provided (2.6) is ruled out (as we do). If $\Upsilon_{\bar{\tau}}^{cate}(\cdot)$ is $a.s.$ constant, it must be equal to $I_{\bar{\tau}}^{ate}$. Thus, we can rewrite H_0^{hom} as

$$H_0^{\text{hom}} : \mathbb{E}^{km}\left[\frac{TQ1\left\{Q \leq \bar{\tau}\right\}}{p_0\left(\mathbf{X}\right)} - \frac{\left(1-T\right)Q1\left\{Q \leq \bar{\tau}\right\}}{1-p_0\left(\mathbf{X}\right)} - I_{\bar{\tau}}^{ate}\middle|\mathbf{X}\right] = 0\ a.s..$$

Note that the aforementioned representation is equivalent to

$$\mathbb{E}^{km}\left[\frac{TQ1\{Q\le\bar{\tau}\}}{p_0(\mathbf{X})}-\frac{(1-T)\,Q1\{Q\le\bar{\tau}\}}{1-p_0(\mathbf{X})}-\left(TI_{\bar{\tau}}^{ate}+(1-T)\,I_{\bar{\tau}}^{ate}\right)\middle|\mathbf{X}\right],$$

implying that we can further characterize H_0^{hom} as

$$H_0^{\text{hom}}:\Upsilon_{\bar{\tau}}^{\text{hom}}(\mathbf{X})=0\;a.s. \tag{S.19}$$

where $\Upsilon_{\bar{\tau}}^{\text{hom}}(\mathbf{X})=\Upsilon_{\bar{\tau}}^{1,\text{hom}}(\mathbf{X})-\Upsilon_{\bar{\tau}}^{0,\text{hom}}(\mathbf{X})$, with

$$\begin{aligned}\Upsilon_{\bar{\tau}}^{1,\text{hom}}(\mathbf{X}) &= \mathbb{E}^{km}\left[T\left(\frac{Q1\{Q\le\bar{\tau}\}}{p_0(\mathbf{X})}-I_{\bar{\tau}}^{ate}\right)\middle|\mathbf{X}\right],\\ &= \mathbb{E}^{km}\left[T\left(\frac{Q1\{Q\le\bar{\tau}\}}{p_0(\mathbf{X})}-(2T-1)\,I_{\bar{\tau}}^{ate}\right)\middle|\mathbf{X}\right]\end{aligned} \tag{S.20}$$

and analogously,

$$\begin{aligned}\Upsilon_{\bar{\tau}}^{0,\text{hom}}(\mathbf{X}) &= \mathbb{E}^{km}\left[(1-T)\left(\frac{Q1\{Q\le\bar{\tau}\}}{1-p_0(\mathbf{X})}+I_{\bar{\tau}}^{ate}\right)\middle|\mathbf{X}\right]\\ &= \mathbb{E}^{km}\left[(1-T)\left(\frac{Q1\{Q\le\bar{\tau}\}}{1-p_0(\mathbf{X})}-(2T-1)\,I_{\bar{\tau}}^{ate}\right)\middle|\mathbf{X}\right].\end{aligned} \tag{S.21}$$

From (S.19), (S.20) and (S.21), we have that H_0^{hom} can be written as a "standard" conditional moment restrictions. To conclude the proof, we just need to show that (S.19) is true if and only if $I_{\bar{\tau}}^{\text{hom}}(\mathbf{x})=0$ *a.e.* in $\mathcal{X}_X$. This follows from the exact same arguments as the proof of Lemma 2. ■

Before stating the next Lemma, we need to introduce some additional notation. Let

$$\begin{aligned}\eta_{\bar{\tau}}^{\text{hom}}(\mathbf{x}) &= \eta_{\bar{\tau},1}^{\text{hom}}(\mathbf{x})-\eta_{\bar{\tau},0}^{\text{hom}}(\mathbf{x}),\\ \eta_{\bar{\tau}}^{ate} &= \eta_{\bar{\tau},1}^{ate}-\eta_{\bar{\tau},0}^{ate},\end{aligned}$$

with

$$\begin{aligned}\eta_{\bar{\tau},t}^{\text{hom}}(\mathbf{x}) &= \xi_{\bar{\tau},t}^{\text{hom}}(Q,\mathbf{X},T;\mathbf{x})\,\gamma_{t,0}(Q)\,\delta+\gamma_{t,1,\bar{\tau}}^{\text{hom}}(Q;\mathbf{x})\,(1-\delta)-\gamma_{t,2,\bar{\tau}}^{\text{hom}}(Q;\mathbf{x}),\\ \eta_{\bar{\tau},t}^{ate} &= \xi_{\bar{\tau},t}^{ate}(Q,\mathbf{X},T)\,\gamma_{t,0}(Q)\,\delta+\gamma_{t,1,\bar{\tau}}^{ate}(Q)\,(1-\delta)-\gamma_{t,2,\bar{\tau}}^{ate}(Q),\end{aligned}$$

$t\in\{0,1\}$, where

$$\begin{aligned}\xi_{\bar{\tau},1}^{\text{hom}}(Q,\mathbf{X},T;x) &= T\left(\frac{Q1\{Q\le\bar{\tau}\}}{p_0(\mathbf{X})}-I_{\bar{\tau}}^{ate}\right)1\{\mathbf{X}\le\mathbf{x}\},\\ \xi_{\bar{\tau},1}^{ate}(Q,\mathbf{X},T) &= \frac{TQ1\{Q\le\bar{\tau}\}}{p_0(\mathbf{X})},\\ \xi_{\bar{\tau},0}^{\text{hom}}(Q,\mathbf{X},T;x) &= (1-T)\left(\frac{Q1\{Q\le\bar{\tau}\}}{p_0(\mathbf{X})}+I_{\bar{\tau}}^{ate}\right)1\{\mathbf{X}\le\mathbf{x}\}\end{aligned}$$

$$\xi^{ate}_{\bar{\tau},0}(Q,\mathbf{X},T) = \frac{(1-T)\,Q1\{Q\leq\bar{\tau}\}}{1-p_0(\mathbf{X})},$$

$\gamma_{t,0}$ is defined in (3.1), $\gamma^{\text{hom}}_{t,1,\bar{\tau}}(Q;x)$ ($\gamma^{ate}_{t,1,\bar{\tau}}(Q)$) and $\gamma^{\text{hom}}_{t,2,\bar{\tau}}(Q;x)$ ($\gamma^{ate}_{t,2,\bar{\tau}}(Q)$) and are defined as (3.2) and (3.3), respectively, but with ξ_t replaced by $\xi^{\text{hom}}_{\bar{\tau},t}$ ($\xi^{ate}_{\bar{\tau},t}$). Furthermore, denote $\alpha^{ate}_{\bar{\tau}}(\mathbf{X}) = \alpha^{ate}_{\bar{\tau},1}(\mathbf{X}) - \alpha^{ate}_{\bar{\tau},0}(\mathbf{X})$, with

$$\begin{aligned}
\alpha^{ate}_{\bar{\tau},1}(\mathbf{X}) &= -\frac{\mathbb{E}\left(Y(1)1\{Y(1)\leq\bar{\tau}\}\,|\mathbf{X}\right)}{p_0(\mathbf{X})},\\
\alpha^{ate}_{\tau,0}(\mathbf{X}) &= \frac{\mathbb{E}\left(Y(0)1\{Y(0)\leq\bar{\tau}\}\,|\mathbf{X}\right)}{1-p_0(\mathbf{X})}.
\end{aligned}$$

Lemma S.8 *Under the same assumptions as in Theorem 6, we have that, uniformly in* $\mathbf{x}\in\mathcal{X}_X$*,*

$$\begin{aligned}
\sqrt{n}\left(\hat{I}^{\text{hom}}_{\bar{\tau},n}(\mathbf{x}) - I^{\text{hom}}_{\bar{\tau}}(\mathbf{x})\right) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\eta^{\text{hom}}_{\bar{\tau},i}(\mathbf{x}) - I^{\text{hom}}_{\bar{\tau}}(\mathbf{x})\right] + \alpha^{cate}_{\bar{\tau}}(\mathbf{X}_i;\mathbf{x})\left(T_i - p_0(\mathbf{X}_i)\right)\\
&\quad - \left[\left(\eta^{ate}_{\bar{\tau},i} - I^{ate}_{\bar{\tau}}\right) + \alpha^{ate}_{\bar{\tau}}(\mathbf{X}_i)\left(T_i - p_0(\mathbf{X}_i)\right)\right]\mathbb{E}\left[1\{\mathbf{X}\leq\mathbf{x}\}\right] + o_{\mathbb{P}}(1), \quad \text{(S.22)}
\end{aligned}$$

Proof of Lemma S.8: With some abuse of notation, let

$$\begin{aligned}
I^{0,\text{hom}}_{\bar{\tau}}\left(\mathbf{x},p,I^{ate}\right) &= \mathbb{E}\left[(1-T)\left(\frac{Y1\{Y\leq\bar{\tau}\}}{1-p(\mathbf{X})} + I^{ate}\right)1\{\mathbf{X}\leq\mathbf{x}\}\right],\\
I^{1,\text{hom}}_{\bar{\tau}}\left(\mathbf{x},p,I^{ate}\right) &= \mathbb{E}\left[T\left(\frac{Y1\{Y\leq\bar{\tau}\}}{p(\mathbf{X})} - I^{ate}\right)1\{\mathbf{X}\leq\mathbf{x}\}\right],\\
I^{0,\text{hom}}_{\bar{\tau},n}\left(\mathbf{x},p,I^{ate}\right) &= \mathbb{E}^{km}_{n}\left[(1-T)\left(\frac{Y1\{Y\leq\bar{\tau}\}}{1-p(\mathbf{X})} + I^{ate}\right)1\{\mathbf{X}\leq\mathbf{x}\}\right]\\
I^{1,\text{hom}}_{\bar{\tau},n}\left(\mathbf{x},p,I^{ate}\right) &= \mathbb{E}^{km}_{n}\left[T\left(\frac{Y1\{Y\leq\bar{\tau}\}}{p(\mathbf{X})} - I^{ate}\right)1\{\mathbf{X}\leq\mathbf{x}\}\right],
\end{aligned}$$

and denote the true (restricted) ATE by $I^{ate}_{0,\bar{\tau}}$.

Then, notice that

$$\begin{aligned}
\sqrt{n}\left(\hat{I}^{\text{hom}}_{\bar{\tau},n}(\mathbf{x}) - I^{\text{hom}}_{\bar{\tau}}(\mathbf{x})\right) &= \sqrt{n}\left(I^{1,\text{hom}}_{\bar{\tau},n}\left(\mathbf{x},\hat{p}_n,\hat{I}^{ate}_{\bar{\tau},n}\right) - I^{1,\text{hom}}_{\bar{\tau}}\left(\mathbf{x},p_0,I^{ate}_{0,\bar{\tau}}\right)\right)\\
&\quad -\sqrt{n}\left(I^{0,\text{hom}}_{\bar{\tau},n}\left(\mathbf{x},\hat{p}_n,\hat{I}^{ate}_{\bar{\tau},n}\right) - I^{0,\text{hom}}_{\bar{\tau}}\left(\mathbf{x},p_0,I^{ate}_{0,\bar{\tau}}\right)\right).
\end{aligned}$$

Therefore, we can work with $\sqrt{n}\left(I^{t,\text{hom}}_{\bar{\tau},n}\left(\mathbf{x},\hat{p}_n,\hat{I}^{ate}_{\bar{\tau},n}\right) - I^{t,\text{hom}}_{\bar{\tau}}\left(\mathbf{x},p_0,I^{ate}_{0,\bar{\tau}}\right)\right)$ for each $t\in\{0,1\}$, separately. Given that these two functionals have symmetric construction, we only provide detailed arguments for the linear representation of $\sqrt{n}\left(I^{1,\text{hom}}_{\bar{\tau},n}\left(\mathbf{x},\hat{p}_n,\hat{I}^{ate}_{\bar{\tau},n}\right) - I^{1,\text{hom}}_{\bar{\tau}}\left(\mathbf{x},p_0,I^{ate}_{0,\bar{\tau}}\right)\right)$. To consider the most general case, we set $\bar{\tau}=\tau$, implying that $1\{Y\leq\bar{\tau}\} = 1$ *a.s.* and $1\{Q\leq\bar{\tau}\} = 1$ *a.s.*. This way we avoid (even more) cumbersome notation, and drop the index $\bar{\tau}$.

Notice that

$$\sqrt{n}\left(I^{1,\text{hom}}_{n}\left(\mathbf{x},\hat{p}_n,\hat{I}^{ate}_{n}\right) - I^{1,\text{hom}}\left(\mathbf{x},p_0,I^{ate}_{0}\right)\right)$$

$$
\begin{aligned}
&= \sqrt{n}\left(\mathbb{E}_n^{km}\left[T\left(\frac{Q}{\hat{p}_n(\mathbf{X})}-\hat{I}_n^{ate}\right)1\{\mathbf{X}\le\mathbf{x}\}\right]\right.\\
&\quad\left.-\mathbb{E}\left[T\left(\frac{Y}{p_0(\mathbf{X})}-I_0^{ate}\right)1\{\mathbf{X}\le\mathbf{x}\}\right]\right)\\
&= \sqrt{n}\left(\mathbb{E}_n^{km}\left[T\left(\frac{Q}{\hat{p}_n(\mathbf{X})}-I_0^{ate}\right)1\{\mathbf{X}\le\mathbf{x}\}\right]-I^{1,\mathrm{hom}}\left(\mathbf{x},p_0,I_0^{ate}\right)\right)\\
&\quad -\sqrt{n}\left(\hat{I}_n^{ate}-I_0^{ate}\right)\mathbb{E}_n^{km}\left[T1\{\mathbf{X}\le\mathbf{x}\}\right]\\
&= \mathbb{A}_n^{\mathrm{hom}}-\mathbb{B}_n^{\mathrm{hom}}.
\end{aligned}
$$

As in the proof of Lemma 3 and Lemma S.6, we

$$
\begin{aligned}
&\mathbb{A}_n^{\mathrm{hom}} =\\
&\sqrt{n}\int\varphi_{\mathbf{x},p_0}^{1,\mathrm{hom}}\left(\bar{y},\bar{\mathbf{x}},\bar{t}\right)\left[F_n^{km}\left(d\bar{y},d\bar{\mathbf{x}},d\bar{t}\right)-F\left(d\bar{y},d\bar{\mathbf{x}},d\bar{t}\right)\right]\\
&+\sqrt{n}\int\left[\varphi_{\mathbf{x},\hat{p}_n}^{1,\mathrm{hom}}\left(\bar{y},\bar{\mathbf{x}},\bar{t}\right)-\varphi_{\mathbf{x},p_0}^{1,\mathrm{hom}}\left(\bar{y},\bar{\mathbf{x}},\bar{t}\right)\right]\left(F_n^{km}\left(d\bar{y},d\bar{\mathbf{x}},d\bar{t}\right)-F\left(d\bar{y},d\bar{\mathbf{x}},d\bar{t}\right)\right)\\
&+\sqrt{n}\int\left[\varphi_{\mathbf{x},\hat{p}_n}^{1,\mathrm{hom}}\left(\bar{y},\bar{\mathbf{x}},\bar{t}\right)-\varphi_{\mathbf{x},p_0}^{1,\mathrm{hom}}\left(\bar{y},\bar{\mathbf{x}},\bar{t}\right)\right]F\left(d\bar{y},d\bar{\mathbf{x}},d\bar{t}\right)\\
= \;&\mathbb{A}_{1n}^{\mathrm{hom}}+\mathbb{A}_{2n}^{\mathrm{hom}}+\mathbb{A}_{3n}^{\mathrm{hom}},
\end{aligned}
$$

where, for a generic $p(\cdot)$,

$$
\varphi_{\mathbf{x},p}^{1,\mathrm{hom}}\left(\bar{y},\bar{\mathbf{x}},\bar{t}\right)=\bar{t}\left(\frac{\bar{y}}{p(\bar{\mathbf{x}})}-I_0^{ate}\right)1\{\bar{\mathbf{x}}\le\mathbf{x}\}.
$$

Then, following the same type of arguments as in the proof of Lemma 3, we have that, uniformly in $\mathbf{x}\in\mathcal{X}_X$,

$$
\begin{aligned}
\mathbb{A}_{1n}^{\mathrm{hom}} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\eta_i^{1,\mathrm{hom}}(\mathbf{x})-I^{1,\mathrm{hom}}(\mathbf{x})\right)+o_{\mathbb{P}}(1),\\
A_{2n}^{\mathrm{hom}} &= o_{\mathbb{P}}(1),\\
\mathbb{A}_{3n}^{\mathrm{hom}} &= \frac{-1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathbb{E}\left(Y(1)\,|\mathbf{X}_i\right)1\{\mathbf{X}_i\le\mathbf{x}\}}{p_0(\mathbf{X}_i)}\left(T_i-p_0(\mathbf{X}_i)\right)+o_{\mathbb{P}}(1),
\end{aligned}
$$

implying that

$$
\mathbb{A}_n^{\mathrm{hom}}=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\eta_i^{1,\mathrm{hom}}(\mathbf{x})-I^{1,\mathrm{hom}}(\mathbf{x})\right)+\alpha_{\bar{\tau},1}^{cate}(\mathbf{X}_i;\mathbf{x})\left(T_i-p_0(\mathbf{X}_i)\right)+o_{\mathbb{P}}(1), \tag{S.23}
$$

uniformly in $x\in\mathcal{X}_X$.

Next, we have to show that

$$
\mathbb{B}_n^{\mathrm{hom}}=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\eta_i^{ate}-I_0^{ate}\right)+\alpha^{ate}(\mathbf{X}_i)\left(T_i-p_0(\mathbf{X}_i)\right)\right]\mathbb{E}\left[T1\{\mathbf{X}\le x\}\right]+o_{\mathbb{P}}(1). \tag{S.24}
$$

To this end, notice that

$$
\begin{aligned}
&\sqrt{n}\left(\hat{I}_n^{ate}-I_0^{ate}\right)= \\
&\sqrt{n}\int\varphi_{p_0}^{ate}\left(\bar{\mathbf{x}},\bar{t}\right)\left[F_n^{km}\left(d\bar{y},d\bar{\mathbf{x}},d\bar{t}\right)-F\left(d\bar{y},d\bar{\mathbf{x}},d\bar{t}\right)\right] \\
&+\sqrt{n}\int\left[\varphi_{\hat{p}_n}^{ate}\left(\bar{\mathbf{x}},\bar{t}\right)-\varphi_{p_0}^{ate}\left(\bar{\mathbf{x}},\bar{t}\right)\right]\left(F_n^{km}\left(d\bar{y},d\bar{\mathbf{x}},d\bar{t}\right)-F\left(d\bar{y},d\bar{\mathbf{x}},d\bar{t}\right)\right) \\
&+\sqrt{n}\int\left[\varphi_{\hat{p}_n}^{ate}\left(\bar{\mathbf{x}},\bar{t}\right)-\varphi_{p_0}^{ate}\left(\bar{\mathbf{x}},\bar{t}\right)\right]F\left(d\bar{y},d\bar{\mathbf{x}},d\bar{t}\right) \\
=\;&\mathbb{B}_{1n}^{ate}+\mathbb{B}_{2n}^{ate}+\mathbb{B}_{3n}^{ate},
\end{aligned}
$$

here, for a generic $p\left(\cdot\right)$,

$$
\varphi_p^{ate}\left(\bar{y},\bar{\mathbf{x}},\bar{t}\right)=\left(\frac{\bar{t}\bar{y}}{p\left(\bar{\mathbf{x}}\right)}-\frac{\left(1-\bar{t}\right)\bar{y}}{1-p\left(\bar{\mathbf{x}}\right)}\right).
$$

Then, following the same type of arguments as in the proof of Lemma 3, we have that

$$
\mathbb{B}_{1n}^{ate} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\eta_i^{ate}-I_0^{ate}\right)+o_{\mathbb{P}}\left(1\right), \tag{S.25}
$$

$$
\mathbb{B}_{2n}^{ate} = o_{\mathbb{P}}\left(1\right), \tag{S.26}
$$

$$
\mathbb{B}_{3n}^{ate} = \frac{-1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\mathbb{E}\left(Y\left(1\right)|\mathbf{X}_i\right)}{p_0\left(\mathbf{X}_i\right)}+\frac{\mathbb{E}\left(Y\left(0\right)|\mathbf{X}_i\right)}{1-p_0\left(\mathbf{X}_i\right)}\right]\left(T_i-p_0\left(\mathbf{X}_i\right)\right)+o_{\mathbb{P}}\left(1\right). \tag{S.27}
$$

Finally, by the uniform law of large numbers in Corollary 1.5 of Stute (1993),

$$
\mathbb{E}_n^{km}\left[T1\left\{\mathbf{X}\le\mathbf{x}\right\}\right]=\mathbb{E}\left[T1\left\{\mathbf{X}\le\mathbf{x}\right\}\right]+o_{\mathbb{P}}\left(1\right), \tag{S.28}
$$

uniformly in $\mathbf{x}\in\mathcal{X}_X$. Thus, (S.24) follows from (S.25)-(S.28).

From (S.23) and (S.24), it follows that

$$
\begin{aligned}
&\sqrt{n}\left(I_n^{1,\mathrm{hom}}\left(\mathbf{x},\hat{p}_n,\hat{I}_n^{ate}\right)-I^{1,\mathrm{hom}}\left(\mathbf{x},p_0,I_0^{ate}\right)\right) \\
=\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\left[\eta_i^{1,\mathrm{hom}}\left(\mathbf{x}\right)-I^{1,\mathrm{hom}}\left(\mathbf{x}\right)\right]+\alpha_1^{cate}\left(\mathbf{X}_i;\mathbf{x}\right)\left(T_i-p_0(\mathbf{X}_i)\right)\right. \\
&\left.-\left[\left(\eta_i^{ate}-I_{\bar{\tau}}^{ate}\right)+\alpha^{ate}\left(\mathbf{X}_i\right)\left(T_i-p_0(\mathbf{X}_i)\right)\right]\mathbb{E}\left[T1\left\{\mathbf{X}\le\mathbf{x}\right\}\right]\right)+o_{\mathbb{P}}\left(1\right)
\end{aligned} \tag{S.29}
$$

Following exactly the same arguments for the untreated sub-group, we have that

$$
\begin{aligned}
&\sqrt{n}\left(I_n^{0,\mathrm{hom}}\left(\mathbf{x},\hat{p}_n,\hat{I}_n^{ate}\right)-I^{1,\mathrm{hom}}\left(\mathbf{x},p_0,I_0^{ate}\right)\right) \\
=\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\left[\eta_i^{0,\mathrm{hom}}\left(\mathbf{x}\right)-I^{0,\mathrm{hom}}\left(\mathbf{x}\right)\right]+\alpha_0^{cate}\left(\mathbf{X}_i;\mathbf{x}\right)\left(T_i-p_0(\mathbf{X}_i)\right)\right. \\
&\left.-\left[\left(\eta_i^{ate}-I_0^{ate}\right)+\alpha^{ate}\left(\mathbf{X}_i\right)\left(T_i-p_0(\mathbf{X}_i)\right)\right]\mathbb{E}\left[\left(1-T\right)1\left\{\mathbf{X}\le\mathbf{x}\right\}\right]\right)+o_{\mathbb{P}}\left(1\right).
\end{aligned} \tag{S.30}
$$

By subtracting (S.30) from (S.29), and putting -$\left[\left(\eta_{\bar{\tau},i}^{ate}-I_{\bar{\tau}}^{ate}\right)+\alpha_{\bar{\tau}}^{ate}\left(\mathbf{X}_i\right)\left(T_i-p_0(\mathbf{X}_i)\right)\right]$ as a common factor, we have that, uniformly in $\mathbf{x}\in\mathcal{X}_X$,

$$\sqrt{n}\left(\hat{I}_n^{\text{hom}}\left(\mathbf{x}\right)-I^{\text{hom}}\left(\mathbf{x}\right)\right)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\eta_i^{\text{hom}}\left(\mathbf{x}\right)-I^{\text{hom}}\left(\mathbf{x}\right)\right]+\alpha^{cate}\left(\mathbf{X}_i;\mathbf{x}\right)\left(T_i-p_0(\mathbf{X}_i)\right)$$
$$-\left[\left(\eta_i^{ate}-I^{ate}\right)+\alpha^{ate}\left(\mathbf{X}_i\right)\left(T_i-p_0(\mathbf{X}_i)\right)\right]\mathbb{E}\left[1\left\{\mathbf{X}\leq\mathbf{x}\right\}\right]+o_{\mathbb{P}}\left(1\right),$$

concluding the proof. ■

Proof of Theorem 6: Once we have proved the validity of the linear representation (S.22), the proof of the weak converge of the process $\sqrt{n}\left(\hat{I}_{\bar{\tau},n}^{\text{hom}}-I_{\bar{\tau}}^{\text{hom}}\right)(x)$ under H_0^{hom}, H_1^{hom} and $H_{1,n}^{\text{hom}}$ follows the same steps of Theorems 1, 2 and 3**,** and the validity of the bootstrap follows the reasoning of Theorem 4 in a routine fashion. Details are omitted.■

S.2.5 Proofs: Testing within the Local Treatment Effect setup

In this subsection we prove Theorem 7. We proceed as in the previous subsections by presenting and proving several auxiliary lemmas, that are the key to establish the proof of the theorems.

Lemma S.9 *Suppose Assumptions 4.1-4.2 hold. Assume that the parametric family $w\left(\mathbf{X},\mathbf{x}\right)$ satisfy Assumption A.1, and let $\bar{\tau}\leq\tau$. Then, we have that*

$$H_0^{ldte}:\Upsilon^{ldte}\left(y|\mathbf{X}\right)=0\ a.s.\ \forall y\in\left[-\infty,\tau\right]$$

is true if and only if

$$I^{ldte}(y,\mathbf{x})=0\ a.e\ in\ \mathcal{W},$$

where $I^{ldte}(y,\mathbf{x})=I^{1,ldte}(y,\mathbf{x})-I^{0,ldte}(y,\mathbf{x})$, with, for $t\in\{0,1\}$,

$$I^{t,ldte}\left(y,\mathbf{x}\right)\equiv(2t-1)\left\{\mathbb{E}^{km}\left[\frac{1\left\{Q\leq y\right\}}{q_0\left(\mathbf{X}\right)}1\left\{\mathbf{X}\leq\mathbf{x}\right\}\middle|T=t,Z=1\right]\mathbb{P}\left(T=t,Z=1\right)\right.$$
$$\left.-\mathbb{E}^{km}\left[\frac{1\left\{Q\leq y\right\}}{1-q_0\left(\mathbf{X}\right)}1\left\{\mathbf{X}\leq\mathbf{x}\right\}|T=t,Z=0\right]\mathbb{P}\left(T=t,Z=0\right)\right\}.$$

Proof of Lemma S.9: From Theorem 3.1 of Abadie (2003), we have that, in the absence of censoring,

$$F_{Y(1)|\mathbf{X}}\left(y|\mathbf{X},pop=comp\right)=\frac{1}{\Gamma_1\left(\mathbf{X}\right)}\left(\mathbb{E}\left[\frac{TZ1\left\{Y\leq y\right\}}{q_0\left(X\right)}\middle|\mathbf{X}\right]\right.$$
$$\left.-\mathbb{E}\left[\frac{T\left(1-Z\right)1\left\{Y\leq y\right\}}{1-q_0\left(\mathbf{X}\right)}\middle|\mathbf{X}\right]\right)\quad a.s.,$$

and

$$F_{Y(0)|\mathbf{X}}\left(y|\mathbf{X},pop=comp\right)=\frac{1}{\Gamma_0\left(\mathbf{X}\right)}\left(\mathbb{E}\left[\left.\frac{\left(1-T\right)\left(1-Z\right)1\left\{Y\leq y\right\}}{1-q_0\left(\mathbf{X}\right)}\right|\mathbf{X}\right]\right.$$
$$\left.-\mathbb{E}\left[\left.\frac{\left(1-T\right)Z1\left\{Y\leq y\right\}}{q_0\left(\mathbf{X}\right)}\right|\mathbf{X}\right]\right)\quad a.s.,$$

where

$$\begin{aligned}\Gamma_1\left(\mathbf{X}\right)&=\mathbb{E}\left[\left.\left(\frac{TZ}{q_0\left(\mathbf{X}\right)}-\frac{T\left(1-Z\right)}{1-q_0\left(\mathbf{X}\right)}\right)\right|\mathbf{X}\right],\\ \Gamma_0\left(\mathbf{X}\right)&=\mathbb{E}\left[\left.\left(\frac{\left(1-T\right)\left(1-Z\right)}{1-q_0\left(\mathbf{X}\right)}-\frac{\left(1-T\right)Z}{q_0\left(\mathbf{X}\right)}\right)\right|\mathbf{X}\right],\end{aligned}$$

and $\Gamma_1\left(\mathbf{X}\right)=\Gamma_0\left(\mathbf{X}\right)=P\left(T\left(1\right)>T\left(0\right)|\mathbf{X}\right)>0$ $a.s.$ from Assumption 4.1, see Lemma 2.1 of Abadie (2003).

From these results, we have that H_0^{ldte} is true if and only if

$$\Upsilon^{1,ldte}\left(y|\mathbf{X}\right)-\Upsilon^{0,dte}\left(y|\mathbf{X}\right)=0\ a.s.\ \forall y\in\left[-\infty,\tau\right],$$

where

$$\begin{aligned}\Upsilon^{1,ldte}\left(y|\mathbf{X}\right)&=\mathbb{E}\left[\left.\frac{TZ1\left\{Y\leq y\right\}}{q_0\left(\mathbf{X}\right)}\right|\mathbf{X}\right]\\ &\quad-\mathbb{E}\left[\left.\frac{T\left(1-Z\right)1\left\{Y\leq y\right\}}{1-q_0\left(\mathbf{X}\right)}\right|\mathbf{X}\right],\\ \Upsilon^{0,dte}\left(y|\mathbf{X}\right)&=\mathbb{E}\left[\left.\frac{\left(1-T\right)\left(1-Z\right)1\left\{Y\leq y\right\}}{1-q_0\left(\mathbf{X}\right)}\right|\mathbf{X}\right]\\ &\quad-\mathbb{E}\left[\left.\frac{\left(1-T\right)Z1\left\{Y\leq y\right\}}{q_0\left(\mathbf{X}\right)}\right|\mathbf{X}\right].\end{aligned}$$

As in Lemma 1, we have that under Assumption 4.2, each of these conditional expectations are identified. That is,

$$\begin{aligned}\Upsilon^{1,ldte}\left(y|\mathbf{X}\right)&=\mathbb{E}^{km}\left[\left.\frac{TZ1\left\{Q\leq y\right\}}{q_0\left(\mathbf{X}\right)}\right|\mathbf{X}\right]\\ &\quad-\mathbb{E}^{km}\left[\left.\frac{T\left(1-Z\right)1\left\{Q\leq y\right\}}{1-q_0\left(\mathbf{X}\right)}\right|\mathbf{X}\right]\quad a.s.,\\ \Upsilon^{0,dte}\left(y|\mathbf{X}\right)&=\mathbb{E}\left[\left.\frac{\left(1-T\right)\left(1-Z\right)1\left\{Q\leq y\right\}}{1-q_0\left(\mathbf{X}\right)}\right|\mathbf{X}\right]\\ &\quad-\mathbb{E}^{km}\left[\left.\frac{\left(1-T\right)Z1\left\{Q\leq y\right\}}{q_0\left(\mathbf{X}\right)}\right|\mathbf{X}\right]\quad a.s..\end{aligned}$$

From the same arguments as in the proof of Lemma 2, we have that H_0^{ldte} is true if and only if

$$I^{1,ldte}\left(y,\mathbf{x}\right)-I^{0,ldte}\left(y,\mathbf{x}\right)=0\ a.e\ \text{in}\ \mathcal{W},$$

where

$$\begin{aligned}
I^{1,ldte}(y,\mathbf{x}) &= \mathbb{E}^{km}\left[\frac{TZ1\{Q\le y\}1\{\mathbf{X}\le\mathbf{x}\}}{q_0(\mathbf{X})}\right] \\
&\quad -\mathbb{E}^{km}\left[\frac{T(1-Z)1\{Q\le y\}1\{\mathbf{X}\le\mathbf{x}\}}{1-q_0(\mathbf{X})}\right], \\
I^{0,ldte}(y,x) &= \mathbb{E}^{km}\left[\frac{(1-T)(1-Z)1\{Q\le y\}1\{\mathbf{X}\le\mathbf{x}\}}{1-q_0(\mathbf{X})}\right] \\
&\quad -\mathbb{E}^{km}\left[\frac{(1-T)Z1\{Q\le y\}1\{\mathbf{X}\le\mathbf{x}\}}{1-q_0(\mathbf{X})}\right].
\end{aligned}$$

Finally, to conclude the proof, note that by the Law of Total Probability we can rewrite $I^{t,ldte}(y,\mathbf{x})$, $t\in\{0,1\}$, as

$$\begin{aligned}
I^{t,ldte}(y,\mathbf{x}) \equiv (2t-1)\Bigg\{&\mathbb{E}^{km}\left[\frac{1\{Q\le y\}}{q_0(\mathbf{X})}1\{\mathbf{X}\le\mathbf{x}\}\middle| T=t,Z=1\right]\mathbb{P}(T=t,Z=1) \\
&-\mathbb{E}^{km}\left[\frac{1\{Q\le y\}}{1-q_0(\mathbf{X})}1\{\mathbf{X}\le\mathbf{x}\}|T=t,Z=0\right]\mathbb{P}(T=t,Z=0)\Bigg\}.
\end{aligned}$$

■

Before stating the next Lemma, we need to introduce some additional notation. For $t\in\{0,1\}$, $z\in\{0,1\}$, let $H_{tz}(y)=\mathbb{P}(Q\le y,T=t,Z=z)$, $H_{tz,0}(y)=\mathbb{P}(Q\le y,\delta=0,T=t,Z=z)$, and $H_{tz,11}(y,x)=\mathbb{P}(Q\le y,\mathbf{X}\le\mathbf{x},T=t,Z=z,\delta=1)$. Define

$$\gamma_{tz,0}(\bar{y})=\exp\left\{\int_0^{\bar{y}-}\frac{H_{tz,0}(d\bar{w})}{1-H_{tz}(\bar{w})}\right\},$$

and put

$$\gamma_{tz,1}(\bar{y};y,\mathbf{x})=\frac{1}{1-H_{tz}(\bar{y})}\int 1\{\bar{y}<\bar{w}\}\,\xi_{tz}^{ldte}(\bar{w},\bar{\mathbf{x}},t,z;y,\mathbf{x})\,\gamma_{tz,0}(\bar{w})\,H_{tz,11}(d\bar{w},d\bar{\mathbf{x}})$$

and

$$\gamma_{tz,2}(\bar{y};y,\mathbf{x})=\int\int\frac{1\{\bar{v}<\bar{y},\bar{v}<\bar{w}\}\,\xi_{tz}^{ldte}(\bar{w},\bar{\mathbf{x}},t,z;y,\mathbf{x})}{[1-H_{tz}(\bar{v})]^2}\gamma_{tz,0}(\bar{w})\,H_{tz,0}(d\bar{v})\,H_{tz,11}(d\bar{w},d\bar{\mathbf{x}}),$$

where

$$\begin{aligned}
\xi_{11}^{ldte}(Q,\mathbf{X},T,Z;y,\mathbf{x}) &= \frac{TZ1\{Q\le y\}1\{\mathbf{X}\le\mathbf{x}\}}{q_0(\mathbf{X})}, \\
\xi_{10}^{ldte}(Q,\mathbf{X},T,Z;y,\mathbf{x}) &= \frac{T(1-Z)1\{Q\le y\}1\{\mathbf{X}\le\mathbf{x}\}}{1-q_0(\mathbf{X})}, \\
\xi_{01}^{ldte}(Q,\mathbf{X},T,Z;y,x) &= \frac{(1-T)Z1\{Q\le y\}1\{\mathbf{X}\le\mathbf{x}\}}{q_0(X)}, \\
\xi_{00}^{ldte}(Q,\mathbf{X},T,Z;y,x) &= \frac{(1-T)(1-Z)1\{Q\le y\}1\{\mathbf{X}\le\mathbf{x}\}}{1-q_0(\mathbf{X})}.
\end{aligned}$$

Let

$$\eta^{ldte}(y,\mathbf{x}) = \left(\eta_{11}^{ldte}(y,\mathbf{x}) - \eta_{10}^{ldte}(y,\mathbf{x})\right) - \left(\eta_{\bar{\tau},00}^{ldte}(y,\mathbf{x}) - \eta_{\bar{\tau},01}^{ldte}(y,\mathbf{x})\right),$$

where for $t \times z \in \{0,1\}^2$,

$$\eta_{tz}^{ldte}(y,\mathbf{x}) = \xi_{tz}^{ldte}(Q,\mathbf{X},T,Z;y,\mathbf{x})\,\gamma_{tz,0}(Q)\,\delta + \gamma_{tz,1}^{ldte}(Q;y,\mathbf{x})(1-\delta) - \gamma_{tz,2}^{ldte}(Q;y,\mathbf{x}).$$

Furthermore, denote

$$\begin{aligned}\alpha^{ldte}(\mathbf{X};y,\mathbf{x}) &= \left(\alpha_{11}^{ldte}(\mathbf{X};y,\mathbf{x}) - \alpha_{10}^{ldte}(\mathbf{X};y,\mathbf{x})\right) \\ &\quad - \left(\alpha_{00}^{ldte}(\mathbf{X};y,\mathbf{x}) - \alpha_{01}^{ldte}(\mathbf{X};y,\mathbf{x})\right),\end{aligned}$$

with

$$\begin{aligned}\alpha_{11}^{ldte}(\mathbf{X};y,\mathbf{x}) &= -\frac{\mathbb{E}\left(T(1)\,1\{Y(1)\le y\}\,|\mathbf{X}\right)1\{\mathbf{X}\le\mathbf{x}\}}{q_0(\mathbf{X})}, \\ \alpha_{10}^{ldte}(\mathbf{X};y,\mathbf{x}) &= \frac{\mathbb{E}\left(T(0)\,1\{Y(1)\le y\}\,|\mathbf{X}\right)1\{\mathbf{X}\le\mathbf{x}\}}{1-q_0(\mathbf{X})}, \\ \alpha_{01}^{ldte}(\mathbf{X};y,\mathbf{x}) &= -\frac{\mathbb{E}\left((1-T(1))\,1\{Y(0)\le y\}\,|\mathbf{X}\right)1\{\mathbf{X}\le\mathbf{x}\}}{q_0(\mathbf{X})}, \\ \alpha_{00}^{ldte}(\mathbf{X};y,\mathbf{x}) &= \frac{\mathbb{E}\left((1-T(0))\,1\{Y(0)\le y\}\,|\mathbf{X}\right)1\{\mathbf{X}\le\mathbf{x}\}}{1-q_0(\mathbf{X})}.\end{aligned}$$

Lemma S.10 *Under the same assumptions as in Theorem 6, we have that, uniformly in* $(y,\mathbf{x}) \in \mathcal{W}$,

$$\begin{aligned}&\sqrt{n}\left(\hat{I}_n^{ldte}(y,\mathbf{x}) - I^{ldte}(y,\mathbf{x})\right) \\ &\quad = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\eta_i^{ldte}(\mathbf{x}) - I^{ldte}(\mathbf{x})\right] + \alpha^{ldte}(\mathbf{X}_i;y,\mathbf{x})(Z_i - q_0(\mathbf{X}_i)) + o_{\mathbb{P}}(1), \quad \text{(S.31)}\end{aligned}$$

Proof of Lemma S.10: With some abuse of notation, let

$$\begin{aligned}I^{00,ldte}(y,\mathbf{x},q) &= \mathbb{E}\left[\frac{(1-T)(1-Z)\,1\{Y\le y\}\,1\{\mathbf{X}\le\mathbf{x}\}}{1-q(\mathbf{X})}\right], \\ I^{01,ldte}(y,\mathbf{x},q) &= \mathbb{E}\left[\frac{(1-T)\,Z1\{Y\le y\}\,1\{\mathbf{X}\le\mathbf{x}\}}{q(\mathbf{X})}\right] \\ I^{10,ldte}(y,\mathbf{x},q) &= \mathbb{E}\left[\frac{T(1-Z)\,1\{Y\le y\}\,1\{\mathbf{X}\le\mathbf{x}\}}{1-q(\mathbf{X})}\right] \\ I^{11,ldte}(y,\mathbf{x},q) &= \mathbb{E}\left[\frac{TZ1\{Y\le y\}\,1\{\mathbf{X}\le\mathbf{x}\}}{q(\mathbf{X})}\right], \\ I_n^{00,ldte}(y,\mathbf{x},q) &= \mathbb{E}_n^{km}\left[(1-T)(1-Z)\,\frac{1\{Q\le y\}\,1\{\mathbf{X}\le\mathbf{x}\}}{1-q(\mathbf{X})}\right] \\ I_n^{01,ldte}(y,\mathbf{x},q) &= \mathbb{E}_n^{km}\left[\frac{(1-T)\,Z1\{Q\le y\}\,1\{\mathbf{X}\le\mathbf{x}\}}{q(\mathbf{X})}\right]\end{aligned}$$

$$
\begin{aligned}
I_n^{10,ldte}(y,\mathbf{x},q) &= \mathbb{E}_n^{km}\left[\frac{T(1-Z)1\{Q\le y\}1\{\mathbf{X}\le\mathbf{x}\}}{1-q(\mathbf{X})}\right]\\
I_n^{11,ldte}(y,\mathbf{x},q) &= \mathbb{E}_n^{km}\left[T\frac{1\{Q\le y\}1\{\mathbf{X}\le\mathbf{x}\}}{q(\mathbf{X})}\right].
\end{aligned}
$$

Then, notice that

$$
\begin{aligned}
\sqrt{n}\left(\hat{I}_n^{ldte}(y,x)-I^{ldte}(y,x)\right) &= \sqrt{n}\left(I_n^{11,ldte}(y,x,\hat{q}_n)-I^{11,ldte}(y,x,q_0)\right)\\
&\quad-\sqrt{n}\left(I_n^{10,ldte}(y,x,\hat{q}_n)-I^{10,ldte}(y,x,q_0)\right)\\
&\quad+\sqrt{n}\left(I_n^{01,ldte}(y,x,\hat{q}_n)-I^{01,ldte}(y,x,q_0)\right)\\
&\quad-\sqrt{n}\left(I_n^{00,ldte}(y,x,\hat{q}_n)-I^{00,ldte}(y,x,q_0)\right).
\end{aligned}
\tag{S.32}
$$

Therefore, we can work with $\sqrt{n}\left(I^{tz,ldte}(y,\mathbf{x},\hat{q}_n)-I^{tz,ldte}(y,\mathbf{x},q_0)\right)$ for each $t\times z\in\{0,1\}^2$, separately. Given that these four functionals have symmetric construction, we only provide detailed arguments for the linear representation of $\sqrt{n}\left(I^{11,ldte}(y,\mathbf{x},\hat{q}_n)-I^{11,ldte}(y,\mathbf{x},q_0)\right)$.

As in the proof of Lemma 3 but now with q playing the role of p, we have that

$$
\begin{aligned}
&\sqrt{n}\left(I_n^{11,ldte}(y,\mathbf{x},\hat{q}_n)-I^{11,ldte}(y,\mathbf{x},q_0)\right)\\
=\ &\sqrt{n}\int\varphi_{y,\mathbf{x},q_0}^{11,ldte}(\bar{y},\bar{\mathbf{x}},\bar{t},\bar{z})\left[F_n^{km}(d\bar{y},d\bar{\mathbf{x}},d\bar{t},tz)-F(d\bar{y},d\bar{\mathbf{x}},d\bar{t},d\bar{z})\right]\\
&+\sqrt{n}\int\left[\varphi_{y,\mathbf{x},\hat{q}_n}^{11,ldte}(\bar{y},\bar{\mathbf{x}},\bar{t},\bar{z})-\varphi_{y,\mathbf{x},q_0}^{11,ldte}(\bar{y},\bar{\mathbf{x}},\bar{t},\bar{z})\right]\left(F_n^{km}(d\bar{y},d\bar{\mathbf{x}},d\bar{t},d\bar{z})-F(d\bar{y},d\bar{\mathbf{x}},d\bar{t},d\bar{z})\right)\\
&+\sqrt{n}\int\left[\varphi_{y,\mathbf{x},\hat{q}_n}^{11,ldte}(\bar{y},\bar{\mathbf{x}},\bar{t},\bar{z})-\varphi_{y,\mathbf{x},q_0}^{11,ldte}(\bar{y},\bar{\mathbf{x}},\bar{t},\bar{z})\right]F(d\bar{y},d\bar{\mathbf{x}},d\bar{t},d\bar{z})\\
=\ &\mathbb{A}_{1n}^{ldte}+\mathbb{A}_{2n}^{ldte}+\mathbb{A}_{3n}^{ldte},
\end{aligned}
$$

where, for a generic $q(\cdot)$, $\varphi_{y,\mathbf{x},p}^{11,ldte}(\bar{y},\bar{\mathbf{x}},\bar{t},\bar{z})=\bar{z}\bar{t}1\{\bar{y}\le y\}1\{\bar{\mathbf{x}}\le\mathbf{x}\}/q(\bar{\mathbf{x}})$.

Then, following the same type of arguments as in the proof of Lemma 3, we have that, uniformly in $\mathbf{x}\in\mathcal{X}_X$,

$$
\begin{aligned}
\mathbb{A}_{2n}^{ldte} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\eta_{11,i}^{ldte}(y,\mathbf{x})-I^{11,ldte}(y,\mathbf{x},q_0)\right)+o_{\mathbb{P}}(1),\\
\mathbb{A}_{2n}^{ldte} &= o_{\mathbb{P}}(1),\\
\mathbb{A}_{3n}^{ldte} &= \frac{-1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\mathcal{E}\left(T(1)1\{Y(1)\le y\}|\mathbf{X}_i\right)1\{\mathbf{X}_i\le\mathbf{x}\}}{q_0(\mathbf{X}_i)}\left(Z_i-q_0(\mathbf{X}_i)\right)+o_{\mathbb{P}}(1).
\end{aligned}
$$

Repeating the same arguments for $tz\in\{10,01,00\}$, one established the asymptotic linear representation of each $\sqrt{n}\left(I^{tz,ldte}(y,\mathbf{x},\hat{q}_n)-I^{tz,ldte}(y,\mathbf{x},q_0)\right)$. The proof is completed by simply plugging these asymptotic linear representations into (S.32). ■

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