Database of known results on analytic number theory exponents

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Chapter 1

Introduction

This is the LaTeX "Blueprint" form of the *analytic number theory exponent database (ANTEDB)*, which is an ongoing project to record (both in a human-readable and computer-executable formats) the latest known bounds, conjectures, and other relationships concerning several exponents of interest in analytic number theory. It can be viewed as an expansion of the paper [279]. Currently, the database is recording information on the following exponents:

- Exponent pairs (k, ℓ).
- The exponential sum function $\beta(\alpha)$ dual to exponent pairs.
- The growth exponent $\mu(\sigma)$ of the zeta function $\zeta(\sigma + it)$.
- The moment exponents $M(\sigma, A)$ of the zeta function.
- Large value exponents $\mathrm{LV}(\sigma, \tau)$ for Dirichlet polynomials $\sum_{n \in [N,2N]} a_n n^{-it}$.
- Large value exponents $\mathrm{LV}_\zeta(\sigma, \tau)$ of the zeta polynomials $\sum_{n \in I} n^{-it}$.
- Large value additive energy exponents $\mathrm{LV}^*(\sigma, \tau)$, $\mathrm{LV}^*_\zeta(\sigma, \tau)$ for Dirichlet and zeta polynomials.
- Zero density exponents $\mathrm{A}(\sigma)$ for the zeta function.
- Zero density additive energy exponents $\mathrm{A}^*(\sigma)$ for the zeta function.
- The regions $\mathcal{E}$, $\mathcal{E}_\zeta$ of exponent tuples $(\sigma, \tau, \rho, \rho^*, s)$ recording possible large values, large value additive energy, and double zeta sums for Dirichlet and zeta polynomials.
- Exponents α_k for the Dirichlet divisor problem.
- The primitive Pythagorean triple exponent θ_{Pythag}.
- Exponents θ_{PNT}, $\theta_{\mathrm{PNT-AA}}$ for the prime number theorem in all or almost all short intervals.
- Exponents related to prime gaps, such as the maximal prime gap exponent θ_{gap}, the prime gap second moment exponent $\theta_{\mathrm{gap},2}$ as well as extremal results on small and large prime gaps (including narrow prime k-tuples).

- Results on the de Bruijn-Newman constant Λ.
- Error terms in the prime number theorem, and in the prime number theorem in arithmetic progressions.
- Zero-free regions for the Riemann zeta function and L-functions.
- Brun-Titchmarsh type theorems and Linnik's constant.
- Goldbach and Waring type problems.
- The Gauss circle problem and its generalizations.

By an exponent, we mean one or more real numbers, possibly depending on other exponent parameters, that occur as an exponent in an analytic number theory estimate, for instance as the exponent in some scale parameter T that bounds some other quantity of interest. (See also [63] for a recent discussion of a similar class of exponents.)
Possible future directions for expansion include

- Exponents for L-functions (in both q and T aspects).
- Zero density exponents for L-functions (this topic is currently claimed).
- More exponents relating to prime gaps.
- Exponents relating to sieve theory.
- Integration with the TME-EMT project.
- Log-free estimates, or estimates with explicit constants (this topic is currently claimed).
- Lean certification of some of the calculations in the database.
- Upper and lower bounds on gaps between zeroes of zeta or L-functions (assuming RH if necessary), for instance incorporating results obtained via the Montgomery-Odlyzko method.
- Character sum bounds (such as the Polya-Vinogradov and Burgess bounds) and the least quadratic residue problem.
- Level of distribution of the primes and other multiplicative functions, possibly with restrictions on the moduli.
- Error terms in the Titchmarsh divisor problem.
- The proportion of zeroes on the critical line, and estimates on mollifiers for the zeta function.
- Vinogradov mean value type theorems.

The database aims to enumerate, as comprehensively as possible, all the various known or conjectured facts about these exponents, including "trivial" or "obsolete" such facts. Of particular interest are implications that allow new bounds on exponents to be established from existing bounds on other exponents.
Each of the facts in the database can be supported with a reference, or one or more proofs, or with executable code in Python; ideally one should have all three (and with a preference

for proofs that rely as much as possible on other facts in the database). In the future we could also expand this database to support as many of these facts as possible with formal derivations in proof assistant languages such as Lean.

In order to facilitate the dependency tree of the python code, as well as to assist readers who wish to derive the facts in this database from first principles, the blueprint is arranged in linear order. Thus, the statement and proof of a proposition in the blueprint is only permitted to use propositions and definitions that are located earlier in the blueprint, although we do allow forward-referencing references in the remarks. As a consequence, the material relating to a single topic will not necessarily be located in a single chapter, but could be spread out over multiple chapters, depending on how much advanced material is needed to state or prove the required results. Additionally, a single proposition may occur multiple times in the blueprint, if it has multiple proofs with varying prerequisites. In the future, one could hope to implement a search feature that will allow the reader to locate all propositions of relevance to a given topic (e.g., all propositions whose statement involves the concept of an exponent pair).

This is intended to be a living database, and we hope to gain community support for updating it. As such, corrections, suggestions, and new contributions are very welcome, either via email to one of us (Terence Tao, Timothy Trudgian, or Andrew Yang), or by a direct pull request to the Github repository. Instructions for contributing can be found here.

A paper describing the ANTEDB, and the new bounds that were already obtained as a result of compiling the database, can be found at [270].

Chapter 2

Basic notation

We freely assume the axiom of choice in this blueprint.
Throughout this blueprint we adopt following notation. If θ is a real number, then we write

$$e(\theta) := e^{2\pi i\theta}$$

where i is the imaginary unit. The indicator function $1_I(n)$ of a set I is defined to equal 1 when $n \in I$, and 0 otherwise.
We adopt the convention that an empty supremum is $-\infty$, and an empty infimum is $+\infty$. Thus, for instance, $\sup_{\sigma_0 \leq \sigma \leq \sigma_1} f(\sigma)$ would equal $-\infty$ if $\sigma_1 < \sigma_0$. Related to this, we also adopt the convention that $N^{-\infty} = 0$ when $N > 1$.
The cardinality of a finite set W will be denoted $|W|$.
A sequence $a_n, n \in I$ of real or complex numbers indexed by some index set is said to be 1-*bounded* if $|a_n| \leq 1$ for all $n \in I$. Similarly, a set W of real numbers is said to be 1-*separated* if $|t - t'| \geq 1$ for all distinct $t, t' \in W$. One can define more general notions of λ-bounded or λ-separated for other $\lambda > 0$ in the obvious fashion.

2.1 Asymptotic (or "cheap nonstandard") notation

It is convenient to use a "cheap nonstandard analysis" framework for asymptotic notation, in the spirit of [269], as this will reduce the amount of "epsilon management" one has to do in the arguments. This framework is inspired by nonstandard analysis, but we will avoid explicitly using such nonstandard constructions as ultraproducts in the discussion below, relying instead on the more familiar notion of sequential limits.
In this framework, we assume there is some ambient index parameter i, which ranges over some ambient sequence of natural numbers going to infinity. All mathematical objects X (numbers, sequences, sets, functions, etc.), will either be *fixed* - i.e., independent of i - or *variable* - i.e., dependent on i. (These correspond to the notions of *standard* and *non-standard* objects in nonstandard analysis.) Of course, fixed objects can be considered as special cases of variable objects, in which the dependency is constant. By default, objects should be understood to be variable if not explicitly declared to be fixed. For emphasis, we shall sometimes write $X = X_{\mathrm{i}}$ to explicitly indicate that an object X is variable; however, to reduce clutter, we shall generally omit explicit mention of the parameter i in most of our arguments. We will often reserve the right to refine the ambient sequence to a subsequence as needed, usually in order to apply a compactness theorem such as the Bolzano–Weierstrass

theorem; we refer to this process as "passing to a subsequence if necessary". When we say that a statement involving variable objects is true, we mean that it is true for all i in the ambient sequence. For instance, a variable set E of real numbers is a set $E = E_{\mathrm{i}}$ indexed by the ambient parameter i, and by an element of such a set, we mean a variable real number $x = x_{\mathrm{i}}$ such that $x_{\mathrm{i}} \in E_{\mathrm{i}}$ for all i in the ambient sequence.

We isolate some special types of variable numerical quantities $X = X_{\mathrm{i}}$ (which could be a natural number, real number, or complex number):

- X is *bounded* if there exists a fixed C such that $|X| \leq C$. In this case we also write $X = O(1)$.
- X is *unbounded* if $|X_{\mathrm{i}}| \to \infty$ as $\mathrm{i} \to \infty$; equivalently, for every fixed C, one has $|X| \geq C$ for i sufficiently large.
- X is *infinitesimal* if $|X_{\mathrm{i}}| \to 0$ as $\mathrm{i} \to \infty$; equivalently, for every fixed $\varepsilon > 0$, one has $|X| \leq \varepsilon$ for i sufficiently large. In this case we also write $X = o(1)$.

Note that any quantity X will be either bounded or unbounded, after passing to a subsequence if necessary; similarly, by the Bolzano–Weierstrass theorem, any bounded (variable) quantity X will be of the form $X_0 + o(1)$ for some fixed X_0, after passing to a subsequence if necessary. Thus, for instance, if $T, N > 1$ are (variable) quantities with $N = T^{O(1)}$ (or equivalently, $T^{-C} \leq N \leq T^C$ for some fixed C), then, after passing to a subsequence if necessary, we may write $N = T^{\alpha + o(1)}$ for some fixed real number α. Note that any further passage to subsequences do not alter these concepts; quantities that are bounded, unbounded, or infinitesimal remain so under any additional restriction to subsequences.

We observe the *underspill principle*: if X, Y are (variable) real numbers, then the relation

$$X \leq Y + o(1)$$

is equivalent to the relation

$$X \leq Y + \varepsilon + o(1)$$

holding for all fixed $\varepsilon > 0$.

We can develop other standard asymptotic notation in the natural fashion: given two (variable) quantities X, Y, we write $X = O(Y)$, $X \ll Y$, or $Y \gg X$ if $|X| \leq CY$ for some fixed C, and $X = o(Y)$ if $|X| \leq cY$ for some infinitesimal c. We also write $X \asymp Y$ for $X \ll Y \ll X$.

A convenient property of this asymptotic formalism, analogous to the property of *ω-saturation* in nonstandard analysis, is that certain asymptotic bounds are automatically uniform in variable parameters.

Proposition 2.1 (Automatic uniformity)**.** *Let $E = E_{\mathrm{i}}$ be a non-empty variable set, and let $f = f_{\mathrm{i}} : E \to \mathbf{C}$ be a variable function.*

(i) Suppose that $f(x) = O(1)$ for all (variable) $x \in E$. Then after passing to a subsequence if necessary, the bound is uniform, that is to say, there exists a fixed C such that $|f(x)| \leq C$ for all $x \in E$.

(ii) Suppose that $f(x) = o(1)$ for all (variable) $x \in E$. Then after passing to a subsequence if necessary, the bound is uniform, that is to say, there exists an infinitesimal c such that $|f(x)| \leq c$ for all $x \in E$.

Proof. We begin with (i). Suppose that there is no uniform bound. Then for any fixed natural number n, one can find arbitrarily large i_n in the sequence and $x_{\mathrm{i}_n} \in E_{\mathrm{i}_n}$ such that

$|f_{\mathrm{i}_n}(x_{\mathrm{i}_n})| \geq n$. Clearly one can arrange matters so that the sequence i_n is increasing. If one then restricts to this sequence and sets x to be the variable element x_{i_n} of E, then $f(x)$ is unbounded, a contradiction.

Now we prove (ii). We can assume for each fixed n that there exists i_n in the ambient sequence such that $|f_{\mathrm{i}}(x_{\mathrm{i}})| \leq 1/n$ for all $\mathrm{i} \geq \mathrm{i}_n$ and $x_{\mathrm{i}} \in E_{\mathrm{i}}$, since if this were not the case one can construct an $x = x_{\mathrm{i}} \in E$ such that $|f_{\mathrm{i}}(x_{\mathrm{i}})| \geq 1/n$ for i sufficiently large, contradicting the hypothesis. Again, we may take the i_n to be increasing. Restricting to this sequence, and writing $c_{\mathrm{i}_n} := 1/n$, we see that $c = o(1)$ and $|f(x)| \leq c$ for all $x \in E$, as required. □

Remark 2.2. *It is important in Proposition 2.1 that the hypotheses in (i), (ii) are assumed for all* variable $x \in E$*, rather than merely all* fixed $x \in E$*. For instance, let $E = \mathbf{R}$ and consider the variable function $f_{\mathrm{i}}(x) := x/\mathrm{i}$. Then $f(x) = o(1)$ for any fixed $x \in E$, but the decay rate is not uniform, and we do not have $f(x) = o(1)$ for all variable $x \in E$ (e.g., $x_{\mathrm{i}} := \mathrm{i}$ is a counterexample).*

Remark 2.3. *There are two caveats to keep in mind when using this asymptotic formalism. Firstly, the law of the excluded middle is only valid after passing to subsequences. For instance, it is possible for a nonstandard natural number to neither be even or odd, since it could be even for some* i *and odd for others. However, one can pass to a subsequence in which it becomes either even or odd. Secondly, one cannot combine the "external" concepts of asymptotic notation with the "internal" framework of (variable) set theory. For instance, one cannot view the collection of all bounded (variable) real numbers as a variable set, since the notion of boundedness is not "pointwise" to each index* i*, but instead describes the "global" behavior of this index set. Thus, for instance, set builder notation such as $\{x : x = O(1)\}$ should be avoided.*

Chapter 3

Basic Fourier estimates

Lemma 3.1 (L^2 integral estimate). *Let $\xi_1, \dots, \xi_R$ be real numbers that are $1/N$-separated. Then for any interval I of length T, and any sequence $a_1, \dots, a_R$ of complex numbers one has*

$$\int_I |\sum_{r=1}^R a_r e(\xi_r t)|^2 \, dt = (T + O(N)) \sum_{r=1}^R |a_r|^2.$$

Proof. We adapt the proof of [149, Theorem 9.1]. Without loss of generality we may normalize $\sum_{r=1}^R |a_r|^2 = 1$. From the Plancherel identity we have

$$\int_{\mathbf{R}} |\sum_{r=1}^R a_r e(\xi_r t)|^2 |\hat{\psi}((t-t_0)/N)|^2 \, dt = N \tag{3.1}$$

whenever $t_0 \in \mathbf{R}$ and ψ is a smooth function supported on $[-1/4, 1/4]$ of L^2 norm 1. By suitable choice of ψ, this implies that

$$\int_J |\sum_{r=1}^R a_r e(\xi_r t)|^2 \, dt \ll N \tag{3.2}$$

whenever J is an interval of length N. If one integrates (3.1) for all $t_0 \in I$, we see that

$$\int_I |\sum_{r=1}^R a_r e(\xi_r t)|^2 \, dt = T - \int_{\mathbf{R}} |\sum_{r=1}^R a_r e(\xi_r t)|^2 \left(\int_I |\hat{\psi}((t-t_0)/N)|^2 \, dt_0 - 1_I(t) \right) \, dt.$$

Since $\hat{\psi}$ is rapidly decreasing and has L^2 norm 1, one can compute

$$\int_I |\hat{\psi}((t-t_0)/N)|^2 \, dt_0 - 1_I(t) \ll (1 + \operatorname{dist}(t, \partial I)/N)^{-10}$$

and hence by (3.2) and the triangle inequality

$$\int_{\mathbf{R}} |\sum_{r=1}^R a_r e(\xi_r t)|^2 \left(\int_I |\hat{\psi}((t-t_0)/N)|^2 \, dt_0 - 1_I(t) \right) \, dt \ll N$$

giving the claim. □

Chapter 4

Exponential sum growth exponents

4.1 Phase functions

Definition 4.1 (Phase function)**.** *A* phase function *is a (variable) smooth function* $F\colon [1,2] \to \mathbf{R}$*. A phase function* F *will be called a* model phase function *if there exists a fixed exponent* $\sigma > 0$ *with the property that*

$$F^{(p+1)}(u) - \frac{d^p}{du^p} u^{-\sigma} = o(1) \tag{4.1}$$

for all (variable) $u \in [1,2]$ *and all fixed* $p \geq 0$*, where* $F^{(p+1)}$ *denotes the* $(p+1)^{\mathrm{st}}$ *derivative of* F*.*

For instance, $u \mapsto \log u$ is a model phase function (with $\sigma = 1$), and for any fixed $\sigma \neq 1$, $u \mapsto u^{1-\sigma}/(1-\sigma)$ is also a model phase function. Informally, a model phase function is a function which asymptotically behaves like $u \mapsto \log u$ (for $\sigma = 1$) or $u \mapsto u^{1-\sigma}/(1-\sigma)$ (for $\sigma \neq 1$), up to constants. This turns out to be a good class for exponential sum estimates, as it is stable under Weyl differencing and Legendre transforms, which show up in the van der Corput A-process and B-process respectively.
Note from Proposition 2.1 that the $o(1)$ decay rate in (4.1) can be made uniform, after passing to a subsequence if necessary.

4.2 Exponential sum exponent

The main purpose of this chapter is to introduce and establish the basic properties of the following exponent function.

Definition 4.2 (Exponent sum growth exponent)**.** *For any fixed* $\alpha \geq 0$*, let* $\beta(\alpha) \in \mathbf{R}$ *denote the least possible (fixed) exponent for which the following claim holds: whenever* $N, T \geq 1$ *are (variable) quantities with* T *unbounded and* $N = T^{\alpha+o(1)}$*,* F *is a model phase function, and* $I \subset [N, 2N]$ *is an interval, then*

$$\sum_{n \in I} e(TF(n/N)) \ll T^{\beta(\alpha)+o(1)}.$$

Implemented at ***bound_beta.py*** *as:*
`Bound_beta`

It is easy to see that the set of possible candidates for $\beta(\alpha)$ is closed (thanks to underspill), non-empty, and bounded from below, so β is well-defined as a (fixed) function from $[0, +\infty)$ to $\mathbf{R}$. Specializing to the logarithmic phase $F(u) = \log u$, and performing a complex conjugation, we see in particular that

$$\sum_{n \in I} n^{-iT} \ll T^{\beta(\alpha)+o(1)} \tag{4.2}$$

whenever T is unbounded, $N = T^{\alpha+o(1)}$, and I is an interval in $[N, 2N]$. Thus it is clear that knowledge of β is of relevance to understanding the Riemann zeta function.
The quantity $\beta(\alpha)$ can also be formulated without asymptotic notation, but at the cost of introducing some "epsilon and delta" parameters:

Lemma 4.3 (Non-asymptotic definition of β)**.** *Let $\alpha \geq 0$ and $\overline{\beta} \in \mathbf{R}$ be fixed. Then the following are equivalent:*

(i) $\beta(\alpha) \leq \overline{\beta}$.

(ii) For every (fixed) $\varepsilon > 0$ and $\sigma > 0$ there exists (fixed) $\delta > 0$, $P \geq 1$, $C \geq 1$ with the following property: if $T \geq C$, $T^{\alpha-\delta} \leq N \leq T^{\alpha+\delta}$ are (fixed) real numbers, $I \subset [N, 2N]$ is a (fixed) interval, and F is a (fixed) phase function such that

$$\left| F^{(p+1)}(u) - \frac{d^p}{du^p} u^{-\sigma} \right| \leq \delta \tag{4.3}$$

for all (fixed) $0 \leq p \leq P$ and $u \in [1, 2]$, then

$$|\sum_{n \in I} e(TF(n/N))| \leq CT^{\overline{\beta}+\varepsilon}.$$

Proof. It is easy to see that (ii) implies (i) by expanding out all the definitions (and using Proposition 2.1 to resolve any uniformity issues). Conversely, suppose that (ii) fails. Carefully negating all the quantifiers, we conclude that there exists a fixed $\varepsilon, \sigma > 0$ such that for any (fixed) natural number i, one can find real numbers $T = T_{\mathrm{i}} \geq \mathrm{i}$, $T^{\alpha-1/\mathrm{i}} \leq N = N_{\mathrm{i}} \leq T^{\alpha+1/\mathrm{i}}$, an interval $I = I_{\mathrm{i}} \subset [N_{\mathrm{i}}, 2N_{\mathrm{i}}]$, and a phase function $F = F_{\mathrm{i}}$ such that

$$|F_{\mathrm{i}}^{(p+1)}(u) - \frac{d^p}{du^p} u^{-\sigma}| \leq 1/\mathrm{i}$$

for all (fixed) $0 \leq p \leq \mathrm{i}$ and $u \in [1, 2]$, but that

$$|\sum_{n \in I} e(TF(n/N))| \geq \mathrm{i}T^{\overline{\beta}+\varepsilon}.$$

But then $F = F_{\mathrm{i}}$ is a model phase function which gives a counterexample to the claim $\beta(\alpha) \leq \overline{\beta}$. ☐

We will however work with the asymptotic formulation of β throughout this database, as it makes the proofs somewhat cleaner.
We record the trivial bounds on β:

Lemma 4.4 (Trivial bounds on β)**.** *For any fixed $\alpha > 1$, we have*

$$\beta(\alpha) = \alpha - 1.$$

For fixed $0 \leq \alpha \leq 1$, we have

$$\frac{\alpha}{2} \leq \beta(\alpha) \leq \alpha.$$

In particular

$$\beta(0) = 0. \tag{4.4}$$

Implemented at `bound_beta.py` *as:*
`trivial_beta_bound_1`
`trivial_beta_bound_2`

Proof. Let $T > 1$ be unbounded, $N = T^{\alpha+o(1)}$, $I \subset [N, 2N]$ an interval, and F a model phase function.
For $\alpha > 1$, the Euler–Maclaurin formula (see e.g. [277, (2.1.2)]) gives

$$\left|\sum_{N \leq n \leq 2N} n^{iT}\right| = \left|\frac{2^{1+iT} - 1}{1 + iT} N^{1+iT} + O(1)\right| \asymp \frac{N}{T} \tag{4.5}$$

which gives the lower bound $\beta(\alpha) \geq \alpha - 1$; applying the Euler–Maclaurin formula for model phase functions F then gives the matching upper bound.
The triangle inequality bound

$$\sum_{n \in I} e(TF(n/N)) \ll N$$

gives the upper bound $\beta(\alpha) \leq \alpha$. Next, if $0 \leq \alpha \leq 1$, then from Lemma 3.1 (and the $\gg 1/N$-separated nature of the $F(n/N)$ for model phase functions F, after passing to a subsequence if necessary) that

$$\int_{T}^{2T} \left|\sum_{n \in [N,2N]} e(tF(n/N))\right|^2 dt \asymp TN$$

for $N = cT^{\alpha}$ for c a fixed small enough constant, which by the pigeonhole principle implies that

$$\left|\sum_{n \in [N,2N]} e(tF(n/N))\right|^2 dt \gg N^{1/2} = T^{\alpha/2}$$

for at least one $t \asymp T$, giving the claim. □

As we shall see, the exponent pair conjecture is equivalent to the lower bound here being sharp, thus it is conjectured that

$$\beta(\alpha) = \begin{cases} \alpha/2, & 0 \leq \alpha \leq 1 \\ \alpha - 1, & \alpha > 1 \end{cases}.$$

Note the discontinuity at 1. Despite this, we have:

Lemma 4.5 (Upper semicontinuity)**.** *β is an upper semicontinuous function.*

Proof. Routine from the definition. □

We record the classical bounds on β:

Lemma 4.6 (Van der Corput A process for β)**.** *If $0 \le \alpha \le 2/3$ and $h \ge 0$ then*

$$2\beta(\alpha) \le \max\left(2\alpha - h, 2h, \alpha - h + \sup_{2\alpha-1\le h'\le h}\left((h'+1-\alpha)\beta\left(\frac{\alpha}{h'+1-\alpha}\right) + h'\right)\right).$$

Implemented at `bound_beta.py` *as:*
`apply_van_der_corput_process_for_beta(bounds)`

Proof. By definition, there exists an unbounded T, $N = T^{\alpha+o(1)}$, F a model phase function, and $I \subset [N, 2N]$ such that

$$\sum_{n\in I} e(TF(n/N)) = T^{\beta(\alpha)+o(1)}.$$

Applying [144, (2.54)] with $H := T^h$, as well as the ensuing computations to dispose of the $j \ll T/N^2$ terms, one then has

$$T^{2\beta(\alpha)+o(1)} \ll N^2H^{-1} + H^2 + NH^{-1} \sum_{T/N^2 \ll j \ll H} |\sum_{n\in I\cap I-j} e(T(F((n+j)/N) - F(n/N)))|$$

and hence by the pigeonhole principle

$$T^{2\beta(\alpha)+o(1)} \ll N^2H^{-1} + H^2 + T^{o(1)}NH^{-1} \sum_{j=T^{h'+o(1)}} |\sum_{n\in I\cap I-h} e(T(F((n+j)/N) - F(n/N)))|$$

for some $2\alpha - 1 \le h' \le h$ (one can delete this term if $h < 2\alpha - 1$). One can verify that $-\frac{1}{\sigma}\frac{N}{j}(F(u+j/N) - F(u))$ is a model phase function. Thus, by Definition 4.2, one has

$$\sum_{n\in I\cap I-h} e(T(F((n+j)/N) - F(n/N))) \ll (T^{1+h'+o(1)}/N)^{\beta(\alpha/(h'+1-\alpha))+o(1)},$$

and the claim follows after evaluating all terms as powers of T. □

Proposition 4.7 (Van der Corput inequality)**.** *For any natural number $k \ge 2$ and any $\alpha > 0$, one has*

$$\beta(\alpha) \le \max\left(\alpha + \frac{1-k\alpha}{2^k-2}, (1-2^{2-k})\alpha - \frac{1-\alpha}{2^k-2}\right).$$

Thus for instance when $k = 2$ we have

$$\beta(\alpha) \le \max\left(\frac{1}{2}, \frac{2\alpha-1}{2}\right),$$

so in particular

$$\beta(1) = \frac{1}{2}, \tag{4.6}$$

by Lemma 4.4, when $k = 3$ one has

$$\beta(\alpha) \leq \max\left(\frac{1+3\alpha}{6}, \frac{6\alpha-1}{3}\right),$$

and when $k = 4$ one has

$$\beta(\alpha) \leq \max\left(\frac{10\alpha+1}{14}, \frac{29\alpha-2}{28}\right).$$

This form of upper bound of $\beta(\alpha)$ - as the maximum of a finite number of linear functions of α - is extremely common in the literature.

Proof. Follows from [149, Theorem 8.20]. It is also possible to prove this by induction on k using Lemma 4.6. □

Corollary 4.8 (Optimizing the van der Corput inequality)**.** *For any $\alpha > 0$ one has*

$$\beta(\alpha) \leq \inf_{k \in \mathbf{N}: k \geq 2} \alpha + \frac{1-k\alpha}{2^k - 2}.$$

Thus for instance

$$\beta(\alpha) \leq \min\left(\frac{1}{2}, \frac{1+3\alpha}{6}, \frac{10\alpha+1}{14}\right).$$

Proof. Let $\beta_k(\alpha) = \alpha + (1-k\alpha)/(2^k - 2)$ and

$$\alpha_k = \frac{2^k}{(k-1)2^k + 2}.$$

Via a routine computation, $\beta_{k+1}(\alpha) \leq \beta_k(\alpha)$ for $\alpha \geq \alpha_k$ and any $k \geq 2$. Thus, to verify that $\beta(\alpha) \leq \beta_k(\alpha)$ for $0 \leq \alpha \leq 1/2$, it suffices to just show that the same result holds for $0 \leq \alpha \leq \alpha_k$. However, for $0 \leq \alpha \leq \alpha_k$ and $k \geq 2$, we have

$$0 \leq \alpha \leq \alpha_k \leq \frac{2^{k+1}}{(k-3)2^k + 8}$$

which rearranges to give

$$\alpha + \frac{1-k\alpha}{2^k - 2} \geq (1 - 2^{2-k})\alpha - \frac{1-\alpha}{2^k - 2}, \qquad (0 \leq \alpha \leq \alpha_k, k \geq 2)$$

which completes the proof in view of Proposition 4.7. See Figure 4.1. □

We can remove the role of I in the definition of β:

Lemma 4.9. *In Definition 4.2, one can take the interval I to be $[N, 2N]$.*

Proof. Suppose that $\alpha, \overline{\beta}$ are fixed quantities such that the bounds in Definition 4.2 hold just for $I = [N, 2N]$, thus whenever $T > 1$ is unbounded, $N = T^{\alpha + o(1)}$, and F is a model phase function one has

$$\sum_{N \leq n \leq 2N} e(TF(n/N)) \ll T^{\overline{\beta} + o(1)}. \tag{4.7}$$

Our task is then to show that

$$\sum_{n \in I} e(TfF(n/N)) \ll T^{\overline{\beta} + o(1)}$$

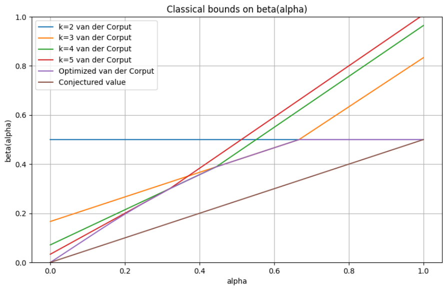

Figure 4.1: The bounds in Proposition 4.7 for $k = 2, 3, 4, 5$, compared against the optimized bound in Corollary 4.8.

under the same hypotheses. Similarly with $\alpha = 1$ we can use the proof of Lemma 4.7 to obtain $\overline{\beta} \geq 1/2$, and we are again done. Thus we may assume that $\alpha < 1$.
For $n \in [N, 2N]$, the constraint $n \in I$ is equivalent to restricting $F(n/N)$ to an interval J of length $O(1)$, which we can also smooth out by $O(1/N)$ without affecting the sum. Applying a Fourier expansion and the triangle inequality, we can thus bound the left-hand side by

$$\ll T^{o(1)} + \int_{-N^{1+o(1)}}^{N^{1+o(1)}} \left| \sum_{n \in [N,2N]} e(TF(n/N) - tF(n/N)) \right| \frac{dt}{1 + |t|}.$$

Since $\alpha > 1$, we have $|t - T| \leq T/2$ for all t in the integral if T is large enough. From hypothesis (4.7) (with T replaced by $T - t$)we have

$$\left| \sum_{n \in [N,2N]} e(TF(n/N) - tF(n/N)) \right| \ll T^{\overline{\beta}+o(1)}$$

for all such t, and the claim follows. See also Sargos [258, p 310]. □

Lemma 4.10 (Reflection). *For any $0 < \alpha < 1$, we have $\beta(\alpha) - \frac{\alpha}{2} = \beta(1-\alpha) - \frac{1-\alpha}{2}$, or equivalently $\beta(1 - \alpha) = \frac{1}{2} - \alpha + \beta(\alpha)$.*

TODO: implement this in python

Proof. This is the van der Corput B-process. See e.g., [129, p 370]. □

4.3 Known bounds on β

We remark that this corollary also follows from Proposition 5.10.

Theorem 4.11 (1989 Watt bound). *For any $3/7 \leq \alpha \leq 1/2$, one has*

$$\beta(\alpha) \leq \frac{89}{560} + \frac{1}{2}\alpha.$$

Recorded in `literature.py` *as:*
`add_beta_bound_watt_1989()`

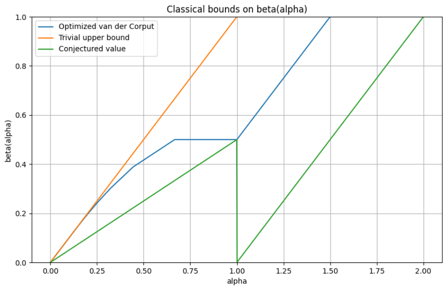

Figure 4.2: The bound in Corollary 4.8, compared against the trivial upper and lower bounds in Lemma 4.4.

Proof. See [293, Theorem 5]. □

Theorem 4.12 (1991 Huxley–Kolesnik bound)**.** *For any* $2/5 \le \alpha \le 1/2$ *one has*

$$\beta(\alpha) \le \max\left(\frac{1+8\alpha}{22}, \frac{11+112\alpha}{158}, \frac{1+17\alpha}{22}\right).$$

Recorded in `literature.py` *as:*
`add_beta_bound_huxley_kolesnik_1991()`

Proof. See [132, Theorem 3]. Note that the paper contains an error, however this result was reinstated with the corrections given in [133]. □

Theorem 4.13 (1993 Huxley bound)**.** *For any* $0 \le \alpha \le 49/114$*, one has*

$$\beta(\alpha) \le \max\left(\frac{13}{60} + \frac{7}{20}\alpha, \frac{11}{120} + \frac{13}{20}\alpha\right).$$

Furthermore, for any $49/114 \le \alpha \le 1/2$*, one has*

$$\beta(\alpha) \le \frac{89}{570} + \frac{1}{2}\alpha.$$

Recorded in `literature.py` *as:*
`add_beta_bound_huxley_1993()`

Proof. See [128, Theorem 1]. □

Theorem 4.14 (Second 1993 Huxley bound)**.** *If* $0 \le \alpha \le 1$*, then* $\beta(\alpha)$ *is bounded by*

$$\begin{aligned} &\frac{1}{146}(13+94\alpha) \text{ for } \alpha \le \frac{87}{275} \\ &\frac{1}{244}(11+191\alpha) \text{ for } \frac{87}{275} \le \alpha \le \frac{423}{1295} \\ &\frac{1}{1282}(89+908\alpha) \text{ for } \frac{423}{1295} \le \alpha \le \frac{227}{601} \\ &\frac{1}{280}(29+173\alpha) \text{ for } \frac{227}{601} \le \alpha \le \frac{12}{31} \\ &\frac{1}{128}(4+103\alpha) \text{ for } \frac{12}{31} \le \alpha \le 1. \end{aligned}$$

Recorded in `literature.py` *as:*
`add_beta_bound_huxley_1993_3()`

Proof. See [128, Theorem 3]. □

Theorem 4.15 (1995 Sargos bound)**.** *[258, Théorème 2.4, Lemme 2.6] For any* $0 \le \alpha \le 1$*, one has*

$$\beta(\alpha) \le \max\left(\alpha + \frac{3(1-4\alpha)}{40}, \frac{7}{8}\alpha, \frac{1}{3}\alpha - \frac{1-4\alpha}{6}, 0\right)$$

and

$$\beta(\alpha) \le \max\left(\alpha + \frac{1-4\alpha}{14}, \frac{5}{6}\alpha, \frac{1}{3}\alpha - \frac{1-4\alpha}{6}, 0\right).$$

Recorded in `literature.py` *as:*
`add_beta_bound_sargos_1995()`

Theorem 4.16 (1996 Huxley table)**.** *One can bound* $\beta(\alpha)$ *by* $\beta_0(\alpha)$ *for* $X \le \alpha \le Y$ *for* β_0, X, Y *given by Tables 4.3, 4.3.*

Recorded in `literature.py` *as:*
`add_beta_bound_huxley_1996()`

`add_beta_bound_huxley_1996_2()`

Proof. See [129, Table 17.1, Table 19.2] (and also [279, §3.0.2, 3.0.3] for some verification of the technical conditions on the phase). □

Theorem 4.17 (2001 Huxley–Kolesnik bound)**.** *For any* $2/5 \le \alpha \le 1/2$ *one has*

$$\beta(\alpha) \le \max\left(\frac{7}{80} + \frac{79}{120}\alpha, \frac{3}{32} + \frac{103}{160}\alpha, \frac{9}{40} + \frac{13}{40}\alpha\right).$$

Recorded in `literature.py` *as:*
`add_beta_bound_huxley_kolesnik_2001()`

Proof. See [134, Theorem 1]. □

Table 4.1: Huxley table 17.1.

$\beta_0(\alpha)$	X	Y
$\frac{4+39\alpha}{60}$	$\frac{7}{12}$	$\frac{517}{873} = 0.5922\ldots$
$\frac{29+42\alpha}{120}$	$\frac{65}{114}$	$\frac{7}{12} = 0.5833\ldots$
$\frac{89+285\alpha}{570}$	$\frac{49}{114}$	$\frac{65}{114} = 0.5701\ldots$
$\frac{11+78\alpha}{120}$	$\frac{5}{12}$	$\frac{49}{114} = 0.4298\ldots$
$\frac{13+21\alpha}{60}$	$\frac{356}{873}$	$\frac{5}{12} = 0.4166\ldots$
$\frac{4+103\alpha}{128}$	$\frac{12}{31}$	$\frac{356}{873} = 0.4546\ldots$
$\frac{29+173\alpha}{280}$	$\frac{227}{601}$	$\frac{12}{31} = 0.3870\ldots$
$\frac{89+908\alpha}{1282}$	$\frac{423}{1295}$	$\frac{227}{601} = 0.3777\ldots$
$\frac{11+191\alpha}{244}$	$\frac{87}{275}$	$\frac{423}{1295} = 0.3266\ldots$
$\frac{13+94\alpha}{146}$	$\frac{1424}{4747}$	$\frac{87}{275} = 0.3163\ldots$
$\frac{4+235\alpha}{264}$	$\frac{120}{419}$	$\frac{1424}{4747} = 0.2999\ldots$
$\frac{49+1351\alpha}{1614}$	$\frac{967}{3428}$	$\frac{120}{419} = 0.2863\ldots$
$\frac{29+464\alpha}{600}$	$\frac{199}{716}$	$\frac{967}{3428} = 0.2820\ldots$
$\frac{89+2243\alpha}{2706}$	$\frac{19}{74}$	$\frac{199}{716} = 0.2779\ldots$
$\frac{11+428\alpha}{492}$	$\frac{161}{646}$	$\frac{19}{74} = 0.2567\ldots$
$\frac{13+253\alpha}{318}$	$\frac{2848}{12173} = 0.2339\ldots$	$\frac{161}{646} = 0.2492\ldots$

Theorem 4.18 (2002 Robert–Sargos bound). *For any $\alpha > 0$ one has*

$$\beta(\alpha) \le \max\left(\alpha + \frac{1-4\alpha}{13}, -\frac{7(1-4\alpha)}{13}\right).$$

Recorded in `literature.py` *as:*
`add_beta_bound_robert_sargos_2002()`

Proof. See [254, Theorem 1]. □

Theorem 4.19 (Sargos 2003 bound). *For any $\alpha > 0$ one has*

$$\beta(\alpha) \le \max\left(\alpha + \frac{1-8\alpha}{204}, -\frac{95(1-8\alpha)}{204}\right)$$

and

$$\beta(\alpha) \le \max\left(\alpha + \frac{7(1-9\alpha)}{2640}, -\frac{1001(1-9\alpha)}{2640}\right).$$

Recorded in `literature.py` *as:*
`add_beta_bound_sargos_2003()`

Proof. See [259, Theorems 3, 4]. □

Theorem 4.20 (Huxley bound). *For any $1/3 \le \alpha \le 1/2$, one has*

$$\beta(\alpha) \le \max\left(\frac{37+59\alpha}{170}, \frac{63+449\alpha}{690}\right).$$

Table 4.2: Huxley table 19.2.

$\beta_0(\alpha)$	X	Y
$\frac{89+285\alpha}{570}$	$\frac{106822}{246639}$	$\frac{139817}{246639} = 0.5668\ldots$
$\frac{2387+17972\alpha}{27290}$	$\frac{675}{1574}$	$\frac{106822}{246639} = 0.4331\ldots$
$\frac{2819+19177\alpha}{29855}$	$\frac{699371}{1647930}$	$\frac{675}{1574} = 0.4288\ldots$
$\frac{11897+88442\alpha}{134680}$	$\frac{156527}{370694}$	$\frac{699371}{1647930} = 0.4243\ldots$
$\frac{113+897\alpha}{1345}$	$\frac{263}{638}$	$\frac{156527}{370694} = 0.4222\ldots$
$\frac{491+3624\alpha}{5530}$	$\frac{143}{349}$	$\frac{263}{638} = 0.4122\ldots$
$\frac{569+1053\alpha}{2800}$	$\frac{307}{761}$	$\frac{143}{349} = 0.4097\ldots$
$\frac{1273+2484\alpha}{6410}$	$\frac{68682}{171139}$	$\frac{307}{761} = 0.4034\ldots$
$\frac{4+103\alpha}{128}$	$\frac{12}{31}$	$\frac{68682}{171139} = 0.4013\ldots$
$\frac{29+173\alpha}{280}$	$\frac{227}{601} = 0.3777\ldots$	$\frac{12}{31} = 0.3870\ldots$

Recorded in `literature.py` as:
`add_beta_bound_huxley_2005()`

Proof. See [131, Proposition 1, Theorem 1]. □

Theorem 4.21 (2016 Robert bound)**.** *For any* $0 < \alpha \le 3/7$ *one has*

$$\beta(\alpha) \le \max\left(\alpha + \frac{1-4\alpha}{12}, \frac{11}{12}\alpha\right).$$

Recorded in `literature.py` as:
`add_beta_bound_robert_2016()`

Proof. See [251, Theorem 1]. □

Theorem 4.22 (Second 2016 Robert bound)**.** *If* $k \ge 4$ *and* $\alpha \ge -(1-k\alpha)\frac{k-1}{2k-3}$ *then*

$$\beta(\alpha) \le \alpha + \max(\frac{1-k\alpha}{2(k-1)(k-2)}, -\frac{1}{2(k-1)(k-2)}).$$

Recorded in `literature.py` as:
`add_beta_bound_robert_2016_2(Constants.BETA_TRUNCATION)`

Proof. See [252, Theorem 10]. □

Theorem 4.23 (2017 Heath-Brown bound)**.** *For any* $\alpha > 0$ *and any natural number* $k \ge 3$ *one has*

$$\beta(\alpha) \le \alpha + \max\left(\frac{1-k\alpha}{k(k-1)}, -\frac{\alpha}{k(k-1)}, -\frac{2\alpha}{k(k-1)} - \frac{2(1-k\alpha)}{k^2(k-1)}\right).$$

Recorded in `literature.py` as:
`add_beta_bound_heath_brown_2017(Constants.BETA_TRUNCATION)`

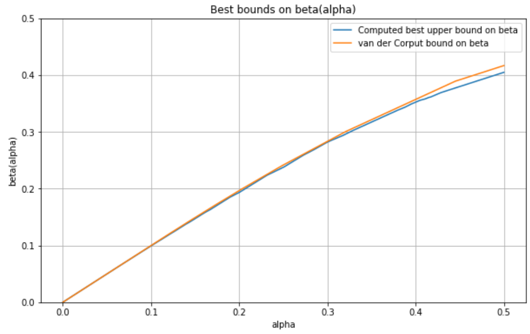

Figure 4.3: The bounds in Proposition 4.7, compared against the best-known bound on $\beta(\alpha)$.

Proof. See [113, Theorem 1]. □

Theorem 4.24 (2017 Bourgain bound)**.** *One has*

$$\beta(\alpha) \leq \begin{cases} \dfrac{2}{9} + \dfrac{1}{3}\alpha, & \dfrac{1}{3} < \alpha \leq \dfrac{5}{12}, \\[2ex] \dfrac{1}{12} + \dfrac{2}{3}\alpha, & \dfrac{5}{12} < \alpha \leq \dfrac{3}{7}, \\[2ex] \dfrac{13}{84} + \dfrac{1}{2}\alpha, & \dfrac{3}{7} < \alpha \leq \dfrac{1}{2}. \end{cases}$$

Recorded in `literature.py` *as:*
`add_beta_bound_bourgain_2017()`

Proof. See [23, Equation (3.18)]. □

Theorem 4.25 (2020 Heath-Brown bound)**.** *[58, Theorem 11.2] If α is fixed with $1 \leq 4\alpha - 1 \leq 2$ (i.e., $1/2 \leq \alpha \leq 3/4$), then*

$$\beta(\alpha) \leq \max\left(\alpha\left(1 - \frac{4\alpha - 1}{4(4\alpha - 1) + 8}\right), \frac{8}{9}\alpha\right).$$

TODO: implement this in python

Theorem 4.26 (Combined bound)**.** *For $X \leq \alpha \leq Y$, one has $\beta(\alpha) \leq \beta_0(\alpha)$, where β_0, X, Y are given by Table 4.3.*

Proof. See [279, Table 3]. □

Recorded in `literature.py` *as:*
`add_beta_bound_trudgian_yang_2024()`

Table 4.3: Bounds on $\beta(\alpha)$ of the form $\beta(\alpha) \le \beta_0(\alpha)$, $(X \le \alpha \le Y)$

$\beta_0(\alpha)$	X	Y	Reference
$\frac{13}{414} + \frac{346}{414}\alpha$	0	$\frac{2848}{12173} = 0.2339\ldots$	Exponent pair $A^2(\frac{13}{84}, \frac{55}{84})$
$\frac{13}{318} + \frac{253}{318}\alpha$	$\frac{2848}{12173}$	$\frac{161}{646} = 0.2492\ldots$	Theorem 4.16
$\frac{11}{492} + \frac{107}{123}\alpha$	$\frac{161}{646}$	$\frac{19}{74} = 0.2567\ldots$	Theorem 4.16
$\frac{89}{2706} + \frac{2243}{2706}\alpha$	$\frac{19}{74}$	$\frac{199}{716} = 0.2779\ldots$	Theorem 4.16
$\frac{29}{600} + \frac{58}{75}\alpha$	$\frac{199}{716}$	$\frac{967}{3428} = 0.2820\ldots$	Theorem 4.16
$\frac{49}{1614} + \frac{1351}{1614}\alpha$	$\frac{967}{3428}$	$\frac{120}{419} = 0.2863\ldots$	Theorem 4.16
$\frac{1}{66} + \frac{235}{264}\alpha$	$\frac{120}{419}$	$\frac{1328}{4447} = 0.2986\ldots$	Theorem 4.16
$\frac{13}{194} + \frac{139}{194}\alpha$	$\frac{1328}{4447}$	$\frac{104}{343} = 0.3032\ldots$	Exponent pair $A(\frac{13}{84}, \frac{55}{84})$
$\frac{13}{146} + \frac{47}{73}\alpha$	$\frac{104}{343}$	$\frac{87}{275} = 0.3163\ldots$	Theorem 4.16
$\frac{11}{244} + \frac{191}{244}\alpha$	$\frac{87}{275}$	$\frac{423}{1295} = 0.3266\ldots$	Theorem 4.16
$\frac{89}{1282} + \frac{454}{641}\alpha$	$\frac{423}{1295}$	$\frac{227}{601} = 0.3777\ldots$	Theorem 4.16
$\frac{29}{280} + \frac{173}{280}\alpha$	$\frac{227}{601}$	$\frac{12}{31} = 0.3870\ldots$	Theorem 4.16
$\frac{1}{32} + \frac{103}{128}\alpha$	$\frac{12}{31}$	$\frac{1508}{3825} = 0.3942\ldots$	Theorem 4.16
$\frac{18}{199} + \frac{521}{796}\alpha$	$\frac{1508}{3825}$	$\frac{62831}{155153} = 0.4049\ldots$	Exponent pair $D(\frac{13}{84}, \frac{55}{84})$
$\frac{569}{2800} + \frac{1053}{2800}\alpha$	$\frac{62831}{155153}$	$\frac{143}{349} = 0.4097\ldots$	Theorem 4.16
$\frac{491}{5530} + \frac{1812}{2765}\alpha$	$\frac{143}{349}$	$\frac{263}{638} = 0.4122\ldots$	Theorem 4.16
$\frac{113}{1345} + \frac{897}{1345}\alpha$	$\frac{263}{638}$	$\frac{1673}{4038} = 0.4143\ldots$	Theorem 4.16
$\frac{2}{9} + \frac{1}{3}\alpha$	$\frac{1673}{4038}$	$\frac{5}{12} = 0.4166\ldots$	Theorem 4.24
$\frac{1}{12} + \frac{2}{3}\alpha$	$\frac{5}{12}$	$\frac{3}{7} = 0.4285\ldots$	Theorem 4.24
$\frac{13}{84} + \frac{1}{2}\alpha$	$\frac{3}{7}$	$\frac{1}{2}$	Theorem 4.24

Chapter 5

Exponent pairs

Definition 5.1 (Exponent pair)**.** *An exponent pair is a (fixed) element* (k, ℓ) *of the triangle*

$$\{(k, \ell) \in \mathbf{R}^2 : 0 \le k \le 1/2 \le \ell \le 1, k + \ell \le 1\} \tag{5.1}$$

with the following property: for all model phase functions F*, all* $T \ge N \ge 1$*, and all intervals* $I \subset [N, 2N]$*, one has*

$$\sum_{n \in I} e(TF(n/N)) \ll (T/N)^{k+o(1)} N^{\ell+o(1)} \tag{5.2}$$

whenever $T \ge N \ge 1$*,* I *is an interval in* $[N, 2N]$*, and* $F \in \mathcal{U}$*.*

Implemented at `exponent_pair.py` *as:*
`Exp_pair`

One can formulate the notion of an exponent pair without recourse to asymptotic notation:

Lemma 5.2 (Non-asymptotic definition of exponent pair)**.** *Let* (k, ℓ) *be a fixed element of* (5.1)*. Then the following are equivalent:*

- *(i)* (k, ℓ) *is an exponent pair.*
- *(ii) For every (fixed)* $\varepsilon > 0$ *there exist (fixed)* $C, P > 0$ *such that, whenever* $T \ge N \ge 1$*,* $I \subset [N, 2N]$*, and* F *is a phase function obeying* (4.3) *for all (fixed)* $0 \le p \le P$ *and* $u \in [1, 2]$*, then*
$$|\sum_{n \in I} e(TF(n/N))| \le C(T/N)^{k+\varepsilon} N^{\ell+\varepsilon}.$$

The proof of this lemma is similar to that of Lemma 4.3 and is omitted.
Exponent pairs are closely related to the function β from the previous chapter:

Lemma 5.3 (Duality between exponent pairs and β)**.** *Let* (k, ℓ) *be in the triangle* (5.1)*. Then the following are equivalent:*

- *(i)* (k, ℓ) *is an exponent pair.*
- *(ii)* $\beta(\alpha) \le k + (\ell - k)\alpha$ *for all* $0 \le \alpha \le 1$*.*

Implemented at `exponent_pair.py` *as:*
`exponent_pairs_to_beta_bounds()`
`beta_bounds_to_exponent_pairs()`

Thus exponent pairs are dual to the convex hull of the graph of β. But β is not known to be convex, so one could have bounds on β that do not directly correspond to exponent pairs. We remark that in the case $\ell - k \geq 1/2$, one only needs to check the case $0 \leq \alpha \leq 1/2$ in (ii) above, since the remaining regime $1/2 \leq \alpha \leq 1$ then follows from Lemma 4.10 and some algebra. Conversely, if $\ell - k \leq 1/2$, one only needs to check the region $1/2 \leq \alpha \leq 1$.

Proof. If (i) holds, then for any $0 < \alpha < 1$, any unbounded $T \geq 1$, any $N = T^{\alpha+o(1)}$, interval $I \subset [N, 2N]$, and model phase function F, we have from (i) that

$$\sum_{n\in I} e(TF(n/N)) \ll (T/N)^{k+o(1)} N^{\ell+o(1)} = T^{k+(\ell-k)\alpha+o(1)}.$$

From Definition 4.2 we conclude that $\beta(\alpha) \leq k + (\ell - k)\alpha$. Also since (k, ℓ) lies in (5.1), we see from (4.4), (4.6) that we also have $\beta(\alpha) \leq k + (\ell - k)\alpha$ for $\alpha = 0, 1$.
Now suppose that (ii) holds. Let F, T, N, I be as in Definition 5.1. By underspill it suffices to show that

$$\sum_{n\in I} e(TF(n/N)) \ll (T/N)^{k+\varepsilon+o(1)} N^{\ell+\varepsilon+o(1)}$$

for any fixed $\varepsilon > 0$. We may assume that $T \leq N^{1/\varepsilon+1}$, since the claim follows from the trivial bound $\sum_{n\in I} e(TF(n/N)) \ll N$ otherwise. We may also assume that N is unbounded, since the claim is clear for N bounded; this forces T to be unbounded as well.
By passing to a subsequence we may assume that $N = T^{\alpha+o(1)}$ for some fixed $0 \leq \alpha \leq 1$. By Definition 4.2 we then have

$$\sum_{n\in I} e(TF(n/N)) \ll T^{\beta(\alpha)+o(1)}$$

and hence by (ii)

$$\sum_{n\in I} e(TF(n/N)) \ll (T/N)^{k+o(1)} N^{\ell+o(1)}$$

giving the claim. □

Corollary 5.4 (Exponent pairs closed and convex)**.** *The set of exponent pairs is closed and convex.*

Proof. Immediate from Lemma 5.3. □

Proposition 5.5 (Trivial exponent pairs)**.** $(0, 1)$ *and* $(1/2, 1/2)$ *are exponent pairs.*

Implemented at `exponent_pair.py` *as:*
`trivial_exp_pair`

Proof. Immediate from Lemma 5.3 and Lemma 4.4. □

Conjecture 5.6 (Exponent pairs conjecture)**.** $(0, 1/2)$ *is an exponent pair. (Equivalently, by Corollary 5.4 and Proposition 5.5, every point in the triangle* (5.1) *is an exponent pair.)*

Implemented at ***exponent_pair.py*** *as:*
`exponent_pair_conjecture`

Lemma 5.7. *The exponent pair conjecture is equivalent to $\beta(\alpha) = \alpha/2$ holding true for all $0 \le \alpha \le 1$.*

Proof. Clear from Lemma 5.3 and Lemma 4.4. ☐

Proposition 5.8 (Van der Corput A-process)**.** *If (k, ℓ) is an exponent pair, then so is*

$$A(k, \ell) := \left(\frac{k}{2k+2}, \frac{\ell}{2k+2} + \frac{1}{2} \right).$$

Recorded in ***literature.py*** *as:*
`A_transform_hypothesis`

Proof. See [144, Lemma 2.8]. It can also be deduced from Lemma 4.6 and Lemma 5.3. ☐

Proposition 5.9 (Van der Corput B-process)**.** *If (k, ℓ) is an exponent pair, then so is*

$$B(k, \ell) := \left(\ell - \frac{1}{2}, k + \frac{1}{2} \right).$$

Recorded in ***literature.py*** *as:*
`B_transform_hypothesis`

Proof. See [144, Lemma 2.9]. Alternatively, this can be derived from Lemma 4.10 and Lemma 5.3. ☐

5.1 Known exponent pairs

Proposition 5.10 (Classical van der Corput exponent pairs)**.** *For any natural number $k \ge 2$,*

$$A^{k-2}B(0,1) = \left(\frac{1}{2^k - 2}, 1 - \frac{k-1}{2^k - 2} \right)$$

is an exponent pair. In particular,

$$\left(\frac{1}{2}, \frac{1}{2} \right), \left(\frac{1}{6}, \frac{2}{3} \right), \left(\frac{1}{14}, \frac{11}{14} \right)$$

are exponent pairs.

Proof. Follows by induction from Proposition 5.8 and Proposition 5.5; alternatively, follows from (and is equivalent to) Corollary 4.8 and Lemma 5.3. ☐

Derived in ***derived.py*** *as:*
`van_der_corput_pair()`

Corollary 5.11 (Additional exponent pairs)**.** *The pairs*

$$\left(\frac{13}{31}, \frac{16}{31}\right), \left(\frac{4}{11}, \frac{6}{11}\right), \left(\frac{2}{7}, \frac{4}{7}\right), \left(\frac{5}{24}, \frac{15}{24}\right), \left(\frac{4}{18}, \frac{11}{18}\right)$$

are all exponent pairs.

Derived in `derived.py` *as:*

```
best_proof_of_exponent_pair(frac(13, 31), frac(16, 31))
best_proof_of_exponent_pair(frac(4, 11), frac(6, 11))
best_proof_of_exponent_pair(frac(2, 7), frac(4, 7))
best_proof_of_exponent_pair(frac(5, 24), frac(15, 24))
best_proof_of_exponent_pair(frac(4, 18), frac(11, 18))
```

Proof. We have $(2/7, 4/7) = BA(1/6, 2/3)$, $(4/18, 11/18) = BABA(1/6, 2/3)$, and $(13/31, 16/31) = BAB^2A^2(1/6, 2/3)$, so these cases follow from Propositions 5.10, 5.8, 5.9. Finally, $(4/11, 6/11)$ is a convex combination of $(1/2, 1/2)$ and $(2/7, 4/7)$, and $(5/24, 15/24)$ is a convex combination of $(1/6, 2/3)$ and $(4/18, 11/18)$, so these cases follow from Corollary 5.4. □

Theorem 5.12 (Exponent pairs on the line of symmetry)**.** $(k, k + 1/2)$ *is an exponent pair for*

(i) $k = 9/56$ *[135, Theorem 1];*

(ii) $k = 89/560$ *[293, Theorem 6];*

(iii) $k = 17/108$ *[132, p. 467];*

(iv) $k = 89/570$ *[128, p. 40];*

(v) $k = 32/205$ *[131, Theorem 1];*

(vi) $k = 13/84$ *[23, p. 206].*

Recorded in `literature.py` *as:*

```
add_literature_exponent_pairs()
```

Theorem 5.13 (Exponent pairs from the Bombieri–Iwaniec method)**.** *The following pairs are exponent pairs:*

(i) $(\frac{2}{13}, \frac{35}{52})$ *[136];*

(ii) $(\frac{6299}{43860}, \frac{29507}{43860})$ *[129, Table 17.3];*

(iii) $(\frac{771}{8116}, \frac{1499}{2029})$ *[258, p. 285];*

(iv) $(\frac{21}{232}, \frac{173}{232})$ *[258, p. 286];*

(v) $(\frac{1959}{21656}, \frac{16135}{21656})$ *[258, p. 286];*

(vi) $(\frac{516247}{6629696}, \frac{5080955}{6629696})$ *[134], [129, Table 19.2], [254].*

Recorded in `literature.py` *as:*

```
add_literature_exponent_pairs()
```

Theorem 5.14 (Exponent pairs from derivative tests)**.** $(k, 1-mk)$ *is an exponent pair when*

(i) $k = \frac{1}{13}$ *and* $m = 3$ *[254, Theorem 1];*

(ii) $k = \frac{1}{204}$ *and* $m = 7$ *[259, p. 231];*

(iii) $k = \frac{1}{130}$ *and* $m = 8$ *[249, (1.1)];*

(iv) $k = \frac{7}{2640}$ *and* $m = 8$ *[259, p. 231];*

(v) $k = \frac{1}{716}$ *and* $m = 9$ *[259, p. 231];*

(vi) $k = \frac{1}{649}$ *and* $m = 9$ *[253];*

(vii) $k = \frac{7}{4540}$ *and* $m = 9$ *[249, (1.2)];*

(viii) $k = \frac{1}{615}$ *and* $m = 9$ *[249, (1.1)];*

(ix) $k = \frac{1}{915}$ *and* $m = 10$ *[250, Théorème 2].*

Recorded in `literature.py` *as:*
`add_literature_exponent_pairs()`

Theorem 5.15 (Huxley sequence)**.** *[129, Table 17.3] For any integer* $m \geq 1$*, the pair*

$$\left(\frac{169}{1424 \times 2^m - 338}, 1 - \frac{169}{1424 \times 2^m - 338}\frac{712m + 1577}{712}\right)$$

is an exponent pair.

Recorded in `literature.py` *as:*
`add_huxley_exponent_pairs(Constants.EXP_PAIR_TRUNCATION)`

Theorem 5.16 (1996 Heath–Brown sequence)**.** *[277, (6.17.4)] For any integer* $m \geq 3$*, the pair*

$$\left(\frac{1}{25m^2(m-2)\log m}, 1 - \frac{1}{25m^2 \log m}\right)$$

is an exponent pair.

(Currently not implemented in python due to the irrational exponents.)

Theorem 5.17 (2017 Heath–Brown sequence)**.** *[113, Theorem 2] For any integer* $m \geq 3$*, the pair*

$$(p_m, q_m) := \left(\frac{2}{(m-1)^2(m+2)}, 1 - \frac{3m-2}{m(m-1)(m+2)}\right)$$

is an exponent pair.

Recorded in `literature.py` *as:*
`add_heath_brown_exponent_pairs(Constants.EXP_PAIR_TRUNCATION)`

Proof. This follows from Theorem 4.23 and Lemma 5.3, after some computation. □

Theorem 5.18 (Sargos C-process). *[259, Theorem 5] If (k, ℓ) is an exponent pair, then so is*

$$\left(\frac{k}{12(1+4k)}, \frac{11(1+4k)+\ell}{12(1+4k)}\right).$$

Recorded in `literature.py` *as:*
`C_transform_hypothesis`

The following process is not quite a process to automatically transform one exponent pair to another, but it often achieves this in practice:

Theorem 5.19 (Sargos D-process). *[258, Theorem 7.1] If (k, ℓ) is an exponent pair, then one has*

$$\beta(\alpha) \leq \max\left(k_1 + \alpha(\ell_1 - k_1), \frac{1}{12} + \frac{2}{3}\alpha\right)$$

for all $0 \leq \alpha \leq 1$, where $(k_1, \ell_1) = D(k, \ell)$ is the pair

$$D(k, \ell) := \left(\frac{5k+\ell+2}{8(5k+3\ell+2)}, \frac{29k+21\ell+10}{8(5k+3\ell+2)}\right).$$

Recorded in `literature.py` *as:*
`D_transform_hypothesis`

Theorem 5.20. *[279, Lemma 1.1] The following are exponent pairs:*

$$(k_1, \ell_1) := \left(\frac{4742}{38463}, \frac{35731}{51284}\right)$$
$$(k_2, \ell_2) := \left(\frac{18}{199}, \frac{593}{796}\right)$$
$$(k_3, \ell_3) := \left(\frac{2779}{38033}, \frac{58699}{76066}\right)$$
$$(k_4, \ell_4) := \left(\frac{715}{10238}, \frac{7955}{10238}\right).$$

Recorded in `literature.py` *as:*
`add_literature_exponent_pairs()`

Proof. For the pair $(18/199, 593/796)$, apply Theorem 5.19 with the pair $(13/84, 55/84)$ from Theorem 5.12 to conclude that

$$\beta(\alpha) \leq 18/199 + 521\alpha/796$$

for all $0 \leq \alpha \leq 1/2$, from which the claim follows from Lemma 5.3 (and Lemma 4.10). The remaining pairs come from Lemma 5.3 and the remaining components of Theorem 4.26. □

Corollary 5.21 (Set of exponent pairs). *[279, Theorem 1.3] Let H be the convex hull $(0,1)$, $(1/2, 1/2)$, and of (k_n, ℓ_n) for $n \in \mathbf{Z}$, where $(k_0, \ell_0) := 13/84$, (k_n, ℓ_n) for $n = 1, 2, 3, 4$ is defined by Theorem 5.20, $(k_n, \ell_n) := A(k_{n-4}, \ell_{n-4})$ for $5 \leq n \leq 8$, $(k_n, \ell_n) := (p_n, q_n)$ for $n > 9$ (with (p_n, q_n) defined by Theorem 5.17), and $(k_{-n}, \ell_{-n}) := B(k_n, \ell_n)$ for $n \geq 0$. Then all elements of H are exponent pairs.*

Indeed, as of [279] the set H represented all known exponent pairs, until Theorem 5.22 below.

Proof. Clear from Corollary 5.4, Proposition 5.5, 5.20, and Theorem 5.17. □

The following new exponent pairs were derived using this database:

Theorem 5.22 (New exponent pairs)**.** *The following are exponent pairs:*

$$\left(\frac{89}{1282}, \frac{997}{1282}\right), \quad \left(\frac{652397}{9713986}, \frac{7599781}{9713986}\right), \quad \left(\frac{10769}{351096}, \frac{609317}{702192}\right), \quad \left(\frac{89}{3478}, \frac{15327}{17390}\right).$$

Derived in ***derived.py*** *as:*

```
prove_exponent_pair(frac(89,1282), frac(997,1282))
prove_exponent_pair(frac(652397,9713986), frac(7599781,9713986))
prove_exponent_pair(frac(10769,351096), frac(609317,702192))
prove_exponent_pair(frac(89,3478), frac(15327,17390))
```

Proof. Using the bounds on $\beta(\alpha)$ collected in Table 5.1, one may verify (after a tedious calculation) that for each of the claimed exponent pairs (k, ℓ) in the lemma statement, one has $\beta(\alpha) \leq k + (\ell - k)\alpha$ for $0 \leq \alpha \leq 1/2$. The result then follows from Lemma 4.10 and Lemma 5.3.

□

Furthermore, more exponent pairs can be derived upon incorporating Lemma 4.6.

Theorem 5.23 (Cushing (2025) exponent pairs)**.** *The following are exponent pairs:*

$$\left(\frac{311}{4822}, \frac{3799}{4822}\right), \quad \left(\frac{80219}{1298878}, \frac{515638}{649439}\right).$$

Implemented at ***examples.py*** *as:*

```
beta_bound_examples2()
```

In summary, the current set of known exponent pairs is the convex hull with vertices $(0, 1)$, $(1/2, 1/2)$ and the points (k_n, ℓ_n) for $n \in \mathbf{Z}$ that are recorded in Table 5.1.

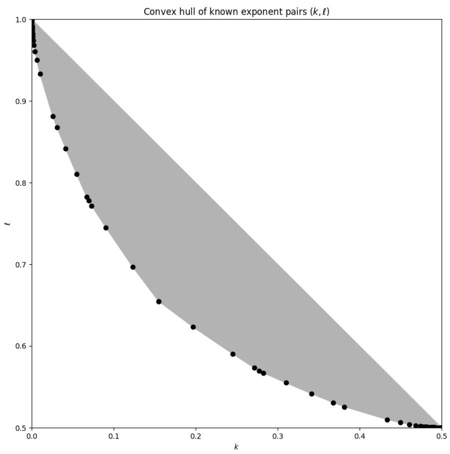

Figure 5.1: The convex hull of known exponent pairs, whose vertices (k_n, ℓ_n) are given in Table 5.1.

Table 5.1: Bounds on $\beta(\alpha)$

$\beta(\alpha)$ bound	α range	Reference
$\frac{1}{20}+\frac{3}{4}\alpha$	$0 \le \alpha < \frac{1}{4}$	Theorem 4.23 with $k=5$
$\frac{19}{20}\alpha$	$\frac{1}{4} \le \alpha < \frac{890}{3277}$	Theorem 4.23 with $k=5$
$\frac{89}{2706}+\frac{2243}{2706}\alpha$	$\frac{890}{3277} \le \alpha < \frac{199}{716}$	Table 4.3
$\frac{1}{66}+\frac{235}{264}\alpha$	$\frac{120}{419} \le \alpha < \frac{754}{2579}$	Table 4.3
$\frac{9}{217}+\frac{1389}{1736}\alpha$	$\frac{754}{2579} \le \alpha < \frac{251324}{841245}$	Exponent pair $(\frac{9}{217},\frac{1461}{1736}) = AD(\frac{13}{84},\frac{55}{84})$
$\frac{2371}{43205}+\frac{52209}{69128}\alpha$	$\frac{251324}{841245} \le \alpha < \frac{861996}{2811205}$	Exponent pair $(\frac{2371}{43205},\frac{280013}{345640})$ $= A(\frac{4742}{38463},\frac{35731}{51284})$ and Theorem 5.20
$\frac{13}{146}+\frac{47}{73}\alpha$	$\frac{861996}{2811205} \le \alpha < \frac{87}{275}$	Table 4.3
$\frac{11}{244}+\frac{191}{244}\alpha$	$\frac{87}{275} \le \alpha < \frac{423}{1295}$	Table 4.3
$\frac{89}{1282}+\frac{454}{641}\alpha$	$\frac{423}{1295} \le \alpha < \frac{227}{601}$	Table 4.3
$\frac{715}{10238}+\frac{3620}{5119}\alpha$	$\frac{227}{601} \le \alpha < \frac{227}{601}$	Exponent pair $(\frac{715}{10238},\frac{7955}{10238})$ in Theorem 5.20
$\frac{29}{280}+\frac{173}{280}\alpha$	$\frac{227}{601} \le \alpha < \frac{12}{31}$	Table 4.3
$\frac{1}{32}+\frac{103}{128}\alpha$	$\frac{12}{31} \le \alpha < \frac{1508}{3825}$	Table 4.3
$\frac{18}{199}+\frac{521}{796}\alpha$	$\frac{1508}{3825} \le \alpha < \frac{62831}{155153}$	Exponent pair $(\frac{18}{199},\frac{593}{796}) = D(\frac{13}{84},\frac{55}{84})$
$\frac{569}{2800}+\frac{1053}{2800}\alpha$	$\frac{62831}{155153} \le \alpha < \frac{143}{349}$	Table 4.3
$\frac{1}{12}+\frac{2}{3}\alpha$	$\frac{5}{12} \le \alpha < \frac{3}{7}$	Theorem 4.24
$\frac{13}{84}+\frac{1}{2}\alpha$	$\frac{3}{7} \le \alpha \le \frac{1}{2}$	Theorem 4.24

Table 5.2: Vertices of the convex hull of known exponent pairs.

n	(k_n, ℓ_n)	Reference
0	$\left(\frac{13}{84}, \frac{55}{84}\right)$	[23, p. 307]
1	$\left(\frac{4742}{38463}, \frac{35731}{51284}\right)$	Theorem 5.20
2	$\left(\frac{18}{199}, \frac{593}{796}\right)$	Theorem 5.20
3	$\left(\frac{2779}{38033}, \frac{58699}{76066}\right)$	Theorem 5.20
4	$\left(\frac{89}{1282}, \frac{997}{1282}\right)$	Theorem 5.22
5	$\left(\frac{311}{4822}, \frac{3799}{4822}\right)$	Theorem 5.23
6	$\left(\frac{80219}{1298878}, \frac{515638}{649439}\right)$	Theorem 5.23
7	$\left(\frac{9}{217}, \frac{1461}{1736}\right)$	$A(k_2, \ell_2)$
8	$\left(\frac{10769}{351096}, \frac{609317}{702192}\right)$	Theorem 5.22
9	$\left(\frac{89}{3478}, \frac{15327}{17390}\right)$	Theorem 5.22
$n \geq 10$	(p_{n-4}, q_{n-4}), where $(p_m, q_m) = \left(\frac{2}{(m-1)^2(m+2)}, 1 - \frac{3m-2}{m(m-1)(m+2)}\right)$	Theorem 5.17
$n < 0$	$B(k_{-n}, \ell_{-n})$	Proposition 5.9

Chapter 6

Growth exponents for the Riemann zeta function

Definition 6.1 (Growth rate of zeta)**.** *For any fixed $\sigma \in \mathbf{R}$, let $\mu(\sigma)$ denote the least possible (fixed) exponent for which one has the bound*

$$|\zeta(\sigma + it)| \ll |t|^{\mu(\sigma)+o(1)}$$

for all unbounded t.

One can check that for each σ, the set of possible candidates for $\mu(\sigma)$ is closed (by underspill), non-empty, and bounded from below, so that $\mu(\sigma)$ is well-defined as a real number. An equivalent definition without asymptotic notation, is that $\mu(\sigma)$ is the least real number such that for every $\varepsilon > 0$ there exists $C > 0$ such that

$$|\zeta(\sigma + it)| \leq C|t|^{\mu(\sigma)+\varepsilon}$$

for all t with $|t| \geq C$; equivalently, one has

$$\mu(\sigma) = \limsup_{|t|\to\infty} \frac{\log |\zeta(\sigma + it)|}{\log |t|}.$$

Implemented at ***bound_mu.py*** *as:*
`Bound_mu`

Lemma 6.2 (Trivial bound)**.** *One has $\mu(\sigma) = 0$ for all $\sigma \geq 1$.*

Implemented at ***bound_mu.py*** *as:*
`apply_trivial_mu_bound()`

Proof. Immediate from the absolute convergence of the Dirichlet series for both $\zeta(s)$ and $1/\zeta(s)$; see e.g., [144, Theorem 1.9]. □

Lemma 6.3 (Convexity)**.** *μ is convex.*

Implemented at ***bound_mu.py*** *as:*
`bound_mu_convexity()`

Proof. Immediate from the Phragmén–Lindelöf principle; see e.g., [144, §A.8]. □

Lemma 6.4 (Functional equation)**.** *One has* $\mu(1-\sigma) = \mu(\sigma) + \sigma - 1/2$ *for all* $0 \le \sigma \le 1/2$*.*

Implemented at `bound_mu.py` *as:*
`apply_functional_equation()`

Proof. Immediate from the functional equation for ζ and asymptotics of the Gamma function; see e.g., [144, (1.23), (1.25)]. □

Lemma 6.5 (Left of critical strip)**.** *One has* $\mu(\sigma) = 1/2 - \sigma$ *for* $\sigma \le 0$*.*

Implemented at `bound_mu.py` *as:*
`apply_trivial_mu_bound()`

Proof. Immediate from Lemmas 6.2, 6.4. □

Lemma 6.6 (Convexity bounds)**.** *One has* $\max(0, 1/2-\sigma) \le \mu(\sigma) \le (1-\sigma)/2$ *for* $0 \le \sigma \le 1$*.*

Implemented at `bound_mu.py` *as:*
`apply_trivial_mu_bound()`

Proof. Immediate from Lemma 6.2, Lemma 6.5, and Lemma 6.6. □

6.1 Connection with exponent pairs and dual exponent pairs

Lemma 6.7 (Connection with dual exponent pairs)**.** *For any* $1/2 \le \sigma \le 1$*, one has*

$$\mu(\sigma) \le \sup_{0 \le \alpha \le 1/2} \beta(\alpha) - \alpha\sigma.$$

Proof. Let t be unbounded. From the Riemann–Siegel formula (see [144, Theorem 4.1]) one has

$$\zeta(\sigma + it) \ll \left| \sum_{n \le \sqrt{t/2\pi}} \frac{1}{n^{\sigma+it}} \right| + |t|^{1/2-\sigma} \left| \sum_{n \le \sqrt{t/2\pi}} \frac{1}{n^{1-\sigma-it}} \right| + O(1).$$

From dyadic decomposition and Definition 4.2 (and Lemma 2.1) one has for any fixed $\varepsilon > 0$ that

$$\sum_{t^\varepsilon \le n \le \sqrt{t/2\pi}} \frac{1}{n^{\sigma+it}} \ll |t|^{\sup_{\varepsilon \le \alpha \le 1/2} \beta(\alpha) - \alpha\sigma + o(1)},$$

while from the triangle inequality one has the crude bound

$$\sum_{n < t^\varepsilon} \frac{1}{n^{\sigma+it}} \ll |t|^\varepsilon.$$

Combining the bounds and using underspill, we conclude that

$$\sum_{n \leq \sqrt{t/2\pi}} \frac{1}{n^{\sigma+it}} \ll |t|^{\sup_{0\leq\alpha\leq 1/2} \beta(\alpha)-\alpha\sigma+o(1)}.$$

A similar argument gives

$$\sum_{n \leq \sqrt{t/2\pi}} \frac{1}{n^{1-\sigma-it}} \ll |t|^{\sup_{0\leq\alpha\leq 1/2} \beta(\alpha)-\alpha(1-\sigma)+o(1)}$$

Since $\sigma \geq 1/2$ and $\alpha \leq 1/2$, one has $(1/2-\sigma)-\alpha(1-\sigma) \leq -\alpha\sigma$, and hence

$$\zeta(\sigma+it) \ll |t|^{\sup_{0\leq\alpha\leq 1/2} \beta(\alpha)-\alpha\sigma+o(1)}$$

giving the claim. □

We remark that this inequality is morally an equality (indeed, it would be if one would restrict the model phases in Definition 4.2 to purely the logarithmic phase $u \mapsto \log u$). The following form of Lemma 6.7 is convenient for applications:

Corollary 6.8 (Exponent pairs and μ)**.** *If (k,ℓ) is an exponent pair, then*

$$\mu(\ell-k) \leq k.$$

Implemented at ***`bound_mu.py`*** *as:*
`exponent_pair_to_mu_bound(exp_pair)`

Proof. Immediate from Lemma 6.7 and Lemma 5.3. See also [144, (7.57)]. □

Conjecture 6.9 (Lindelöf hypothesis)**.** *One has $\mu(1/2)=0$.*

Implemented at ***`bound_mu.py`*** *as:*
`bound_mu_Lindelof()`

Lemma 6.10. *The exponent pair conjecture implies the Lindelöf hypothesis.*

Proof. Immediate from Corollary 6.8. □

Proposition 6.11 (Conjectured value of μ)**.** *We have the lower bound*

$$\mu(\sigma) \geq \max\left(0, \frac{1}{2}-\sigma\right) \tag{6.1}$$

for all $\sigma \in \mathbf{R}$, and equality holds everywhere in (6.1) *if and only if the Lindelöf hypothesis holds.*

We remark that this proposition explains why there are no further lower bounds on μ in the literature beyond (6.1); all the remaining known results revolve around upper bounds.

Proof. Clearly equality in (6.1) implies the Lindelöf hypothesis, while from the trivial bounds in Propositions 6.2, 6.5 and convexity (Lemma 6.6) one we see that the Lindelöf hypothesis implies the upper bound

$$\mu(\sigma) \leq \max\left(0, \frac{1}{2} - \sigma\right)$$

for all σ. So it suffices to establish the lower bound unconditionally. By the functional equation (Proposition 6.4) it suffices to do this for $\sigma \geq 1/2$; in fact by convexity it suffices to establish the claim when $1/2 < \sigma < 1$. In this regime, the L^2 mean value theorem (see [144, Theorem 1.11]) gives

$$\int_0^T |\zeta(\sigma + it)|^2 \, dt \asymp T$$

for large T, giving the claim. □

6.2 Known bounds on μ

Theorem 6.12 (Historical bounds)**.** *The upper bounds on $\mu(\sigma)$ given by Table 6.2 are known.*

TODO: supplement as many of these citations as possible with derivations from other exponents and relations in the database
Recorded in `literature.py` *as:*
`add_literature_bounds_mu()`

Some additional bounds are recorded in [279] by combining various exponential sum estimates.

Theorem 6.13. *[279, Theorems 2.4-2.6] We have*

$$\mu(\sigma) \leq \begin{cases} (31 - 36\sigma)/84, & \frac{1}{2} \leq \sigma < \frac{88225}{153852} = 0.5734\ldots, \\ (220633 - 251324\sigma)/620612, & \frac{88225}{153852} \leq \sigma < \frac{521}{796} = 0.6545\ldots, \\ (1333 - 1508\sigma)/3825, & \frac{521}{796} \leq \sigma < \frac{53141}{76066} = 0.6986\ldots, \\ (405 - 454\sigma)/1202, & \frac{53141}{76066} \leq \sigma < \frac{3620}{5119} = 0.7071\ldots, \\ (49318855 - 52938216\sigma)/170145110, & \frac{3620}{5119} \leq \sigma < \frac{52209}{69128} = 0.7552\ldots, \\ (471957 - 502648\sigma)/1682490, & \frac{52209}{69128} \leq \sigma < \frac{1389}{1736} = 0.8001\ldots, \\ (2841 - 3016\sigma)/10316, & \frac{1389}{1736} \leq \sigma < \frac{134765}{163248} = 0.8255\ldots, \\ (859 - 908\sigma)/3214, & \frac{134765}{163248} \leq \sigma < \frac{18193}{21906} = 0.8305\ldots, \\ 5(8707 - 9067\sigma)/180277, & \frac{18193}{21906} \leq \sigma < \frac{249}{280} = 0.8892\ldots, \\ (29 - 30\sigma)/130, & \frac{249}{280} \leq \sigma \leq \frac{9}{10}. \end{cases}$$

Furthermore, for $1/2 \leq \sigma \leq 1$, we have

$$\mu(\sigma) \leq \frac{2}{13}\sqrt{10}(1 - \sigma)^{3/2} = 0.4865\ldots(1 - \sigma)^{3/2},$$

and

$$\mu(\sigma) \leq \frac{2}{3^{3/2}}(1 - \sigma)^{3/2} + \frac{103}{300}(1 - \sigma)^2, \qquad \frac{117955}{118272} \leq \sigma \leq 1.$$

Table 6.1: Historical bounds on $\mu(\sigma)$ for $1/2 \leq \sigma \leq 1$, and the exponent pair generating them (if applicable).

Reference	Results	Exponent pair
Hardy–Littlewood (1923) [97]	$\mu(1/2) \leq 1/6$	(1/6, 2/3)
Walfisz (1924) [287]	$\mu(1/2) \leq 193/988$	
Titchmarsh (1932) [273]	$\mu(1/2) \leq 27/164$	
Phillips (1933) [232]	$\mu(1/2) \leq 229/1392$	
Titchmarsh (1942) [276]	$\mu(1/2) \leq 19/116$	
Min (1949) [216]	$\mu(1/2) \leq 15/92$	
Haneke (1962) [92]	$\mu(1/2) \leq 6/37$	
Kolesnik (1973) [170]	$\mu(1/2) \leq 173/1067$	
Kolesnik (1982) [172]	$\mu(1/2) \leq 35/216$	
Kolesnik (1985) [173]	$\mu(1/2) \leq 139/858$	
Bombieri–Iwaniec (1985) [18]	$\mu(1/2) \leq 9/56$	$(9/56, 1/2 + 9/56)$
Watt (1989) [293]	$\mu(1/2) \leq 89/560$	$(89/560, 1/2 + 89/560)$
Huxley–Kolesnik (1991) [132]	$\mu(1/2) \leq 17/108$	$(17/108, 1/2 + 17/108)$
Huxley (1993) [128]	$\mu(1/2) \leq 89/570$	$(89/570, 1/2 + 89/570)$
Huxley (1996) [129]	$\mu(1934/3655) \leq 6299/43860$	
Sargos (2003) [259]	$\mu(49/51) \leq 1/204$, $\mu(361/370) \leq 1/370$	
Huxley (2005) [131]	$\mu(1/2) \leq 32/205$	$(32/205, 1/2 + 32/205)$
Lelechenko (2014) [177]	$\mu(3/5) \leq 1409/12170$, $\mu(4/5) \leq 3/71$	
Bourgain (2017) [23]	$\mu(1/2) \leq 13/84$	$(13/84, 1/2 + 13/84)$
Heath-Brown (2017) [113]	$\mu(\sigma) \leq \frac{8}{63}\sqrt{15}(1-\sigma)^{3/2}$ for $1/2 \leq \sigma \leq 1$	
Heath-Brown (2020) [58]	$\mu(11/15) \leq 1/15$	

Recorded in `literature.py` *as:*
`add_literature_bounds_mu()`

Additionally, the series of exponent pairs in Theorem 5.17 imply the following bounds on $\mu(\sigma)$ close to $\sigma = 1$.

Theorem 6.14 (Heath-Brown [113] μ bounds)**.** *For any integer $k \geq 3$, one has*

$$\mu\left(1 - \frac{3k^2 - 3k + 2}{k(k-1)^2(k+2)}\right) \leq \frac{2}{(k-1)^2(k+2)}.$$

Proof. Follows from substituting Theorem 5.17 into (6.8). □

The new exponent pairs in Theorem 5.22 may be used to obtain sharper bounds on $\mu(\sigma)$ in certain ranges. The current sharpest bounds on $\mu(\sigma)$ are recorded in Table 6.2 and graphed in Figure 6.1.
Derived in `derived.py` *as:*
`compute_best_mu_bound()`

Table 6.2: Current sharpest known bound on $\mu(\sigma)$ for $1/2 \leq \sigma \leq 1$

Upper bound on $\mu(\sigma)$	Range of σ	Reference
$\mu(\sigma) \leq \frac{31}{84} - \frac{3}{7}\sigma$	$\frac{1}{2} \leq \sigma \leq \frac{88225}{153852} = 0.5734\ldots$	Theorem 6.13
$\mu(\sigma) \leq \frac{220633}{620612} - \frac{62831}{155153}\sigma$	$\frac{88225}{153852} \leq \sigma \leq \frac{521}{796} = 0.6545\ldots$	Theorem 6.13
$\mu(\sigma) \leq \frac{1333}{3825} - \frac{1508}{3825}\sigma$	$\frac{521}{796} \leq \sigma \leq \frac{53141}{76066} = 0.6986\ldots$	Theorem 6.13
$\mu(\sigma) \leq \frac{405}{1202} - \frac{227}{601}\sigma$	$\frac{53141}{76066} \leq \sigma \leq \frac{454}{641} = 0.7082\ldots$	Theorem 6.13
$\mu(\sigma) \leq \frac{779}{2590} - \frac{423}{1295}\sigma$	$\frac{454}{641} \leq \sigma \leq \frac{1744}{2411} = 0.7234\ldots$	Theorem 5.22, Corollary 6.8
$\mu(\sigma) \leq \frac{179}{622} - \frac{96}{311}\sigma$	$\frac{1744}{2411} \leq \sigma \leq \frac{951057}{1298878} = 0.7322\ldots$	Theorem 5.23, Corollary 6.8
$\mu(\sigma) \leq \frac{157319}{560830} - \frac{251324}{841245}\sigma$	$\frac{951057}{1298878} \leq \sigma \leq \frac{1389}{1736} = 0.8001\ldots$	Theorem 5.22, Corollary 6.8
$\mu(\sigma) \leq \frac{2841}{10316} - \frac{754}{2579}\sigma$	$\frac{1389}{1736} \leq \sigma \leq \frac{587779}{702192} = 0.8370\ldots$	Theorem 6.13
$\mu(\sigma) \leq \frac{1691}{6554} - \frac{890}{3277}\sigma$	$\frac{587779}{702192} \leq \sigma \leq \frac{7441}{8695} = 0.8557\ldots$	Theorem 5.22, Corollary 6.8
$\mu(\sigma) \leq \frac{29}{130} - \frac{3}{13}\sigma$	$\frac{7441}{8695} \leq \sigma \leq \frac{277}{300} = 0.9233\ldots$	Theorem 5.22, Theorem 6.14
$\mu(\sigma) \leq \lambda\mu_n + (1-\lambda)\mu_{n+1}$ $\mu_n = \frac{2}{(n-1)^2(n+2)}$ $\lambda = (\sigma_{n+1} - \sigma)/(\sigma_{n+1} - \sigma_n)$	$\sigma_n \leq \sigma \leq \sigma_{n+1}$ $\sigma_n = 1 - \frac{3n^2 - 3n + 2}{n(n-1)^2(n+2)}, \quad (n \geq 7)$	Theorem 6.14

6.3 Connection to the Riemann hypothesis

It is well known that the Riemann hypothesis implies the Lindelöf hypothesis. Here is a sharper version, essentially due to Backlund [2]:

Lemma 6.15 (Growth exponent and zeroes)**.** *Let $1/2 \leq \sigma_0 < 1$ be fixed. Then the assertion $\mu(\sigma_0) = 0$ is equivalent to the assertion that for any fixed $\varepsilon > 0$ and unbounded $T > 0$, the number of zeroes $\sigma + it$ of the zeta function with $\sigma \geq \sigma_0 + \varepsilon$ and $T \leq t \leq T + 1$ is $o(\log T)$.*

Proof. This is a routine adaptation of Theorem 2 of `https://terrytao.wordpress.com/2015/03/01`. ☐

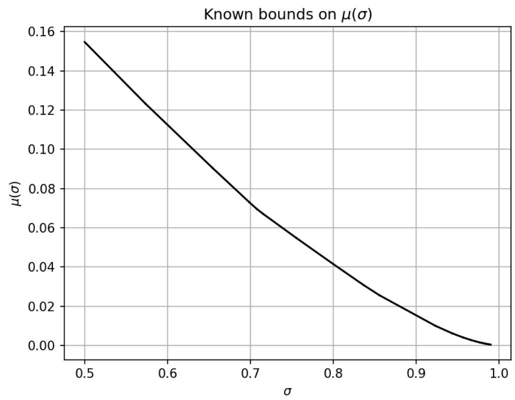

Figure 6.1: Current sharpest known bound on $\mu(\sigma)$ for $1/2 \leq \sigma \leq 1$.

Chapter 7

Large value estimates

The theory of zero density estimates for the Riemann zeta function (and other L-functions) rests on the study of what will be called *large value patterns* in this blueprint.

Definition 7.1 (Large value pattern). *A* large value pattern *is a tuple* $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$, *where* $N > 1$ *and* $T, V > 0$ *are real numbers,* a_n *is a* 1*-bounded sequence on* $[N, 2N]$*,* J *is an interval of length* T*, and* W *is a* 1*-separated subset of* J *such that*

$$\left|\sum_{n\in[N,2N]} a_n n^{-it}\right| \geq V \tag{7.1}$$

for all $t \in W$.
A Zeta large value pattern *is a large value pattern in which* $J = [T, 2T]$ *and* $a_n = 1_I(n)$ *for some interval* $I \subset [N, 2N]$.

The choice of interval J is not too important for a large value pattern, since one can translate J and W by any shift t_0 if we also modulate the coefficients a_n by n^{it_0} to compensate. However, this modulation freedom is not available for zeta large value patterns, as it destroys the form $a_n = 1_I(n)$ of the coefficients. The cardinality $|W|$ of W is traditionally called R in the literature.
It is common in the literature to relax the 1-boundedness hypothesis on a_n slightly, for instance to $a_n \ll T^{o(1)}$, but this does not significantly affect the analysis here. Similarly, the 1-separation hypothesis is sometimes strengthened slightly to a λ-separation hypothesis for some $\lambda = T^{o(1)}$, but again this does not make much difference. For some estimates, the uniform bound on a_n can be relaxed to an ℓ^2 hypothesis $\sum_{n\in[N,2N]} |a_n|^2 \ll N$ (and this second moment is traditionally called G in the literature), but we will not study such relaxations systematically here, as they are less relevant for the theory of zero density estimates.

Definition 7.2 (Large value exponent). *Let* $1/2 \leq \sigma \leq 1$ *and* $\tau \geq 0$ *be fixed. We define* $\mathrm{LV}(\sigma, \tau)$ *to be the least fixed quantity for which the following claim is true: whenever* $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ *is a large value pattern with* $N > 1$ *unbounded,* $T = N^{\tau+o(1)}$*, and* $V = N^{\sigma+o(1)}$*, then*

$$|W| \ll N^{\mathrm{LV}(\sigma,\tau)+o(1)}.$$

Implemented at `large_values.py` *as:*
`Large_Value_Estimate`

One can check that the set of possible candidates for $\mathrm{LV}(\sigma,\tau)$ is closed (by underspill), non-empty, and bounded from below, so $\mathrm{LV}(\sigma,\tau)$ is well-defined as a real number. As usual, we have an equivalent non-asymptotic definition:

Lemma 7.3 (Asymptotic form of large value exponent)**.** *Let $1/2 \le \sigma \le 1$, $\tau \ge 0$, and $\rho \ge 0$ be fixed. Then the following are equivalent:*

(i) $\mathrm{LV}(\sigma,\tau) \le \rho$.

(ii) For every (fixed) $\varepsilon > 0$ there exists $C, \delta > 0$ such that if $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ is a large value pattern with $N \ge C$ and $N^{\tau-\delta} \le T \le N^{\tau+\delta}$, $N^{\sigma-\delta} \le V \le N^{\sigma+\delta}$, then one has

$$|W| \le CN^{\rho+\varepsilon}.$$

The proof of Lemma 7.3 is similar to that of Lemma 4.3, and is left to the reader.

Lemma 7.4 (Basic properties)**.** *(i) (Monotonicity in σ) For any $\tau \ge 0$, $\sigma \mapsto \mathrm{LV}(\sigma,\tau)$ is upper semicontinuous and monotone non-increasing.*

(ii) (Huxley subdivision) For any $1/2 \le \sigma \le 1$ and $\tau' \ge \tau$ one has

$$\mathrm{LV}(\sigma,\tau) \le \mathrm{LV}(\sigma,\tau') \le \mathrm{LV}(\sigma,\tau) + \tau' - \tau.$$

In particular, $\tau \mapsto \mathrm{LV}(\sigma,\tau)$ is Lipschitz continuous.

(iii) ($\tau = 0$ endpoint) One has $\mathrm{LV}(\sigma,0) = 0$ for all $1/2 \le \sigma \le 1$, and hence by (ii) $0 \le \mathrm{LV}(\sigma,\tau) \le \tau$ for all $1/2 \le \sigma \le 1$ and $\tau \ge 0$.

TODO: implement Huxley subdivision as a way to transform a large values estimate into a better estimate

Proof. All claims are clear except perhaps for the upper bound

$$\mathrm{LV}(\sigma,\tau') \le \mathrm{LV}(\sigma,\tau) + \tau' - \tau,$$

but this follows because any interval of length $N^{\tau'+o(1)}$ may be subdivided into $N^{\tau'-\tau+o(1)}$ intervals of length $N^{\tau+o(1)}$, so on applying Definition 7.2 to each subinterval and summing (using Lemma 2.1 to ensure uniformity), one obtains the claim. ☐

Lemma 7.5 (Lower bound)**.** *For any $1/2 < \sigma \le 1$ and $\tau \ge 0$, one has $\mathrm{LV}(\sigma,\tau) \ge \min(2-2\sigma,\tau)$, while for $\sigma = 1/2$ one has $\mathrm{LV}(\sigma,\tau) = \tau$.*

Proof. In view of Lemma 7.4(ii), it suffices to show that $\mathrm{LV}(\sigma, 2-2\sigma) \ge 2-2\sigma$. By definition, it suffices to find a large value pattern $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ is a large value pattern with N unbounded, $V = N^{\sigma+o(1)}$, $T = N^{2-2\sigma+o(1)}$, and $|W| \gg N^{2-2\sigma-o(1)}$.
In the endpoint case $\sigma = 1$ one can achieve this by setting $a_n = 1$ for all n and taking $W = \{0\}$, so now we assume that $1/2 < \sigma < 1$.
We use the probabilistic method. We divide $[N, 2N]$ into $\asymp N^{2-2\sigma}$ intervals I of length $\asymp N^{2\sigma-1}$. On each interval I, we choose a_n to equal some randomly chosen sign $\epsilon_I \in \{-1,+1\}$, with the ϵ_I chosen independently in I. If $t = o(N^{2-2\sigma})$, then $\sum_{n\in I} a_n n^{-it}$ is equal to ϵ_I times a deterministic quantity $c_{t,I}$ of magnitude $\asymp N^{2\sigma-1}$ (the point being that the phase $t\log n$ is close to constant in this range). By the Chernoff bound, we thus see that for any such t, $\sum_{n\in[N,2N]} a_n n^{it}$ will have size $\gg N^{(2\sigma-1)+(2-2\sigma)/2} = N^{\sigma}$ with probability $\gg 1$. By

linearity of expectation, we thus see that with positive probability, a $\gg 1$ fraction of integers t with $t = o(N^{2-2\sigma})$ will have this property, giving the claim.
Finally, let $\sigma = 1/2$. In this case we just take each a_n to be a random sign, then by the Chernoff bound one has for each t that $|\sum_{n\in[N,2N]} a_n n^{it}| \asymp N^{1/2}$ with positive probability, which by linearity of expectation as before gives the lower bound $\mathrm{LV}(\sigma,\tau) \geq \tau$, while the upper bound is trivial from Lemma 7.4(iii). ☐

We conjecturally have a complete description of the function LV:

Conjecture 7.6 (Montgomery conjecture)**.** *One has*

$$\mathrm{LV}(\sigma,\tau) \leq 2 - 2\sigma \tag{7.2}$$

for all fixed $1/2 < \sigma \leq 1$ *and* $\tau \geq 0$*. Equivalently (by Lemma 7.4(ii), (iii) and Lemma 7.5), one has* $\mathrm{LV}(\sigma,\tau) = \min(2-2\sigma,\tau)$ *for all* $1/2 < \sigma \leq 1$ *and* $\tau \geq 0$*.*

Implemented at ***large_values.py*** *as:*
`montgomery_conjecture`

We refer to [19] for further discussion of this conjecture, including some counterexamples to strong versions of the conjecture in which certain epsilon losses are omitted. In view of this conjecture, we do not expect any further lower bounds on $\mathrm{LV}(\sigma,\tau)$ to be proven, and the literature is instead focused on upper bounds.
The following application of subdivision is useful:

Lemma 7.7 (Subdivision and the Montgomery conjecture)**.** *If* σ *is fixed, and the Montgomery conjecture holds for all fixed* $\tau < \tau_0$*, then*

$$\mathrm{LV}(\sigma,\tau) \leq \max(2-2\sigma, \tau - \tau_0 + 2 - 2\sigma) \tag{7.3}$$

for all fixed $\tau \geq 0$*.*

Proof. Clear from Lemma 7.4(ii). ☐

The following basic property of $\mathrm{LV}(\sigma,\tau)$ is extremely useful in applications:

Lemma 7.8 (Raising to a power)**.** *For any* $1/2 \leq \sigma \leq 1$*,* $\tau \geq 0$*, and natural number* k*, one has*

$$\mathrm{LV}(\sigma, k\tau) \leq k\mathrm{LV}(\sigma,\tau).$$

Implemented at ***large_values.py*** *as:*
`raise_to_power_hypothesis()`

Proof. Let $(N,T,V,(a_n)_{n\in[N,2N]},J,W)$ be a large value pattern with $T = N^{k\tau+o(1)}$ and $V = N^{\sigma+o(1)}$, Raising (7.1) the k^{th} power, we conclude that

$$\left|\sum_{n\in[N^k,2^kN^k]} b_n n^{-it}\right| \geq V^k$$

for all $t \in W$, where b_n is the Dirichlet convolution of k copies of a_n, and thus is bounded by $N^{o(1)}$ thanks to divisor bounds. Subdividing $[N^k, 2^kN^k]$ into k intervals of the form $[N', 2N']$ for $N' \asymp N^k$ and applying Definition 7.2 (with N,T,V replaced by N',T,V^k) we conclude that

$$|W| \ll N^{k\mathrm{LV}(\sigma,\tau)+o(1)}$$

and the claim then follows. ☐

7.1 Known upper bounds on $\mathrm{LV}(\sigma, \tau)$

Similarly to upper bounds on $\beta(\alpha)$, upper bounds on $\mathrm{LV}(\sigma, \tau)$ in the literature (also known as *large value theorems*) tend to be piecewise linear functions of σ and τ. Such bounds often tend to be convex initially (i.e., the maximum of several linear functions), but when one combines multiple large value theorems together, the bound is usually neither convex nor concave, though it often remains piecewise linear, and continuous in τ (though jump discontinuities in σ are possible).
Listed below are some examples of such bounds.

Theorem 7.9 (L^2 mean value theorem)**.** *For any fixed* $1/2 \leq \sigma \leq 1$ *and* $\tau \geq 0$ *one has*

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, 1 + \tau - 2\sigma).$$

In particular, the Montgomery conjecture (7.2) *holds for* $\tau \leq 1$*.*

Implemented at `large_values.py` *as:*
`large_value_estimate_L2`

Proof. Let $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ be a large value pattern with $T = N^{\tau+o(1)}$, $V = N^{\sigma+o(1)}$. Applying [149, Theorem 9.4] (with N, T replaced with $2N$, $2T$ respectively and taking $a_n = 0$ for $n < N$) one has

$$|W|V^2 \leq \sum_{t\in W} \left| \sum_{N\leq n\leq 2N} a_n n^{-it} \right|^2 \ll N^{1+o(1)}(T + N)$$

from which the result follows. □

Theorem 7.10 (Montgomery large values theorem)**.** *If* $1/2 \leq \sigma \leq 1$ *and* $\tau \geq 0$ *is such that*

$$\sup_{1\leq\tau'\leq\tau} \beta(1/\tau')\tau' < 2\sigma - 1 \tag{7.4}$$

(this condition is vacuous for $\tau < 1$*) then the Montgomery conjecture* (7.2) *holds for this choice of parameters.*

For a stronger version of this inequality, see Lemma 8.12.

Proof. Set $\rho := \mathrm{LV}(\sigma, \tau)$; we may assume without loss of generality that $\rho \geq 0$. Then by Definition 7.2, we can find a large value pattern $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ with $N > 1$ unbounded, $T = N^{\tau+o(1)}$, $V = N^{\sigma+o(1)}$, and $|W| = N^{\rho+o(1)}$. From (7.1) we have

$$\sum_{t\in W} \left| \sum_{n\in[N,2N]} a_n n^{it} \right| \geq |W|V$$

hence for some 1-bounded coefficients c_t

$$\left| \sum_{t\in W} c_t \sum_{n\in[N,2N]} a_n n^{it} \right| \geq |W|V$$

We apply the Halász argument. Interchanging the summations and applying Cauchy–Schwarz, we conclude that

$$|W|V \leq N^{1/2} \left| \sum_{t,t' \in W} c_t \overline{c_{t'}} \sum_{n \in [N,2N]} n^{i(t-t')} \right|^{1/2}$$

hence on squaring and using the triangle inequality

$$N^{2\rho} \ll N^{1-2\sigma+o(1)} \sum_{t,t' \in W} \left| \sum_{n \in [N,2N]} n^{i(t-t')} \right|.$$

In the case $|t-t'| \leq N^{1-\varepsilon}$ for any fixed $\varepsilon > 0$, one can use Lemma 4.4 to obtain the bound

$$\sum_{n \in [N,2N]} n^{i(t-t')} \ll N^{o(1)} \frac{N}{1+|t-t'|}.$$

The total contribution of this case can then be bounded by $N^{1+o(1)}R = N^{1+\rho+o(1)}$, thanks to the 1-separation. In the remaining cases $|t-t'| \geq N^{1-o(1)}$, we use Definition 4.2 to see that

$$\sum_{n \in [N,2N]} n^{i(t-t')} \ll N^{\sup_{1 \leq \tau' \leq \tau} \beta(1/\tau')\tau' + o(1)}$$

and thus

$$N^{2\rho} \ll N^{2-2\sigma+\rho+o(1)} + N^{2\rho+1-2\sigma+\sup_{1 \leq \tau' \leq \tau} \beta(1/\tau')\tau' + o(1)}.$$

By hypothesis, the second term on the right-hand side is asymptotically smaller than the left-hand side, and so we obtain $\rho \leq 2 - 2\sigma$ as required. □

Corollary 7.11 (Converting an exponent pair to a large values theorem)**.** *If (k,ℓ) is an exponent pair, and $1/2 \leq \sigma \leq 1$, and $\tau \geq 0$ are fixed, then*

$$\mathrm{LV}(\sigma,\tau) \leq \max\left(2-2\sigma, 2-2\sigma+\tau-\frac{2\sigma+k-\ell-1}{k}\right).$$

In particular, the Montgomery conjecture holds for $\tau \leq \frac{2\sigma+k-\ell-1}{k}$.

One can also obtain a similar implication starting from a bound on μ: see Lemma 8.13.

Proof. By Lemma 7.7 it suffices to prove the latter claim. From Lemma 5.3 one has $\beta(1/\tau')\tau' \leq k\tau' + (\ell - k)$ and so the condition (7.4) holds whenever

$$\tau < \frac{2\sigma+k-\ell-1}{k}.$$

The claim follows. □

Theorem 7.12 (Huxley large values theorem)**.** *[122, Equation (2.9)] Let $1/2 \leq \sigma \leq 1$ and $\tau \geq 0$ be fixed. Then one has*

$$\mathrm{LV}(\sigma,\tau) \leq \max(2-2\sigma, 4+\tau-6\sigma).$$

In particular, one has the Montgomery conjecture for $\tau \leq 4\sigma - 2$.

Recorded in `literature.py` *as:*
`add_huxley_large_values_estimate()`

Proof. Apply Corollary 7.11 with the pair $(k, \ell) = (1/2, 1/2)$ from Lemma 5.10. □

Theorem 7.13 (Heath-Brown large values theorem, preliminary form)**.** *Let* $1/2 \leq \sigma \leq 1$ *and* $\tau \geq 0$ *be fixed. If* $\mathrm{LV}(\sigma, \tau) \leq \rho$ *then*

$$\mathrm{LV}(\sigma, \tau) \leq \max\left(2 - 2\sigma, \frac{11}{12}\rho + \frac{3}{2} + \frac{\tau}{6} - 2\sigma\right)$$

Proof. Follows from [107, Lemma 1]. □

Theorem 7.14 (Heath-Brown large values theorem, optimized)**.** *Let* $1/2 \leq \sigma \leq 1$ *and* $\tau \geq 0$ *be fixed. One has*

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, 10 + \tau - 13\sigma).$$

In particular, the Montgomery conjecture holds for $\tau \leq 11\sigma - 8$*.*

Recorded in `literature.py` *as:*
`add_heath_brown_large_values_estimate()`

Proof. By Lemma 7.7 it suffices to show that $\mathrm{LV}(\sigma, \tau) \leq 2 - 2\sigma$ for $\tau \leq 11\sigma - 8$. From the previous theorem, and setting $\rho = \mathrm{LV}(\sigma, \tau)$, we have either

$$\mathrm{LV}(\sigma, \tau) \leq 2 - 2\sigma$$

or

$$\mathrm{LV}(\sigma, \tau) \leq \frac{11}{12}\mathrm{LV}(\sigma, \tau) + \frac{3}{2} + \frac{\tau}{6} - 2\sigma.$$

The latter bound can be rearranged as

$$\mathrm{LV}(\sigma, \tau) \leq 2\tau + 18 - 24\sigma$$

and thus

$$\mathrm{LV}(\sigma, \tau) \leq \min(2 - 2\sigma, 2\tau + 18 - 24\sigma),$$

and the claim follows. (See also the arguments in the first paragraph of [107, p. 226].) □

Lemma 7.15 (Second Heath-Brown large values theorem)**.** *If* $3/4 < \sigma \leq 1$ *and* $\tau \geq 0$ *are fixed, then*

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, k\tau + k(2 - 4\sigma), 2\tau/3 + k(12 - 16\sigma)/3)$$

for any positive integer k*.*

Proof. Let $(N, T, V, (a_n)_{n \in [N, 2N]}, J, W)$ be a large value pattern with $N \geq 1$ be unbounded, $T = N^{\tau + o(1)}$, $V = N^{\sigma + o(1)}$, and $|W| = N^{\rho + o(1)}$. By [106, Lemma 6] we have

$$(|W|V)^2 \ll T^{o(1)}(|W|N + |W|^2 N^{1/2} + |W|^{2-1/2k} T^{1/2} + |W|^{2-3/8k} N^{1/2} T^{1/4k})N$$

and thus

$$2(\rho + \sigma) \leq \max(\rho + 1, 2\rho + 1/2, (2 - 1/2k)\rho + \tau/2, (2 - 3/8k)\rho + 1/2 + \tau/4k).$$

Since $\sigma > 3/4$, we can delete the second term $2\rho + 1/2$ on the right-hand side. Solving for ρ, we conclude that

$$\rho \leq \max(2 - 2\sigma, k\tau + k(2 - 4\sigma), 2\tau/3 + k(12 - 16\sigma)/3),$$

and taking suprema in ρ, we obtain the claim. □

Theorem 7.16 (Jutila large values theorem)**.** *For any integer $k \geq 1$, one has*

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, \tau + (4 - 2/k) - (6 - 2/k)\sigma, \tau + (6 - 8\sigma)k).$$

Thus for instance with $k = 2$ we have

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, \tau + 3 - 5\sigma, \tau + 12 - 16\sigma)$$

and with $k = 3$ we have

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, \tau + \frac{10 - 16\sigma}{3}, \tau + 18 - 24\sigma).$$

In particular, the Montgomery conjecture holds for

$$\tau \leq \min((4 - 2/k)\sigma - (2 - 2/k), (8k - 2)\sigma - 6k + 2).$$

Recorded in `literature.py` as:
`add_jutila_large_values_estimate(Constants.LARGE_VALUES_TRUNCATION)`

Proof. See [160, (1.4)] (setting $V = N^{\sigma + o(1)}$, $T = N^{\tau + o(1)}$, and $G \leq N$). We remark that this form is an optimized form of the inequality after (3.2) in Jutila's paper, which in our notation would read that

$$2\mathrm{LV}(\sigma, \tau) + 2\sigma \leq \max\left(2 + \rho, \frac{3}{2} + \left(2 - \frac{1}{k}\right)\rho + \rho + \frac{1}{2k}\max\left(k(\tau - 1), \frac{\rho + \tau}{2}\right), 2\rho + 1\right)$$

whenever $\mathrm{LV}(\sigma, \tau) \leq \rho$. The optimization follows from Lemma 7.7 and routine calculations. □

Some additional large values theorems are established in Chapter 10.

Chapter 8

Large value theorems for zeta partial sums

Now we study a variant of the exponent $\mathrm{LV}(\sigma, \tau)$, specialized to the Riemann zeta function.

Definition 8.1 (Large value zeta exponent). *Let $1/2 \leq \sigma \leq 1$ and $\tau \geq 0$ be fixed. We define $\mathrm{LV}_\zeta(\sigma, \tau) \in [-\infty, +\infty)$ to be the least (fixed) exponent for which the following claim is true: if $(N, T, V, (a_n)_{n \in [N,2N]}, J, W)$ is a zeta large value pattern with N is unbounded, $T = N^{\tau + o(1)}$, and $V = N^{\sigma + o(1)}$, then $|W| \ll N^{\rho + o(1)}$.*

Implemented at `large_values.py` as:
`Large_Value_Estimate`

We will primarily be interested in the regime $\tau \geq 2$ (as this is the region connected to the Riemann-Siegel formula for $\zeta(\sigma + it)$), but for sake of completeness we develop the theory for the entire range $\tau \geq 0$. (The range $0 \leq \tau \leq 1$ can be worked out exactly by existing tools, while the region $1 < \tau < 2$ can be reflected to the region $2 < \tau < \infty$ by Poisson summation.)
As usual, we have a non-asymptotic formulation of $\mathrm{LV}_\zeta(\sigma, \tau)$:

Lemma 8.2 (Asymptotic form of large value exponent at zeta). *Let $1/2 \leq \sigma \leq 1$, $\tau \geq 0$, and $\rho \geq 0$ be fixed. Then the following are equivalent:*

(i) $\mathrm{LV}_\zeta(\sigma, \tau) \leq \rho$.

(ii) For every $\varepsilon > 0$ there exists $C, \delta > 0$ such that if $(N, T, V, (a_n)_{n \in [N,2N]}, J, W)$ is a zeta large value pattern with $N > C$, $N^{\tau - \delta} \leq T \leq N^{\tau + \delta}$, and $N^{\sigma - \delta} \leq V \leq N^{\sigma + \delta}$, then one has

$$|W| \leq C N^{\rho + \varepsilon}.$$

The proof of Lemma 8.2 proceeds as in previous sections and is omitted.

Lemma 8.3 (Basic properties). *(i) (Monotonicity in σ) For any $\tau \geq 0$, $\sigma \mapsto \mathrm{LV}_\zeta(\sigma, \tau)$ is upper semicontinuous and monotone non-increasing.*

(ii) (Trivial bound) For any $1/2 \leq \sigma \leq 1$ and $\tau \geq 0$, we have $\mathrm{LV}_\zeta(\sigma, \tau) \leq \tau$.

(iii) (Domination by large values) We have $\mathrm{LV}_\zeta(\sigma,\tau) \leq \mathrm{LV}(\sigma,\tau)$ *for all* $1/2 \leq \sigma \leq 1$ *and* $\tau \geq 0$*.*

(iv) (Reflection) For $1/2 \leq \sigma \leq 1$ *and* $\tau > 1$*, one has*

$$\sup_{\sigma \leq \sigma' \leq 1} \mathrm{LV}_\zeta\left(\frac{1}{2} + \frac{1}{\tau-1}(\sigma' - \frac{1}{2}), \frac{\tau}{\tau-1}\right) + \frac{1}{\tau-1}(\sigma'-\sigma) = \frac{1}{\tau-1} \sup_{\sigma \leq \sigma' \leq 1} (\mathrm{LV}_\zeta(\sigma',\tau) + \sigma' - \sigma).$$

Implemented at ***zeta_large_values.py*** *as:*
`get_trivial_zlv()`

We note that in practice, bounds for $\mathrm{LV}_\zeta(\sigma',\tau) + \sigma'$ are monotone decreasing[1] in σ', so the reflection property in Lemma 8.3(iv) morally simplifies[2] to

$$\mathrm{LV}_\zeta\left(\frac{1}{2} + \frac{1}{\tau-1}(\sigma - \frac{1}{2}), \frac{\tau}{\tau-1}\right) = \frac{1}{\tau-1}\mathrm{LV}_\zeta(\sigma,\tau). \tag{8.1}$$

TODO: implement a python method for reflection

Proof. The claims (i), (ii) are obvious. The claim (iii) is clear by setting $a_n = 1_I$ in Definition 7.2.
Now we turn to (iv). By symmetry it suffices to prove the upper bound. Actually it suffices to just show

$$\mathrm{LV}_\zeta\left(\frac{1}{2} + \frac{1}{\tau-1}(\sigma - \frac{1}{2}), \frac{\tau}{\tau-1}\right) \leq \frac{1}{\tau-1} \sup_{\sigma \leq \sigma' \leq 1} (\mathrm{LV}_\zeta(\sigma',\tau) + \sigma' - \sigma)$$

as this easily implies the general upper bound.
Let $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ be a zeta large value pattern with N unbounded, $T = N^{\frac{\tau}{\tau-1}+o(1)}$, and $V = N^{\frac{1}{2}+\frac{1}{\tau-1}(\sigma-\frac{1}{2})+o(1)}$. By definition, it suffices to show the bound

$$|W| \ll N^{\frac{1}{\tau-1}(\mathrm{LV}_\zeta(\sigma',\tau)+\sigma'-\sigma)+o(1)}. \tag{8.2}$$

for some $\sigma \leq \sigma' \leq 1$. By definition, $a_n = 1_I(n)$. By a Fourier expansion of $(n/N)^{1/2}$ in $\log n$, we can bound

$$|\sum_{n\in I} n^{-it_r}| \ll_A N^{1/2} \int_{\mathbf{R}} |\sum_{n\in I} n^{-1/2-it}|(1+|t-t_r|)^{-A}\ dt$$

and hence by the pigeonhole principle, we can find $t' = t + O(N^{o(1)})$ for each $t \in W$ such that

$$|\sum_{n\in I} n^{-1/2-it'}| \gg N^{-1/2-o(1)}V$$

for $t \in W$. By refining W by $N^{o(1)}$ if necessary, we may assume that the t' are 1-separated.
Now we use the approximate functional equation

$$\zeta(1/2+it') = \sum_{n\leq x} n^{-1/2-it'} + \chi(1/2+it') \sum_{m \leq t'/2\pi x} m^{-1/2+it'} + O(N^{-1/2}) + O((T/N)^{-1/2})$$

[1]This reflects the fact that large value theorems usually relate to p^{th} moment bounds for $p \geq 1$ (e.g., $p = 2, 4, 6, 12$) rather than for $0 < p < 1$.

[2]Alternatively, one can redefine LV_ζ to use smooth cutoffs in the n variable rather than rough cutoffs $1_I(n)$, in which case one can obtain the analogue of (8.1) rigorously, but we will not do so here.

for $x \sim N$; see [144, Theorem 4.1]. Applying this to the two endpoints of I and subtracting, we conclude that

$$\sum_{n \in I} n^{-1/2-it'} = \chi(1/2+it') \sum_{m \in J_{t'}} m^{-1/2+it'} + O(N^{-1/2}) + O((T/N)^{-1/2})$$

where $J_{t'} := \{m : t'/2\pi m \in I\}$. Since $\chi(1/2+it')$ has magnitude one, we conclude that

$$|\sum_{m \in J_r} m^{-1/2-it'}| \gg N^{-1/2-o(1)} V.$$

Writing $M := T/N = N^{\frac{1}{\tau-1}+o(1)}$, we see that $J_r \subset [M/10, 10M]$ and

$$|\sum_{m \in J_r} (M/m)^{1/2} m^{-it'}| \gg M^{1/2} N^{-1/2-o(1)} V = M^{\sigma+o(1)}.$$

Performing a Fourier expansion of $(M/m)^{1/2} 1_{J_r}(m)$ (smoothed out at scale $O(1)$) in $\log m$, we can bound

$$|\sum_{m \in J_r} (M/m)^{1/2} m^{-it'}| \ll \int_{T/10}^{10T} |\sum_{m \in [M/10,10M]} m^{-it_1}| (1+|t_1-t'|)^{-1}\, dt_1 + T^{-10}$$

and hence

$$\int_{T/10}^{10T} |\sum_{m \in [M/10,10M]} m^{-it_1}| (1+|t_1-t'|)^{-1}\, dt_1 \gg M^{\sigma+o(1)}.$$

If we let E denote the set of $t_1 \in [T/10, 10T]$ for which $|\sum_{m \in [M/10,10M]} m^{-it_1}| \geq M^{\sigma-o(1)}$ for a suitably chosen $o(1)$ error, then we have

$$\int_E |\sum_{m \in [M/10,10M]} m^{-it_1}| (1+|t_1-t'|)^{-1}\, dt \gg M^{\sigma+o(1)}.$$

Summing in t', we obtain

$$\int_E |\sum_{m \in [M/10,10M]} m^{-it_1}|\, dt_1 \gg M^{\sigma+o(1)} R$$

and so by dyadic pigeonholing we can find $M^{\sigma-o(1)} \ll V'' \ll M$ and a 1-separated subset W'' of E such that

$$|\sum_{m \in [M/10,10M]} m^{-it''}|\, dt \asymp V''$$

for all $t'' \in W''$, and

$$V''|W''| \gg M^{\sigma+o(1)} |W|.$$

By passing to a subsequence we may assume that $V'' = M^{\sigma'+o(1)}$ for some $\sigma \leq \sigma' \leq 1$. Partitioning $[M/10, 10M]$ into a bounded number of intervals each of which lies in a dyadic range $[M', 2M']$ for some $M' \asymp M$, and using Definition 8.1, we have

$$|W''| \ll M^{\mathrm{LV}_\zeta(\sigma',\tau)+o(1)}$$

and (8.2) follows. $\square$

Note in comparison with $\mathrm{LV}(\sigma,\tau)$, that $\mathrm{LV}_\zeta(\sigma,\tau)$ can be $-\infty$, and is indeed conjectured to do so whenever $\sigma > 1/2$ and $\tau \geq 1$. Indeed:

Lemma 8.4 (Characterization of negative infinite value)**.** *Let $1/2 \leq \sigma \leq 1$ and $\tau \geq 0$ be fixed. Then the following are equivalent:*

(i) $\mathrm{LV}_\zeta(\sigma,\tau) = -\infty$.

(ii) $\mathrm{LV}_\zeta(\sigma,\tau) < 0$.

(iii) There exists a fixed $\varepsilon > 0$ such that if N is unbounded and I is a subinterval of $[N,2N]$, then one has

$$\sum_{n\in I} n^{-it} \ll N^{\sigma-\varepsilon+o(1)}$$

whenever $|t| = N^{\tau+o(1)}$.

Proof. Clearly (i) implies (ii). If (iii) holds, then in any zeta large value pattern $(N,T,V,(a_n)_{n\in[N,2N]},J,W)$ with N unbounded and $V = N^{\sigma+o(1)}$, W is necessarily empty, giving (i). Conversely, if (i) fails, then there must be $(N,T,V,(a_n)_{n\in[N,2N]},J,W)$ with N unbounded and $V = N^{\sigma+o(1)}$ with W non-empty, contradicting (ii). $\square$

Corollary 8.5. *If $\tau \geq 0$ is fixed then $\mathrm{LV}_\zeta(\sigma,\tau) = -\infty$ whenever $\sigma > \tau\beta(1/\tau)$ is fixed. For instance, by (4.6), one has $\mathrm{LV}_\zeta(\sigma,1) = -\infty$ whenever $\sigma > 1/2$ is fixed.*

Proof. Suppose one has data N, I obeying the hypotheses of Lemma 8.4(iii), then by (4.2) (with $\alpha = 1/\tau$) one has

$$\sum_{n\in I} n^{-it} \ll |t|^{\beta(1/\tau)+o(1)} = N^{\tau\beta(1/\tau)+o(1)}$$

and the claim follows from Lemma 8.4. $\square$

Corollary 8.6. *If $\tau > 0$ and $1/2 \leq \sigma_0 \leq 1$ are fixed, then $\mathrm{LV}_\zeta(\sigma,\tau) = -\infty$ whenever $\sigma > \sigma_0 + \tau\mu(\sigma_0)$.*

Proof. From Definition 6.1 one has

$$\zeta(\sigma_0 + it) \ll |t|^{\mu(\sigma_0)+o(1)}$$

for unbounded t. By standard arguments (see [144, (8.13)]), this implies that

$$\sum_{n\in I} \frac{1}{n^{\sigma_0+it}} \ll |t|^{\mu(\sigma_0)+o(1)}$$

for unbounded N, if $I \subset [N,2N]$ and $|t| = N^{\tau+o(1)}$. By partial summation this gives

$$\sum_{n\in I} n^{-it} \ll N^{\sigma_0}|t|^{\mu(\sigma_0)+o(1)} = N^{\sigma_0+\tau\mu(\sigma_0)+o(1)}.$$

The claim now follows from Lemma 8.4. $\square$

Corollary 8.7. *If (k,ℓ) is an exponent pair, then $\mathrm{LV}_\zeta(\sigma,\tau) = -\infty$ whenever $1/2 \leq \sigma \leq 1$, $\tau \geq 0$ are fixed quantities with $\sigma > k\tau + \ell - k$.*

Proof. Immediate from Corollary 8.5 and Lemma 5.3; alternatively, one can use Corollary 8.6 and Corollary 6.8. ☐

Corollary 8.8. *Assuming the Lindelof hypothesis, one has* $\mathrm{LV}_\zeta(\sigma,\tau) = -\infty$ *whenever* $\sigma > 1/2$ *and* $\tau \geq 1$.

Proof. Apply Corollary 8.6 with $\sigma_0 = 1/2$, so that $\mu(\sigma_0)$ vanishes from the Lindelof hypothesis. ☐

For completeness, we now work out the values of $\mathrm{LV}_\zeta(\sigma,\tau)$ in the remaining cases not covered by the above corollary.

Lemma 8.9 (Value at $\sigma = 1/2$). *One has* $\mathrm{LV}_\zeta(1/2,\tau) = \tau$ *for all* $\tau \geq 1$.

Proof. The upper bound $\mathrm{LV}_\zeta(1/2,\tau) \leq \tau$ follows from Lemma 8.3(ii), so it suffices to prove the lower bound. Accordingly, let N be unbounded, let $T = CN$ for a large fixed constant C, and set $I := [N, 2N]$. In the case $\sigma = 1$, we see from the L^2 mean value theorem (Lemma 3.1) that the expression $\sum_{n\in I} n^{-it}$ has an L^2 mean of $\asymp N^{1/2}$ for $t \in [T, 2T]$; on other hand, from (4.6) we also have an L^∞ norm of $O(N^{1/2+o(1)})$. We conclude that $|\sum_{n\in I} n^{-it}| \gg N^{1/2+o(1)}$ for t in a subset of $[T, 2T]$ of measure $T^{1-o(1)}$, and hence on a 1-separated subset of cardinality $\gg T^{1-o(1)}$. This gives the claim $\mathrm{LV}(1/2, 1) \geq 1$.
Next, we establish the $\tau \geq 2$ case. Let N be unbounded, set $T := N^\tau$, and set $I := [N, 2N]$. From Lemma 3.1 we see that the L^2 mean of $\sum_{n\in I} n^{-it}$ is $\asymp N^{1/2}$. Also, by squaring this Dirichlet series and applying Lemma 3.1 again we see that the L^4 mean is $O(N^{1/2+o(1)})$. We may now argue as before to give the desired claim $\mathrm{LV}(1/2,\tau) \geq \tau$.
Finally we need to handle the case $1 < \tau < 2$. By Lemma 8.3(iv) with $\sigma = 1/2$ we have

$$\mathrm{LV}_\zeta\left(\frac{1}{2}, \frac{\tau}{\tau-1}\right) = \frac{1}{\tau-1} \sup_{1/2\leq\sigma'\leq 1} (\mathrm{LV}_\zeta(\sigma',\tau) + \sigma' - 1/2).$$

By the $\tau \geq 2$ case, the left-hand side is at least $\tau/(\tau-1)$, thus

$$\sup_{1/2\leq\sigma'\leq 1} (\mathrm{LV}_\zeta(\sigma',\tau) + \sigma' - 1/2) \geq \tau.$$

On the other hand, from Theorem 7.9 and Lemma 8.3(iii) we have

$$\mathrm{LV}_\zeta(\sigma',\tau) + \sigma' - 1/2 \leq \tau + 1/2 - \sigma'.$$

We conclude that the supremum is in fact attained asymptotically at $\sigma' = 1/2$, in the sense that

$$\limsup_{\sigma'\to 1/2^+} \mathrm{LV}_\zeta(\sigma',\tau) + \sigma' - 1/2 \geq \tau.$$

By the monotonicity of LV_ζ in σ, this implies that $\mathrm{LV}_\zeta(1/2,\tau) \geq \tau$, as required. ☐

Lemma 8.10 (Value at $\tau < 1$). *If* $0 \leq \tau < 1$, *then* $\mathrm{LV}_\zeta(\sigma,\tau)$ *is equal to* $-\infty$ *for* $\sigma > 1-\tau$ *and equal to* τ *for* $\sigma \leq 1-\tau$.

Proof. The first claim follows from Corollary 8.5 and Lemma 4.4. For the second claim, it suffices by Lemma 8.3(ii) to establish the lower bound $\mathrm{LV}_\zeta(\sigma,\tau) \geq \tau$. But this is clear from (4.5). ☐

One can use exponent pairs to control $\mathrm{LV}_\zeta(\sigma,\tau)$:

Lemma 8.11 (From exponent pairs to zeta large values estimate)**.** *[144, Theorem 8.2] If (k,ℓ) is an exponent pair with $k>0$, then for any $1/2 \le \sigma \le 1$ and $\tau \ge 0$ one has*

$$\mathrm{LV}_\zeta(\sigma,\tau) \le \max\left(\tau - 6(\sigma-1/2), \frac{k+\ell}{k}\tau - \frac{2(1+2k+2\ell)}{k}(\sigma-1/2)\right).$$

By applying this lemma to the exponent pairs in Corollary 5.11, one recovers the bounds in [144, Corollary 8.1, 8.2] (up to epsilon losses in the exponents).
A useful connection between large values estimates and large values estimates for the zeta function is the following strengthening of Theorem 7.10.

Lemma 8.12 (Halász–Montgomery inequality)**.** *For any $1/2 \le \sigma \le 1$ and $\tau \ge 0$, we have*

$$\mathrm{LV}(\sigma,\tau) \le \max\left(2-2\sigma, 1-2\sigma + \sup_{\substack{1\le\tau'\le\tau \\ \max(1/2,2\sigma-1)\le\sigma'\le 1}} \sigma' + \mathrm{LV}_\zeta(\sigma',\tau')\right).$$

Note from Lemma 8.5 one could also impose the restriction $\sigma' \le \tau'\beta(1/\tau')$ in the supremum if desired, at which point one recovers Theorem 7.10. Similarly, from Corollary 8.6 one could also impose the restriction $\sigma' \le \sigma_0 + \tau'\mu(\sigma_0)$ for any fixed $1/2 \le \sigma_0 \le 1$.

Proof. It suffices to show that

$$\mathrm{LV}(\sigma,\tau) \le \max\left(2-2\sigma, 1-2\sigma + \sup_{\substack{1\le\tau'\le\tau \\ 1/2\le\sigma'\le 1}} \sigma' + \min(\mathrm{LV}_\zeta(\sigma',\tau'), \mathrm{LV}(\sigma,\tau))\right)$$

since the terms with $\sigma' < 2\sigma-1$ are less than the left-hand side and can thus be dropped. We repeat the proof of Lemma 7.10. We can find a large value pattern $(N,T,V,(a_n)_{n\in[N,2N]},J,W)$ with N unbounded, $V = N^{\sigma+o(1)}$, $T = N^{\tau+o(1)}$, and $|W| = N^{\mathrm{LV}(\sigma,\tau)+o(1)}$, and we have

$$|W|V \le N^{1/2}\left|\sum_{t,t'\in W} c_t\overline{c_{t'}} \sum_{n\in[N,2N]} n^{i(t-t')}\right|^{1/2}$$

for some 1-bounded c_t, and hence by the triangle inequality

$$|W|V \le N^{1/2}|W|^{1/2} \sup_{t'}\left|\sum_{t\in W}|\sum_{n\in[N,2N]} n^{i(t-t')}|\right|^{1/2}$$

which we rearrange as

$$|W| \le N^{1-2\sigma+o(1)} \sup_{t'} \sum_{t\in W}|\sum_{n\in[N,2N]} n^{i(t-t')}|.$$

As in the proof of Lemma 7.10, the contribution of the case $|t-t'| \le N^{1-\varepsilon}$ to the right-hand side is $N^{2-2\sigma+o(1)}$, so we can restrict attention to the case $|t-t'| \ge N^{1-o(1)}$. By a dyadic decomposition and the pigeonhole principle, we may then assume that

$$|W| \le N^{1-2\sigma+o(1)} \sum_{t\in W:|t-t'|\asymp T'} |\sum_{n\in[N,2N]} n^{i(t_r-t_{r'})}|$$

for some $N^{1-o(1)} \ll T' \ll T$ and some r'; by passing to a subsequence we may assume that $T' = N^{\tau'+o(1)}$ for some $1 \leq \tau' \leq \tau$. By further dyadic decomposition, we may also assume that $|\sum_{n\in[N,2N]} n^{i(t-t')}| \asymp N^{\sigma'+o(1)}$ for some $\sigma' \leq 1$; the cardinality of the sum is then bounded both by $|W|$ and by $N^{\mathrm{LV}_\zeta(\sigma',\tau')+o(1)}$, hence

$$|W| \leq N^{1-2\sigma+\sigma'+\min(\mathrm{LV}(\sigma,\tau),\mathrm{LV}_\zeta(\sigma',\tau'))+o(1)}.$$

The case $\sigma' < 1/2$ is dominated by that of $\sigma' = 1/2$. The claim now follows. □

Corollary 8.13 (Converting a bound on μ to a large values theorem)**.** *If $1/2 \leq \sigma \leq 1$, $\sigma' \leq 1$, and $\tau \geq 0$ are fixed, then*

$$\mathrm{LV}(\sigma,\tau) \leq \max\left(2-2\sigma, 2-2\sigma+\tau-\frac{2\sigma-1-\sigma'}{\mu(\sigma')}\right).$$

In particular, the Montgomery conjecture holds for $\tau \leq \frac{2\sigma-1-\sigma'}{\mu(\sigma')}$.

Proof. By Lemma 7.7 it suffices to verify the claim for $\tau < \frac{2\sigma-1-\sigma'}{\mu(\sigma')}$. The claim now follows from Lemma 8.12 and Corollary 8.6. □

Theorem 8.14 (Halász-Turán large values theorem)**.** *[91, Theorem 1] On the Lindelöf hypothesis, one has the Montgomery conjecture whenever $\sigma > 3/4$.*

Proof. Immediate from Corollary 8.13, since $\mu(1/2) = 0$ in this case. □

Theorem 8.15 (First Ivic large values theorem)**.** *[144, Lemma 8.2] If $\tau \geq 0$ and $1/2 < \sigma < \sigma' < 1$ are fixed, then*

$$\mathrm{LV}(\sigma',\tau) \leq \max(2-2\sigma', \tau - f(\sigma)(\sigma'-\sigma))$$

where $f(\sigma)$ is equal to

$$\frac{2}{3-4\sigma} \text{ for } 1/2 < \sigma \leq 2/3;$$
$$\frac{10}{7-8\sigma} \text{ for } 2/3 \leq \sigma \leq 11/14;$$
$$\frac{34}{15-16\sigma} \text{ for } 11/14 \leq \sigma \leq 13/15;$$
$$\frac{98}{31-32\sigma} \text{ for } 13/15 \leq \sigma \leq 57/62;$$
$$\frac{5}{1-\sigma} \text{ for } 57/62 \leq \sigma < 1.$$

In particular, the Montgomery conjecture holds for this choice of σ' if

$$\tau \leq \sup_{1/2<\sigma<\sigma'} f(\sigma)(\sigma'-\sigma) + 2 - 2\sigma'.$$

Proof. We set θ to equal

$$(3\sigma-2)/(2\sigma-1) \text{ for } 1/2 < \sigma \leq 2/3;$$

$$(9\sigma-6)/(4\sigma-1) \text{ for } 2/3 \leq \sigma \leq 11/14;$$

$$(25\sigma - 16)/(8\sigma + 1) \text{ for } 11/14 \leq \sigma \leq 13/15;$$
$$(65\sigma - 40)/(16\sigma + 9) \text{ for } 13/15 \leq \sigma \leq 57/62;$$
$$(12\sigma - 7)/(2\sigma + 3) \text{ for } 57/62 \leq \sigma < 1,$$

and then from the bounds $\mu(1/2) \leq 1/6$, $\mu(5/7) \leq 1/14$, $\mu(5/6) \leq 1/30$ one can bound $\mu(\theta)$ by the quantity $c(\theta)$, defined to equal

$$1/2 - \theta \text{ for } \theta \leq 0$$
$$(3 - 4\theta)/6 \text{ for } 0 \leq \theta \leq 1/2$$
$$(7 - 8\theta)/18 \text{ for } 1/2 \leq \theta \leq 5/7$$
$$(15 - 16\theta)/50 \text{ for } 5/7 \leq \theta \leq 5/6$$
$$(1 - \theta)/5 \text{ for } 5/6 \leq \theta \leq 1.$$

By Corollary 8.13, we have $\mathrm{LV}(\sigma', \tau) \leq 2 - 2\sigma'$ for

$$\tau \leq \frac{2\sigma' - 1 - \theta}{c(\theta)}.$$

The right-hand side can be computed to equal $f(\sigma)(\sigma' - \sigma) + 2 - 2\sigma'$, giving the claim. □

Another typical application of the Halász-Montgomery inequality is

Lemma 8.16 (Second Ivic large values theorem)**.** *[144, (11.40)] For any* $1/2 \leq \sigma \leq 1$ *and* $\tau \geq 0$*, one has*

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, \tau + 9 - 12\sigma, 3\tau + 19(3 - 4\sigma)/2).$$

In particular, optimizing using subdivision (Lemma 7.7) we have

$$\mathrm{LV}(\sigma, \tau) \leq \max\left(2 - 2\sigma, \tau + 9 - 12\sigma, \tau - \frac{84\sigma - 65}{6}\right).$$

This implies the Montgomery conjecture for

$$\tau \leq \min\left(10\sigma - 7, 12\sigma - \frac{53}{6}\right).$$

Proof. Write $\rho := \mathrm{LV}(\sigma, \tau)$, and let $\varepsilon > 0$ be arbitrary. By Lemma 8.12, we may assume without loss of generality that

$$\rho \leq \max(2 - 2\sigma, 1 - 2\sigma + \sigma' + \min(\rho, \mathrm{LV}_\zeta(\sigma', \tau'))) + \varepsilon$$

for some $1/2 \leq \sigma' \leq 1$ and $1 \leq \tau' \leq \tau$. On the other hand, from Lemma 8.11 applied to the exponent pair $(2/7, 4/7)$ from Lemma 5.11, and bounding τ' by τ, one has

$$\mathrm{LV}_\zeta(\sigma', \tau') \leq \max(\tau - 6(\sigma' - 1/2), 3\tau - 19(\sigma' - 1/2))$$

and thus on taking convex combinations

$$\min(\rho, \mathrm{LV}_\zeta(\sigma', \tau')) \leq \max(\frac{5}{6}\rho + \frac{1}{6}\tau - (\sigma' - 1/2), \frac{18}{19}\rho + \frac{3}{19}\tau - (\sigma' - 1/2)),$$

hence ρ is bounded by either $2 - 2\sigma$, $1 - 2\sigma + \frac{5}{6}\rho + \frac{1}{6}\tau + \frac{1}{2}$, or $1 - 2\sigma + \frac{18}{19}\rho + \frac{3}{19}\tau + \frac{1}{2}$. The claim then follows after solving for ρ. □

Chapter 9

Moment growth for the zeta function

Definition 9.1 (Zeta moment exponents)**.** *For fixed $\sigma \in \mathbf{R}$ and $A \geq 0$, we define $M(\sigma, A)$ to be the least (fixed) exponent for which the bound*

$$\int_T^{2T} |\zeta(\sigma + it)|^A \, dt \ll T^{M(\sigma,A)+o(1)}$$

holds for all unbounded $T > 1$.

Such moments may be interpreted as the "average" order of the Riemann zeta function. It is not difficult to show that $M(\sigma, A)$ is a well-defined (fixed) real number. A non-asymptotic definition is that it is the least constant such that for every $\varepsilon > 0$ there exists $C > 0$ such that

$$\int_T^{2T} |\zeta(\sigma + it)|^A \, dt \leq CT^{M(\sigma,A)+\varepsilon}$$

holds for all $T \geq C$.

Lemma 9.2 (Basic properties of $M(\sigma, A)$)**.**

(i) $M(\sigma, A)$ is convex in σ.

(ii) For any σ, $a(M(\sigma, 1/a) - 1)$ is convex non-increasing in a.

(iii) $M(\sigma, A) = 1$ for all $A \geq 0$ and $\sigma \geq 1$.

(iv) $M(\sigma, A) \geq 1$ for all $A \geq 0$ and $1/2 \leq \sigma \leq 1$.

(v) $M(\sigma, 0) = 1$ for all σ.

(vi) $M(1 - \sigma, A) = M(1 - \sigma, A) + (1/2 - \sigma)A$ for all $\sigma \in \mathbf{R}$ and $A \geq 0$.

(vii) For any σ, $a(M(\sigma, 1/a) - 1)$ converges to $\mu(\sigma)$ as $a \to 0$. In particular (by previous properties), $M(\sigma, A) \leq A\mu(\sigma) + 1$ for all $\sigma \geq 0$ and $A \geq 0$, and also $M(\sigma, A) \leq M(\sigma, A_0) + \mu(\sigma)(A - A_0)$ for $\sigma \geq 0$ and $A \geq A_0 \geq 0$.

Proof. The claim (i) follows from the Phragmen-Lindelöf principle. The claim (ii) follows from Hölder. The claim (iii) follows from standard upper and lower bounds on $\zeta(\sigma+it)$ for $\sigma \geq 1$. The claim (iv) follows from (i)-(iii), and (v) is trivial. The claim (vi) follows easily from the functional equation.
For (vii), the bound $M(\sigma, A) \leq A\mu(\sigma) + 1$ is trivial, which implies that

$$\lim_{a\to 0} a(M(\sigma, 1/a) - 1) \leq \mu(\sigma).$$

Suppose for contradiction that

$$\lim_{a\to 0} a(M(\sigma, 1/a) - 1) < \mu(\sigma),$$

thus there is $\delta > 0$ such that

$$M(\sigma, A) \leq A(\mu(\sigma) - \delta) + 1$$

for all $A \geq 0$. By convexity, this gives

$$M(\sigma + \varepsilon, A) \leq A(\mu(\sigma) - \delta/2) + 1$$

for all sufficiently small ε, and then by the Cauchy integral formula and Hölder's inequality we can conclude that

$$|\zeta(\sigma + \varepsilon/2 + it)| \ll |t|^{\mu(\sigma)-\delta/2+O(1/A)+o(1)}$$

for unbounded $|t|$, leading to

$$\mu(\sigma + \varepsilon/2) \leq \mu(\sigma) - \delta/2 + O(1/A).$$

Sending A to infinity and ε to zero, we obtain a contradiction. □

Corollary 9.3 (Relationship with Lindelöf hypothesis)**.** *If the Lindelöf hypothesis holds, then $M(\sigma, A) = 1+\max(1/2-\sigma, 0)A$ for all $\sigma \in \mathbf{R}$ and $A \geq 0$. Conversely, if $M(1/2, A) = 1$ for arbitrarily large $A \geq 0$, then the Lindelöf hypothesis is true.*

Note from Lemma 9.2 that we always have the lower bound $M(\sigma, A) \geq 1+\max(1/2-\sigma, 0)A$. Thus there are not expected to be any further lower bound results for $M(\sigma, A)$, and we focus now on upper bounds. Compared to the pointwise estimates $\mu(\sigma)$ of $\zeta(\sigma + it)$, which are currently open for all $0 < \sigma < 1$, more are known about moment estimates $M(\sigma, A)$. In particular,

Lemma 9.4. *One has $M(1/2, A) = 1$ for all $0 \leq A \leq 4$.*

Proof. Follows from Hölder's inequality and the standard estimates

$$\int_T^{2T} |\zeta(1/2 + it)|^2\ dt = T^{1+o(1)}$$

and

$$\int_T^{2T} |\zeta(1/2 + it)|^4\ dt = T^{1+o(1)}$$

for any unbounded $T > 1$, due to [95] and [96] respectively. □

From Lemma 9.2 and Lemma 9.4 we may restrict attention to the region $1/2 \leq \sigma \leq 1$ and $A \geq 4$.

9.1 Relationship to zeta large value estimates

We can relate $M(\sigma, A)$ to $\mathrm{LV}_\zeta(\sigma, \tau)$:

Lemma 9.5. *If* $1/2 \leq \sigma_0 \leq 1$ *and* $A \geq 1$*, then*

$$M(\sigma_0, A) = \sup_{\tau \geq 2; \sigma \geq 1/2} (A(\sigma - \sigma_0) + \mathrm{LV}_\zeta(\sigma, \tau))/\tau. \tag{9.1}$$

In particular, one has

$$\mathrm{LV}_\zeta(\sigma, \tau) \leq \tau M(\sigma_0, A) - A(\sigma - \sigma_0)$$

whenever $\sigma \geq 1/2$ *and* $\tau \geq 2$.

Proof. We first show the lower bound, or equivalently that

$$A(\sigma - \sigma_0) + \mathrm{LV}_\zeta(\sigma, \tau) \leq \tau M(\sigma_0, A) - A(\sigma - \sigma_0)$$

whenever $\tau \geq 2$ and $\sigma \geq 1/2$. Accordingly, let N be unbounded, $T = N^{\tau + o(1)}$, $I \subset [N, 2N]$, and W be a 1-separated subset of $[T, 2T]$ such that

$$|\sum_{n \in I} n^{-it}| \gg N^{\sigma + o(1)}$$

for $t \in W$. By standard Fourier analysis (or by Perron's formula and contour shifting), this gives

$$\int_{T/2}^{3T} |\zeta(\sigma_0 + it')| \frac{dt}{1 + |t' - t|} \gg N^{\sigma - \sigma_0 + o(1)}$$

and hence by Hölder

$$\int_{T/2}^{3T} |\zeta(\sigma_0 + it')|^A \frac{dt'}{1 + |t' - t|} \gg N^{A(\sigma - \sigma_0) + o(1)}$$

so on summing in r

$$\int_{T/2}^{3T} |\zeta(\sigma_0 + it')|^A \, dt' \gg |W| N^{A(\sigma - \sigma_0) + o(1)}.$$

By Definition 9.1, the left-hand side is $\ll T^{M(\sigma_0, A) + o(1)}$. Since $T = N^{\alpha + o(1)}$, we obtain

$$|W| \ll N^{\tau M(\sigma_0, A) - A(\sigma - \sigma_0)},$$

giving the claim.

For the converse bound, let M be the right-hand side of (9.1). From Lemma 8.9 we have $M \geq 1$. By [144, §8.1] it will suffice to show that for any $V > 0$ and any 1-separated $W \subset [T, 2T]$ with

$$|\zeta(\sigma_0 + it)| \geq V$$

for all $t \in W$, one has

$$|W| \ll T^{M + o(1)} V^{-A}.$$

The claim is clear if $V \geq T^C$ or $V \leq T^C$ for some sufficiently large C, so we may assume that $V = T^{O(1)}$. We also clearly can assume $|W| \geq 1$. Using the Riemann–Siegel formula [144, Theorem 4.1] and dyadic decomposition, we have either

$$|\sum_{n\in I} \frac{1}{n^{\sigma_0+it}} \gg T^{-o(1)}V$$

or

$$T^{1/2-\sigma_0}|\sum_{n\in I} \frac{1}{n^{1-\sigma_0-it}}| \gg T^{-o(1)}V$$

for some $I \subset [N, 2N]$ and $1 \leq N \ll T^{1/2}$, and all $t \in W$. In either case, we can perform summation by parts and conclude that

$$|\sum_{n\in I'} n^{-it}| \gg T^{-o(1)}VN^{\sigma_0}$$

or

$$|\sum_{n\in I'} n^{-it}| \gg T^{-o(1)}VN^{1-\sigma_0}T^{\sigma_0-1/2}$$

for some I' in $[N, 2N]$ and all $t \in W$. As $\sigma_0 \geq 1/2$, the letter hypothesis is stronger than the former, so we may assume the former. If $N = T^{o(1)}$ then this would imply that $V \ll T^{o(1)}$, and we would be done from the trivial bound $R \ll T$ since $M \geq 1$. Hence, after passing to a subsequence, we can assume that $N = T^{1/\tau+o(1)}$ for some $2 < \tau < \infty$. We can also assume that $V = N^{\sigma-\sigma_0+o(1)}$ for some $\sigma \in \mathbf{R}$. If $\sigma \leq \sigma_0$ then $V \ll T^{o(1)}$ and we are done as before, so we may assume $\sigma > \sigma_0$; in particular, $\sigma \geq 1/2$. From Lemma 8.2 we have

$$|W| \ll N^{\mathrm{LV}_\zeta(\sigma,\tau)+o(1)}$$

and hence by definition of M

$$|W| \ll N^{M\tau-A(\sigma-\sigma_0)+o(1)} = T^{M+o(1)}V^{-A}$$

as required. □

Corollary 9.6 (Fourth moment bound)**.** *One has* $\mathrm{LV}_\zeta(\sigma,\tau) \leq \tau - 4(\sigma - 1/2)$ *for all* $1/2 \leq \sigma \leq 1$ *and* $\tau \geq 2$.

Proof. Apply Lemma 9.5 with $\sigma_0 = 1/2$ and $A = 4$, using Lemma 9.2(iv). □

We have an important twelfth moment estimate of Heath-Brown:

Theorem 9.7 (Heath-Brown twelfth moment estimate)**.** *[103]* $M(1/2, 12) \leq 2$. *Equivalently (by Lemma 9.5), one has* $\mathrm{LV}_\zeta(\sigma,\tau) \leq 2\tau - 12(\sigma - 1/2)$ *for all* $\tau \geq 2$ *and* $1/2 \leq \sigma \leq 1$.

Proof. From Lemma 8.11 with the exponent pair $(1/2, 1/2)$ from Lemma 5.10 we have

$$\mathrm{LV}_\zeta(\sigma,\tau) \leq \min(\tau - 6(\sigma - 1/2), 2\tau - 12(\sigma - 1/2)).$$

If $2\tau - 12(\sigma - 1/2) \geq 0$, the claim is immediate; if instead $2\tau - 12(\sigma - 1/2) < 0$, use Lemma 8.4. □

We also have a variant bound, which is slightly better when τ is close to $6(\sigma - 1/2)$:

Theorem 9.8 (Auxiliary Heath-Brown estimate)**.** *For $\tau \geq 2$ and $1/2 \leq \sigma \leq 1$, one has*

$$\mathrm{LV}_\zeta(\sigma,\tau) \leq \min(\tau - 6(\sigma - 1/2), 5\tau - 32(\sigma - 1/2)).$$

Proof. Let $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ be a zeta large value pattern with N, $V = N^{\sigma+o(1)}$, $T = N^{\tau+o(1)}$ and $W = N^{\mathrm{LV}_\zeta(\sigma,\tau)+o(1)}$. Our task is to show that

$$|W| \ll T^{o(1)}(T(N^{-1/2}V)^{-6} + T^5(N^{-1/2}V)^{-32}).$$

Write $a(n) = 1_I(n)$. By a Fourier analytic expansion we can bound

$$N^{-1/2}|\sum_{n\in I} n^{-it}| \ll T^{o(1)} \int_{T/4}^{3T} |\zeta(1/2 + it_1)| \frac{dt_1}{1 + |t_1 - t|} + N^{-\varepsilon}$$

for some fixed $\varepsilon > 0$ and all $t \in W$, hence

$$\int_{T/4}^{3T} |\zeta(1/2 + it_1)| \frac{dt}{1 + |t_1 - t|} \gg T^{-o(1)} N^{-1/2} V.$$

In particular, we can truncate to large values of $\zeta(1/2 + it_1)$, in the sense that

$$\int_{T/4}^{3T} |\zeta(1/2 + it_1)| 1_{|\zeta(1/2+it_1)| \geq T^{-o(1)} N^{-1/2} V} \frac{dt_1}{1 + |t_1 - t|} \gg T^{-o(1)} N^{-1/2} V.$$

Summing in t and using the 1-separation to bound the sum of $1/(1 + |t_1 - t|)$ by $T^{o(1)}$, we conclude that

$$\int_{T/4}^{3T} |\zeta(1/2 + it_1)| 1_{|\zeta(1/2+it_1)| \geq T^{-o(1)} N^{-1/2} V}\ dt_1 \gg T^{-o(1)} R N^{-1/2} V.$$

Hence by dyadic pigeonholing we have

$$V' \int_{T/4}^{3T} |\zeta(1/2 + it_1)| 1_{|\zeta(1/2+it_1)| \asymp V'}\ dt \gg T^{-o(1)} R N^{-1/2} V$$

for some $V' \geq T^{-o(1)} N^{-1/2} V$, and thus

$$|\zeta(1/2 + it')| \asymp V'$$

for all t' in some 1-separated subset W' of $[T/4, 3T]$ with

$$|W'| \gg T^{-o(1)} |W| N^{-1/2} V / V'.$$

Applying [103, Theorem 2] (treating different cases using the bounds [103, (7), (8), (9)]), we have the bound

$$|W'| \ll T^{o(1)}(T(V')^{-6} + T^5(V')^{-32})$$

and thus

$$|W| \ll T^{o(1)}(T(N^{-1/2}V)^{-1}(V')^{-5} + T^5(N^{-1/2}V)^{-1}(V')^{-31})$$

and the claim now follows from the lower bound on V'. □

9.2 Known moment growth bounds

Lemma 9.9 (Ivic's table of moment bounds)**.** *[144, Theorem 8.4] We have $M(\sigma, A) = 1$ when A is equal to*

$$\begin{aligned}
&\frac{4}{3-4\sigma} \textit{ for } 1/2 < \sigma \le 5/8;\\
&\frac{10}{5-6\sigma} \textit{ for } 5/8 < \sigma \le 35/54;\\
&\frac{19}{6-6\sigma} \textit{ for } 35/54 < \sigma \le 41/60;\\
&\frac{2112}{859-948\sigma} \textit{ for } 41/60 < \sigma \le 3/4;\\
&\frac{12408}{4537-4890\sigma} \textit{ for } 3/4 \le \sigma \le 5/6;\\
&\frac{4324}{1031-1044\sigma} \textit{ for } 5/6 \le \sigma \le 7/8;\\
&\frac{98}{31-32\sigma} \textit{ for } 7/8 \le \sigma \le 0.91591\dots;\\
&\frac{24\sigma-9}{(4\sigma-1)(1-\sigma)} \textit{ for } 0.91591\dots \le \sigma < 1.
\end{aligned}$$

Additionally, for $\sigma = 2/3$ one can take $A = 9.6187\dots$, for $\sigma = 7/10$ one can take $A = 11$, and for $\sigma = 5/7$ one can take $A = 12$.

Proof. This is a computation using Lemma 8.11, Theorem 8.15, and Lemma 9.5; see [144] for details. ☐

Theorem 9.10 (Moment bounds for $\sigma = 1/2$)**.** *[279, Theorems 2.1, 2.2] We have*

$$M(1/2, A) \le \begin{cases}
(16A+35)/114, & \frac{866}{65} \le A < 14,\\
(176677A+358428)/1246476, & 14 \le A < \frac{122304}{7955} = 15.37\dots,\\
(779A+1398)/5422, & \frac{122304}{7955} \le A < \frac{910020}{58699} = 15.50\dots,\\
3(1661A+2856)/34532, & \frac{910020}{58699} \le A < \frac{9604}{593} = 16.19\dots,\\
(405277A+677194)/2800950, & \frac{9604}{593} \le A < \frac{629068}{35731} = 17.60\dots,\\
(40726597A+64268678)/280113282, & \frac{629068}{35731} \le A < \frac{13789}{709} = 19.44\dots,\\
3(46A+49)/926, & \frac{13789}{709} \le A < \frac{204580}{10333} = 19.79\dots,\\
(3475A+3236)/23168, & \frac{204580}{10333} \le A < \frac{4252}{195} = 21.80\dots,\\
7(39945A+33704)/1857036, & \frac{4252}{195} \le A < \frac{812348}{30267} = 26.83\dots,\\
(37A+24)/244, & \frac{812348}{30267} \le A < \frac{440}{13} = 33.84\dots,\\
(31A-24)/196, & \frac{440}{13} \le A < \frac{203087}{4742} = 42.82\dots,\\
7(31519A-33704)/1385180, & \frac{203087}{4742} \le A < \frac{3516129}{65729} = 53.49\dots,\\
1+13(A-6)/84, & \frac{3516129}{65729} \le A.
\end{cases}$$

and also

$$M(1/2, 12+\delta) \le 2 + \frac{\delta}{8} + \frac{3\sqrt{510}}{7568}\delta^{3/2}, \qquad 0 < \delta \le \frac{86}{65}.$$

In particular, we have

$$M(1/2,13) \leq 2.1340, \qquad M(1/2,14) \leq 2.2720, \qquad M(1/2,15) \leq 2.4137,$$
$$M(1/2,16) \leq 2.5570, \qquad M(1/2,17) \leq 2.7016, \qquad M(1/2,18) \leq 2.8466.$$

9.3 Large values of ζ moments

It is also of interest to control large values of the moments.

Definition 9.11 (Mixed moments)**.** *For fixed $1/2 \leq \sigma \leq 1$, $A \geq 0$, and $h \geq 0$, let $M(\sigma, A, \geq h)$ be the least (fixed) exponent for which the bound*

$$\int_{0 \leq t \leq T: |\zeta(\sigma+it)| \geq T^h} |\zeta(\sigma+it)|^A \, dt \ll T^{M(\sigma,A,h)+o(1)}$$

holds for unbounded T. Similarly, let $M(\sigma, A, \leq h)$ be the least exponent for which

$$\int_{0 \leq t \leq T: |\zeta(\sigma+it)| < T^h} |\zeta(\sigma+it)|^A \, dt \ll T^{M(\sigma,A,h)+o(1)}$$

holds for unbounded T.

Lemma 9.12 (Mixed moments and large values of zeta)**.** *If $1/2 \leq \sigma_0 \leq 1$, $A \geq 1$, and $h \geq 0$ are fixed, then*

$$M(\sigma_0, A, \geq h) \leq \sup_{\tau \geq 2; \sigma \geq 1/2, h\tau} (A(\sigma - \sigma_0) + \mathrm{LV}_\zeta(\sigma, \tau))/\tau. \tag{9.2}$$

and

$$M(\sigma_0, A, \leq h) \leq \sup_{\tau \geq 2; \sigma \leq 1/2, h\tau} (A(\sigma - \sigma_0) + \mathrm{LV}_\zeta(\sigma, \tau))/\tau. \tag{9.3}$$

That is to say, any bound of the form

$$\mathrm{LV}_\zeta(\sigma, \tau) \leq M\tau - A(\sigma - \sigma_0)$$

whenever $\tau \geq 2$ and $\sigma \geq 1/2, h\tau$, gives rise to a bound

$$M(\sigma_0, A, \geq h) \leq M.$$

Similarly for $M(\sigma_0, A, \geq h)$, in which we replace the condition $\sigma \geq h\tau$ by $\sigma \leq h\tau$.

Proof. This is a routine modification of the proof of Lemma 9.5. □

Corollary 9.13 (Mixed moments and exponent pairs)**.** *If (k, ℓ) is an exponent pair with $k > 0$, then*

$$M(1/2, 6, \geq h) \leq 1$$

and

$$M\left(1/2, \frac{2(1+2k+2\ell)}{k}, \leq h\right) \leq \frac{k+\ell}{k}$$

where

$$h := \frac{\ell}{2+4l-2k}.$$

Proof. From Lemma 8.11 with the exponent pair (k,ℓ) we have

$$\mathrm{LV}_\zeta(\sigma,\tau) \le \min(\tau - 6(\sigma-1/2), \frac{k+\ell}{k}\tau - \frac{2(1+2k+2\ell)}{k}(\sigma-1/2)).$$

In particular, for $\sigma - 1/2 \le h\tau$ one has

$$\mathrm{LV}_\zeta(\sigma,\tau) \le \tau - 6(\sigma-1/2)$$

and for $\sigma - 1/2 \ge h\tau$ one has

$$\frac{k+\ell}{k}\tau - \frac{2(1+2k+2\ell)}{k}(\sigma-1/2).$$

The claim then follows from Lemma 9.12. ☐

Corollary 9.14 (Specific mixed moments)**.** *[144, (8.56)]* $M(1/2, 6, \ge 11/72) \le 1$ *and* $M(1/2, 24, \le 11/72) \le 15/4$.

Proof. Apply Corollary 9.13 with the exponent pair $(4/18, 11/18) = BABA(1/6, 2/3)$ from Corollary 5.11. ☐

Lemma 9.15 (Large value theorems from mixed moment bounds)**.** *[20, Proposition 2] Suppose that* $M(1/2, A, \ge h) \le 1$ *for some* $A \ge 4$ *and* $h \ge 0$*. Then one has*

$$\mathrm{LV}(\sigma,\tau) \le \max(\alpha + 2 - 2\sigma, -\alpha + \tau + A/2 - 2A(\sigma - 1/2))$$

whenever $1/2 \le \sigma \le 1$*,* $\tau > 0$*, and* $0 \le \alpha \le 1 - \sigma$ *is such that*

$$\sigma - \frac{1}{2} > \frac{\tau h}{2} + \frac{1}{4}.$$

Lemma 9.16 (Zero density theorems from mixed moment bounds)**.** *[20, Proposition 5] Suppose that* $M(1/2, 6, \ge h) \le 1$ *for some* $h \ge 0$*. Then for any* $1/2 \le \alpha < \sigma < 1$*, one has*

$$\mathrm{A}(\sigma) \le \max\left(\frac{4\mu(\alpha)}{\sigma-\alpha}, \frac{3}{8\sigma-5}, \frac{6h}{4\sigma-3}\right).$$

It is remarked in [20] that this proposition could lead to some improvements in current zero density estimate bounds.

Lemma 9.17 (Chen-Debruyne-Vidas large values theorem)**.** *[32, Lemma A.1] Let* $1/2 \le \sigma \le 1$ *and* $\tau \ge \frac{30\sigma-11}{8}$ *be fixed. Let* q_0, A_0, q_1, A_1, h *be fixed quantities such that* $M(1/2, q_0, \ge h) \le A_0$ *and* $M(1/2, q_0, \le h) \le A_1$*. Suppose that* $\rho \le \mathrm{LV}(\sigma,\tau)$ *is such that*

$$\frac{24(1-\sigma)}{30\sigma-11}\tau \le \rho \le 1.$$

Then for any $\alpha_1 \ge 0$ *and* $0 \le \alpha_2 \le \tau$*, one has*

$$\rho \le \max(2-2\sigma+\alpha_2, -2\alpha_1-(A_0-1)\alpha_2+A_0\tau+(3-4\sigma)q_0/2, -2\alpha_1+(A_1-1)\alpha_2+A_1\tau+(3-4\sigma)q_1/2, 8\alpha_1/7+4\alpha_2/7+16(1$$

In [32] this lemma is applied with $(q_0, A_0) = (6, 1)$ and $(q_1, A_1) = (19, 3)$ with $h = 2/13$, which follows from Corollary 9.13 applied to the exponent pair $(2/7, 4/7) = BA(1/6, 2/3)$ from Corollary 5.11.

Chapter 10

Large value additive energy

10.1 Additive energy

Definition 10.1 (Additive energy)**.** *Let W be a finite set of real numbers. The* additive energy $E_1(W)$ *of such a set is defined to be the number of quadruples* $(t_1, t_2, t_3, t_4) \in W$ *such that*

$$|t_1 + t_2 - t_3 - t_4| \leq 1.$$

We remark that in additive combinatorics, the variant $E_0(W)$ of the additive energy is often studied, in which $t_1+t_2-t_3-t_4$ is not merely required to be 1-bounded, but in fact vanishes exactly. However, this version of additive energy is less relevant for analytic number theory applications.

Lemma 10.2 (Basic properties of additive energy)**.** *(i) If W is a finite set of reals, then*

$$E_1(W) \asymp \int_{\mathbf{R}} |\#\{(t_1, t_2) \in W : |t_1 + t_2 - x| \leq 1\}|^2 \, dx.$$

More generally, for any $r > 0$ we have

$$E_1(W) \asymp r^{O(1)} \int_{\mathbf{R}} |\#\{(t_1, t_2) \in W : |t_1 + t_2 - x| \leq r\}|^2 \, dx.$$

(ii) If W is a finite set of reals, then

$$E_1(W) \asymp \int_{-1}^{1} |\sum_{t \in W} e(t\theta)|^4 \, d\theta.$$

(iii) If $W_1, \dots, W_k$ are finite sets of reals, then

$$E_1(W_1 \cup \dots \cup W_k)^{1/4} \ll E_1(W_1)^{1/4} + \dots + E_1(W_k)^{1/4}.$$

(iv) If W is 1*-separated and contained in an interval of length $T \geq 1$, then*

$$(\#W)^2, (\#W)^4/T \ll E_1(W) \ll (\#W)^3.$$

(v) If W is contained in an interval I, which is in turn split into K equally sized subintervals $J_1, \dots, J_K$, then

$$E_1(W)^{1/3} \ll \sum_{k=1}^{K} E_1(W \cap J_k)^{1/3}.$$

Note that the lower bound of $(\#W)^4/T$ would be expected to be attained if the set W is distributed "randomly" and is reasonably large (of size $\gg \sqrt{T}$). So getting upper bounds of the additive energy of similar strength to this lower bound can be viewed as a statement of "pseudorandomness" (or "Gowers uniformity") of this set.

Proof. For (i), we just prove the first estimate, as the second follows from the first by several applications of the triangle inequality. The right-hand side can be expanded as

$$\sum_{t_1,t_2,t_3,t_4 \in W} |\{x : |t_1 + t_2 - x|, |t_3 + t_4 - x| \le 1\}|.$$

Every quadruple contributing to $E_1(W)$ then contributes $\gg 1$ to the right-hand side, giving the upper bound. To get the matching lower bound, note that

$$\sum_{t_1,t_2,t_3,t_4 \in W} |\{x : |t_1 + t_2 - x|, |t_3 + t_4 - x| \le 1/2\} \le E_1(W)$$

and hence

$$E_1(W) \gg \int_{\mathbf{R}} |\#\{(t_1,t_2) \in W : |t_1 + t_2 - x| \le 1/2\}|^2 \, dx.$$

The upper bound then follows from the triangle inequality.
For (ii), we can upper bound the indicator function of $[-1, 1]$ by the Fourier transform of a non-negative bump function φ, so that the right-hand side is bounded by

$$\sum_{t_1,t_2,t_3,t_4 \in W} \varphi(t_1 + t_2 - t_3 - t_4)$$

which is then bounded by $O(E_1(W))$ by choosing the support of φ appropriately. The lower bound is established similarly (using the arguments in (i) to adjust the error tolerance 1 in the constraint $|t_1 + t_2 - x| \le 1$ as necessary.)
For (iii), first observe we may remove duplicates and assume that the W_i are disjoint, then we can use (ii) and the triangle inequality.
For (iv), the first lower bound comes from considering the diagonal case $t_1 = t_3, t_2 = t_4$ and the upper bound comes from observing that once t_1, t_2, t_3 are fixed, there are only $O(1)$ choices for t_4 thanks to the 1-separated hypothesis. Finally, observe that

$$\int_{\mathbf{R}} |\#\{(t_1,t_2) \in W : |t_1 + t_2 - x| \le 1\}| \, dx = (2\#W)^2$$

hence by Cauchy–Schwarz

$$\int_{\mathbf{R}} |\#\{(t_1,t_2) \in W : |t_1 + t_2 - x| \le 1\}|^2 \, dx \gg (\#W)^2/T$$

and the claim follows from (i).

For (v), write $a_k := E_1(W \cap J_k)^{1/4}$. Each tuple (t_1, t_2, t_3, t_4) that contributes to $E_1(W)$ is associated to a tuple $J_{k_1}, J_{k_2}, J_{k_3}, J_{k_4}$ of intervals with $k_1 + k_2 - k_3 - k_4 = O(1)$. By modifying the proof of (ii), the total contribution of such a tuple of intervals is

$$\ll \int_{\mathbf{R}} \prod_{j=1}^{4} |\sum_{t \in W \cap J_{k_j}} e(t\theta)| \, d\theta$$

which by Cauchy–Schwarz is bounded by

$$\ll a_{k_1} a_{k_2} a_{k_3} a_{k_4}.$$

Thus we see that

$$E_1(W) \ll \sum_{m=O(1)} a * a * \tilde{a} * \tilde{a}(m)$$

where $\tilde{a}_k := a_{-k}$ and $*$ denotes convolution on the integers. By Young's inequality we then have

$$E_1(W) \ll \|a\|_{\ell^{4/3}}^4$$

and the claim follows.
We remark that (v) can also be proven using [42, Lemma 4.8, (4.2)]. □

We will also study the following related quantity. Given a set W and a scale $N > 1$, let $S(N, W)$ denote the *double zeta sum*

$$S(N, W) := \sum_{t, t' \in W} \left| \sum_{n \in [N, 2N]} n^{-i(t-t')} \right|^2. \tag{10.1}$$

We caution that this normalization differs from the one in [144], where $n^{-1/2-i(t-t')}$ is used in place of $n^{-i(t-t')}$. This sum may also be rearranged as

$$S(N, W) = \sum_{n, m \in [N, 2N]} |R_W(n/m)|^2 \tag{10.2}$$

where R_W is the exponential sum

$$R_W(x) := \sum_{t \in W} x^{it}.$$

From the first formula it is clear that $S(N, W)$ is monotone non-decreasing in W, and from the second formula one has the triangle inequality

$$S\left(N, \bigcup_{i=1}^{k} W_i\right)^{1/2} \leq \sum_{i=1}^{k} S(N, W_i)^{1/2} \tag{10.3}$$

when the W_i are disjoint, and hence also when they are not assumed to be disjoint, thanks to the monotonicity.
The following Cauchy–Schwarz inequality is also useful:

Lemma 10.3 (Cauchy–Schwarz and double ζ-sums)**.** *[22, Lemma 3.4] If W, W' are finite sets of reals, $N > 1$, and a_n is a 1-bounded sequence for $n \in [N, 2N]$, then*

$$\sum_{t \in W, t' \in W'} \left| \sum_{n \in [N,2N]} a_n n^{-it} \right|^2 \leq S(N,W)^{1/2} S(N,W')^{1/2}. \tag{10.4}$$

In particular

$$\sum_{t \in W} \left| \sum_{n \in [N,2N]} a_n n^{-it} \right|^2 \leq S(N,W).$$

Proof. The left-hand side of (10.4) can be rewritten as

$$\sum_{n,m \in [N,2N]} a_n \overline{a_m} \overline{R_W}(n/m) R_{W'}(n/m).$$

The claim is now immediate from (10.2) and the Cauchy–Schwarz inequality. □

To relate $S(N,W)$ to $E_1(W)$, we first observe the following lemma, implicit in [104] and made more explicit in [88, Lemma 11.4].

Lemma 10.4 (Energy controlled by third moment)**.** *Suppose that $(N, T, V, (a_n)_{n \in [N,2N]}, J, W)$ is a large value pattern with $T \geq 1$ and $1 \leq N \ll T^{O(1)}$. Then*

$$V^2 E_1(W) \ll T^{o(1)} \sum_{n,m \in [N,2N]} |R_W(n/m)|^3 + T^{-50}.$$

Proof. By hypothesis, we have

$$V^2 E_1(W) \leq \sum_{t_1,t_2,t_3,t_4 \in W : |t_1+t_2-t_3-t_4| \leq 1} \left| \sum_{n \in [N,2N]} a_n n^{-it_4} \right|^2.$$

By standard Fourier arguments (see [88, Lemma 11.3]), we can bound

$$\left| \sum_{n \in [N,2N]} a_n n^{-it_4} \right| \ll T^{o(1)} \int_{t : |t-t_4| \leq T^{o(1)}} \left| \sum_{n \in [N,2N]} a_n n^{-it} \right| \, dt + T^{-100}.$$

Since each t_1, t_2, t_3 generates at most $O(1)$ choices for t_4, we conclude that

$$V^2 E_1(W) \ll T^{o(1)} \sum_{t_1,t_2,t_3 \in W} \int_{s : |s| \leq T^{o(1)}} \left| \sum_{n \in [N,2N]} a_n n^{-i(t_1+t_2-t_3+s)} \right|^2 \, ds + T^{-50},$$

The right-hand side can be rewritten as

$$T^{o(1)} \sum_{n,m \in [N,2N]} a_n \overline{a_m} (n/m)^{-is} \overline{R_W}(n/m)^2 R_W(n/m) + T^{-50},$$

and the claim then follows from the triangle inequality. □

Thus, $S(N,W)$ involves a second moment of R_W, while the energy $E_1(W)$ is related to the third moment. Using the trivial bound $|R_W(x)| \leq |W|$ we can then obtain the trivial bound

$$V^2 E_1(W) \ll T^{o(1)}|W|S(N,W) + T^{-50} \tag{10.5}$$

It is then natural to introduce the fourth moment

$$S_4(N,W) := \sum_{n,m\in[N,2N]} |R_W(n/m)|^4$$

since from Hölder's inequality one now has

$$V^2 E_1(W) \ll T^{o(1)} S(N,W)^{1/2} S_4(N,W)^{1/2} + T^{-50} \tag{10.6}$$

(cf. [104, Lemma 3]). The quantity $S_4(N,W)$ can also be expressed as

$$S_4(N,W) = \sum_{t_1,t_2,t_3,t_4\in W} \left| \sum_{n\in[N,2N]} n^{-i(t_1+t_2-t_3-t_4)} \right|^2.$$

One can bound this quantity by an $S(N,W)$ type expression:

Lemma 10.5. *If $W \subset [-T,T]$ is 1-separated and $1 \leq N \ll T^{O(1)}$, then one has*

$$S_4(N,W) \ll T^{o(1)} u^2 S(N,U) + T^{-100}$$

for some $1 \leq u \ll |W|$ and 1-separated subset U of $[-2T,2T]$ with

$$u|U| \ll |W|^2 \tag{10.7}$$

and

$$u^2|U| \ll E_1(W). \tag{10.8}$$

This result appears implicitly in [104, p. 229], and is made more explicit in the proof of [88, Lemma 11.6].

Proof. One can bound

$$S_4(N,W) \ll T^{o(1)} \sum_{t_1,t_2,t_3,t_4\in W} \int_{t=t_1+t_2-t_3-t_4+O(T^{o(1)})} \left| \sum_{n\in[N,2N]} n^{-it} \right|^2 dt + T^{-100},$$

and hence

$$S_4(N,W) \ll T^{o(1)} \sum_{t_1,t_2\in[-2N,2N]\cap\mathbf{Z}} f(t_1)f(t_2) \int_{t=t_1-t_2+O(T^{o(1)})} \left| \sum_{n\in[N,2N]} n^{-it} \right|^2 dt + T^{-100}$$

where f is the counting function

$$f(t) := |\{(t_1,t_2) \in W : |t - t_1 - t_2| \leq 1\}|.$$

Note that f is integer valued and bounded above by $|W|$. By dyadic decomposition, one can then find $1 \leq u \ll |W|$ and a subset U of $[-2N, 2N] \cap \mathbf{Z}$ such that $f(t) \asymp u$ for $t \in U$, and

$$S_4(N, W) \ll T^{o(1)} \sum_{t_1, t_2 \in U} u^2 \int_{t = t_1 - t_2 + O(T^{o(1)})} \left| \sum_{n \in [N, 2N]} n^{-it} \right|^2 dt + T^{-100}$$

which we can rearrange as

$$S_4(N, W) \ll T^{o(1)} u^2 \int_{s = O(T^{o(1)})} \sum_{n, m \in [N, 2M]} (n/m)^{is} |R_U(n/m)|^2 \, ds + T^{-100}$$

and hence by the triangle inequality

$$S_4(N, W) \ll T^{o(1)} v^2 S(N, V) + T^{-100}.$$

Also, by double counting one easily verifies the claims (10.7), (10.8). The claim follows. □

10.2 Large value additive energy region

Because the cardinality $|W|$ and additive energy $E_1(W)$ of a set W are correlated with each other, as well as with the double zeta sum $S(N, W)$, we will not be able to consider them separately, and instead we will need to consider the possible joint exponents for these two quantities. We formalize this via the following set:

Definition 10.6 (Large value energy region)**.** *The* large value energy region $\mathcal{E} \subset \mathbf{R}^5$ *is defined to be the set of all fixed tuples* $(\sigma, \tau, \rho, \rho^*, s)$ *with* $1/2 \leq \sigma \leq 1$, $\tau, \rho, \rho' \geq 0$, *such that there exists a large value pattern* $(N, T, V, (a_n)_{n \in [N, 2N]}, J, W)$ *with* $N > 1$ *unbounded,* $V = N^{\sigma + o(1)}$, $T = N^{\tau + o(1)}$, $V = N^{\sigma + o(1)}$, $|W| = N^{\rho + o(1)}$, $E_1(W) = N^{\rho^* + o(1)}$ *and* $S(N, W) = N^{s + o(1)}$.
We define the zeta large value energy region $\mathcal{E}_\zeta \subset \mathbf{R}^5$ *similarly, but where now* $(N, T, V, (a_n)_{n \in [N, 2N]}, J, W)$ *is required to be a zeta large value pattern.*

Clearly we have

Lemma 10.7 (Trivial containment)**.** *We have* $\mathcal{E}_\zeta \subset \mathcal{E}$.

These regions are related to $\mathrm{LV}(\sigma, \tau)$ and $\mathrm{LV}_\zeta(\sigma, \tau)$ as follows:

Lemma 10.8. *For any fixed* $1/2 \leq \sigma \leq 1, \tau \geq 0$, *we have*

$$\mathrm{LV}(\sigma, \tau) = \sup\{\rho : (\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}\}$$

and

$$\mathrm{LV}_\zeta(\sigma, \tau) = \sup\{\rho : (\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}_\zeta\}$$

In particular, we have $\rho \leq \mathrm{LV}(\sigma, \tau)$ *for all* $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$, *and* $\rho \leq \mathrm{LV}_\zeta(\sigma, \tau)$ *for all* $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}_\zeta$.

Proof. Clear from definition. □

Inspired by this, we can define

Definition 10.9. *For any fixed* $1/2 \leq \sigma \leq 1, \tau \geq 0$*, we define*

$$\mathrm{LV}^*(\sigma,\tau) := \sup\{\rho^* : (\sigma,\tau,\rho,\rho^*,s) \in \mathcal{E}\}$$

and

$$\mathrm{LV}^*_\zeta(\sigma,\tau) := \sup\{\rho^* : (\sigma,\tau,\rho,\rho^*,s) \in \mathcal{E}_\zeta\}.$$

Thus these exponents are upper bounds for the additive energy of large values of Dirichlet polynomials which may or may not be of zeta function type.
As usual, we have an equivalent non-asymptotic definition of the large value energy region:

Lemma 10.10 (Non-asymptotic form of large value energy region)**.** *Let* $1/2 \leq \sigma \leq 1$*,* $\tau \geq 0$*,* $\rho, \rho^* \geq 0$*, and* $s \in \mathbf{R}$ *be fixed. Then the following are equivalent:*

(i) $(\sigma,\tau,\rho,\rho^*,s) \in \mathcal{E}$.

(ii) For every $\varepsilon > 0$ *there exists* $C, \delta > 0$ *such that there is a large value pattern* $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ *with* $N \geq C$*,* $N^{\tau-\delta} \leq T \leq N^{\tau+\delta}$*,* $N^{\sigma-\delta} \leq V \leq N^{\sigma+\delta}$*,* $N^{\rho-\varepsilon} \leq |W| \leq N^{\rho+\varepsilon}$*,* $N^{\rho^*-\varepsilon} \leq E_1(W) \leq N^{\rho^*+\varepsilon}$*, and* $N^{s-\varepsilon} \leq S(N,W) \leq N^{s+\varepsilon}$*.*

Similarly with $\mathcal{E}$ *replaced by* $\mathcal{E}_\zeta$*, and with* $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ *required to be a zeta large value pattern.*

This lemma is proven by a routine expansion of the definitions, and is omitted.

Lemma 10.11 (Basic properties)**.**

(i) (Monotonicity in σ*) If* $(\sigma,\tau,\rho,\rho^*,s) \in \mathcal{E}$*, then* $(\sigma',\tau',\rho,\rho^*,s) \in \mathcal{E}$ *for all* $1/2 \leq \sigma' \leq \sigma$ *and* $\tau' \geq \tau$*.*

(ii) (Subdivision) If $(\sigma,\tau,\rho,\rho^*,s) \in \mathcal{E}$ *and* $0 \leq \tau' \leq \tau$*, then amongst all tuples* $(\sigma,\tau',\rho',(\rho')^*,s') \in \mathcal{E}$ *with* $\rho' \leq \rho$*,* $(\rho')^* \leq \rho^*$*, and* $s' \leq s$*, there exists a tuple with*

$$\rho \leq \rho' + \tau - \tau';$$

there exists a tuple with

$$\rho^* \leq \rho' + 3\min(\rho-\rho', \tau-\tau');$$

and there exists a tuple with

$$s \leq s' + 2\min(\rho-\rho', \tau-\tau').$$

(But it may not be the same tuple that satisfies all three properties.)

(iii) (Trivial bounds) If $(\sigma,\tau,\rho,\rho^*,s) \in \mathcal{E}$*, one has*

$$2\rho, 4\rho-\tau \leq \rho^* \leq 3\rho.$$

Proof. The claim (i) is trivial, so we turn to (ii). By definition, there exists a large value pattern $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ with $N > 1$ unbounded, $T = N^{\tau+o(1)}$, $V = N^{\sigma+o(1)}$, $|W| = N^{\rho+o(1)}$, $E_1(W) = N^{\rho^*+o(1)}$, and $S(N,W) = N^{s+o(1)}$. We now partition J into $N^{\tau-\tau'+o(1)}$ subintervals I of length $N^{\tau'+o(1)}$, and subdivide W into W_I accordingly. By dyadically pigeonholing, we can then subdivide this collection I of intervals into $N^{o(1)}$ subcollections, where on each subcollection there exists fixed $\rho', (\rho')^*, s'$ such that $|W_I| = N^{\rho'+o(1)}$,

$E_1(W_I) = N^{(\rho')^*+o(1)}$, and $S(N, W_I) = N^{s'+o(1)}$. Since $W_I \subset W$, this forces $\rho' \leq \rho$, $(\rho')^* \leq \rho^*$, and $s' \leq s$. From Definition 10.6 we see that $(\sigma, \tau', \rho', (\rho')^*, s') \in \mathcal{E}$.
By the pigeonhole principle, one of these subcollections must contribute at least $N^{-o(1)}$ of the cardinality of W. Since there are at most $N^{\tau-\tau'+o(1)}$ intervals in this collection, we must have $\rho \leq \rho' + \tau - \tau'$ in this case.
By Lemma 10.2(iii), we also know that a (possibly different subcollection) must contribute at least $N^{-o(1)}$ of the additive energy of W. The number of intervals in this subcollection is at most $\min(N^{\tau-\tau'+o(1)}, N^{\rho-\rho'+o(1)})$. Applying Lemma 10.2(iii) again, we conclude $\rho^* \leq (\rho^*)' + 3\min(\rho - \rho', \tau - \tau')$.
Finally, from (10.3), we know that a (possibly yet another subcollection) must contribute at least $N^{-o(1)}$ of the double zeta sum $S(N, W)$. The number of intervals in this subcollection is at most $\min(N^{\tau-\tau'+o(1)}, N^{\rho-\rho'+o(1)})$. Applying Lemma 10.3(iii) again, we conclude $s \leq s' + 2\min(\rho - \rho', \tau - \tau')$. □

Lemma 10.12 (Raising to a power). *If $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$, and $k \geq 1$, then amongst all tuples $(\sigma, \tau/k, \rho', (\rho')^*, s') \in \mathcal{E}$ with $\rho' \leq \rho/k$, $(\rho')^* \leq \rho^*/k$, and $s' \leq s/k$, there exists a tuple with $\rho' = \rho/k$, there exists a tuple with $(\rho')^* = \rho^*/k$, and there exists a tuple with $s' = s/k$. (These may be three different tuples.)*

Proof. By definition, there exists a large value pattern $(N, T, V, (a_n)_{n \in [N,2N]}, J, W)$ with $N > 1$ unbounded, $T = N^{\tau+o(1)}$, $V = N^{\sigma+o(1)}$, $|W| = N^{\rho+o(1)}$, $E_1(W) = N^{\rho^*+o(1)}$, and $S(N, W) = N^{s+o(1)}$. Observe that

$$\left(\sum_{n \in [N,2N]} a_n n^{-it}\right)^k = \sum_{n \in [N^k, 2^k N^k]} b_n n^{-it}$$

for some coefficients $b_n = O(n^{o(1)})$. In particular, partitioning $[N^k, 2^k N^k]$ into $O(1)$ subintervals $[N', 2N']$ with $N' = N^{k+o(1)}$, we can partition W into $O(1)$ subcollections $W_{N'}$, such that

$$\left|\sum_{n \in [N',2N']} b_n n^{-it}\right| \gg V^k = (N')^{\sigma+o(1)}$$

for all $t \in W_{N'}$. Again by the pigeonhole principle, one of the $W_{N'}$ must have cardinality $N^{\rho+o(1)}$, one must have energy $N^{\rho^*+o(1)}$, and one must have double zeta sum $N^{s+o(1)}$ (but these may be different $W_{N'}$). Each of these $W_{N'}$ then give the different conclusions to the lemma. □

Morally speaking, one should be able to obtain equality in all three conclusions of Lemma 10.12 simultaneously, i.e. that $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ essentially implies $(\sigma, \tau/k, \rho/k, \rho^*/k, s/k) \in \mathcal{E}$. This is because in practice one frequently controls $\mathcal{E}$ by computing a containment region $\mathcal{E}_1$ that possesses precisely the required monotonicity property. Specifically, we have

Lemma 10.13 (Monotonicity criterion). *Let $\mathcal{E}_1$ be the intersection of sets E_i, each of the form*

$$\{(\sigma, \tau, \rho, \rho^*, s) \in \mathbf{R}^5 : \rho \leq f_1(\rho^*, s), \rho^* \leq f_2(\rho, s), s \leq f_3(\rho, \rho^*)\}$$

for some monotonically increasing functions f_1, f_2, f_3 (that possibly also depend on σ and τ).
Suppose amongst all tuples $(\sigma, \tau, \rho', (\rho^*)', s') \in \mathcal{E}_1$ with $\rho' \leq \rho$, $(\rho^*)' \leq \rho^*$ and $s' \leq s$, there exists a tuple with $\rho' = \rho$, a tuple with $(\rho^*)' = \rho^*$ and a tuple with $s' = s$ (not necessarily the same tuple each time). Then, $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}_1$.

Proof. Suppose that $(\sigma, \tau, \rho, (\rho^*)', s') \in \mathcal{E}_1$ for some $(\rho^*)' \leq \rho^*$ and $s \leq s'$ so that also $(\sigma, \tau, \rho, (\rho^*)', s') \in E_i$. Then by definition $\rho \leq f_1((\rho^*)', s')$. Since f_1 is monotonically increasing (with respect to both ρ^* and s), one has $\rho \leq f_1(\rho^*, s)$. Similarly, $(\sigma, \tau, \rho', \rho^*, s') \in \mathcal{E}_1$ implies $\rho^* \leq f_2(\rho, s)$ and $(\sigma, \tau, \rho', (\rho^*)', s) \in \mathcal{E}_1$ implies $s \leq f_3(\rho, \rho^*)$, which together imply $(\sigma, \tau, \rho, \rho^*, s) \in E_i$ by definition. Hence $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}_1$ since $\mathcal{E}_1$ is the intersection of sets E_i. □

Lemma 10.14 (Raising to a power, alternative formulation). *Let k be a positive integer, $\mathcal{E}_1 \subseteq \mathbf{R}^5$ be a set satisfying the monotonicity criterion of Lemma 10.13 and*

$$\mathcal{E}_k := \{(\sigma, \tau, \rho, \rho^*, s) \in \mathbf{R}^5 : (\sigma, \tau/k, \rho/k, \rho^*/k, s/k) \in \mathcal{E}_1\}.$$

If $\mathcal{E} \subseteq \mathcal{E}_1$ then $\mathcal{E} \subseteq \mathcal{E}_k$.

Proof. Suppose that $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E} \subseteq \mathcal{E}_1$. By Lemma 10.12 and Lemma 10.13, $(\sigma, \tau/k, \rho/k, \rho^*/k, s/k) \in \mathcal{E}_1$, hence by definition $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}_k$. □

10.3 Known relations for the large value energy region

Theorem 10.15 (Reflection principle). *[144, §11.5] If $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ with $\sigma \geq 3/4$ and $\tau > 1$, then for any integer $k \geq 1$, either $\rho \leq 2 - 2\sigma$, or there exists $0 < \alpha \leq k(\tau - 1)$ and $(\sigma, \tau/\alpha, \rho/\alpha, \rho^*/\alpha, s'/\alpha) \in \mathcal{E}$ such that*

$$\rho \leq \min(2 - 2\sigma, k(3 - 4\sigma)/2 + s' - 1).$$

Proof. By definition, there exists a large value pattern $(N, T, V, (a_n)_{n \in [N,2N]}, J, W)$ with $N > 1$ unbounded, $T = N^{\tau + o(1)}$, $|W| = N^{\rho + o(1)}$, $E_1(W) = N^{\rho^* + o(1)}$ and $S(N, W) = N^{s + o(1)}$. By [144, (11.58)], one has

$$|W|^2 \ll T^{o(1)} \left(|W| N^{2-2\sigma} + N^{1-2\sigma} |W|^2 + N^{(3-4\sigma)/2} \int_{v = O(T^{o(1)})} \sum_{t, t' \in W} \left| \sum_{n \leq 4T/N} n^{-1/2 + it - it' + iv} \right| \, dv \right).$$

Since $\sigma > 1/2$, the $N^{1-2\sigma}|W|^2$ term can be dropped. Applying Hölder's inequality and dyadic pigeonholing as in [144, (11.59)], we conclude that

$$|W| \ll T^{o(1)} \left(N^{2-2\sigma} + N^{k(3-4\sigma)/2} \left(\sum_{t, t' \in W} \left| \sum_{n \in [N', 2N']} b_n n^{-1/2 + it - it' + iv} \right|^2 \right)^{1/2} \right)$$

for some $v = O(T^{o(1)})$ and coefficients $b_n = O(T^{o(1)})$, and some $N' \ll (4T/N)^k$. After passing to a subsequence if necessary, we may assume that $N' = N^{\alpha + o(1)}$ for some $0 \leq \alpha \leq k(\tau - 1)$. If $\alpha = 0$ then the second term here is negligible compared to the first and we obtain $\rho \leq 2 - 2\sigma$, so suppose that $\alpha > 0$. Using [144, Lemma 11.1] to eliminate the $b_n n^{-1/2+iv}$ coefficients, we conclude that

$$|W| \ll T^{o(1)} (N^{2-2\sigma} + N^{k(3-4\sigma)/2 - 1} S(N', W)).$$

By construction, we have $S(N', W) = (N')^{s'/\alpha + o(1)} = N^{s' + o(1)}$ for some tuple $(\sigma, \tau/\alpha, \rho/\alpha, \rho^*/\alpha, s'/\alpha) \in \mathcal{E}$. The claim follows. □

Heuristically one expects $s \le \max(\rho+1, 2\rho)+1$ (see [144, (11.63)]). There is one easy case in which this is true:

Lemma 10.16. *If $(\sigma,\tau,\rho,\rho^*,s) \in \mathcal{E}$ with $\tau < 1$, then $s \le \max(\rho+1, 2\rho)+1$.*

Recorded in `literature.py` *as:*
`add_lver_ivic_1985()`

Proof. By definition, there exists a large value pattern $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ with $N > 1$ unbounded, $T = N^{\tau+o(1)}$, $|W| = N^{\rho+o(1)}$, $E_1(W) = N^{\rho^*+o(1)}$ and $S(N,W) = N^{s+o(1)}$. By the discussion after [144, (11.63)], we have

$$N^{-1}S(N,W) \ll T^{\varepsilon}(|W|N + |W|^2)$$

for any fixed $\varepsilon > 0$, which gives the claim. . ☐

Another bound is

Lemma 10.17. *[144, Lemma 11.2] If (k,ℓ) is an exponent pair with $k > 0$, and $(\sigma,\tau,\rho,\rho^*,s) \in \mathcal{E}$, then*

$$s \le \max\left(\rho+1, 5\rho/3+\tau/3, \frac{2+3k+4\ell}{1+2k+2\ell}\rho + \frac{k+\ell}{1+2k+2\ell}\tau\right)+1.$$

Proof. By definition, there exists a large value pattern $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ with $N > 1$ unbounded, $T = N^{\tau+o(1)}$, $|W| = N^{\rho+o(1)}$, $E_1(W) = N^{\rho^*+o(1)}$ and $S(N,W) = N^{s+o(1)}$. From [144, Lemma 11.2] we have

$$N^{-1}S(N,W) \ll |W|N + |W|^{5/3}T^{1/3+\varepsilon} + |W|^{\frac{2+3k+4\ell}{1+2k+2\ell}}T^{\frac{k+\ell+\varepsilon}{1+2k+2\ell}}$$

for any fixed $\varepsilon > 0$, which gives the claim. ☐

Finally, we have the useful

Lemma 10.18 (Heath-Brown bound on double sums)**.** *If $(\sigma,\tau,\rho,\rho^*,s) \in \mathcal{E}$, then*

$$s \le \max(\rho+1, 2\rho, 5\rho/4+\tau/2)+1.$$

Note that if $\tau \le 3/2$, the $5\rho/4+\tau/2$ term is bounded by the convex combination $(3/4)(\rho+1)+(1/4)(2\rho)$ and may therefore be omitted.
Recorded in `literature.py` *as:*
`add_lver_heath_brown_1979()`

Proof. By definition, there exists a large value pattern $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ with $N > 1$ unbounded, $T = N^{\tau+o(1)}$, $|W| = N^{\rho+o(1)}$, $E_1(W) = N^{\rho^*+o(1)}$ and $S(N,W) = N^{s+o(1)}$. From [106, Theorem 1] or [144, Lemma 11.5], one has

$$N^{-1}S(N,W) \ll T^{\varepsilon}(|W|N + |W|^2 + |W|^{5/4}T^{1/2}),$$

giving the claim. ☐

Lemma 10.5 can be formulated in terms of the large value energy region as follows.

Lemma 10.19. *If $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$, then there exists $(\sigma, \tau, \rho', (\rho')^*, s') \in \mathcal{E}$ and $0 \leq \kappa \leq \rho$ such that*

$$\kappa + \rho' \leq 2\rho$$

$$2\kappa + \rho' \leq \rho^*$$

and

$$\rho^* + 2\sigma \leq \kappa + (s + s')/2.$$

Proof. By definition, there exists a large value pattern

$$(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$$

with $N \geq 1$ unbounded, $T = N^{\tau+o(1)}$, $V = N^{\sigma+o(1)}$, $|W| = N^{\rho+o(1)}$, $E_1(W) = N^{\rho^*+o(1)}$, and $S(N, W) = N^{s+o(1)}$. From (10.6) we have

$$V^2 E_1(W) \ll T^{o(1)} S(N, W)^{1/2} S_4(N, W)^{1/2} + T^{-50}.$$

By Lemma 10.5, there exists $1 \leq u \ll |W|$ and a 1-separated subset U of $[-2T, 2T]$ such that such that

$$V^2 E_1(W) \ll T^{o(1)} u S(N, W)^{1/2} S(N, U)^{1/2} + T^{-50}$$

with (10.7), (10.8) holding. Since W is non-empty, $E_1(W) \geq 1$ and $V \geq N^{1/2} \geq 1$, so the T^{-50} error here may be discarded. Passing to a subsequence, we may assume that $u = N^{\kappa+o(1)}$ for some $0 \leq \kappa \leq \rho$, and that $|U| = N^{\rho'+o(1)}$ for some $\rho' \geq 0$. Then we have $S_2(N, U) = s'$ for some $(\sigma, \tau, \rho', (\rho')^*, s') \in \mathcal{E}$, and the claim follows. ☐

These bounds on the double zeta sums can be used to control additive energies:

Theorem 10.20 (Heath-Brown relation)**.** *[107, (33)] If $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$, then one has*

$$\rho^* \leq 1 - 2\sigma + \frac{1}{2}\max\left(\rho + 1, 2\rho, \frac{5}{4}\rho + \frac{\tau}{2}\right) + \frac{1}{2}\max\left(\rho^* + 1, 4\rho, \frac{3}{4}\rho^* + \rho + \frac{\tau}{2}\right).$$

Recorded in `literature.py` *as:*
`add_lver_heath_brown_1979b1()`

Proof. By Lemma 10.19 we have

$$\rho^* + 2\sigma \leq \kappa + (\max(\rho + 1, 2\rho, 5\rho/4 + \tau/2) + \max(\rho' + 1, 2\rho', 5\rho'/4 + \tau/2))/2 + 1$$

for some $0 \leq \kappa \leq \rho$ with

$$\kappa + \rho' \leq 2\rho$$

$$2\kappa + \rho' \leq \rho^*$$

In particular,

$$2\kappa + 5\rho'/4 \leq 3\rho^*/4 + \rho$$

and the claim follows after moving the κ inside the second maximum and performing some algebra. ☐

Corollary 10.21 (Simplified Heath-Brown relation)**.** *If $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ and $\tau \leq 3/2$, then*

$$\rho^* \leq \max(3\rho + 1 - 2\sigma, \rho + 4 - 4\sigma, 5\rho/2 + (3 - 4\sigma)/2).$$

Recorded in `literature.py` *as:*
`add_lver_heath_brown_1979b2()`

This result essentially appears as [107, Lemma 3].

Proof. Apply the previous result. For $\tau \leq 3/2$ we observe that $5\rho/4 + \tau/2$ is less than $5\rho/4 + 3/4$, which is a convex combination of $\rho + 1$ and 2ρ. Similarly $3\rho^*/4 + \rho + \tau/2$ is less than $3\rho^*/4 + \rho + 3/4$, which is a convex combination of $\rho^* + 1$ and 4ρ. We conclude that

$$\rho^* \leq 1 - 2\sigma + \max(\rho + 1, 2\rho)/2 + \max(\rho^* + 1, 4\rho)/2.$$

Thus ρ^* is less than one of

$$1 - 2\sigma + (\rho + \rho^* + 2)/2, 1 - 2\sigma + (5\rho + 1)/2, 1 - 2\sigma + (2\rho + \rho^* + 1)/2, 1 - 2\sigma + (6\rho)/2;$$

solving for ρ^*, we conclude

$$\rho^* \leq \max(4 - 4\sigma + \rho, (3 - 4\sigma)/2 + 5\rho/2, 3 - 4\sigma + 2\rho, 1 - 2\sigma + 3\rho).$$

But since $\sigma \geq 1/2$, $3 - 4\sigma + 2\rho$ is less than $5/2 - 3\sigma + 2\rho$, which is the mean of $4 - 4\sigma + \rho$ and $1 - 2\sigma + 3\rho$. Thus

$$\rho^* \leq \max(4 - 4\sigma + \rho, (3 - 4\sigma)/2 + 5\rho/2, 1 - 2\sigma + 3\rho),$$

which gives the claim. □

Similarly, using Lemma 10.17 and Lemma 10.19, one has

Theorem 10.22. *If (k, ℓ) be an exponent pair with $k > 0$ and $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$, then*

$$\begin{aligned}\rho^* \leq 1 - 2\sigma + \frac{1}{2}\max\left(\rho + 1, \frac{5}{3}\rho + \frac{\tau}{3}, \frac{2 + 3k + 4\ell}{1 + 2k + 2\ell}\rho + \frac{k + \ell}{1 + 2k + 2\ell}\tau\right) \\ + \frac{1}{2}\max\left(\rho^* + 1, 4\rho, \frac{3}{4}\rho^* + \rho + \frac{\tau}{2}\right).\end{aligned}$$

Implemented at `additive_energy.py` *as:*
`ep_to_lver(eph)`

Proof. By Lemma 10.19 and Lemma 10.17, there exists some $(\sigma, \tau, \rho', (\rho^*)', s') \in \mathcal{E}$ satisfying

$$0 \leq \kappa \leq \rho, \qquad \kappa + \rho' \leq 2\rho, \qquad 2\kappa + \rho' \leq \rho^* \tag{10.9}$$

and

$$\begin{aligned}\rho^* + 2\sigma \leq \kappa + \frac{1}{2}\max\left(\rho + 1, 5\rho/3 + \tau/3, \frac{2 + 3k + 4\ell}{1 + 2k + 2\ell}\rho + \frac{k + \ell}{1 + 2k + 2\ell}\tau\right) \\ + \frac{1}{2}\max\left(\rho' + 1, 2\rho', \frac{5}{4}\rho' + \frac{\tau}{2}\right) + 1.\end{aligned}$$

The result follows by moving the κ term into the second maximum and applying (10.9). □

Lemma 10.23 (Second Heath-Brown relation)**.** *If $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ then*

$$\rho \leq \max(2 - 2\sigma, \rho^*/4 + \max(\tau/4 + k(3 - 4\sigma)/4, k\tau/4 + k(1 - 2\sigma)/2))$$

for any positive integer k.

Recorded in ***literature.py*** *as:*
`add_lver_heath_brown_1979c(K)`

Proof. By definition, there exists a large value pattern $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ with $N > 1$ unbounded, $T = N^{\tau+o(1)}$, $|W| = N^{\rho+o(1)}$, $E_1(W) = N^{\rho^*+o(1)}$ and $S(N, W) = N^{s+o(1)}$. From [107, Lemma 4], we have

$$|W| \ll T^\varepsilon \left(N^{2-2\sigma} + E_1(W)^{1/4}(T^{1/4}N^{k(3-4\sigma)/4} + T^{k/4}N^{k(1-2\sigma)/2})\right)$$

for any fixed $\varepsilon > 0$, giving the claim. □

Lemma 10.24 (Guth-Maynard relation)**.** *If* $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ *then*

$$\rho \leq \max(2 - 2\sigma, 1 - 2\sigma + \max(S_1, S_2, S_3)/3)$$

where S_1, S_2, S_3 *are real numbers with*

$$S_1 \leq -10,$$

$$S_2 \leq \max(2 + 2\rho, \tau + 1 + (2 - 1/k)\rho, 2 + 2\rho + (\tau/2 - 3\rho/4)/k)$$

for any positive integer k *and*

$$S_3 \leq 2\tau + \rho/2 + \rho^*/2$$

and also

$$S_3 \leq \max(2\tau + 3\rho/2, \tau + 1 + \rho/2 + \rho^*/2).$$

Recorded in ***literature.py*** *as:*
`add_lver_guth_maynard_2024a()`

Proof. By definition, there exists a large value pattern $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ with $N > 1$ unbounded, $T = N^{\tau+o(1)}$, $|W| = N^{\rho+o(1)}$, $E_1(W) = N^{\rho^*+o(1)}$ and $S(N, W) = N^{s+o(1)}$. By [88, Proposition 4.6, (5.5)], one may bound

$$|W| \ll N^{2-2\sigma} + N^{1-2\sigma}(S_1 + S_2 + S_3)^{1/3}$$

for three expressions S_1, S_2, S_3 defined after [88, (5.5)]. From [88, Proposition 5.1] we have

$$S_1 \ll T^{-10}.$$

From [88, Proposition 6.1] we have

$$S_2 \ll T^{o(1)}(N^2|W|^2 + TN|W|^{2-1/k} + N^2|W|^2(\frac{T^{1/2}}{|W|^{3/4}})^{1/k}).$$

From [88, Proposition 8.1] we have

$$S_3 \ll T^{2+o(1)}|W|^{1/2}E_1(W)^{1/2}$$

while from [88, Proposition 10.1] we have

$$S_3 \ll T^{2+o(1)}|W|^{3/2} + T^{1+o(1)}N|W|^{1/2}E_1(W)^{1/2}.$$

Combining all these bounds, we obtain the claim. □

Lemma 10.25 (Second Guth-Maynard relation)**.** *[88, Lemma 1.7] If* $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ *then*

$$\rho^* \leq \rho + s - 2\sigma.$$

In particular, from Lemma 10.18 we see for $\tau \leq 3/2$ *that*

$$\rho^* \leq \max(3\rho + 1 - 2\sigma, 2\rho + 2 - 2\sigma).$$

Recorded in `literature.py` *as:*
`add_lver_guth_maynard_2024b()`

Proof. By definition, we can find a large value pattern $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ with $N > 1$ unbounded, $T = N^{\tau+o(1)}$, $V = N^{\sigma+o(1)}$, $|W| = N^{\rho+o(1)}$, $E_1(W) = N^{\rho^*+o(1)}$, and $S(N, W) = N^{s+o(1)}$. From (10.5) one has

$$V^2 E_1(W) \ll T^{o(1)}|W|S(N, W) + T^{-50}.$$

Since W is non-empty, $E_1(W) \geq 1$, and $V \gg 1$, so the T^{-50} error can be discarded. The claim then follows. □

Lemma 10.26 (Third Guth-Maynard relation)**.** *If* $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ *and* $1 \leq \tau \leq 4/3$*, then*

$$\rho^* \leq \max(\rho + 4 - 4\sigma, 21\rho/8 + \tau/4 + 1 - 2\sigma, 3\rho + 1 - 2\sigma).$$

Recorded in `literature.py` *as:*
`add_lver_guth_maynard_2024c()`

Proof. By definition, there exists a large value pattern $(N, T, V, (a_n)_{n\in[N,2N]}, J, W)$ with $N > 1$ unbounded, $T = N^{\tau+o(1)}$, $|W| = N^{\rho+o(1)}$, $E_1(W) = N^{\rho^*+o(1)}$ and $S(N, W) = N^{s+o(1)}$. Applying [88, Proposition 11.1], we conclude that

$$E_1(W) \ll T^{o(1)}(|W|N^{4-4\sigma} + |W|^{21/8}T^{1/4}N^{1-2\sigma} + |W|^3N^{1-2\sigma}),$$

giving the claim. □

We can put this all together to prove the Guth–Maynard large values theorem.

Theorem 10.27 (Guth–Maynard large values theorem)**.** *[88, Theorem 1.1] One has*

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, 18/5 - 4\sigma, \tau + 12/5 - 4\sigma).$$

Recorded in `literature.py` *as:*
`add_guth_maynard_large_values_estimate()`
Derived in `derived.py` *as:*
`prove_guth_maynard_large_values_theorem()`
`prove_guth_maynard_lvt_from_intermediate_lemmas()`

Proof. For $\sigma \le 7/10$ this follows from Lemma 7.9, and for $\sigma \ge 8/10$ it follows from Lemma 7.12. Thus we may assume that $7/10 \le \sigma \le 8/10$. By subdivision (Lemma 7.7) it then suffices to treat the case $\tau = 6/5$, that is to say to show that

$$\rho \le 18/5 - 4\sigma$$

whenever $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ with $\tau = 6/5$ and $7/10 \le \sigma \le 8/10$.
Applying Lemma 10.24 and discarding the very negative S_1 term, we have

$$\rho \le \max(2 - 2\sigma, 1 - 2\sigma + \max(S_2, S_3)/3)$$

where S_2, S_3 are real numbers with

$$S_2 \le \max(2 + 2\rho, \tau + 1 + (2 - 1/k)\rho, 2 + 2\rho + (\tau/2 - 3\rho/4)/k)$$

for any positive integer k and

$$S_3 \le 2\tau + \rho/2 + \rho^*/2$$

and also

$$S_3 \le \max(2\tau + 3\rho/2, \tau + 1 + \rho/2 + \rho^*/2).$$

From the latter bound and Lemma 10.26, one has

$$S_3 \le \max(2\tau + 3\rho/2, \tau + \rho + 3 - 2\sigma, \tau + 2\rho + 3/2 - \sigma, 9\tau/8 + 29\rho/16 + 3/2 - \sigma).$$

Inserting this and the S_2 bound (with $k = 4$) into the bound for ρ and simplifying (using $\tau = 6/5$), we eventually obtain the desired bound $\rho \le 18/5 - 4\sigma$. ☐

We also record a variant of that theorem from the same paper:

Theorem 10.28 (Additional Guth–Maynard large values estimate)**.** *For any* $1/2 < \sigma < 1$, $\tau \ge 1$, *and natural number* k *one has*

$$\mathrm{LV}(\sigma, \tau) \le \max\Big(2 - 2\sigma, 5 - 6\sigma, (4 - 6\sigma)\frac{k}{k+1} + \frac{k}{k+1}\tau, (5 - 6\sigma)\frac{4k}{4k+3} + \frac{2}{4k+3}\tau, 2 - 4\sigma + \frac{4}{3}\tau, 3 - 4\sigma + \frac{\tau}{2}, \frac{9}{2} - 7\sigma + \tau, \frac{72 - 112\sigma}{19} + \frac{18}{19}\tau\Big). \tag{10.10}$$

If one specializes to the case $\sigma \ge 7/10$ *and* $1 \le \tau \le 6/5$, *one then has*

$$\mathrm{LV}(\sigma, \tau) \le \max\Big(2 - 2\sigma, 3 - 4\sigma + \frac{\tau}{2}, (4 - 6\sigma)\frac{k}{k+1} + \frac{k}{k+1}\tau, (5 - 6\sigma)\frac{4k}{4k+3} + \frac{2}{4k+3}\tau\Big). \tag{10.11}$$

and also

$$\mathrm{LV}(\sigma, \tau) \le \max\Big(2 - 2\sigma, 3 - 4\sigma + \frac{\tau}{2}, \frac{46 - 60\sigma}{5} + \frac{30\sigma - 21}{5}\tau\Big). \tag{10.12}$$

Recorded in `literature.py` *as:*
`add_guth_maynard_large_values_estimate2(Constants.LARGE_VALUES_TRUNCATION)`
Derived in `derived.py` *as:*
`prove_guth_maynard_intermediate_lvt()`
`prove_guth_maynard_intermediate_lvt2()`

Proof. (Sketch) The first bound (10.10) is [88, (12.1)], and is proven by the same methods used to prove Theorem 10.27. The second and third bounds (10.11), (10.12) are in [88, Proposition 12.1]. The second bound (10.11) follows from the $k = 3$ case of Theorem 7.16 when $\sigma \geq 39/50$, and from (10.10) together with Theorems 7.9, 7.12 for $\sigma < 39/50$. The third bound (10.12) follows from Theorems 7.9, 7.12 when $\tau \leq 4 - 4\sigma$, and for $\tau > 4 - 4\sigma$ the bound follows from (10.11) after optimizing in k (see the proof of [88, Proposition 12.1] for details). ☐

Now we turn to another application of double zeta sums to large value theorems.

Theorem 10.29 (Bourgain large values theorem)**.** *[21] Let* $1/2 < \sigma < 1$ *and* $\tau > 0$*, and let* $\rho := \mathrm{LV}(\sigma, \tau)$*. Let* $\alpha_1, \alpha_2 \geq 0$ *be real numbers. Then either*

$$\rho \leq \max(\alpha_2 + 2 - 2\sigma, -\alpha_2 + 2\tau + 4 - 8\sigma, -2\alpha_1 + \tau + 12 - 16\sigma) \tag{10.13}$$

or else there exists $s \geq 0$ *such that*

$$\begin{gathered}\frac{1}{2}\max(\rho + 2, 2\rho + 1, 5\rho/4 + \tau/2 + 1) + \frac{1}{2}\max(s + 2, 2s + 1, 5s/4 + \tau/2 + 1) \geq \\ \max(-2\alpha_1 + 2\sigma + s + \rho, -\alpha_1 - \alpha_2/2 + 2\sigma + s/2 + 3\rho/2).\end{gathered} \tag{10.14}$$

Proof. By Definition 7.2, we can find a large value pattern $(N, T, V, (a_n)_{n\in[N,2N]}, J, R)$ with $N > 1$ unbounded, $N \geq 1$, $T = N^{\tau+o(1)}$, $|R| = N^{\rho+o(1)}$, $V = N^{\sigma+o(1)}$; we use R here instead of W to be consistent with the notation from [21]. Now set $\delta_1 := N^{-\alpha_1}$, $\delta_2 := N^{-\alpha_2}$. From [21, (4.41), (4.42)], one has the inequality

$$|R| \leq |R^{(1)}| + |R^{(2)}|$$

for certain sets $R^{(1)}$ and $R^{(2)}$ with the former set obeying the bound

$$|R^{(1)}| \ll \delta_2^{-1} N^2 V^{-2} + \delta_2 T^2 N^4 V^{-8} + \delta_1^2 T N^{12} V^{-16}.$$

Hence, we either have

$$|R| \ll \delta_2^{-1} N^2 V^{-2} + \delta_2 T^2 N^4 V^{-8} + \delta_1^2 T N^{12} V^{-16}$$

which implies (10.13), or else

$$|R| \ll |R^{(2)}|. \tag{10.15}$$

Henceforth we assume that (10.15) holds. From [21, (4.53), (4.54)] we may upper bound

$$T^{-\varepsilon}\delta'(\delta'')^2 V^2 |S||R^{(2)}| + T^{-\varepsilon}\delta_1 V^2 |S|^{1/2} \sum_{\alpha} |R_\alpha|^{3/2} \tag{10.16}$$

by

$$\ll T^{\varepsilon} S(N, R^{(2)})^{1/2} S(N, S)^{1/2} \tag{10.17}$$

for arbitrarily small fixed ε, some $\delta', \delta'' > 0$ with $\delta' > T^{-\varepsilon}(\delta_1/\delta'')^2$ (see [21, (4.37)]), some set S (which will be non-empty by [21, (4.47)]), and some sets R_α defined in [21, (4.39)], where the double zeta sums $S(N, W)$ are defined in (10.2). Applying Lemma 10.18, the latter expression is bounded by

$$\ll T^{\varepsilon}(|R|N^2 + |R|^2 N + |R|^{5/4} T^{1/2} N)^{1/2} (|S|N^2 + |S|^2 N + |S|^{5/4} T^{1/2} N)^{1/2}.$$

Meanwhile, from [21, (4.57)], the expression (10.16) is bounded from below by

$$\gg T^{-2\varepsilon}(\delta_1^2 V^2|S||R| + \delta_1\delta_2^{1/2} V^2 |S|^{1/2}|R|^{3/2}).$$

After passing to a subsequence, we can ensure that $|S| = N^{s+o(1)}$ for some $s > 0$. Combining these bounds and writing all expressions as powers of N, we obtain the claim (after sending $\varepsilon \to 0$). □

Corollary 10.30 (Bourgain large values theorem, simplified version)**.** *[21, Lemma 4.60] Let the notation be as above, but additionally assume $\rho \leq \min(1, 4-2\tau)$. Then*

$$\rho \leq \max(\alpha_2+2-2\sigma, \alpha_1+\alpha_2/2+2-2\sigma, -\alpha_2+2\tau+4-8\sigma, -2\alpha_1+\tau+12-16\sigma, 4\alpha_1+2+\max(1, 2\tau-2)-4\sigma).$$

In [21] this bound is only established in the case $\tau \leq 3/2$ (in which case the condition on ρ simplifies to $\rho \leq 1$, and the final term $4\alpha_1+2+\max(1, 2\tau-2)-4\sigma$ simplifies to $4\alpha_1+3-4\sigma$), but the argument extends to the $\tau > 3/2$ case without significant difficulty.

Proof. With $\rho \leq \min(1, 4-2\tau)$, $5\rho/4+\tau/2+1$ and $2\rho+1$ are both bounded by $\rho+2$, hence

$$\max(\rho+2, 2\rho+1, 5\rho/4+\tau/2+1) = \rho+2.$$

Furthermore, $5s/4+\tau/2+1$ is a convex combination of $s+2$ and $2s+2\tau-2$, hence

$$\max(s+2, 2s+1, 5s/4+\tau/2+1) \leq \max(s+2, 2s+\max(1, 2\tau-2)).$$

Thus (10.14) simplifies to

$$(\rho+2)/2+\max(s+2, 2s+\max(1, 2\tau-2))/2 \geq \max(-2\alpha_1+2\sigma+s+\rho, -\alpha_1-\alpha_2/2+2\sigma+s/2+3\rho/2).$$

Thus either

$$(\rho+2)/2+(s+2)/2 \geq -\alpha_1-\alpha_2/2+2\sigma+s/2+3\rho/2$$

or

$$(\rho+2)/2+(2s+\max(1, 2\tau-2))/2 \geq -2\alpha_1+2\sigma+s+\rho.$$

In both cases we may eliminate s and solve for ρ to obtain

$$\rho \leq \alpha_1+\alpha_2/2+2-2\sigma$$

or

$$\rho \leq 4\alpha_1+2+\max(1, 2\tau-2)-4\sigma,$$

giving the claim. □

With the aid of computer assistance, one is able to produce an optimized version of the above large values theorem. We have

Corollary 10.31 (Bourgain large values theorem, optimized version)**.** *For each row $(\rho_0, \alpha_1, \alpha_2, \mathcal{S})$ of Table 10.1, one has*

$$\rho \leq \rho_0(\sigma, \tau), \qquad (\sigma, \tau) \in \mathcal{S}.$$

Recorded in `literature.py` *as:*
`add_bourgain_large_values_estimate()`

Derived in `derived.py` *as:*
`prove_bourgain_large_values_theorem()`

Proof. Follows from substituting the specified values of α_1 and α_2 and a routine calculation. $\square$

The preprint of Kerr [165] contains additional large value theorems:

Lemma 10.32 (Kerr large values theorem)**.**

(i) [165, Theorem 2] Let $3/4 < \sigma \leq 1$, $0 \leq \tau \leq 3/2$, and $0 \leq \rho \leq \mathrm{LV}(\sigma, \tau), 1$ be fixed. Then for any fixed $\alpha \geq 0$, one has

$$\rho \leq \max(2 - 2\sigma + \alpha, 2\tau + 4 - 8\sigma - \alpha, \tau/3 + 16/3 - 20\sigma/3 + \alpha/3, 2\tau/3 + 9 - 12\sigma).$$

(ii) [165, Theorem 3] Under the same hypotheses as (i), we have for any fixed integer $k \geq 2$ obeying $-\alpha < 4k\sigma - (\tau + 3k - 1)$ and $-\alpha < 1 + \frac{1}{k-1} - \tau$ that

$$\rho \leq \max(2 - 2\sigma + \alpha, \tau/3 + (3k + 4)/3 - (4k + 4)\sigma/3 + \alpha/3).$$

(iii) [165, Theorem 4] If $25/32 < \sigma \leq 1$, $\tau \geq 0$, and $0 \leq \rho \leq \mathrm{LV}(\sigma, \tau), 1, 4 - 2\tau$ are fixed, then for any fixed α with

$$26 - 32\sigma - \tau < -\alpha < 16\sigma - 11 - \tau$$

one has

$$\rho \leq \max(2 - 2\sigma + \alpha, 2\tau + 4 - 8\sigma - \alpha, -\tau + 8 - 8\sigma + 2\alpha, 10 - 12\sigma + 2\alpha/3).$$

(iv) [165, Theorem 5] If $1/2 < \sigma \leq 1$, $0 \leq \tau \leq 3/2$, and $0 \leq \rho \leq \mathrm{LV}(\sigma, \tau), 1$ are fixed, then for any fixed α with

$$-\alpha < -\tau + 8\sigma - 5$$

one has

$$\rho \leq \max(2 - 2\sigma + \alpha, 4\tau/3 + 23/3 - 12\sigma - 2\alpha/3, 2\tau/3 + 14/3 - 20/3).$$

Table 10.1: Bounds on LV(σ,τ) for $1/2 \le \sigma \le 1$ and $\tau \ge 1$

$\rho_0(\sigma,\tau)$	α_1	α_2	$\mathcal{S}$
$\frac{16}{3} - \frac{20}{3}\sigma + \frac{1}{3}\tau$	$\frac{10}{3} - \frac{14}{3}\sigma + \frac{\tau}{3}$	0	$-1+\tau \ge 0,$ $10-14\sigma+\tau \ge 0,$ $4+4\sigma-5\tau \ge 0,$ $-11+16\sigma-\tau \ge 0.$
$5 - 7\sigma + \frac{3}{4}\tau$	$\frac{7}{2} - \frac{9}{2}\sigma + \frac{\tau}{8}$	$-1-\sigma+\frac{5}{4}\tau$	$8-8\sigma-\tau \ge 0,$ $-16+20\sigma+\dfrac{1}{3}\tau \ge 0,$ $-6+10\sigma-\dfrac{7}{6}\tau \ge 0,$ $-4-4\sigma+5\tau \ge 0.$
$3-5\sigma+\tau$	$\frac{9}{2} - \frac{11}{2}\sigma$	$1-3\sigma+\tau$	$-8+8\sigma+\tau \ge 0,$ $2-6\sigma+2\tau \ge 0,$ $-10+14\sigma-\dfrac{2}{3}\tau \ge 0,$ $6-2\sigma-2\tau \ge 0.$
$-4\sigma+2\tau$	0	$1-3\sigma+\tau$	$-6+2\sigma+2\tau \ge 0,$ $-12+12\sigma+\tau \ge 0,$ $1-\sigma \ge 0.$
$8-12\sigma+\frac{4}{3}\tau$	$2-2\sigma-\frac{1}{6}\tau$	$6-10\sigma+\frac{4}{3}\tau$	$15-21\sigma+\tau \ge 0,$ $12-12\sigma-\tau \ge 0,$ $-\dfrac{3}{2}+\tau \ge 0,$ $6-10\sigma+\dfrac{7}{6}\tau \ge 0.$
$2-2\sigma$	0	0	$1-\sigma \ge 0,$ $-10+14\sigma-\tau \ge 0,$ $-1+\tau \ge 0,$ $-2+6\sigma-2\tau \ge 0.$
$9-12\sigma+\frac{2}{3}\tau$	$\frac{3}{2}-2\sigma+\frac{1}{6}\tau$	$11-16\sigma+\tau$	$\dfrac{3}{2}-\tau \ge 0,$ $-\dfrac{1}{2}+\sigma \ge 0,$ $-1+\tau \ge 0,$ $11-16\sigma+\tau \ge 0,$ $16-20\sigma-\dfrac{1}{3}\tau \ge 0.$

Chapter 11

Zero density theorems

Definition 11.1 (Zero density exponents)**.** *For $\sigma \in \mathbf{R}$ and $T > 0$, let $N(\sigma, T)$ denote the number of zeroes ρ of the Riemann zeta function with $\mathrm{Re}(\rho) \geq \sigma$ and $|\mathrm{Im}(\rho)| \leq T$.*
If $1/2 \leq \sigma < 1$ is fixed, we define the zero density exponent $\mathrm{A}(\sigma) \in [-\infty, \infty)$ to be the infimum of all (fixed) exponents A for which one has

$$N(\sigma - \delta, T) \ll T^{A(1-\sigma)+o(1)}$$

whenever T is unbounded and $\delta > 0$ is infinitesimal.

The shift by δ is for technical convenience, it allows for $\mathrm{A}(\sigma)$ to control (very slightly) the zeroes to the left of $\mathrm{Re}s = \sigma$. In non-asymptotic terms: $\mathrm{A}(\sigma)$ is the infimum of all A such that for every $\varepsilon > 0$ there exists $C, \delta > 0$ such that

$$N(\sigma - \delta, T) \leq CT^{A(1-\sigma)+\varepsilon}$$

whenever $T \geq C$.

Lemma 11.2 (Basic properties of A)**.** *(i) $\sigma \mapsto (1-\sigma)\mathrm{A}(\sigma)$ is non-increasing and left-continuous, with $\mathrm{A}(1/2) = 2$.*

(ii) If the Riemann hypothesis holds, then $\mathrm{A}(\sigma) = -\infty$ for all $1/2 < \sigma \leq 1$.

Proof. The claim (i) is clear using the Riemann-von Mangoldt formula [144, Theorem 1.7] and the functional equation. The claim (ii) is also clear. □

Remark 11.3. *One can ask what happens if one omits the δ shift. Thus, define $\mathrm{A}_0(\sigma)$ to be the infimum of all fixed exponents A for which $N(\sigma, T) \ll T^{A(1-\sigma)+o(1)}$ for unbounded T. Then it is not difficult to see that*

$$\lim_{\sigma' \to \sigma^+} \mathrm{A}(\sigma) \leq \mathrm{A}_0(\sigma) \leq \mathrm{A}(\sigma)$$

for any fixed $1/2 < \sigma < 1$; thus A_0 is basically the same exponent at A, except possibly at jump discontinuities of the left-continuous function A, in which case it could theoretically take on a different value. (But we do not expect such discontinuities to actually exist.) Thus there is not a major difference between $\mathrm{A}(\sigma)$ and $\mathrm{A}_0(\sigma)$, but the former has some very slight technical advantages (such as the aforementioned left continuity).

The quantity $\|\mathrm{A}\|_\infty := \sup_{1/2\le\sigma<1} \mathrm{A}(\sigma)$ is of particular importance to the theory of primes in short intervals; see Section 15. From Lemma 11.2 we have $\|\mathrm{A}\|_\infty \ge 2$. It is conjectured that this is an equality.

Conjecture 11.4 (Density hypothesis)**.** *One has* $\|\mathrm{A}\|_\infty = 2$. *Equivalently,* $\mathrm{A}(\sigma) \le 2$ *for all* $1/2 \le \sigma < 1$.

Indeed, the Riemann hypothesis implies the stronger assertion that $\mathrm{A}(\sigma) = -\infty$ for all $12 < \sigma < 1$. However, for many applications to the prime numbers in short intervals, the density hypothesis is almost as powerful; see Section 15.
Upper bounds on $\mathrm{A}(\sigma)$ can be obtained from large value theorems via the following relation.

Lemma 11.5 (Zero density from large values)**.** *Let* $1/2 < \sigma < 1$. *Then*

$$\mathrm{A}(\sigma)(1-\sigma) \le \max(\sup_{\tau\ge 2} \mathrm{LV}_\zeta(\sigma,\tau)/\tau, \limsup_{\tau\to\infty} \mathrm{LV}(\sigma,\tau)/\tau).$$

Proof. Write the right-hand side as B, then $B \ge 0$ (from Lemma 7.4(iii)) and we have

$$\mathrm{LV}_\zeta(\sigma,\tau) \le B\tau \tag{11.1}$$

for all $\tau \ge 1$, and

$$\mathrm{LV}(\sigma,\tau) \le (B+\varepsilon)\tau \tag{11.2}$$

whenever $\varepsilon > 0$ and τ is sufficiently large depending on ε (and σ). It would suffice to show, for any $\varepsilon > 0$, that $N(\sigma - o(1), T) \ll T^{B+O(\varepsilon)+o(1)}$ as $T \to \infty$.
By dyadic decomposition, it suffices to show for large T that the number of zeroes with real part at least $\sigma - o(1)$ and imaginary part in $[T, 2T]$ is $\ll T^{B+O(\varepsilon)+o(1)}$. From the Riemann-von Mangoldt theorem, there are only $O(\log T)$ zeroes whose imaginary part is within $O(1)$ of a specified ordinate $t \in [T, 2T]$, so it suffices to show that given some zeroes $\sigma_r + it_r$, $r = 1, \ldots, R$ with $\sigma - o(1) \le \sigma_r < 1$ and $t_r \in [T, 2T]$ 1-separated, that $R \ll T^{B+O(\varepsilon)+o(1)}$.
Suppose that one has a zero $\sigma_r + it_r$ of this form. Then by a standard approximation to the zeta function [144, Theorem 1.8], one has

$$\sum_{n\le T} \frac{1}{n^{\sigma_r+it_r}} \ll T^{-1/2}.$$

Let $0 < \delta_1 < \varepsilon$ be a small quantity (independent of T) to be chosen later, and let $0 < \delta_2 < \delta_1$ be sufficiently small depending on δ_1, δ_2. By the triangle inequality, and refining the sequence t_r by a factor of at most 2, we either have

$$\left| \sum_{T^{\delta_1}\le n\le T} \frac{1}{n^{\sigma_r+it_r}} \right| \gg T^{-\delta_2}$$

for all r, or

$$\sum_{n\le T^{\delta_1}} \frac{1}{n^{\sigma_r+it_r}} \ll T^{-\delta_2} \tag{11.3}$$

for all r.
Suppose we are in the former ("Type I") case, we perform a smooth partition of unity, and conclude that

$$\left| \sum_{T^{\delta_1}\le n\le T} \frac{\psi(n/N)}{n^{\sigma_r+it_r}} \right| \gg T^{-\delta_2-o(1)}$$

for some fixed bump function ψ supported on $[1/2, 1]$, and some $T^{\delta_1} \ll N \ll T$.
We divide into several cases depending on the size of N. First suppose that $N \ll T^{1/2}$. The variable n is restricted to the interval $I := [\max(N/2, T^{\delta_1}), N]$. We have

$$\left|\sum_{n\in I} \psi(n/N)(n/N)^{-\sigma_r} n^{-it_r}\right| \gg N^{\sigma} T^{-\delta_2 - o(1)}.$$

Performing a Fourier expansion of $\psi(n/N)(n/N)^{-\sigma_r}$ in $\log n$ and using the triangle inequality, we can bound

$$\sum_{n\in I} \psi(n/N)(n/N)^{-\sigma_r} n^{-it_r} \ll_A \int_{\mathbf{R}} \left|\sum_{n\in I} \frac{1}{n^{it}}\right| (1 + |t - t_r|)^{-A}\, dt$$

for any $A > 0$, so by the triangle inequality we conclude that

$$\left|\sum_{n\in I} n^{-it'_r}\right| \gg N^{\sigma} T^{-\delta_2 - o(1)}$$

for some $t'_r = t_r + O(T^{o(1)})$. By refining the t_r by a factor of $T^{o(1)}$ if necessary, we may assume that the t'_r are 1-separated, and by passing to a subsequence we may assume that $T = N^{\tau + o(1)}$ for some $2 \le \tau \le 1/\delta_1$, then we conclude that

$$\left|\sum_{n\in I} \frac{1}{n^{it'_r}}\right| \gg N^{\sigma - \delta_2/\delta_1 + o(1)}$$

for all remaining r. By Definition 8.1 we then have (for δ_2 small enough)

$$R \ll N^{\mathrm{LV}_\zeta(\sigma,\tau) + \varepsilon + o(1)} \ll T^{\mathrm{LV}_\zeta(\sigma,\tau)/\tau + \varepsilon + o(1)}$$

and the claim follows in this case from (11.1).
In the case $N \asymp T$, a standard application of the Euler–Maclaurin formula (see e.g., [277, (2.1.2)]) yields

$$\sum_{T^{\delta_1} \le n \le T} \frac{\psi(n/N)}{n^{\sigma_r + it_r}} \ll T^{-\sigma_r}$$

which leads to a contradiction. So the only remaining case is when $T^{1/2} \ll N \ll o(T)$. Here we can ignore the cutoffs on n and obtain

$$\left|\sum_{n} \psi(n/N)(n/N)^{-\sigma_r} n^{-it_r}\right| \gg N^{\sigma} T^{-\delta_2 - o(1)}.$$

Applying the van der Corput B-process (see, e.g., [149, §8.3]) or the approximate functional equation we have

$$\sum_{n} \psi(n/N)(n/N)^{-\sigma_r} n^{-it_r} = e\Big(\frac{t_r}{2\pi}\log\frac{t_r}{2\pi} - \frac{t_r}{2\pi} + \frac{1}{8}\Big) \sum_{m} \psi(2\pi t_r/mN)(2\pi t_r/mN)^{-\sigma_r} m^{it_r} (2\pi m^2/t_r)^{-1/2} + O(T^{o(1)})$$

and thus

$$\left|\sum_{m} \psi(2\pi t_r/mN)(2\pi t_r/mN)^{1-\sigma_r} m^{-it_r}\right| \gg M^{1/2} N^{\sigma - 1/2} T^{-\delta_2 - o(1)};$$

where $M := 2\pi T/N \ll N^{1/2}$. In particular

$$\sum_{m \in [M/10, 10M]} \psi(2\pi t_r/mN)(2\pi t_r/mN)^{1-\sigma_r} m^{-it_r} \gg M^\sigma T^{-\delta_2 - o(1)};$$

since $N \gg T^{1/2}$ and $\sigma \geq 1/2$. Performing a Fourier expansion as before, we conclude that

$$\sum_{m \in [M/10, 10M]} m^{-it'_r} \ll M^\sigma T^{-\delta_2 - o(1)}$$

for some $t'_r = t_r + O(T^{o(1)})$, and one can argue as in the $N \ll T^{1/2}$ case (partitioning $[M/10, 10M]$ into $O(1)$ intervals each contained in some $[M', 2M']$ with $M' \ll T^{1/2}$).
Now suppose instead we are in the latter ("Type II") case (11.3). We multiply both sides of (11.3) by the mollifier $\sum_{m \leq T^{\delta_2/2}} \frac{1}{m^{\sigma_r + it_r}}$ to obtain

$$\left| 1 + \sum_{T^{\delta_2/2} \leq n \leq T^{\delta_1 + \delta_2/2}} \frac{a_n}{n^{\sigma_r + it_r}} \right| = o(1)$$

where a_n is some sequence with $a_n \ll T^{o(1)}$. By dyadic decomposition and the pigeonhole principle, and refining the t_r by a factor of $O(T^{o(1)})$ as needed, we can then find an interval I in $[N, 2N]$ with $T^{\delta_2/2} \ll N \ll T^{\delta_1 + \delta_2/2}$ such that

$$\left| \sum_{n \in I} \frac{a_n}{n^{\sigma_r + it_r}} \right| \gg T^{-o(1)}$$

and hence by Fourier expansion of $\frac{1}{n^{\sigma_r}}$ in $\log n$

$$\left| \sum_{n \in I} \frac{a_n}{n^{it'_r}} \right| \gg N^{\sigma_r} T^{-o(1)}$$

for some $t'_r = t_r + O(T^{o(1)})$; by refining the t_r by a further factor of $T^{o(1)}$ we may assume that the t'_r are also 1-separated; we can also pigeonhole so that $T = N^{\tau + o(1)}$ for some $\frac{1}{\delta_1 + \delta_2/2} \leq \tau \leq \frac{1}{\delta_2/2}$. Applying Lemma 7.3, we conclude that

$$R \ll N^{\mathrm{LV}(\sigma,\tau) + o(1)} = T^{\mathrm{LV}(\sigma,\tau)/\tau + o(1)}$$

and the claim follows in this case from (11.2). □

Recently, a partial converse to the above lemma was established:

Lemma 11.6 (Large values from zero density)**.** *[206, Theorem 1.2] If $\tau > 0$ and $1/2 \leq \sigma \leq 1$ are fixed, then*

$$\mathrm{LV}_\zeta(\sigma, \tau)/\tau \leq \max\left(\frac{1}{2}, \sup_{\sigma \leq \sigma' \leq 1} \mathrm{A}(\sigma')(1 - \sigma') + \frac{\sigma' - \sigma}{2} \right).$$

Proof. Let $N \geq 1$ be unbounded, $T = N^{\tau + o(1)}$, and $I \subset [N, 2N]$ be an interval, and $t_1, \dots, t_R \in [T, 2T]$ be 1-separated with

$$\left| \sum_{n \in I} \frac{1}{n^{it_r}} \right| \gg N^{\sigma - o(1)}$$

uniformly for all r. By [206, Theorem 1.2], we have for any fixed $\delta > 0$ that

$$R \ll T^{\delta} \sup_{\sigma-\delta \leq \sigma' \leq 1} T^{\frac{\sigma'-\sigma}{2}} N(\sigma', O(T)) + T^{\frac{1-\sigma}{2}+\delta}.$$

Using Definition 11.1, we conclude that

$$R \ll T^{\max(\frac{1}{2}, \sup_{\sigma-\delta \leq \sigma' \leq 1} \mathrm{A}(\sigma')(1-\sigma') + \frac{\sigma'-\sigma}{2}) + O(\delta)}$$

and thus

$$\mathrm{LV}_{\zeta}(\sigma, \tau) \leq \tau \max(\frac{1}{2}, \sup_{\sigma-\varepsilon \leq \sigma' \leq 1} \mathrm{A}(\sigma')(1-\sigma') + \frac{\sigma'-\sigma}{2}) + O(\delta).$$

Here the implied constant in the $O()$ notation is understood to be uniform in δ. Letting δ go to zero, and using left-continuity of A, we obtain the claim. □

The suprema in Lemma 11.5 require unbounded values of τ, but thanks to the ability to raise to a power, we can reduce to a bounded range of τ. Here is a basic such reduction, suited for machine-assisted proofs:

Corollary 11.7. *Let $1/2 < \sigma < 1$ and $\tau_0 > 0$. Then*

$$\mathrm{A}(\sigma)(1-\sigma) \leq \max\left(\sup_{2 \leq \tau < \tau_0} \mathrm{LV}_{\zeta}(\sigma, \tau)/\tau, \sup_{\tau_0 \leq \tau \leq 2\tau_0} \mathrm{LV}(\sigma, \tau)/\tau \right)$$

with the convention that the first supremum is $-\infty$ if it is vacuous (i.e., if $\tau_0 < 2$).

Implemented at ***zero_density_estimate.py*** *as:*
`lv_zlv_to_zd(hypotheses, sigma_interval, tau0)`

Proof. Denote the right-hand side by B, thus

$$\mathrm{LV}(\sigma, \tau) \leq B\tau$$

for all $\tau_0 \leq \tau \leq 2\tau_0$, and

$$\mathrm{LV}_{\zeta}(\sigma, \tau) \leq B\tau \tag{11.4}$$

whenever $2 \leq \tau < \tau_0$. From Lemma 7.8 we then have

$$\mathrm{LV}(\sigma, \tau) \leq B\tau$$

for all $k\tau_0 \leq \tau \leq 2k\tau_0$ and natural numbers k. Note that the intervals $[k\tau_0, 2k\tau_0]$ cover all of $[\tau_0, \infty)$, hence we have

$$\mathrm{LV}(\sigma, \tau) \leq B\tau$$

for all $\tau \geq \tau_0$. In particular

$$\limsup_{\tau \to \infty} \mathrm{LV}(\sigma, \tau)/\tau \leq B.$$

Also, combining the previous estimate with (11.4) using Lemma 8.3(iii) we have

$$\mathrm{LV}_{\zeta}(\sigma, \tau) \leq B\tau \tag{11.5}$$

for all $\tau \geq 2$. By Lemma 8.3(iv), this implies that

$$\mathrm{LV}_\zeta\left(\frac{1}{2} + \frac{1}{\tau - 1}(\sigma - \frac{1}{2}), \frac{\tau}{\tau - 1}\right) \leq B\frac{\tau}{\tau - 1}$$

for $\tau \geq 2$. Thus

$$\sup_{\tau \geq 2} \frac{\mathrm{LV}_\zeta(\sigma, \tau)}{\tau} \leq B.$$

The claim now follows from Lemma 11.5. □

For machine assisted proofs, one can simply take τ_0 to be a sufficiently large quantity, e.g., $\tau_0 = 3$ for σ not too close to 1, and larger for σ approaching 1, to recover the full power of Lemma 11.5. However, the amount of case analysis required increases with τ_0. For human written proofs, the following version of Corollary 11.7 is more convenient:

Corollary 11.8. *Let $1/2 < \sigma < 1$ and $\tau_0 > 0$. Then*

$$\mathrm{A}(\sigma)(1 - \sigma) \leq \max\left(\sup_{2 \leq \tau < 4\tau_0/3} \mathrm{LV}_\zeta(\sigma, \tau)/\tau, \sup_{2\tau_0/3 \leq \tau \leq \tau_0} \mathrm{LV}(\sigma, \tau)/\tau\right).$$

Implemented at ***zero_density_estimate.py*** *as:*
`lv_zlv_to_zd2(hypotheses, sigma_interval, tau0)`

Proof. Applying Corollary 11.7 with τ replaced by $4\tau_0/3$, it suffices to show that

$$\sup_{4\tau_0/3 \leq \tau \leq 8\tau_0/3} \mathrm{LV}(\sigma, \tau)/\tau \leq \sup_{2\tau_0/3 \leq \tau \leq \tau_0} \mathrm{LV}(\sigma, \tau)/\tau.$$

But this follows from Lemma 7.8, since the intervals $[2k\tau_0/3, k\tau_0]$ for $k = 2, 3$ cover all of $[4\tau_0/3, 8\tau_0/3]$. □

The following special case of the above corollary is frequently used in practice to assist with the human readability of zero density proofs:

Corollary 11.9. *Let $1/2 < \sigma < 1$ and $\tau_0 > 0$. Suppose that one has the bounds*

$$\mathrm{LV}(\sigma, \tau) \leq (3 - 3\sigma)\frac{\tau}{\tau_0} \tag{11.6}$$

for $2\tau_0/3 \leq \tau \leq \tau_0$, and

$$\mathrm{LV}_\zeta(\sigma, \tau) \leq (3 - 3\sigma)\frac{\tau}{\tau_0} \tag{11.7}$$

for $2 \leq \tau < 4\tau_0/3$. Then $\mathrm{A}(\sigma) \leq \frac{3}{\tau_0}$.

The reason why this particular special case is convenient is because the inequality

$$2 - 2\sigma \leq (3 - 3\sigma)\frac{\tau}{\tau_0} \tag{11.8}$$

obviously holds for $\tau \geq 2\tau_0/3$. That is to say, we automatically verify (11.6) in regimes where the Montgomery conjecture holds. In fact, we can do a bit better, thanks to subdivision:

Corollary 11.10. *Let $1/2 < \sigma < 1$ and $\tau_0 > 0$. Suppose that one has the bound* (11.7) *for $2 \le \tau < 4\tau_0/3$, and the Montgomery conjecture $\mathrm{LV}(\sigma,\tau) \le 2-2\sigma$ whenever $0 \le \tau \le \tau_0+\sigma-1$. Then $\mathrm{A}(\sigma) \le \frac{3}{\tau_0}$.*

Proof. We may assume that $\tau_0 \ge 3-3\sigma$, since otherwise the claim follows from the Riemann–von Mangoldt bound

$$\mathrm{A}(\sigma)(1-\sigma) \le \mathrm{A}(1/2)(1-1/2) = 1.$$

By Lemma 7.4(ii) we have

$$\mathrm{LV}(\sigma,\tau) \le \max(2-2\sigma, 3-3\sigma+\tau-\tau_0)$$

for all $\tau \ge 0$. But both expressions on the right-hand side are bounded by $(3-3\sigma)\frac{\tau}{\tau_0}$ for $2\tau_3 \le \tau \le \tau_0$ and $\tau_0 \ge 3-3\sigma$, so the claim follows from Corollary 11.9. □

11.1 Known zero density bounds

Let us see some examples of these corollaries in action.

Theorem 11.11. *The Montgomery conjecture implies the density hypothesis.*

Proof. Apply Corollary 11.9 with $\tau_0 = 3/2$ (so that (11.7) is vacuously true). □

Theorem 11.12. *The Lindelof hypothesis implies the density hypothesis, and also that $\mathrm{A}(\sigma) \le 0$ for $3/4 < \sigma \le 1$.*

Proof. The first result is proved in [138], and the second result is due to [91]. We will apply Corollary 11.8. From Corollary 8.8 we see that $\mathrm{LV}_\zeta(\sigma,\tau) = -\infty$ whenever $\sigma > 1/2$ and $\tau \ge 1$, so for any choice of τ_0 we have

$$\sup_{2\le\tau<4\tau_0/3} \mathrm{LV}_\zeta(\sigma,\tau)/\tau = -\infty.$$

From Theorem 7.9 and Lemma 7.8 we have

$$\mathrm{LV}(\sigma,\tau) \le \max((2-2\sigma)k, \tau+(1-2\sigma)k) \quad (11.9)$$

for any natural number k and $\tau \ge 1$; setting k to be the integer part of τ we conclude in particular that

$$\mathrm{LV}(\sigma,\tau) \le (2-2\sigma)\tau + O(1),$$

and hence by taking τ_0 large enough, we can make $\sup_{2\tau_0/3\le\tau\le\tau_0} \mathrm{LV}(\sigma,\tau)/\tau$ bounded by $2-2\sigma+\varepsilon$ for any $\varepsilon > 0$. This already gives the density hypothesis bound $\mathrm{A}(\sigma) \le 2$. For $\sigma > 3/4$, we may additionally apply Theorem 8.14 to make $\sup_{2\tau_0/3\le\tau\le\tau_0} \mathrm{LV}(\sigma,\tau)/\tau$ arbitrarily small, giving the bound $\mathrm{A}(\sigma) \le 0$. □

There are similar results assuming weaker versions of the Lindelof hypothesis. For instance, we have

Theorem 11.13 (Ingham's first bound)**.** *[137] (See also [277]) For any $1/2 < \sigma < 1$, we have*

$$\mathrm{A}(\sigma) \le 2 + 4\mu(1/2).$$

Proof. We give here a proof (somewhat different from the original proof) that passes through Corollary 11.7. We apply Corollary 11.7 with τ_0 chosen so that $\mu(1/2)\tau_0 < \sigma - 1/2$. From Corollary 8.6 we then have

$$\mathrm{A}(\sigma)(1-\sigma) \leq \sup_{\tau_0 \leq \tau \leq 2\tau_0} \mathrm{LV}(\sigma,\tau)/\tau.$$

For any integer $k \geq 0$ and $k \leq \tau \leq k+1$, we see from (11.9) that

$$\mathrm{LV}(\sigma,\tau) \leq (2-2\sigma)(k+1)$$

and

$$\mathrm{LV}(\sigma,\tau) \leq \tau + (1-2\sigma)k;$$

multiplying the first inequality by $2\sigma - 1$, the second by $2-2\sigma$, and summing, we conclude that

$$\mathrm{LV}(\sigma,\tau) \leq (\tau + 2\sigma - 1)(2-2\sigma);$$

inserting this bound we have

$$\mathrm{A}(\sigma) \leq 2 + \frac{2\sigma-1}{\tau_0}.$$

Optimizing in τ_0, we obtain the claim. □

Theorem 11.14 (Ingham's second bound)**.** *[138] For any* $1/2 < \sigma < 1$*, one has* $\mathrm{A}(\sigma) \leq \frac{3}{2-\sigma}$*.*

Recorded in `literature.py` *as:*
`add_zero_density_ingham_1940()`
Derived in `derived.py` *as:*
`prove_zero_density_ingham_1940()`
`prove_zero_density_ingham_1940_v2()`

Proof. We apply Corollary 11.10 with $\tau_0 := 2 - \sigma$. Here we have $4\tau_0/3 < 2$ since $\sigma > 1/2$, so the claim (11.7) is automatic; and the Montgomery conjecture hypothesis follows from Theorem 7.9. □

Either of Theorem 11.13 or Theorem 11.14 implies an older result of Carlson [29] that $\mathrm{A}(\sigma) \leq 4\sigma$ for $1/2 < \sigma < 1$. *Recorded in* `literature.py` *as:*
`add_zero_density_carlson_1921()`

Theorem 11.15 (Huxley bound)**.** *[122] For any* $1/2 < \sigma < 1$*, one has* $\mathrm{A}(\sigma) \leq \frac{3}{3\sigma-1}$*. (In particular, the density hypothesis holds for* $\sigma \geq 5/6$*.)*

Recorded in `literature.py` *as:*
`add_zero_density_huxley_1972()`
Derived in `derived.py` *as:*
`prove_zero_density_huxley_1972()`
`prove_zero_density_huxley_1972_v2()`

Proof. We apply Corollary 11.10 with $\tau_0 := 3\sigma - 1$. The Montgomery conjecture hypothesis follows from Theorem 7.12. So it remains to show that (11.7) holds for $2 \leq \tau < 4\tau_0/3$. For $\sigma \leq 5/6$ we have $4\tau_0/3 \leq 2$, so the claim is vacuously true in this case. For $\sigma > 5/6$ we use Corollary 8.6 and the bound $\mu(1/2) \leq 1/6$ from Table 6.2 to conclude that $\mathrm{LV}_\zeta(\sigma, \tau) = -\infty$ whenever $\sigma > 1/2 + \tau/6$, but this is precisely $\tau < 6\sigma - 3$. Since $6\sigma - 3 > 4\tau_0/3$ when $\sigma > 5/6$, we obtain the claim. □

Theorem 11.16 (Guth–Maynard bound)**.** *For any* $1/2 < \sigma < 1$*, one has* $\mathrm{A}(\sigma) \leq \frac{15}{3+5\sigma}$.

Recorded in ***literature.py*** *as:*
`add_zero_density_guth_maynard_2024()`
Derived in ***derived.py*** *as:*
`prove_zero_density_guth_maynard_2024()`
`prove_zero_density_guth_maynard_2024_v2()`

Proof. We may assume that $7/10 < \sigma < 8/10$, since the bound follows from the Ingham and Huxley bounds otherwise. We apply Corollary 11.9 with $\tau_0 := \frac{3+5\sigma}{5}$. We have $4\tau_0/3 < 2$, so the claim (11.7) is vacuous and we only need to establish (11.6). We split into the subranges $13/5 - 2\sigma \leq \tau < \tau_0$ and $2\tau_0/3 \leq \tau \leq 13/5 - 2\sigma$. In the former range we use Theorem 10.27 (and (11.8)), and reduce to showing that

$$18/5 - 4\sigma \leq (3 - 3\sigma)\frac{\tau}{\tau_0},$$

and

$$\tau + 12/5 - 4\sigma \leq (3 - 3\sigma)\frac{\tau}{\tau_0}$$

for $13/5 - 2\sigma \leq \tau < \tau_0$. The first inequality follows from

$$18/5 - 4\sigma \leq (3 - 3\sigma)\frac{13/5 - 2\sigma}{\tau_0} \tag{11.10}$$

which one can numerically check holds in the range $7/10 < \sigma < 8/10$. Finally, the third inequality is obeyed with equality when $\tau = \tau_0$ and the right-hand side has a larger slope in τ than the left (since $\tau_0 \geq 3 - 3\sigma$), so the claim follows as well.
In the remaining region $2\tau_0/3 \leq \tau \leq 13/5 - 2\sigma$, we use Theorem 7.9 and (11.8) to reduce to showing that

$$\tau + 1 - 2\sigma \leq (3 - 3\sigma)\frac{\tau}{\tau_0}$$

in this range. This follows again from (11.10) which guarantees the inequality at the right endpoint $\tau = 13/5 - 2\sigma$. □

Theorem 11.17 (Jutila zero density theorem)**.** *[160] The zero density hypothesis is true for* $\sigma \geq 11/14$.

Derived in ***derived.py*** *as:*
`prove_zero_density_jutila_1977()`
`prove_zero_density_jutila_1977_v2()`

Proof. We apply Corollary 11.8 with $\tau_0 := 3/2$, then it suffices to show that

$$\mathrm{LV}(\sigma, \tau) \leq (2 - 2\sigma)\tau$$

for all $1 \leq \tau \leq 3/2$.
From the $k = 3$ case of Theorem 7.16 we have

$$\mathrm{LV}(\sigma, \tau) \leq \max\left(2 - 2\sigma, \tau + \frac{10 - 16\sigma}{3}, \tau + 18 - 24\sigma\right).$$

But all terms on the right-hand side can be verified to be less than or equal to $(2 - 2\sigma)\tau$ when $1 \leq \tau \leq 3/2$ and $\sigma \geq 11/14$, giving the claim. □

In fact, we can do better:

Theorem 11.18 (Heath-Brown zero density theorem)**.** *[107] For* $11/14 \leq \sigma < 1$*, one has* $\mathrm{A}(\sigma) \leq \frac{9}{7\sigma - 1}$ *(in particular, this recovers Theorem 11.17). For any* $3/4 \leq \sigma \leq 1$*, one has* $\mathrm{A}(\sigma) \leq \max(\frac{3}{10\sigma - 7}, \frac{4}{4\sigma - 1})$ *(which is a superior bound when* $\sigma \geq 20/23$*).*

Recorded in ***literature.py*** *as:*
`add_zero_density_heathbrown_1979()`
Derived in ***derived.py*** *as:*
```
prove_zero_density_heathbrown_1979a()
prove_zero_density_heathbrown_1979b()
prove_zero_density_heathbrown_1979a_v2()
prove_zero_density_heathbrown_1979b_v2()
```

Proof. For the first estimate, we apply Corollary 11.9 with $\tau_0 := \frac{7\sigma - 1}{3}$. To verify (11.6), we apply the $k = 3$ version of Theorem 7.16, which gives

$$\mathrm{LV}(\sigma, \tau) \leq \max\left(2 - 2\sigma, \tau + \frac{10 - 16\sigma}{3}, \tau + 18 - 24\sigma\right).$$

When $\sigma \geq 11/14$ one has $18 - 24\sigma \leq \frac{10 - 16\sigma}{3}$, so by (11.8) we need to show that

$$\tau + \frac{10 - 16\sigma}{3} \leq (3 - 3\sigma)\frac{\tau}{\tau_0}$$

for $2\tau_0/3 \leq \tau \leq \tau_0$. This holds with equality at $\tau = \tau_0$, hence holds for $\tau \leq \tau_0$ as well since $\tau_0 \geq 3 - 3\sigma$. As for (11.7), we invoke Theorem 9.7 and reduce to showing that

$$2\tau + 6 - 12\sigma \leq (3 - 3\sigma)\frac{\tau}{\tau_0}$$

for $2 \leq \tau \leq 4\tau_0/3$. Since $6 - 12\sigma$ is negative, the ratio of the left-hand side and right-hand side is increasing in τ, so it suffices to verify this claim at the endpoint $\tau = 4\tau_0/3$. The claim then simplifies to $\tau_0 \leq \frac{3}{4}(4\sigma - 1)$, which one can verify from the choice of τ_0 and the hypothesis $\sigma \geq 11/14$.
For the second estimate, we take $\tau_0 := \min(10\sigma - 7, \frac{3}{4}(4\sigma - 1))$. To verify (11.6), we now use Theorem 7.14 and (11.8), and reduce to showing that

$$\tau + 10 - 13\sigma \leq (3 - 3\sigma)\frac{\tau}{\tau_0}$$

for $2\tau_0/3 \leq \tau \leq \tau_0$. The inequality holds at $\tau = \tau_0$ since $\tau_0 \leq 10\sigma - 7$, and hence for all smaller τ since $\tau_0 \geq 3 - 3\sigma$. As for (11.7), we can repeat the previous arguments since $\tau_0 \leq \frac{3}{4}(4\sigma - 1)$. □

With the aid of computer assistance, we were able to strengthen the second claim here. We first need a lemma:

Lemma 11.19. $(3/40, 31/40)$ *is an exponent pair. In particular, by Corollary 6.8,* $\mu(7/10) \leq 3/40$.

Derived in ***derived.py*** *as:*
```
best_proof_of_exponent_pair(frac(3,40), frac(31,40))
```

Proof. This can be derived from the Watt exponent pair $W := (89/560, 1/2 + 89/560)$ from Theorem 5.12 as well as the A and B transforms and convexity (Lemmas 5.4, 5.8, 5.9) after observing that

$$(3/40, 31/40) = xyAW + (1-x)yABAW + (1-y)W$$

with $x = 37081/40415$ and $y = 476897/493711$. (One could of course also use more recent exponent pairs that are stronger, such as the Bourgain exponent pair $(13/84, 1/2 + 13/84)$.) We remark that one could also obtain this result from Lemma 5.3, after observing that the required bound $\beta(\alpha) \leq 3/40 + 7\alpha/10$ can be derived from Theorem 4.16 (as well as the classical bounds in Corollary 4.8). We also note that the corollary $\mu(7/10) \leq 3/40 = 0.075$ is immediate from [279, Theorem 2.4], which in fact gives the slightly stronger bound $\mu(7/10) \leq 218/3005 = 0.07254\ldots$. □

Theorem 11.20 (Improved Heath-Brown zero density theorem)**.** *For any* $7/10 < \sigma \leq 1$, *one has* $\mathrm{A}(\sigma) \leq \frac{3}{10\sigma - 7}$.

Derived in ***derived.py*** *as:*
```
prove_zero_density_heathbrown_extended()
```

Proof. We apply Corollary 11.10 with $\tau_0 := 10\sigma - 7$. The claim (11.6) again follows from Theorem 7.14 and (11.8) as in the proof of Theorem 11.18. Meanwhile, from Lemma 11.19 and Corollary 8.7 we have $\mathrm{LV}_\zeta(\sigma, \tau) = -\infty$ whenever $\sigma > \frac{3}{40}\tau + \frac{7}{10}$, or equivalently $\tau < \frac{4}{3}(10\sigma - 7)$, which then immediately gives (11.7). □

Theorem 11.21 (Bourgain result on density hypothesis)**.** *The density hypothesis holds for* $\sigma > 25/32$.

Recorded in ***literature.py*** *as:*
```
add_zero_density_bourgain_2000()
```

Proof. The arguments below are a translation of the original arguments of Bourgain [21] to our notational framework.
In view of Theorem 11.17 (or Theorem 11.18), we may assume that $25/32 < \sigma < 11/14$. Set $\rho := \mathrm{LV}(\sigma, \tau)$. As in the proof of Theorem 11.17, it suffices to show that

$$\rho \leq (2 - 2\sigma)\tau \tag{11.11}$$

for all $1 \leq \tau \leq 3/2$.

From the $k = 3$ case of Theorem 7.16 we have

$$\rho \leq \max\left(2 - 2\sigma, \tau + \frac{10 - 16\sigma}{3}, \tau + 18 - 24\sigma\right) \tag{11.12}$$

which in the $\sigma < 11/14$ regime simplifies to

$$\rho \leq \max(2 - 2\sigma, \tau + 18 - 24\sigma) \tag{11.13}$$

and this already suffices unless

$$\tau \geq \frac{24\sigma - 18}{2\sigma - 1}. \tag{11.14}$$

In the regime $\sigma > 25/32$ and $\tau \leq 3/2$, the bound (11.13) certainly implies

$$\rho \leq \min(1, 4 - 2\tau)$$

and also

$$\max(1, 2\tau - 2) = 1$$

so we may invoke Corollary 10.30 to conclude that

$$\rho \leq \max(\alpha_2 + 2 - 2\sigma, \alpha_1 + \alpha_2/2 + 2 - 2\sigma, -\alpha_2 + 2\tau + 4 - 8\sigma, -2\alpha_1 + \tau + 12 - 16\sigma, 4\alpha_1 + 3 - 4\sigma) \tag{11.15}$$

for any $\alpha_1, \alpha_2 \geq 0$.

We now divide into cases. First suppose that $\tau \leq \frac{4(1+\sigma)}{5}$. In this case we set $\alpha_1 := \frac{\tau}{3} - \frac{2}{3}(7\sigma - 5)$ (which can be checked to be nonnegative using (11.14) and $\sigma \geq 25/32$) and $\alpha_2 = 0$, and one can check that (11.15) implies (11.11) in this case (with some room to spare).

Now suppose that $\tau > \frac{4(1+\sigma)}{5}$. In this case we choose $\alpha_1 = \frac{\tau}{8} - \frac{9\sigma-7}{2}$ and $\alpha_2 = \frac{5\tau}{4} - (1 + \sigma)$, which can be checked to be nonnegative using the hypotheses on σ, τ. In this case one can again check that (11.15) implies (11.11). □

We can improve this bound as follows:

Theorem 11.22 (Improved Bourgain density hypothesis bound)**.** *For* $17/22 \leq \sigma \leq 4/5$, *one has* $\mathrm{A}(\sigma) \leq \max(\frac{2}{9\sigma-6}, \frac{9}{8(2\sigma-1)})$*. Thus one has* $\mathrm{A}(\sigma) \leq \frac{9}{8(2\sigma-1)}$ *for* $38/49 \leq \sigma \leq 4/5$ *and* $\mathrm{A}(\sigma) \leq \frac{2}{9\sigma-6}$ *for* $17/22 \leq \sigma \leq 38/49$.

Derived in ***derived.py*** *as:*
`prove_zero_density_bourgain_improved()`

The arguments can be pushed to some σ below 17/22, but in that range the estimate in Corollary 11.31 becomes superior, so we do not pursue this further.

Proof. We apply Corollary 11.10 with $\tau_0 := \min(\frac{9(3\sigma-2)}{2}, \frac{8(2\sigma-1)}{3})$. For future reference we note that

$$\frac{1}{3} < \frac{3 - 3\sigma}{\tau_0} < \frac{1}{2}. \tag{11.16}$$

It suffices to show that

$$\mathrm{LV}(\sigma, \tau) \leq \frac{\tau}{\tau_0}(3 - 3\sigma) \tag{11.17}$$

for $2\tau_0/3 \leq \tau \leq \tau_0$, as well as

$$\mathrm{LV}_\zeta(\sigma,\tau) \leq \frac{\tau}{\tau_0}(3-3\sigma) \tag{11.18}$$

for $2 \leq \tau < 4\tau_0/3$. For (11.18) we use the twelfth moment bound in Theorem 9.7. Since the slope of $2\tau - 12(\sigma - 1/2)$ in τ exceeds that of $\frac{\tau}{\tau_0}(3-3\sigma)$ by (11.16), it suffices to check the bound at the endpoint, i.e., to show that

$$8\tau_0/3 - 12(\sigma - 1/2) \leq 4 - 4\sigma$$

or equivalently $\tau_0 \leq \frac{3(4\sigma-1)}{4}$, which one can easily check to be the case.
Now we prove (11.17). Set $\rho := \mathrm{LV}(\sigma,\tau)$. From the $k = 3$ case of Theorem 7.16 we again have (11.12), which implies the required bound $\rho \leq \frac{\tau}{\tau_0}(3-3\sigma)$ unless one has

$$\tau \geq -\max(\frac{10-16\sigma}{3}, 18-24\sigma)/(1-\frac{3-3\sigma}{\tau_0}) \tag{11.19}$$

In this regime, one can also check from (11.12) that

$$\rho \leq \min(1, 4-2\tau)$$

(with room to spare) so we may apply Corollary 10.30 to obtain

$$\begin{aligned}\rho \leq \max(&\alpha_2 + 2 - 2\sigma, \alpha_1 + \alpha_2/2 + 2 - 2\sigma, -\alpha_2 + 2\tau + 4 - 8\sigma,\\ &-2\alpha_1 + \tau + 12 - 16\sigma, 4\alpha_1 + 2 + \max(1, 2\tau - 2) - 4\sigma)\end{aligned} \tag{11.20}$$

for any $\alpha_1, \alpha_2 \geq 0$.
We first consider the case when $38/49 \leq \sigma \leq 4/5$, so that $\tau_0 = 8(2\sigma-1)/3$. As in the proof of Theorem 11.21, we set

$$\alpha_2 := \max\left(\frac{5\tau}{4} - (1+\sigma), 0\right)$$

and

$$\alpha_1 := \frac{\tau}{3} - \frac{2}{3}(7\sigma - 5) - \alpha_2/6.$$

With this choice, the expressions $\alpha_1 + \alpha_2/2 + 2 - 2\sigma$ and $-2\alpha_1 + \tau + 12 - 16\sigma$ are both equal to $\frac{\tau}{3} + \frac{16-20\sigma}{3} + \frac{\alpha_2}{3}$, while $-\alpha_2 + 2\tau + 4 - 8\sigma$ is equal to

$$\frac{\tau}{3} + \frac{16-20\sigma}{3} + \frac{\alpha_2}{3} - \frac{4}{3}\left(\alpha_2 - \frac{5\tau}{4} - (1+\sigma)\right)$$

which is less than or equal to the previous expression by definition of α_2. Finally, the expression $4\alpha_1 + 1 + \max(2, 2\tau - 2) - 4\sigma$ is equal to

$$\frac{4\tau}{3} + \frac{46-68\sigma}{3} + \max(1, 2\tau-2) - \frac{2\alpha_2}{3}.$$

Thus it remains to show the bounds

$$\alpha_2 + 2 - 2\sigma \leq \frac{\tau}{\tau_0}(3-3\sigma) \tag{11.21}$$

$$\frac{\tau}{3} + \frac{16-20\sigma}{3} + \frac{\alpha_2}{3} \leq \frac{\tau}{\tau_0}(3-3\sigma) \tag{11.22}$$

and

$$\frac{4\tau}{3} + \frac{46-68\sigma}{3} + \max(1, 2\tau - 2) - \frac{2\alpha_2}{3} \leq \frac{\tau}{\tau_0}(3-3\sigma). \tag{11.23}$$

For (11.21), we trivially have $2 - 2\sigma \leq \frac{\tau}{\tau_0}(3-3\sigma)$ since $\tau \geq 2\tau_0/3$, and the slope of $\frac{5\tau}{4} - (1+\sigma) + 2 - 2\sigma$ in τ certainly exceeds $\frac{3-3\sigma}{\tau_0}$ by (11.16), so it suffices to check the endpoint

$$\frac{5\tau_0}{4} - (1+\sigma) + 2 - 2\sigma \leq 3 - 3\sigma$$

which one can check to be valid for $\sigma \leq 4/5$. Now we turn to (11.22). From (11.16) it suffices to show that

$$\frac{\tau_0}{3} + \frac{16-20\sigma}{3} + \frac{\frac{5\tau_0}{4} - (1+\sigma)}{3} \leq 3 - 3\sigma$$

and

$$\frac{\tau}{3} + \frac{16-20\sigma}{3} + \frac{0}{3} \leq 2 - 2\sigma.$$

The former is an identity, and the latter simplifies to $\tau \geq 14\sigma - 10$, which one can check follows from (11.19) (with some room to spare) in the regime $38/49 \leq \sigma \leq 4/5$, giving the claim. Finally, for (11.23) it suffices to show that

$$\frac{4\tau}{3} + \frac{46-68\sigma}{3} + \max(1, 2\tau_0 - 2) - \frac{2(\frac{5\tau}{4} - (1+\sigma))}{3} \leq \frac{\tau}{\tau_0}(3-3\sigma)$$

which by (11.16) would follow from

$$\frac{4\tau_0}{3} + \frac{46-68\sigma}{3} + \max(1, 2\tau_0 - 2) - \frac{2(\frac{5\tau_0}{4} - (1+\sigma))}{3} \leq 3 - 3\sigma$$

and one can check that this applies for $\sigma \geq 38/49$.

Now suppose that we are in the case $17/22 \leq \sigma \leq 38/49$, so that $\tau_0 = \frac{9(3\sigma-2)}{2} \leq \frac{72}{49} < \frac{3}{2}$ (so in particular $\max(1, 2\tau - 2) = 1$ for $\tau \leq \tau_0$). We set

$$\alpha_2 := \max(11 - 16\sigma + \tau, 0)$$

and

$$\alpha_1 := \frac{\tau}{3} - \frac{2}{3}(7\sigma - 5) - \alpha_2/6.$$

Note that for $\sigma \leq 38/49$, one has

$$(5\tau/4 - (1+\sigma)) - (11 - 16\sigma + \tau) \leq 15\sigma - 12 - \tau_0/4 \leq 0$$

and hence

$$\alpha_2 \geq 5\tau/4 - (1+\sigma).$$

As before, we conclude that the quantities $\alpha_1 + \alpha_2/2 + 2 - 2\sigma$ and $-2\alpha_1 + \tau + 12 - 16\sigma$ are both equal to $\frac{\tau}{3} + \frac{16-20\sigma}{3} + \frac{\alpha_2}{3}$, while $-\alpha_2 + 2\tau + 4 - 8\sigma$ is less than or equal to this quantity. Thus it suffices to show (11.21), (11.22), (11.23) as before.

For (11.21) we argue as before to reduce to showing that

$$11 - 16\sigma + \tau_0 + 2 - 2\sigma \leq 3 - 3\sigma$$

which one can check to be true (with room to spare) for $\sigma \geq 17/22$. For (11.22), we use (11.16) as before to reduce to showing that

$$\frac{\tau_0}{3} + \frac{16 - 20\sigma}{3} + \frac{11 - 16\sigma + \tau}{3} \leq 3 - 3\sigma$$

and

$$\frac{\tau}{3} + \frac{16 - 20\sigma}{3} + \frac{0}{3} \leq 2 - 2\sigma.$$

The first inequality is an identity, and the latter again reduces to $\tau \geq 14\sigma - 10$ which one can check follows from (11.19). For (11.23) it suffices to show that

$$\frac{4\tau}{3} + \frac{46 - 68\sigma}{3} + 1 - \frac{2(11 - 16\sigma + \tau)}{3} \leq \frac{\tau}{\tau_0}(3 - 3\sigma)$$

which by (11.16) follows from

$$\frac{4\tau_0}{3} + \frac{46 - 68\sigma}{3} + 1 - \frac{2(11 - 16\sigma + \tau_0)}{3} \leq 3 - 3\sigma,$$

but this is an identity. □

Theorem 11.23 (Bourgain zero density theorem)**.** *[20, Proposition 3] Let (k, ℓ) be an exponent pair with $k < 1/5$, $\ell > 3/5$, and $15\ell + 20k > 13$. Then, for any $\sigma > \frac{\ell+1}{2(k+1)}$, one has*

$$\mathrm{A}(\sigma) \leq \frac{4k}{2(1+k)\sigma - 1 - \ell}$$

assuming either that $k < \frac{11}{85}$, or that $\frac{11}{85} < k < \frac{1}{5}$ and $\sigma > \frac{144k - 11\ell - 11}{170k - 22}$.

Corollary 11.24 (Special case of Bourgain's zero density theorem)**.** *[20, Corollary 4] One has*

$$\mathrm{A}(\sigma) \leq \frac{4}{30\sigma - 25}$$

for $\frac{15}{16} \leq \sigma \leq 1$ and

$$\mathrm{A}(\sigma) \leq \frac{2}{7\sigma - 5}$$

for $\frac{17}{19} \leq \sigma \leq \frac{15}{16}$.

Recorded in `literature.py` *as:*
`add_zero_density_bourgain_1995()`

Proof. Apply Theorem 11.23 with the classical pairs $(\frac{1}{14}, \frac{11}{14})$ and $(\frac{1}{6}, \frac{2}{3})$ respectively from Proposition 5.10. □

It was remarked in [20] that further zero density estimates could be obtained by using additional exponent pairs. This we do here:

Corollary 11.25 (Optimized Bourgain zero density bound)**.** *One has*

$$A(\sigma) \le \begin{cases} \dfrac{11}{12(4\sigma-3)} & \dfrac{3}{4} < \sigma \le \dfrac{14}{15}, \\ \dfrac{391}{2493\sigma - 2014} & \dfrac{14}{15} < \sigma \le \dfrac{2841}{3016}, \\ \dfrac{22232}{163248\sigma - 134765} & \dfrac{2841}{3016} < \sigma \le \dfrac{859}{908}, \\ \dfrac{356}{2742\sigma - 2279} & \dfrac{859}{908} < \sigma \le \dfrac{1625}{1692}, \\ \dfrac{2609588}{20732766\sigma - 17313767} & \dfrac{1625}{1692} < \sigma \le \dfrac{3334585}{3447984}, \\ \dfrac{75872}{9(81024\sigma - 69517)} & \dfrac{3334585}{3447984} < \sigma \le \dfrac{974605}{1005296}, \\ \dfrac{288}{3616\sigma - 3197} & \dfrac{974605}{1005296} < \sigma \le \dfrac{5857}{6032}, \\ \dfrac{86152}{1447460\sigma - 1311509} & \dfrac{5857}{6032} < \sigma < 1. \end{cases}$$

Implemented at ***zero_density_estimate.py*** *as:*
`bourgain_ep_to_zd()`

Proof. Let $\mathcal{S}(\sigma)$ denote the closure of the region

$$\left\{(k,\ell) : 0 < k < \frac{1}{5}, \frac{3}{5} < \ell < 1, 15\ell + 20k > 13, \frac{\ell+1}{2(k+1)} < \sigma, \right.$$
$$\left. k < \frac{11}{85} \text{ or } \left(k > \frac{11}{85} \text{ and } \frac{144k - 11\ell - 11}{170k - 22} < \sigma\right) \right\}$$

One may verify that $\mathcal{S}(\sigma)$ is a convex polygon for all $3/4 < \sigma < 1$, and thus so is $\mathcal{S}(\sigma) \cap H$, where H is the convex hull of exponent pairs. Thus

$$\min_{(k,\ell)\in\mathcal{S}(\sigma)\cap H} \frac{4k}{2(1+k)\sigma - 1 - \ell}$$

is a convex optimisation problem for each $3/4 < \sigma < 1$. We take the following choices of (k,ℓ) (found with the aid of computer assistance).

$$\left(\frac{11}{85} - \varepsilon, \frac{59}{85} + 2\varepsilon\right), \left(\frac{391}{4595} + \varepsilon, \frac{3461}{4595}\right), \left(\frac{2779}{38033} + \varepsilon, \frac{58699}{76066}\right), \left(\frac{89}{1282} + \varepsilon, \frac{997}{1282}\right),$$
$$\left(\frac{652397}{9713986} + \varepsilon, \frac{7599781}{9713986}\right), \left(\frac{2371}{43205} + \varepsilon, \frac{280013}{345640}\right), \left(\frac{9}{217} + \varepsilon, \frac{1461}{1736}\right), \left(\frac{10769}{351096} + \varepsilon, \frac{609317}{702192}\right).$$

Of these exponent pairs:

- $(\frac{11}{85}, \frac{59}{85})$ is the intersection of the lines $k = 1/5$ and $15\ell + 20k = 13$;
- $(\frac{391}{4595}, \frac{3461}{4595})$ is an intersection of the line $15\ell + 20k = 13$ and the boundary of H;
- all other exponent pairs are vertices of H.

The desired result follows from taking a minimum over the implied bounds. Sharper bounds are possible close to $\sigma = 1$ by choosing other exponent pairs, however it turns out such results are superseded by other zero density estimates. □

Lemma 11.26 (1980 Ivic zero density bound)**.** *[141], [144, Theorem 11.2] We have*

$$\mathrm{A}(\sigma) \leq \frac{4}{2\sigma+1}$$

for $17/18 \leq \sigma \leq 1$*, and*

$$\mathrm{A}(\sigma) \leq \frac{24}{30\sigma-11}$$

for $155/174 \leq \sigma \leq 17/18$*.*

Recorded in `literature.py` *as:*
`add_zero_density_ivic_1980()`

Proof. From Lemma 8.16 we have

$$\mathrm{LV}(\sigma,\tau) \leq \max(2-2\sigma, \tau+9-12\sigma, \tau - \frac{84\sigma-65}{6})$$

for all $\tau \geq 0$. Meanwhile, applying Lemma 8.11 with the exponent pair $(2/7, 4/7)$ we have

$$\mathrm{LV}_\zeta(\sigma,\tau) \leq \max(\tau+(3-6\sigma), 3\tau+19(1/2-\sigma)).$$

We apply Corollary 11.9 with $\tau_0 := \max(\frac{30\sigma-11}{8}, \frac{6\sigma+3}{4})$, and reduce to showing that (11.6) for $2\tau_0/3 \leq \tau \leq \tau_0$ and (11.7) for $2 \leq \tau < 4\tau_0/3$. But this follows from the preceding estimates after routine calculations. □

One can also use bounds on μ to obtain zero density theorems:

Lemma 11.27 (Zero density from μ bound)**.** *[218, Theorem 12.3] If* $1/2 \leq \alpha \leq 1$ *and* $\frac{\alpha+1}{2} \leq \sigma \leq 1$*, then*

$$\mathrm{A}(\sigma) \leq \mu(\alpha)\frac{2(3\sigma-1-2\alpha)}{(2\sigma-1-\alpha)(\sigma-\alpha)}.$$

Corollary 11.28 (1971 Montgomery zero density bound)**.** *[218], [144, Theorem 11.3] For any* $9/10 \leq \sigma \leq 1$ *and* $1/2 \leq \alpha \leq 1$ *one has*

$$\mathrm{A}(\sigma)(1-\sigma) \leq \frac{7}{6}\mu(5\sigma-4).$$

In particular, for $152/155 \leq \sigma \leq 1$*, one has*

$$\mathrm{A}(\sigma) \leq \min(35/36, 1600(1-\sigma)^{1/2}).$$

Recorded in `literature.py` *as:*
`add_zero_density_montgomery_1971()`

Proof. Apply the previous lemma with $\alpha = 5\sigma - 4$. □

Lemma 11.29 (Preliminary large values estimate). *If $m \geq 2$ is an integer, $3/4 < \sigma \leq 1$, and (k, ℓ) is an exponent pair, then*

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, m(2 - 4\sigma) + m\tau, \min(X, Y))$$

where

$$X := 2\tau/3 + 4m(3 - 4\sigma)/3$$

and

$$Y := \max(\tau + 3m(3 - 4\sigma), (k + \ell)\tau/k + k(1 + 2k + 2\ell)(3 - 4\sigma)/k).$$

Proof. See [144, (11.74)]. □

Lemma 11.30 (General zero density estimate). *[144, (11.76), (11.77)] If (k, ℓ) is an exponent pair, and $m \geq 2$ an integer, then*

$$\mathrm{A}(\sigma) \leq \frac{3m}{(3m - 2)\sigma + 2 - m}$$

whenever

$$\sigma \geq \min\Bigg(\frac{6m^2 - 5m + 2}{8m^2 - 7m + 2}, \max\left(\frac{9m^2 - 4m + 2}{12m^2 - 6m + 2}, \frac{3m^2(1 + 2k + 2\ell) - (4k + 2\ell)m + 2k + 2\ell}{4m^2(1 + 2k + 2\ell) - (6k + 4\ell)m + 2k + 2\ell}\right)\Bigg).$$

Implemented at ***zero_density_estimate.py*** *as:*
`ivic_ep_to_zd(exp_pairs, m=2)`

Proof. With the hypothesis on σ, one sees from Lemma 11.29 that

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, \tau - \frac{(4m - 2)\sigma + 2 - 2m}{m} + 2 - 2\sigma)$$

for $0 \leq \tau < \frac{(4m-2)\sigma+2-2m}{m}$, and hence for all $\tau \geq 0$ by by Lemma 7.4(ii). Meanwhile, from Theorem 9.7 one has

$$\mathrm{LV}_\zeta(\sigma, \tau) \leq 2\tau - 12(\sigma - 1/2)$$

for all $\tau \geq 2$. The claim then follows from Corollary 11.9 with $\tau_0 := \frac{(3m-2)\sigma+2-m}{m}$ after a routine calculation. □

Corollary 11.31 (1980-1984 Ivic zero density bound). *[141], [144, Theorem 11.4] One can bound $\mathrm{A}(\sigma)$ by*

$$\frac{3}{2\sigma} \text{ for } \frac{3831}{4791} \leq \sigma \leq 1;$$
$$\frac{9}{7\sigma - 1} \text{ for } \frac{41}{53} \leq \sigma \leq 1;$$
$$\frac{6}{5\sigma - 1} \text{ for } \frac{13}{17} \leq \sigma \leq 1;$$
$$\frac{15}{13\sigma - 3} \text{ for } \frac{127}{167} \leq \sigma \leq 1;$$
$$\frac{9}{8\sigma - 2} \text{ for } \frac{47}{62} \leq \sigma \leq 1;$$

Recorded in `literature.py` *as:*
`add_zero_density_ivic_1980()`
`add_zero_density_ivic_1984()`
Derived in `derived.py` *as:*
`prove_zero_density_ivic_1984()`

Proof. Apply Lemma 11.30 with $m = 2$ and $(k, \ell) = (\frac{97}{251}, \frac{132}{251})$ for the first claim; the remaining claims follow from taking $m = 3, 4, 5, 6$ and the trivial exponent pair $(0, 1)$. □

The first bound has been improved:

Theorem 11.32 (2000 Bourgain zero density theorem)**.** *[22] One has* $\mathrm{A}(\sigma) \leq 3/2\sigma$ *for* $3734/4694 \leq \sigma \leq 1$.

Recorded in `literature.py` *as:*
`add_zero_density_bourgain_2002()`

Lemma 11.33 (Preliminary large values theorem)**.** *If* $1/2 \leq \sigma \leq 1$ *and* $\tau < 8\sigma - 5$*, then*

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, 6\tau/5 + (20 - 32\sigma)/5).$$

Proof. See [144, (11.95)]. □

Corollary 11.34 (Zero density estimates for σ close to 3/4)**.** *[144, Theorem 11.5] One has* $\mathrm{A}(\sigma) \leq \frac{3}{7\sigma - 4}$ *for* $3/4 \leq \sigma \leq 10/13$*, and* $\mathrm{A}(\sigma) \leq \frac{9}{8\sigma - 2}$ *for* $10/13 \leq \sigma \leq 1$.

Proof. For $3/4 \leq \sigma \leq 10/13$, we see from Lemma 11.33 that the bound

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, \tau + 7 - 10\sigma)$$

holds for $0 \leq \tau < 8\sigma - 5$, and hence for all $\tau \geq 0$ by Lemma 7.7. Meanwhile, from Lemma 9.6 we have

$$\mathrm{LV}_\zeta(\sigma, \tau) \leq \tau - 4(\sigma - 1/2)$$

for all $1/2 \leq \sigma \leq 1$ and $\tau \geq 2$. The claim then follows from Corollary 11.9 with $\tau_0 := 7\sigma - 4$ after a routine calculation. Similarly, for $10/13 \leq \sigma \leq 1$, we have

$$\mathrm{LV}(\sigma, \tau) \leq \max(2 - 2\sigma, \tau + \frac{11 - 17\sigma}{3})$$

for $0 \leq \tau < \frac{11\sigma - 5}{3}$, hence for all $\tau \geq 0$ by Lemma 7.4(ii); the claim then follows from Corollary 11.9 with $\tau_0 := \frac{8\sigma - 2}{3}$ after a routine calculation. □

Theorem 11.35 (Pintz zero density theorem)**.** *[237, Theorem 1] If* $k \geq 4$*,* $\ell \geq 3$ *are integers and* $\sigma = 1 - \eta$ *is such that*

$$\frac{1}{k(k+1)} \leq \eta < \frac{1}{k(k-1)} \tag{11.24}$$

and

$$\frac{1}{2\ell(\ell+1)} \leq \eta < \frac{1}{2\ell(\ell-1)} \tag{11.25}$$

then

$$\mathrm{A}(\sigma) \leq \max\left(\frac{3}{\ell(1 - 2(\ell-1)\eta)}, \frac{4}{k(1 - (k-1)\eta)}\right).$$

Recorded in `literature.py` *as:*
`add_zero_density_pintz_2023()`

As a corollary of these bounds one has $A(\sigma) \leq 3\sqrt{2}\eta^{1/2} + 18\eta$ for $\eta < 1/18$; see [237, Theorem 2'].

Proof. We apply Corollary 11.9 with

$$\tau_0 := \min(\ell(1 - 2(\ell - 1)\eta), \frac{3}{4}(k(1 - (k - 1)\eta))) - \varepsilon \tag{11.26}$$

for an arbitrarily small ε. It then suffices to show that (11.6) holds for $2\tau_0/3 \leq \tau \leq \tau_0$ and (11.7) holds for $2 \leq \tau < 4\tau_0/3$.
To prove (11.7), it suffices by Lemma 8.5 to show that $\sigma > \tau\beta(1/\tau)$ for all $2 \leq \tau < 4\tau_0/3$. By (11.26) one has $2 \leq \tau < k(1 - (k - 1)\eta)$. Meanwhile, from Lemma 4.23 one has

$$\tau\beta(1/\tau) \leq 1 + \max\left(\frac{\tau - r}{r(r - 1)}, -\frac{1}{r(r - 1)}, -\frac{2\tau}{r^2(r - 1)}\right) \tag{11.27}$$

for any $r \geq 3$. So by (11.24) it suffices to find $3 \leq r \leq k$ such that

$$\frac{r - \tau}{r(r - 1)}, \frac{2\tau}{r^2(r - 1)} \geq \eta$$

or equivalently

$$\tau \in [\frac{r^2(r - 1)\eta}{2}, r(1 - (r - 1)\eta)].$$

But one can check that these intervals for $3 \leq r \leq k$ cover the entire range $2 \leq \tau < 4\tau_0/3$, as required.
To prove (11.6), it suffices by Lemma 7.10 and (11.8) to show that

$$\sup_{1 \leq \tau \leq \tau_0} \beta(1/\tau)\tau < 2\sigma - 1 = 1 - 2\eta.$$

Using (11.27), (11.25) we obtain the claim whenever

$$\tau \in [r^2(r - 1)\eta + \varepsilon, r(1 - 2(r - 1)\eta) - \varepsilon]$$

for some $3 \leq r \leq \ell$. These cover the range $[18\eta + \varepsilon, \tau_0]$. For the remaining range $[1, 18\eta + \varepsilon]$ we use the van der Corput bound

$$\tau\beta(1/\tau) \leq \frac{\tau}{2} \leq 9\eta$$

from Corollary 4.8, which suffices since $\eta \leq \frac{1}{k(k-1)} \leq \frac{1}{12}$. ∎

The range of the second bound in Lemma 11.26 was recently extended:

Theorem 11.36 (Chen-Debruyne-Vidas density theorem). *[32] For any* $279/314 \leq \sigma \leq 17/18$, *one has* $A(\sigma) \leq \frac{24}{30\sigma - 11}$.

Recorded in `literature.py` *as:*
`add_zero_density_chen_debruyne_vindas_2024()`

The following result appears in an unpublished preprint of Kerr, and is based on the large values theorems in Theorem 10.32:

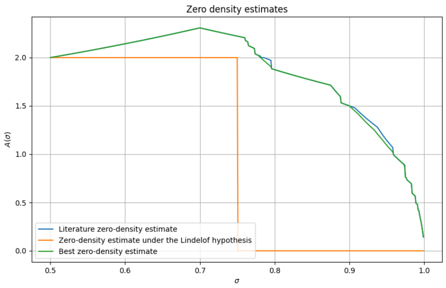

Figure 11.1: The bounds in Table 11.1, compared against the existing literature bounds on $\mathrm{A}(\sigma)$.

Proposition 11.37. *[165, Theorems 6, 7] One has* $\mathrm{A}(\sigma) \leq \frac{3}{2\sigma}$ *for* $\sigma \geq 23/29$, *and*

$$\mathrm{A}(\sigma) \leq \max\left(\frac{36}{138\sigma - 89}, \frac{114\sigma - 79}{(1-\sigma)(138\sigma - 89)}\right)$$

for $127/168 \leq \sigma \leq 107/138$.

The current best known zero density estimates (excepting the unpublished result in Proposition 11.37) are summarized in Table 11.1.
Derived in ***derived.py*** *as:*
`compute_best_zero_density()`

For completeness, we list in Table 11.2 some historical zero density theorems not already covered, which have now been superseded by more recent estimates.
TODO: enter this table into literature.py

11.2 Estimates for σ very close to $1/2$ or 1

Some additional estimates were established for σ sufficiently close to $1/2$ or 1.
Turán [280] introduced the power sum method to establish

$$\mathrm{A}(1-\eta) \leq 2 + \eta^{0.14}$$

for η small enough. Halász and Turán [91] combined this method with the large values approach of Halász [90] to improve the bound to

$$\mathrm{A}(1-\eta) \leq C\eta^{1/2} \tag{11.28}$$

with $C = 12,000$ for sufficiently small η. See [236] for an alternate proof of these results.
The constant C in (11.28) was improved to 1304.37 by Montgomery [218, Theorem 12.3] (see also the remark after [144, (11.97)] for a correction), to 58.05 by Ford [67], to 5.03 by Heath-Brown [113] (the latter exploiting the resolution of the Vinogradov mean value conjecture [24]), and to any $C > 3\sqrt{2} = 4.242\ldots$ in [237]. See also an explicit version at [13].
"Log-free" zero density estimates of the form

$$N(1-\eta, T) \ll T^{B\eta}$$

for various B were established starting with the work of Linnik [190, 191] and developed further in [280], [66], [17], [159], [79], [85], [112]. An explicit version of such estimates may be found in [14].
There is some work establishing bounds on $N(\sigma,T)$ for σ very close to 1/2 (and not necessarily fixed), although these bounds do not make further improvements on $\mathrm{A}(\sigma)$. Specifically, bounds of the form

$$N(\sigma,T) \ll T^{1-\theta(2\sigma-1)} \log T$$

for $T \geq 2$ (say) were established for $\theta = 1/8$ by Selberg [264] (see [267] for an explicit version), any $0 < \theta < 1/2$ by Jutila [161], and any $0 < \theta < 4/7$ by Conrey (claimed in [43], with a full proof given in [9]). Note that the density hypothesis would follow if we could establish the claim for all $0 < \theta < 1$, but an improvement to Ingham's bound (Theorem 11.14) would only occur once θ exceeded 2/3.

11.3 A heuristic for zero density estimates

We can now state a rough heuristic as to what zero density estimates to expect from a given large value theorem:

Heuristic 11.38 (Predicting a zero density estimate from a large value theorem)**.** *Suppose that* $1/2 \leq \sigma \leq 1$ *and* $\tau_0 \geq 1$ *are such that one can prove* $\mathrm{LV}(\sigma,\tau_0) \leq 3-3\sigma$ *(i.e., the Montgomery conjecture holds here with a multiplicative loss of* $3/2$*). Then in principle, one can hope to prove* $\mathrm{A}(\sigma) \leq 3/\tau_0$*. Conversely, if one cannot prove* $\mathrm{LV}(\sigma,\tau_0) \leq 3-3\sigma$*, then the bound* $\mathrm{A}(\sigma) \leq 3/\tau_0$ *is likely out of reach.*

We justify this heuristic as follows, though we stress that the arguments that follow are not fully rigorous. In the first part, we simply apply Corollary 11.9. In practice, the (11.6) is often more delicate than (11.7) and ends up being the limiting factor for the bounds; furthermore, within (11.6), it is the right endpoint $\tau = \tau_0$ of the range $2\tau_0/3 \leq \tau \leq \tau_0$ that ends up being the bottleneck; but this is precisely the claimed criterion $\mathrm{LV}(\sigma,\tau_0) \leq 3-3\sigma$. We remark that in some cases (particularly for σ close to one), the estimate (11.7) ends up being more of the bottleneck than (11.6), and so one should view $3/\tau_0$ here as a theoretical upper limit of methods rather than as a guaranteed bound. (In particular, the need to also establish the bound $\mathrm{LV}_\zeta(\sigma, \frac{4}{3}\tau_0 - \varepsilon) < 4-4\sigma$ for $\varepsilon > 0$ small can sometimes be a more limiting factor.)
Conversely, suppose that

$$\mathrm{LV}(\sigma,\tau_0) > 3-3\sigma, \tag{11.29}$$

but that one still wants to prove the bound $\mathrm{A}(\sigma) \leq 3/\tau_0$. Heuristically, Theorem 11.6 suggests that in order to do this, it is necessary to establish the bound $\mathrm{LV}_\zeta(\sigma,\tau)/\tau \leq \frac{3}{\tau_0}(1-\sigma)$ for all $\tau \geq 2$. In particular, one should show that

$$\mathrm{LV}_\zeta(\sigma, 2\tau_0) \leq 6-6\sigma.$$

Let us consider the various options one has to do this. There are ways to control zeta large values that do not apply to general large value estimates, such as moment estimates of the zeta function, exponent pairs, or control of β and μ. However, at our current level of understanding, these techniques only control $\mathrm{LV}_\zeta(\sigma,\tau)$ for relatively small values of τ, and in practice $2\tau_0$ is too large for these methods to apply; this exponent also tends to be too large for direct application of standard large value theorems to be useful. Hence, the most viable option in practice is raising to a power (Lemma 7.8), using

$$\mathrm{LV}_\zeta(\sigma, 2\tau_0) \leq k\mathrm{LV}_\zeta(\sigma, 2\tau_0/k)$$

for some natural number $k \geq 2$. However, the most natural choice $k = 2$ is blocked due to our hypothesis (11.29), while in practice the $k \geq 3$ choice is blocked because of Lemma 7.5. Hence it appears heuristically quite difficult to establish $\mathrm{A}(\sigma) \leq 3/\tau_0$ with current technology, in the event that (11.29) occurs.
In Table 11.3 we list some examples in which the heuristic can actually be attained. Note that this only covers some, but not all, of the best known zero density estimates in Table 11.1, as there are often other bounds that need to be established that prevent the heuristic limit of $3/\tau_0$ from actually being attained; so one should take the heuristic with a certain grain of salt.
One consequence of Heuristic 11.38 is that, in the regimes where the heuristic is accurate, combining multiple large values theorems together are unlikely to achieve new zero density theorems that could not be accomplished with each large value theorem separately.

11.4 Explicit results

A number of explicit versions of the above zero-density estimates have been established, which are particularly relevant when σ is close to 1/2 or 1, where factors of $T^{o(1)}$ become significant.

Theorem 11.39 ([267]). *For $T \geq 3$ and $1/2 \leq \sigma \leq 0.778$, one has*

$$N(\sigma, 2T) - N(\sigma, T) \leq 5874.051 T^{1-\frac{1}{4}(\sigma-\frac{1}{2})} \log T + 1.107 \log^2 T + 0.345 \log T \log\log T.$$

Sharper bounds for T large can be found in [267].

Theorem 11.40 ([41]). *For every $T \geq 3$ and $1/2 \leq \sigma \leq 5/8$ one has*

$$N(\sigma, T) \leq 8.604 T^{\frac{3(1-\sigma)}{2-\sigma}} \log^3 T + 9.461 \log^2 T + 167.8 \log T.$$

For every $T \geq 3$ and $5/8 \leq \sigma \leq 7/8$ one has

$$N(\sigma, T) \leq 22.44 T^{\frac{3(1-\sigma)}{2-\sigma}} \log^3 T + 8.290 \log^2 T + 147.0 \log T.$$

Theorem 11.41 ([40]). *For every $T \geq 3$ and $\sigma \geq 3/5$, one has*

$$N(\sigma, T) \leq 0.7756 T^{4\sigma(1-\sigma)} \log^{5-2\sigma} T.$$

Further bounds for larger values of σ can be found in [40].

Theorem 11.42 ([243]). *For every $T \geq 3$ and $\sigma \geq 0.52$ one has*

$$N(\sigma, T) \leq 965(3T)^{\frac{8(1-\sigma)}{3}} (\log T)^{5-2\sigma} + 51.5(\log T)^2.$$

The following result is an improvement upon Theorem 11.42.

Theorem 11.43 ([163]). *For each tuple (σ_0, A, B) of Table 11.4, one has*

$$N(\sigma, T) \leq A T^{\frac{8}{3}(1-\sigma)} (\log T)^{5-2\sigma} + B(\log T)^2$$

for each $\sigma_0 \leq \sigma \leq 1$ and $T \geq 3$.

See [163] for further estimates.
The following result is an explicit log-free zero density estimate.

Theorem 11.44 ([14]). *For every $T \geq 3$ and $\sigma \in [0.9927, 1]$, one has*

$$N(\sigma, T) \leq 4.45 \cdot 10^{12} \cdot T^{8(1-\sigma)}.$$

Sharper estimates of the form

$$N(\sigma, T) \leq CT^{B(1-\sigma)}, \qquad \sigma \in [\sigma_0, 1], \quad T \in [T_0, T_1]$$

can be found in [14]. We mention a couple of examples in Table 11.5.

Theorem 11.45 ([13]). *For every $\sigma \in [0.98, 1]$ and $T \geq 3$, one has:*

$$N(\sigma, T) \leq 2.15 \cdot 10^{23} \cdot T^{57.8875(1-\sigma)^{3/2}} (\log T)^{10393/900}.$$

Note that Theorem 11.45 implies the following log-free zero-density bound.

Corollary 11.46 ([14]). *For every $T \geq \exp(6.7 \cdot 10^{12})$ and $\sigma \in [0.98, 1]$, one has*

$$N(\sigma, T) \leq 4.45 \cdot 10^{12} \cdot T^{11.3(1-\sigma)}.$$

Table 11.1: Current best upper bound on A(σ)

A(σ) bound	Range	Reference
$\frac{3}{2-\sigma}$	$\frac{1}{2} \le \sigma \le \frac{7}{10} = 0.7$	Theorem 11.14
$\frac{15}{3+5\sigma}$	$\frac{7}{10} \le \sigma < \frac{19}{25} = 0.76$	Theorem 11.16
$\frac{9}{8\sigma-2}$	$\frac{19}{25} \le \sigma < \frac{127}{167} = 0.7604\ldots$	Corollary 11.31
$\frac{15}{13\sigma-3}$	$\frac{127}{167} \le \sigma < \frac{13}{17} = 0.7647\ldots$	Corollary 11.31
$\frac{6}{5\sigma-1}$	$\frac{13}{17} \le \sigma < \frac{17}{22} = 0.7727\ldots$	Corollary 11.31
$\frac{2}{9\sigma-6}$	$\frac{17}{22} \le \sigma < \frac{41}{53} = 0.7735\ldots$	Theorem 11.22
$\frac{9}{7\sigma-1}$	$\frac{41}{53} \le \sigma < \frac{7}{9} = 0.7777\ldots$	Corollary 11.31
$\frac{9}{8(2\sigma-1)}$	$\frac{7}{9} \le \sigma < \frac{1867}{2347} = 0.7954\ldots$	Theorem 11.22
$\frac{3}{2\sigma}$	$\frac{1867}{2347} \le \sigma < \frac{4}{5} = 0.8$	Theorem 11.32
$\frac{3}{2\sigma}$	$\frac{4}{5} \le \sigma < \frac{7}{8} = 0.875$	Corollary 11.31
$\frac{3}{10\sigma-7}$	$\frac{7}{8} \le \sigma < \frac{279}{314} = 0.8885\ldots$	Theorem 11.18
$\frac{24}{30\sigma-11}$	$\frac{279}{314} \le \sigma < \frac{155}{174} = 0.8908\ldots$	Theorem 11.36
$\frac{24}{30\sigma-11}$	$\frac{155}{174} \le \sigma \le \frac{9}{10} = 0.9$	Theorem 11.26
$\frac{3}{10\sigma-7}$	$\frac{9}{10} < \sigma \le \frac{31}{34} = 0.9117\ldots$	Theorem 11.20
$\frac{11}{48\sigma-36}$	$\frac{31}{34} < \sigma < \frac{14}{15} = 0.9333\ldots$	Corollary 11.25
$\frac{391}{2493\sigma-2014}$	$\frac{14}{15} \le \sigma < \frac{2841}{3016} = 0.9419\ldots$	Corollary 11.25
$\frac{22232}{163248\sigma-134765}$	$\frac{2841}{3016} \le \sigma < \frac{859}{908} = 0.9460\ldots$	Corollary 11.25
$\frac{356}{2742\sigma-2279}$	$\frac{859}{908} \le \sigma < \frac{23}{24} = 0.9583\ldots$	Corollary 11.25
$\frac{3}{24\sigma-20}$	$\frac{23}{24} \le \sigma < \frac{2211487}{2274732} = 0.9721\ldots$	Theorem 11.35
$\frac{86152}{1447460\sigma-1311509}$	$\frac{2211487}{2274732} \le \sigma < \frac{39}{40} = 0.975$	Corollary 11.25
$\frac{2}{15\sigma-12}$	$\frac{39}{40} \le \sigma < \frac{41}{42} = 0.9761\ldots$	Theorem 11.35
$\frac{3}{40\sigma-35}$	$\frac{41}{42} \le \sigma < \frac{59}{60} = 0.9833\ldots$	Theorem 11.35
$\frac{3}{n(1-2(n-1)(1-\sigma))}$	$1-\frac{1}{2n(n-1)} \le \sigma < 1-\frac{1}{2n(n+1)}$ (for integer $n \ge 6$)	Theorem 11.35

Table 11.2: Historical upper bounds on $A(\sigma)$

$A(\sigma)$ bound	Range	Reference
4σ	$\frac{1}{2} \le \sigma \le 1$	Carlson (1921) [29]
2	$4/5 \le \sigma \le 1$	Montgomery (1969) [217]
2	$0.8080 \le \sigma \le 1$	Forti–Viola (1972) [71]
$\frac{39}{115\sigma-75}$	$55/67 \le \sigma \le 189/230$	Huxley (1973) [123]
2	$189/230 \le \sigma \le 78/89$	Huxley (1973) [123]
$\frac{48}{37(2\sigma-1)}$	$78/89 \le \sigma \le 61/74$	Huxley (1973) [123]
$\frac{3}{2\sigma}$	$37/42 \le \sigma \le 1$	Huxley (1975) [124]
$\frac{48}{37(2\sigma-1)}$	$61/74 \le \sigma \le 37/42$	Huxley (1975) [124]
2	$0.80119 \le \sigma \le 1$	Huxley (1975) [124]
2	$4/5 \le \sigma \le 1$	Huxley (1975) [125]
$\frac{6}{5\sigma-1}$	$67/87 \le \sigma \le 1$	Ivić (1979) [146]
$\frac{3}{34\sigma-25}$	$28/37 \le \sigma \le 74/95$	Ivić (1979) [146]
$\frac{9}{7\sigma-1}$	$74/95 \le \sigma \le 1$	Ivić (1979) [146]
$\frac{3}{2\sigma}$	$4/5 \le \sigma \le 1$	Ivić (1979) [146]
$\frac{68}{98\sigma-47}$	$115/166 \le \sigma \le 1$	Ivić (1979) [146]
$\frac{3}{2\sigma}$	$3831/4791 \le \sigma \le 1$	Ivić (1980) [141]
$\frac{9}{7\sigma-1}$	$41/53 \le \sigma \le 1$	Ivić (1980) [141]
$\frac{6}{5\sigma-1}$	$13/17 \le \sigma \le 1$	Ivić (1980) [141]
$\frac{4}{2\sigma+1}$	$17/18 \le \sigma \le 1$	Ivić (1980) [141]
$\frac{24}{30\sigma-11}$	$155/174 \le \sigma \le 17/18$	Ivić (1980) [141]
$\frac{3}{7\sigma-4}$	$3/4 \le \sigma \le 10/13$	Ivić (1983) [142]
$\frac{9}{8\sigma-2}$	$10/13 \le \sigma \le 1$	Ivić (1983) [142]
$\frac{15}{22\sigma-10}$	$10/13 \le \sigma \le 5/6$	Ivić (1984) [143]
$\frac{3k}{(3k-2)\sigma+2-k}$	$\frac{9k^2-3k+2}{12k^2-5k+2} \le \sigma \le 1;\ k \ge 2$	Ivić (1984) [143]
$58.05(1-\sigma)^{1/2}$	$1/2 \le \sigma \le 1$	Ford (2002) [67]
$6.42(1-\sigma)^{1/2}$	$9/10 \le \sigma \le 1$	Heath-Brown (2017) [113]
$3\sqrt{2}(1-\sigma)^{1/2}+18(1-\sigma)$	$17/18 \le \sigma \le 1$	Pintz (2023) [237]

Table 11.3: Examples of large value theorems, the values of τ_0 and $\mathrm{A}(\sigma)$ they suggest, and rigorous zero density theorems that attain the predicted value for at least some ranges of σ.

Large value theorem	Predicted choice of τ_0	Predicted bound $\frac{3}{\tau_0}$ on $\mathrm{A}(\sigma)$	Matching zero density theorem(s)
Theorem 7.9	$2-\sigma$	$\frac{3}{2-\sigma}$	Theorem 11.14
Theorem 7.12	$3\sigma-1$	$\frac{3}{3\sigma-1}$	Theorem 11.15
Theorem 7.14	$10\sigma-7$	$\frac{3}{10\sigma-7}$	Theorems 11.18, 11.20
Theorem 7.16, $k=3$	$\frac{7\sigma-1}{3}$	$\frac{9}{7\sigma-1}$	Theorems 11.18, 11.31
Lemma 11.29, $m=2$	$\frac{4\sigma}{2}$	$\frac{3}{2\sigma}$	Corollary 11.31, Theorem 11.32
Lemma 11.29, $m=3$	$\frac{7\sigma-1}{3}$	$\frac{9}{7\sigma-1}$	Theorems 11.18, Corollary 11.31
Lemma 11.29, $m=4$	$\frac{10\sigma-2}{4}$	$\frac{6}{5\sigma-1}$	Corollary 11.31
Lemma 11.33	$7\sigma-4$	$\frac{3}{7\sigma-4}$	Corollary 11.34
Lemma 11.33	$\frac{8\sigma-2}{3}$	$\frac{9}{8\sigma-2}$	Corollary 11.34
Theorem 10.27	$\frac{5\sigma-3}{3}$	$\frac{15}{5\sigma-3}$	Theorem 11.16

σ_0	A	B
0.75	5.277	4.403
0.80	6.918	3.997
0.85	8.975	3.588
0.90	11.499	3.186
0.95	14.513	2.772
0.98	16.544	2.532

Table 11.4: Some examples of (σ_0, A, B)

Table 11.5: Values of constants C, B

B	C	σ_0	T_0	T_1
1.551	$1.62 \cdot 10^{11}$	0.9927	3	$\exp(6.7 \cdot 10^{12})$
1.551	$1.62 \cdot 10^{11}$	0.985	$\exp(80)$	$\exp(6.7 \cdot 10^{12})$

Chapter 12

Zero density energy theorems

Definition 12.1 (Zero density exponents)**.** *For $1/2 \leq \sigma \leq 1$ and $T > 0$, let $N^*(\sigma, T)$ denote the additive energy $E_1(\Sigma)$ of the imaginary parts of the zeroes ρ of the Riemann zeta function with $\mathrm{Re}(\rho) \geq \sigma$ and $|\mathrm{Im}(\rho)| \leq T$. For fixed $1/2 \leq \sigma \leq 1$, the zero density exponent $A^*(\sigma) \in [-\infty, \infty)$ is the infimum of all exponents A^* for which one has*

$$N^*(\sigma - \delta, T) \ll T^{A^*(1-\sigma)+o(1)}$$

for all unbounded T and infinitesimal $\delta > 0$.

The exponent $\mathrm{A}^*(\sigma)$ is also essentially referred to as $B(\sigma)$ in [104] (though without the technical shift by δ in that reference).
Implemented at ***zero_density_energy_estimate.py*** *as:*
`Zero_Density_Energy_Estimate`

Lemma 12.2 (Basic properties of A^*)**.** *(i) We have the trivial bounds*

$$2\mathrm{A}(\sigma), 4\mathrm{A}(\sigma) - \frac{1}{1-\sigma} \leq \mathrm{A}^*(\sigma) \leq 3\mathrm{A}(\sigma)$$

for any $1/2 \leq \sigma \leq 1$.

(ii) $\sigma \mapsto (1-\sigma)\mathrm{A}^*(\sigma)$ is non-increasing, with $\mathrm{A}^*(1/2) = 6$ and $\mathrm{A}^*(1) = -\infty$.

(iii) If the Riemann hypothesis holds, then $\mathrm{A}^*(\sigma) = -\infty$ for all $1/2 < \sigma \leq 1$.

Implemented at ***zero_density_energy_estimate.py*** *as:*
`add_trivial_zero_density_energy_estimates(hypotheses)`

Proof. The claim (i) follows from Lemma 10.2(iv), and the remaining claims then follow from Lemma 11.2. □

Upper bounds on $\mathrm{A}^*(\sigma)$ can be obtained from large value energy theorems via the following relation.

Lemma 12.3 (Zero density energy from large values energy)**.** *Let $1/2 < \sigma < 1$. Then*

$$\mathrm{A}^*(\sigma)(1-\sigma) \leq \max\left(\sup_{\tau \geq 1} \mathrm{LV}_\zeta^*(\sigma, \tau)/\tau, \limsup_{\tau \to \infty} \mathrm{LV}^*(\sigma, \tau)/\tau\right).$$

Proof. Write the right-hand side as B, then $B \geq 0$ (from Lemma 10.11(iii)) and we have

$$\mathrm{LV}^*_\zeta(\sigma, \tau) \leq B\tau \tag{12.1}$$

for all $\tau \geq 1$, and

$$\mathrm{LV}^*(\sigma, \tau) \leq (B + \varepsilon)\tau \tag{12.2}$$

whenever $\varepsilon > 0$ and τ is sufficiently large depending on ε (and σ). It would suffice to show, for any $\varepsilon > 0$, that $N^*(\sigma, T) \ll T^{B+O(\varepsilon)+o(1)}$ for unbounded T.

By dyadic decomposition, it suffices to show for unbounded T that the additive energy of imaginary parts of zeroes in $[T, 2T]$ is $\ll T^{B+O(\varepsilon)+o(1)}$. As in the proof of Lemma 11.5, we can assume the imaginary parts are 1-separated (here we take advantage of the triangle inequality in Lemma 10.2(iii)).

Suppose that one has a zero $\sigma' + it$ of this form. Then by standard approximations to the zeta function, one has

$$\sum_{n \leq T} \frac{1}{n^{\sigma'+it}} \ll T^{-1}.$$

Let $0 < \delta_1 < \varepsilon$ be a small quantity (independent of T) to be chosen later, and let $0 < \delta_2 < \delta_1$ be sufficiently small depending on δ_1, δ_2. By the triangle inequality, and refining the sequence t' by a factor of at most 2, we either have

$$\left| \sum_{T^{\delta_1} \leq n \leq T} \frac{1}{n^{\sigma'+it}} \right| \gg T^{-\delta_2}$$

for all zeroes, or (11.3) for all zeroes.

Suppose we are in the former ("Type I") case, we can dyadically partition and conclude from the pigeonhole principle that

$$\left| \sum_{n \in I} \frac{1}{n^{\sigma'+it}} \right| \gg T^{-\delta_2 - o(1)}$$

for some interval I in some $[N, 2N]$ with $T^{\delta_1} \ll N \ll T$, with at most $O(\log T)$ different choices for I. Performing a Fourier expansion of $n^{\sigma'}$ in $\log n$ and using the triangle inequality one can then deduce that

$$\left| \sum_{n \in I} \frac{1}{n^{it'}} \right| \gg N^{\sigma'} T^{-\delta_2 - o(1)}$$

for some $t' = t + O(T^{o(1)})$; refining the t by a factor of $T^{o(1)}$ if necessary, we may assume that the t' are 1-separated and that the interval I is independent of t', and by passing to a subsequence we may assume that $T = N^{\tau+o(1)}$ for some $1 \leq \tau \leq 1/\delta_1$, then

$$\left| \sum_{n \in I} \frac{1}{n^{it'}} \right| \gg N^{\sigma - \delta_2/\delta_1 + o(1)}$$

for all t'. If we let Σ' denote the set of such t', then by Definition 10.9 we then have (for δ_2 small enough) we have

$$E_1(\Sigma') \ll N^{\mathrm{LV}^*_\zeta(\sigma,\tau)+\varepsilon+o(1)} \ll T^{\mathrm{LV}^*_\zeta(\sigma,\tau)/\tau+\varepsilon+o(1)}.$$

By Lemma 10.2(i) this implies that the set Σ of imaginary parts of zeroes under consideration also obeys the bound

$$E_1(\Sigma) \ll T^{\mathrm{LV}^*_\zeta(\sigma,\tau)/\tau+\varepsilon+o(1)}.$$

and the claim follows in this case from (12.1).
The Type II case similarly follows from (12.2) exactly as in the proof of Lemma 11.5. $\square$

Corollary 12.4. *Let $1/2 < \sigma < 1$ and $\tau_0 > 0$ be fixed. Then*

$$\mathrm{A}^*(\sigma)(1-\sigma) \leq \max\left(\sup_{2\leq\tau<\tau_0} \mathrm{LV}^*_\zeta(\sigma,\tau)/\tau, \sup_{\tau_0\leq\tau\leq 2\tau_0} \mathrm{LV}^*(\sigma,\tau)/\tau\right)$$

Implemented at ***`zero_density_energy_estimate.py`*** *as:*
`lver_to_energy_bound(LVER, LVER_zeta, sigma_interval)`

Proof. Repeat the proof of Corollary 11.7. $\square$

12.1 Known additive energy bounds

Proposition 12.5 (Additive energy under the Lindelof hypothesis)**.** *Let $1/2 \leq \sigma \leq 1$ be fixed. Then one has*

$$\mathrm{A}^*(\sigma) \leq 8 - 4\sigma$$

and $\mathrm{A}^*(\sigma) \leq 0$ if $\sigma > 3/4$.

Proof. See [104, Lemma 4]. $\square$

Theorem 12.6 (Heath-Brown's additive energy bound)**.** *[107, Theorem 2] Let $1/2 \leq \sigma \leq 1$ be fixed. Then one can bound $A^*(\sigma)$ by*

$$\frac{10-11\sigma}{(2-\sigma)(1-\sigma)} \textit{ for } 1/2 \leq \sigma \leq 2/3;$$
$$\frac{18-19\sigma}{(4-2\sigma)(1-\sigma)} \textit{ for } 2/3 \leq \sigma \leq 3/4;$$
$$\frac{12}{4\sigma-1} \textit{ for } 3/4 \leq \sigma \leq 1.$$

Recorded in ***`literature.py`*** *as:*
`add_zero_density_energy_heath_brown_1979()`
Derived in ***`derived.py`*** *as:*
`prove_heath_brown_energy_estimate()`

Proof. We first suppose that $\sigma \leq 3/4$. Here we apply Corollary 12.4 with $\tau_0 = 2$. The LV^*_ζ supremum is now trivial, so it suffices to show that

$$\rho^* \leq \max\left(\frac{10-11\sigma}{2-\sigma}, \frac{18-19\sigma}{4-2\sigma}\right)\tau \tag{12.3}$$

whenever $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ with $2 \leq \tau \leq 3$. Let k be the first integer for which $1 \leq \tau/k \leq 3/2$, thus $k = 2, 3$ and also $\tau/(k+1) \leq 1$. By Lemma 10.12, there exist tuples

$$\left(\sigma, \frac{\tau}{k}, \rho', \frac{\rho^*}{k}, s'\right), \left(\sigma, \frac{\tau}{k+1}, \rho'', \frac{\rho^*}{k+1}, s''\right) \in \mathcal{E}. \tag{12.4}$$

for some ρ', s', ρ'' and s'' satisfying

$$\rho' \leq \frac{\rho}{k}, \qquad s' \leq \frac{s}{k}, \qquad \rho'' \leq \frac{\rho}{k+1}, \qquad s'' \leq \frac{s}{k+1}.$$

Applying Corollary 10.21 to the former tuple of (12.4) and using $\rho' \leq \rho/k$, we have

$$\frac{\rho^*}{k} \leq \max\left(\frac{3\rho}{k} + 1 - 2\sigma, \frac{\rho}{k} + 4 - 4\sigma, \frac{5\rho}{2k} + \frac{3-4\sigma}{2}\right).$$

Write $\tau' := \tau/k$. Applying Lemma 7.9 to the first tuple of (12.4) one has

$$\rho/k \leq \tau' + 1 - 2\sigma$$

while applying Lemma 7.9 to the second tuple of (12.4) (recalling that $\tau/(k+1) \leq 1$) gives

$$\rho/k = \frac{k+1}{k}\frac{\rho}{k+1} \leq \frac{k+1}{k}(2-2\sigma) \leq 3 - 3\sigma$$

and thus

$$\rho/k \leq \min(\tau' + 1 - 2\sigma, 3 - 3\sigma) \tag{12.5}$$

and

$$\begin{aligned}\rho^*/k \leq \max(&3\min(\tau' + 1 - 2\sigma, 3-3\sigma) + 1 - 2\sigma,\\ &\min(\tau' + 1 - 2\sigma, 3 - 3\sigma) + 4 - 4\sigma,\\ &5\min(\tau' + 1 - 2\sigma, 3 - 3\sigma)/2 + (3-4\sigma)/2).\end{aligned}$$

A tedious calculation shows that for $1 \leq \tau' \leq 3/2$, we have

$$3\min(\tau' + 1 - 2\sigma, 3 - 3\sigma) + 1 - 2\sigma \leq \frac{10 - 11\sigma}{2-\sigma}\tau',$$

$$\min(\tau' + 1 - 2\sigma, 3 - 3\sigma) + 4 - 4\sigma \leq \max\left(\frac{7-7\sigma}{2-\sigma}, 6 - 6\sigma\right)\tau'$$

and

$$5\min(\tau' + 1 - 2\sigma, 3 - 3\sigma)/2 + (3 - 4\sigma)/2 \leq \frac{18 - 19\sigma}{4 - 2\sigma}\tau'.$$

Since

$$\max\left(\frac{7-7\sigma}{2-\sigma}, 6-6\sigma\right) \leq \max\left(\frac{10-11\sigma}{2-\sigma}, \frac{18-19\sigma}{4-2\sigma}\right)$$

we obtain the claim.

Now suppose that $\sigma > 3/4$. From Theorem 11.18 and Lemma 12.2(i) we are already done when $\sigma \geq 25/28$, so we may assume $\sigma < 25/28$.

Here we apply Corollary 12.4 with $\tau_0 = 4\sigma - 1$. To control the LV_ζ^* term, we need to establish

$$\rho^* \leq \frac{12(1-\sigma)}{4\sigma - 1}\tau \tag{12.6}$$

whenever $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}_\zeta$ and $2 \leq \tau < 4\sigma - 1$. We use Lemma 8.3(ii) followed by Lemma 9.7 to give

$$\rho^* \leq 3\rho \leq 3(2\tau - 12(\sigma - 1/2))$$

so the claim reduces to verifying

$$3(2\tau - 12(\sigma - 1/2)) \leq \frac{12(1-\sigma)}{4\sigma - 1}\tau.$$

This holds with equality when $\tau = 4\sigma - 1$, and the slope in τ is higher on the left-hand side for $\sigma > 1/2$, so the claim (12.6) follows.

It remains to establish

$$\rho^* \leq \frac{12(1-\sigma)}{4\sigma - 1}\tau \tag{12.7}$$

whenever $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ and $4\sigma - 1 \leq \tau \leq 2(4\sigma - 1)$. Let k be the first integer for which $(4\sigma - 1)/2 \leq \tau/k \leq 3(4\sigma - 1)/4$, thus $k = 2, 3$ and also $\tau/(k+1) \leq 4\sigma - 1$. By Lemma 10.12, we have (12.4). From Theorem 7.12 we have

$$\rho/k \leq \max(2 - 2\sigma, \tau' + 4 - 6\sigma)$$

and also

$$\rho/k = \frac{k+1}{k}\frac{\rho}{k+1} \leq \frac{k+1}{k}(2 - 2\sigma) \leq 3 - 3\sigma$$

and hence

$$\rho/k \leq \min(\max(2 - 2\sigma, \tau' + 4 - 6\sigma), \tau' + 4 - 6\sigma, 3 - 3\sigma). \tag{12.8}$$

Among other things, this implies that $\rho/k \leq 1$.

From Theorem 10.20 and $\rho' \leq \rho/k$, we have

$$\begin{aligned} \rho^*/k \leq 1 - 2\sigma &+ \frac{1}{2}\max\left(\frac{\rho}{k} + 1, \frac{2\rho}{k}, \frac{5\rho}{4k} + \frac{\tau'}{2}\right) \\ &+ \frac{1}{2}\max\left(\frac{\rho^*}{k} + 1, \frac{4\rho}{k}, \frac{3\rho^*}{4k} + \frac{\rho}{k} + \frac{\tau'}{2}\right) \end{aligned} \tag{12.9}$$

where $\tau' := \tau/k$. This expression is complicated, so we divide into cases. First suppose that $\rho/k + 1 \geq 5\rho/4k + \tau'/2$. In this case the first maximum in the above expression is $\rho/k + 1$, and we simplify to

$$\rho^*/k \leq 3/2 - 2\sigma + \rho/2k + \max(\rho^*/k + 1, 4\rho/k, 3\rho^*/4k + \rho/k + \tau'/2)/2,$$

which after solving for ρ^*/k gives

$$\rho^*/k \leq \max(\rho/2k + 4 - 4\sigma, 5\rho/2k + (3 - 4\sigma)/2, 8\rho/5k + 2\tau'/5 + (12 - 16\sigma)/5).$$

Inserting (12.8), one can verify after a tedious analysis (using the hypothesis $3/4 \leq \sigma < 25/28$) that

$$\rho^*/k \leq \frac{12(1-\sigma)}{4\sigma - 1}\tau' \tag{12.10}$$

as required.

It remains to treat the case where $\rho/k + 1 > 5\rho/4k + \tau'/2$. Using (12.8) one can check that this forces

$$4\sigma - 2 \leq \tau' \leq \frac{3}{4}(4\sigma - 1), \tag{12.11}$$

so that (12.8) now becomes

$$\rho/k \leq 3 - 3\sigma. \tag{12.12}$$

The bound (12.9) becomes

$$\rho^*/k \leq 1 - 2\sigma + (5\rho/4k + \tau'/2)/2 + \max(\rho^*/k + 1, 4\rho/k, 3\rho^*/4k + \rho/k + \tau'/2)/2$$

which simplifies to

$$\rho^*/k \leq \max(5\rho/4k + \tau'/2 + 3 - 4\sigma, 21\rho/8k + \tau'/4 + 1 - 2\sigma, 9\rho/5k + 4\tau'/5 + (8 - 16\sigma)/5).$$

Inserting (12.12) and (12.11), one can eventually show (again using the hypothesis $3/4 \leq \sigma < 25/28$) that (12.10) holds as required. □

We found the following estimates with the use of computer-aided proof discovery, which improve on Theorem 12.6 in various ranges of σ. First, by using Theorem 10.20 in place of Corollary 10.21 in the proof of the previous theorem, it is possible to obtain an improved additive energy estimate for $\sigma \geq 3/4$. A human-readable proof is contained in the following theorem.

Theorem 12.7. *For* $3/4 \leq \sigma \leq 5/6$ *one has*

$$\mathrm{A}^*(\sigma) \leq \max\left(\frac{18 - 19\sigma}{2(3\sigma - 1)(1 - \sigma)}, \frac{4(10 - 9\sigma)}{5(4\sigma - 1)(1 - \sigma)}\right).$$

Derived in `derived.py` *as:*
`prove_improved_heath_brown_energy_estimate()`

Proof. Throughout assume that $3/4 \leq \sigma \leq 5/6$. Choose

$$\tau_0 = 8\sigma - 4.$$

We will show that

$$\rho^*/\tau \leq \begin{cases} \dfrac{18 - 19\sigma}{2(3\sigma - 1)}, & 3/4 \leq \sigma < 4/5, \\ \dfrac{7(1 - \sigma)}{3\sigma - 1}, & 4/5 \leq \sigma \leq 5/6, \end{cases} \tag{12.13}$$

for all $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ for which $\tau_0 \leq \tau \leq 2\tau_0$, and that

$$\rho^*/\tau \leq \max\left(\frac{45 - 46\sigma}{4(4\sigma - 1)}, \frac{4(10 - 9\sigma)}{5(4\sigma - 1)}\right), \tag{12.14}$$

for all $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}_\zeta$ such that $2 \leq \tau \leq \tau_0$. The desired result (i) then follows from Corollary 12.4 and computing the piecewise maximum of (12.13) and (12.14).
First, consider (12.14). Suppose that $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}_\zeta$ with $3/4 \leq \sigma \leq 5/6$ and $2 \leq \tau \leq \tau_0$. Then, from Theorem 9.7 we have

$$\rho \leq 2\tau - 12(\sigma - 1/2). \tag{12.15}$$

Furthermore, by Theorem 7.12 and Lemma 7.8 with $k = 2$, one has $\rho \leq 2\max(2 - 2\sigma, 4 - 6\sigma + \tau/2)$. However since $\tau \leq \tau_0 = 8\sigma - 4$, this simplifies to

$$\rho \leq 4 - 4\sigma. \tag{12.16}$$

Since $\sigma \geq 3/4$, this also implies that $\rho \leq 1$. For future reference we also note that

$$\frac{4}{5} < \max\left(\frac{45-46\sigma}{4(4\sigma-1)}, \frac{4(10-9\sigma)}{5(4\sigma-1)}\right) < 2, \qquad (3/4 \leq \sigma \leq 5/6). \tag{12.17}$$

By Theorem 10.20, one has

$$\rho^* \leq 1 - 2\sigma + \frac{1}{2}\max\left(\rho+1, 2\rho, \frac{5}{4}\rho + \frac{\tau}{2}\right) + \frac{1}{2}\max\left(\rho^*+1, 4\rho, \frac{3}{4}\rho^* + \rho + \frac{\tau}{2}\right).$$

Since $\rho \leq 1$, one has $\rho + 1 \geq 2\rho$. Thus the middle term in the first maximum may be omitted, and we are left with two cases to consider.
Case 1: If $\rho + 1 \geq 5\rho/4 + \tau/2$ then

$$\rho^* \leq 1 - 2\sigma + \frac{\rho+1}{2} + \frac{1}{2}\max\left(\rho^*+1, 4\rho, \frac{3}{4}\rho^* + \rho + \frac{\tau}{2}\right).$$

Solving for ρ^* gives

$$\rho^* \leq \max\left(4 - 4\sigma + \rho, \frac{3}{2} - 2\sigma + \frac{5}{2}\rho, \frac{2}{5}(6 - 8\sigma + \tau + 4\rho)\right).$$

Applying (12.16) to each term,

$$\begin{aligned}\rho^* &\leq \max\left(8 - 8\sigma, \frac{23}{2} - 12\sigma, \frac{2}{5}(22 - 24\sigma + \tau)\right)\\ &\leq \max\left(\frac{45-46\sigma}{4(4\sigma-1)}\tau, \frac{4(10-9\sigma)}{5(4\sigma-1)}\tau\right),\end{aligned}$$

i.e. (12.14) holds. The last inequality may be verified by inspecting the growth rates with respect to τ of each term (using (12.17)), and checking that the desired inequality holds at $\tau = 2$.
Case 2: If $\rho + 1 < 5\rho/4 + \tau/2$, then

$$\rho^* \leq 1 - 2\sigma + \frac{5}{8}\rho + \frac{\tau}{4} + \frac{1}{2}\max\left(\rho^*+1, 4\rho, \frac{3}{4}\rho^* + \rho + \frac{\tau}{2}\right).$$

Solving for ρ gives

$$\rho^* \leq \max\left(3 - 4\sigma + \frac{\tau}{2} + \frac{5}{4}\rho, 1 - 2\sigma + \frac{\tau}{4} + \frac{21}{8}\rho, \frac{8 - 16\sigma + 4\tau + 9\rho}{5}\right).$$

If $\tau \geq 4\sigma - 1$, then apply (12.16) termwise to get

$$\begin{aligned}\rho^* &\leq \max\left(8 - 9\sigma + \frac{\tau}{2}, \frac{23}{2} - \frac{25}{2} + \frac{\tau}{4}, \frac{4}{5}(11 - 13\sigma + \tau)\right)\\ &\leq \max\left(\frac{45-46\sigma}{4(4\sigma-1)}\tau, \frac{4(10-9\sigma)}{5(4\sigma-1)}\tau\right),\end{aligned}$$

since the RHS is increasing faster in τ and at $\tau = 4\sigma - 1$ we have equality.
On the other hand if $\tau < 4\sigma - 1$ then we apply (12.15) termwise to get

$$\begin{aligned}\rho^* &\leq \max\left(\frac{21}{2} - 19\sigma + 3\tau, \frac{67 - 134\sigma + 22\tau}{4}, \frac{2}{5}(31 - 62\sigma + 11\tau)\right)\\ &\leq \max\left(\frac{45-46\sigma}{4(4\sigma-1)}\tau, \frac{4(10-9\sigma)}{5(4\sigma-1)}\tau\right),\end{aligned}$$

by inspecting growth rates in τ and noting that at $\tau = 4\sigma - 1$ one has equality.
Thus we have shown that if $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}_\zeta$ with $3/4 \leq \sigma \leq 5/6$ and $2 \leq \tau \leq 8\sigma - 4$, then

$$\rho^*/\tau \leq \min\left(\frac{45 - 46\sigma}{4(4\sigma - 1)}, \frac{4(10 - 9\sigma)}{5(4\sigma - 1)}\right),$$

which is (12.14).
Now consider (12.13). Suppose that $\tau_0 \leq \tau \leq 2\tau_0$ and $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$. Note that the interval $[\tau_0, 2\tau_0]$ is covered by intervals $I_k := [(4\sigma - 2)k, (4\sigma - 2)(k + 1)]$ with $k = 2, 3$. Suppose that $\tau \in I_k$, and write

$$\tau' := \tau/k.$$

Then, by Theorem 7.12 and $\tau' \geq 4\sigma - 2$ one has

$$\rho/k \leq \max(2 - 2\sigma, 4 - 6\sigma + \tau') = 4 - 6\sigma + \tau'.$$

Also, from Theorem 7.12 and $\tau \leq (4\sigma - 2)(k + 1)$ one has

$$\rho/(k + 1) \leq \max(2 - 2\sigma, 4 - 6\sigma + \tau/(k + 1)) \leq 2 - 2\sigma$$

so that for $k = 2, 3$ one has $\rho/k \leq (2 - 2\sigma)(k + 1)/k \leq 3 - 3\sigma$. In summary,

$$\rho/k \leq \min(3 - 3\sigma, 4 - 6\sigma + \tau'). \tag{12.18}$$

Next, by Lemma 10.12,

$$(\sigma, \tau', \rho', \rho^*/k, s') \in \mathcal{E}$$

for $\tau' := \tau/k$ and some $\rho' \leq \rho/k$ and $s' \leq s/k$.
Applying Theorem 10.20 to this tuple, then applying $\rho' \leq \rho/k$, one has

$$\begin{aligned}\frac{\rho^*}{k} \leq 1 - 2\sigma &+ \frac{1}{2}\max\left(\frac{\rho}{k} + 1, \frac{2\rho}{k}, \frac{5\rho}{4k} + \frac{\tau'}{2}\right) \\ &+ \frac{1}{2}\max\left(\frac{\rho^*}{k} + 1, \frac{4\rho}{k}, \frac{3}{4}\frac{\rho^*}{k} + \frac{\rho}{k} + \frac{\tau'}{2}\right).\end{aligned}$$

By (12.18) one has $\rho/k \leq 1$ (since $\sigma \geq 3/4$) so there are only two cases to consider:
Case 1: $\rho/k + 1 \geq 5\rho/(4k) + \tau'/2$ then

$$\frac{\rho^*}{k} \leq \frac{3}{2} - 2\sigma + \frac{\rho}{2k} + \frac{1}{2}\max\left(\frac{\rho^*}{k} + 1, \frac{4\rho}{k}, \frac{3}{4}\frac{\rho^*}{k} + \frac{\rho}{k} + \frac{\tau'}{2}\right).$$

Solving for ρ^*/k, we get

$$\frac{\rho^*}{k} \leq \max\left(4(1 - \sigma) + \frac{\rho}{k}, \frac{(3 - 4\sigma) + 5\rho/k}{2}, \frac{2}{5}((6 - 8\sigma) + \tau' + \frac{4\rho}{k})\right).$$

If $\tau' \geq 3\sigma - 1$ then (12.18) reduces to $\rho/k \leq 3 - 3\sigma$. Substituting this bound gives

$$\rho^*/k \leq \max(7 - 7\sigma, 9 - 19\sigma/2, 32(1 - \sigma)/5).$$

However the RHS is bounded by

$$\frac{18 - 19\sigma}{6\sigma - 2}\tau' \tag{12.19}$$

for all $\tau' \geq 3\sigma - 1$ since the desired bound holds at $\tau' = 3\sigma - 1$ (where one has equality).

On the other hand, if $4\sigma - 2 \le \tau' \le 3\sigma - 1$ then (12.18) reduces to $\rho/k \le 4 - 6\sigma + \tau'$. Substituting this bound gives

$$\begin{aligned}\rho^*/k &\le \max\left(8 - 10\sigma + \tau', \frac{23 - 34\sigma + 5\tau'}{2}, \frac{2}{5}(22 - 32\sigma + 5\tau')\right)\\ &\le \frac{18 - 19\sigma}{6\sigma - 2}\tau'\end{aligned} \tag{12.20}$$

where the last inequality may be established by checking that it holds at both $\tau' = 4\sigma - 2$ and $\tau' = 3\sigma - 1$ (where one has equality). To summarise, by taking $k = 2, 3$ in the (12.19) and (12.20), one has

$$\rho^* \le \frac{18 - 19\sigma}{6\sigma - 2}\tau, \qquad (\tau_0 \le \tau \le 2\tau_0)$$

in this case.
Case 2: $\rho/k + 1 < 5\rho/(4k) + \tau'/2$ then

$$\rho^*/k \le 1 - 2\sigma + \frac{5\rho}{8k} + \frac{\tau'}{4} + \frac{1}{2}\max\left(\frac{\rho^*}{k} + 1, \frac{4\rho}{k}, \frac{3}{4}\frac{\rho^*}{k} + \frac{\rho}{k} + \frac{\tau'}{2}\right).$$

Solving for ρ^*/k gives

$$\rho^*/k \le \max\left(3 - 4\sigma + \frac{\tau'}{2} + \frac{5}{4}\frac{\rho}{k}, 1 - 2\sigma + \frac{\tau'}{4} + \frac{21}{8}\frac{\rho}{k}, \frac{1}{5}(8 - 16\sigma + 4\tau' + 9\frac{\rho}{k})\right).$$

Proceeding as before, if $\tau' \ge 3\sigma - 1$ then (12.18) becomes $\rho/k \le 3 - 3\sigma$, and substituting gives

$$\rho^*/k \le \max\left(\frac{27 - 31\sigma}{4} + \frac{\tau'}{2}, \frac{71 - 79\sigma}{8} + \frac{\tau'}{4}, 7 - \frac{43}{5}\sigma + \frac{4}{5}\tau'\right).$$

One may verify that the RHS is bounded by

$$\frac{18 - 19\sigma}{6\sigma - 2}\tau'$$

(with some room to spare) by checking at the endpoint $\tau' = 3\sigma - 1$.
Similarly, if $4\sigma - 2 \le \tau' \le 3\sigma - 1$ then using $\rho/k \le 4 - 6\sigma + \tau'$ from (12.18) one has

$$\rho^*/k \le \max\left(8 - \frac{23\sigma}{2} + \frac{7\tau'}{4}, \frac{23}{2} - \frac{71\sigma}{4} + \frac{23\tau'}{8}, \frac{44 - 70\sigma + 13\tau'}{5}\right).$$

One can check that the RHS is bounded by

$$\frac{18 - 19\sigma}{6\sigma - 2}\tau'$$

by checking the required inequalities hold at $\tau' = 4\sigma - 2$ and $\tau' = 3\sigma - 1$ (in each case, with some room to spare).
Combining all the cases, by taking $k = 2, 3$ we have shown that for $3/4 \le \sigma \le 4/5$ and $\tau_0 \le \tau \le 2\tau_0$ one has

$$\rho^* \le \frac{18 - 19\sigma}{6\sigma - 2}\tau$$

which is the first part of (12.13).

The case for $4/5 \le \sigma \le 5/6$ may be treated similarly. Here one needs to verify that

$$\max(7 - 7\sigma, 9 - 19\sigma/2, 32(1-\sigma)/5) \le \frac{7(1-\sigma)}{3\sigma - 1}\tau',$$

$$\max\left(\frac{27 - 31\sigma}{4} + \frac{\tau'}{2}, \frac{71 - 79\sigma}{8} + \frac{\tau'}{4}, 7 - \frac{43}{5}\sigma + \frac{4}{5}\tau'\right) \le \frac{7(1-\sigma)}{3\sigma - 1}\tau'$$

for $\tau' \ge 3\sigma - 1$, and that

$$\max\left(8 - 10\sigma + \tau', \frac{23 - 34\sigma + 5\tau'}{2}, \frac{2}{5}(22 - 32\sigma + 5\tau')\right) \le \frac{7(1-\sigma)}{3\sigma - 1}\tau',$$

$$\max\left(8 - \frac{23\sigma}{2} + \frac{7\tau'}{4}, \frac{23}{2} - \frac{71\sigma}{4} + \frac{23\tau'}{8}, \frac{44 - 70\sigma + 13\tau'}{5}\right) \le \frac{7(1-\sigma)}{3\sigma - 1}\tau'$$

for $4\sigma - 2 \le \tau' \le 3\sigma - 1$. The treatment is analogous to before, so we omit the proof. □

Using Theorem 10.27, it is possible to obtain improved energy estimates near $\sigma = 3/4$, which are given by the next two theorems.

Theorem 12.8. *For $7/10 \le \sigma \le 3/4$, one has*

$$\mathrm{A}^*(\sigma) \le \max\left(\frac{5(18 - 19\sigma)}{2(5\sigma + 3)(1-\sigma)}, \frac{2(45 - 44\sigma)}{(2\sigma + 15)(1-\sigma)}\right).$$

Derived in ***`derived.py`*** *as:*
`prove_zero_density_energy_2()`

Proof. Throughout assume $7/10 \le \sigma \le 3/4$ and take $\tau_0 = 2$ in Corollary 12.4. It suffices to show that if $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ with $2 \le \tau \le 4$, then either

$$\rho^* \le \frac{5(18 - 19\sigma)}{2(5\sigma + 3)}\tau \tag{12.21}$$

or

$$\rho^* \le \frac{2(45 - 44\sigma)}{2\sigma + 15}\tau. \tag{12.22}$$

Note for future reference the crude bounds

$$1 < \frac{5(18 - 19\sigma)}{2(5\sigma + 3)} < 2, \qquad 1 < \frac{2(45 - 44\sigma)}{2\sigma + 15} < \frac{7}{4}. \tag{12.23}$$

Let

$$k := \begin{cases} 2, & 2 \le \tau < 3, \\ 3, & 3 \le \tau \le 4, \end{cases} \qquad \tau' := \tau/k.$$

Via Theorem 7.9 and Lemma 7.8, one has

$$\rho/k \le \max(2 - 2\sigma, 1 - 2\sigma + \tau') \tag{12.24}$$

and via Theorem 10.27 and Lemma 7.8, one has

$$\rho/k \le \max(2 - 2\sigma, 18/5 - 4\sigma, 12/5 - 4\sigma + \tau').$$

Since $\sigma \le 4/5$, we may drop the first term, i.e.

$$\rho/k \le \max(18/5 - 4\sigma, 12/5 - 4\sigma + \tau'). \tag{12.25}$$

Combining (12.24) and (12.25),

$$\rho/k \le \begin{cases} 1 - 2\sigma + \tau', & 1 \le \tau' \le 13/5 - 2\sigma, \\ 18/5 - 4\sigma, & 13/5 - 2\sigma \le \tau' \le 6/5, \\ 12/5 - 4\sigma + \tau', & 6/5 \le \tau' \le 2(\sigma - 1/5) + 2(1-\sigma)/k, \\ (2-2\sigma)(k+1)/k, & 2(\sigma - 1/5) + 2(1-\sigma)/k \le \tau' \le 1 + 1/k. \end{cases} \tag{12.26}$$

One can verify that all intervals are proper for $7/10 \le \sigma \le 3/4$ and $k = 2, 3$.
Since $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$, by Lemma 10.12 one has $(\sigma, \tau', \rho', \rho^*/k, s') \in \mathcal{E}$ for some $\rho' \le \rho/k$ and $s' \le s/k$. Applying Theorem (10.20)to the first tuple followed by $\rho' \le \rho/k$, and noting that $\rho/k + 1 \ge 2\rho/k$ since $\rho/k \le 1$ by (12.26), one obtains

$$\frac{\rho^*}{k} \le 1 - 2\sigma + \frac{1}{2}\max\left(\frac{\rho}{k} + 1, \frac{5\rho}{4k} + \frac{\tau'}{2}\right) + \frac{1}{2}\max\left(\frac{\rho^*}{k} + 1, \frac{4\rho}{k}, \frac{3\rho^*}{4k} + \frac{\rho}{k} + \frac{\tau'}{2}\right). \tag{12.27}$$

First, suppose that $\rho/k + 1 < 5\rho/(4k) + \tau'/2$ so that

$$\frac{\rho^*}{k} \le 1 - 2\sigma + \frac{5\rho}{8k} + \frac{\tau'}{4} + \frac{1}{2}\max\left(\frac{\rho^*}{k} + 1, \frac{4\rho}{k}, \frac{3}{4}\frac{\rho^*}{k} + \frac{\rho}{k} + \frac{\tau'}{2}\right).$$

Solving for ρ^*/k gives

$$\frac{\rho^*}{k} \le \max\left(3 - 4\sigma + \frac{\tau'}{2} + \frac{5}{4}\frac{\rho}{k}, 1 - 2\sigma + \frac{\tau'}{4} + \frac{21}{8}\frac{\rho}{k}, \frac{1}{5}(8 - 16\sigma + 4\tau' + 9\frac{\rho}{k})\right).$$

One may verify that the RHS is bounded by

$$\frac{5(18 - 19\sigma)}{2(5\sigma + 3)}\tau'$$

by substituting each case of (12.26). This involves the tedious verification of the following four inequalities:

$$\max\left(\frac{17}{4} - \frac{13}{2}\sigma + \frac{7}{4}\tau', \frac{29}{8} - \frac{29}{4}\sigma + \frac{23}{8}\tau', \frac{17}{5} - \frac{34}{5}\sigma + \frac{13}{5}\tau'\right) \le \frac{5(18 - 19\sigma)}{2(5\sigma + 3)}\tau' \tag{12.28}$$

for $1 \le \tau' \le 13/5 - 2\sigma$;

$$\max\left(\frac{15}{2} - 9\sigma + \frac{\tau'}{2}, \frac{209}{20} - \frac{25}{2}\sigma + \frac{\tau'}{4}, \frac{202}{25} - \frac{52}{5}\sigma + \frac{4}{5}\tau'\right) \le \frac{5(18 - 19\sigma)}{2(5\sigma + 3)}\tau'$$

for $13/5 - 2\sigma \le \tau' \le 6/5$;

$$\max\left(6 - 9\sigma + \frac{7}{4}\tau', \frac{73}{10} - \frac{25}{2}\sigma + \frac{23}{8}\tau', \frac{148}{25} - \frac{52}{5}\sigma + \frac{13}{5}\tau'\right) \le \frac{5(18 - 19\sigma)}{2(5\sigma + 3)}\tau'$$

for $k = 2, 3$ and $6/5 \le \tau' \le 2(\sigma - 1/5) + 2(1-\sigma)/k$;

$$\max(3 - 4\sigma + \frac{\tau'}{2} + \frac{5(2-2\sigma)}{4}\frac{k+1}{k}, 1 - 2\sigma + \frac{\tau'}{4} + \frac{21(2-2\sigma)}{8}\frac{k+1}{k},$$
$$\frac{1}{5}(8 - 16\sigma + 4\tau' + 9(2-2\sigma)\frac{k+1}{k})) \le \frac{5(18 - 19\sigma)}{2(5\sigma + 3)}\tau'$$

for $k = 2, 3$ and $2(\sigma - 1/5) + 2(1-\sigma)/k \le \tau' \le 1 + 1/k$. For instance, in the case of (12.28), the LHS is increasing faster with respect to τ' than the RHS in view of (12.23), and the inequality holds at the upper limit $\tau' = 13/5 - 2\sigma$ (with some room to spare). The other inequalities may be verified similarly, with the exception of

$$6 - 9\sigma + \frac{7}{4}\tau' \le \frac{5(18-19\sigma)}{2(5\sigma+3)}\tau'$$

which is equivalent to $6 - 9\sigma \le 3(53 - 75\sigma)/(4(5\sigma+3))\tau'$. For $\sigma < 53/75$ the LHS is negative while the RHS is positive so the inequality holds. For $\sigma \ge 53/75$ one may verify that the inequality holds at the lower limit $\tau' = 6/5$.

In the remainder of the proof we assume $\rho/k + 1 \ge 5\rho/(4k) + \tau'/2$ so that (12.27) becomes

$$\frac{\rho^*}{k} \le \frac{3}{2} - 2\sigma + \frac{\rho}{2k} + \frac{1}{2}\max\left(\frac{\rho^*}{k} + 1, \frac{4\rho}{k}, \frac{3}{4}\frac{\rho^*}{k} + \frac{\rho}{k} + \frac{\tau'}{2}\right).$$

Solving for ρ^*/k gives

$$\frac{\rho^*}{k} \le \max\left(4 - 4\sigma + \frac{\rho}{k}, \frac{3 - 4\sigma + 5\rho/k}{2}, \frac{2}{5}(6 - 8\sigma + \tau' + \frac{4\rho}{k})\right). \tag{12.29}$$

The case where $\tau' \ge 6/5$ is simpler so we handle it first. Applying the last two cases of (12.26) it suffices to verify that

$$\max\left(\frac{32}{5} - 8\sigma + \tau', \frac{15}{2} - 12\sigma + \frac{5}{2}\tau', \frac{156}{25} - \frac{48}{5}\sigma + 2\tau'\right) \le \frac{5(18-19\sigma)}{2(5\sigma+3)}\tau'$$

for $k = 2, 3$ and $6/5 \le \tau' \le 2(\sigma - 1/5) + 2(1-\sigma)/k$;

$$\max(4 - 4\sigma + (2-2\sigma)\frac{k+1}{k}, \frac{3-4\sigma}{2} + (5 - 5\sigma)\frac{k+1}{k},$$
$$\frac{2}{5}(6 - 8\sigma + \tau' + (8 - 8\sigma)\frac{k+1}{k})) \le \frac{5(18-19\sigma)}{2(5\sigma+3)}\tau'$$

for $k = 2, 3$ and $2(\sigma - 1/5) + 2(1-\sigma)/k \le \tau' \le 1 + 1/k$. Note also that one has equality when $\tau' = 2(\sigma - 1/5) + 2(1-\sigma)/k$ and $k = 2$.

Lastly, consider the case where $1 \le \tau' \le 6/5$. Applying $\rho/k \le \min(1 - 2\sigma + \tau', 18/5 - 4\sigma)$ from the first two cases of (12.26), one obtains

$$\frac{3 - 4\sigma + 5\rho/k}{2} \le \min\left(4 - 7\sigma + \frac{5}{2}\tau', \frac{21}{2} - 12\sigma\right),$$

$$\frac{2}{5}(6 - 8\sigma + \tau' + \frac{4\rho}{k}) \le \min\left(4 - \frac{32}{5}\sigma + 2\tau', \frac{204}{25} - \frac{48}{5}\sigma + \frac{2}{5}\tau'\right).$$

However one may verify that the RHS of both of the above inequalities are bounded by $5(18-19\sigma)/(2(5\sigma+3))\tau'$ for $1 \le \tau' \le 6/5$ by checking at $\tau' = 13/5 - 2\sigma$. Thus

$$\frac{\rho^*}{k} \le \max\left(\frac{5(18-19\sigma)}{2(5\sigma+3)}\tau', 4 - 4\sigma + \frac{\rho}{k}\right). \tag{12.30}$$

Meanwhile, by Lemma 10.12,

$$(\sigma, \tau/(k-1), \rho'', \rho^*/(k-1), s'') \in \mathcal{E}$$

for some $\rho'' \leq \rho/(k-1)$ and $s'' \leq s/(k-1)$. Applying Theorem 10.20 to this tuple (and applying $\rho'' \leq \rho/(k-1)$) gives

$$\begin{aligned} \frac{\rho^*}{k-1} \leq 1 - 2\sigma + \frac{1}{2}\max(\frac{\rho}{k-1} + 1, \frac{2\rho}{k-1}, \frac{5\rho}{4(k-1)} + \frac{\tau}{2(k-1)}) \\ + \frac{1}{2}\max(\frac{\rho^*}{k-1} + 1, \frac{4\rho}{k-1}, \frac{3\rho^*}{4(k-1)} + \frac{\rho}{k-1} + \frac{\tau}{2(k-1)}). \end{aligned} \tag{12.31}$$

By expanding the first maximum and simplifying, one of the following inequalities must hold:

$$\rho^* \leq (\frac{3}{2} - 2\sigma)(k-1) + \frac{\rho}{2} + \frac{1}{2}\max(\rho^* + k - 1, 4\rho, \frac{3\rho^*}{4} + \rho + \frac{\tau}{2}), \tag{12.32}$$

$$\rho^* \leq (1 - 2\sigma)(k-1) + \rho + \frac{1}{2}\max(\rho^* + k - 1, 4\rho, \frac{3\rho^*}{4} + \rho + \frac{\tau}{2}), \tag{12.33}$$

$$\rho^* \leq (1 - 2\sigma)(k-1) + \frac{5}{8}\rho + \frac{\tau}{4} + \frac{1}{2}\max(\rho^* + k - 1, 4\rho, \frac{3\rho^*}{4} + \rho + \frac{\tau}{2}). \tag{12.34}$$

If (12.32) holds, then solving for ρ^*/k gives

$$\rho^*/k \leq \max((4 - 4\sigma)\frac{k-1}{k} + \frac{\rho}{k}, \frac{(3 - 4\sigma)(k-1)/k + 5\rho/k}{2}, \frac{2}{5}((6 - 8\sigma)\frac{k-1}{k} + \tau' + \frac{4\rho}{k})).$$

For $\tau' \leq 13/5 - 2\sigma$, we apply $\rho/k \leq 1 - 2\sigma + \tau'$ from (12.26) (along with the inequalities $4 - 4\sigma \geq 0$, $3 - 4\sigma \geq 0$, $6 - 8\sigma \geq 0$ and $(k-1)/k \leq 2/3$),

$$\frac{\rho^*}{k} \leq \max\left(\frac{11}{3} - \frac{14}{3}\sigma + \tau', \frac{7}{2} - \frac{19}{3}\sigma + \frac{5}{2}\tau', \frac{16}{5} - \frac{16}{3}\sigma + 2\tau'\right) < \frac{5(18 - 19\sigma)}{2(5\sigma + 3)}\tau'$$

for all $1 \leq \tau' \leq 13/5 - 2\sigma$. The last inequality is verified using (12.23) and checking at both $\tau' = 1$ and $\tau' = 13/5 - 2\sigma$. Similarly, for $13/5 - 2\sigma \leq \tau' \leq 6/5$ we use $\rho/k \leq 18/5 - 4\sigma$ and verify that

$$\frac{\rho^*}{k} \leq \max\left(\frac{94}{15} - \frac{20}{3}\sigma, 10 - \frac{34}{3}\sigma, \frac{184}{25} - \frac{128}{15}\sigma + \frac{2}{5}\tau'\right) < \frac{5(18 - 19\sigma)}{2(5\sigma + 3)}\tau'$$

where the last inequality is verified using (12.23) and checking at the lower limit $\tau' = 13/5 - 2\sigma$.

Suppose now that (12.33) holds. Solving for ρ^*/k gives

$$\frac{\rho^*}{k} \leq \max\left((3 - 4\sigma)\frac{k-1}{k} + 2\frac{\rho}{k}, (1 - 2\sigma)\frac{k-1}{k} + 3\frac{\rho}{k}, \frac{2}{5}((4 - 8\sigma)\frac{k-1}{k} + \tau' + 6\frac{\rho}{k})\right)$$

Similarly to before, applying the first two cases of (12.26) allows one to verify that for $1 \leq \tau' \leq 6/5$,

$$(3 - 4\sigma)\frac{k-1}{k} + 2\frac{\rho}{k} \leq \frac{2}{3}(3 - 4\sigma) + 2\min(1 - 2\sigma + \tau', 18/5 - 4\sigma) \leq \frac{5(18 - 19\sigma)}{2(5\sigma + 3)}\tau'$$

and

$$\frac{2}{5}((4 - 8\sigma)\frac{k-1}{k} + \tau' + 6\frac{\rho}{k}) \leq \frac{2}{5}(\frac{1}{2}(4 - 8\sigma) + \tau' + 6\min(1 - 2\sigma + \tau', 18/5 - 4\sigma)) \leq \frac{5(18 - 19\sigma)}{2(5\sigma + 3)}\tau',$$

each with room to spare. Therefore, for $1 \le \tau' \le 6/5$, one has

$$\frac{\rho^*}{k} \le \max\left(\frac{5(18-19\sigma)}{2(5\sigma+3)}\tau', (1-2\sigma)\frac{k-1}{k} + \frac{3\rho}{k}\right) \tag{12.35}$$

However, if $\tau' \le \sigma + 1/2 - (2\sigma-1)/(2k)$ we may apply $\rho/k \le 1 - 2\sigma + \tau'$ to get

$$(1-2\sigma)\frac{k-1}{k} + \frac{3\rho}{k} \le (1-2\sigma)\frac{k-1}{k} + 3(1-2\sigma+\tau') \le \max\left(\frac{5(18-19\sigma)}{2(5\sigma+3)}, \frac{2(45-44\sigma)}{2\sigma+15}\right)\tau',$$

where by (12.23), the last inequality is verified by checking that it holds at the upper limit $\tau' = \sigma + 1/2 - (2\sigma-1)/(2k)$ for $k = 2, 3$. For $\tau' > \sigma + 1/2 - (2\sigma-1)/(2k)$, we once again apply $\rho/k \le 1 - 2\sigma + \tau'$ to get

$$4 - 4\sigma + \frac{\rho}{k} \le 5 - 6\sigma + \tau' \le \max\left(\frac{5(18-19\sigma)}{2(5\sigma+3)}, \frac{2(45-44\sigma)}{2\sigma+15}\right)\tau',$$

where now the last inequality is verified at the lower limit $\tau' = \sigma + 1/2 - (2\sigma-1)/(2k)$. Therefore, in view of (12.30) and (12.35), one has

$$\frac{\rho^*}{k} \le \max\left(\frac{5(18-19\sigma)}{2(5\sigma+3)}, \frac{2(45-44\sigma)}{2\sigma+15}\right)\tau'$$

in this case, as required.

Lastly, suppose that (12.34) holds. Then solving for ρ^*/k gives

$$\frac{\rho^*}{k} \le \max((3-4\sigma)\frac{k-1}{k} + \frac{\tau'}{2} + \frac{5}{4}\frac{\rho}{k}, (1-2\sigma)\frac{k-1}{k} + \frac{\tau'}{4} + \frac{21}{8}\frac{\rho}{k}, \frac{(8-16\sigma)(k-1)/k + 4\tau' + 9\rho/k}{5}).$$

Proceeding as before, we use $\rho/k \le 1 - 2\sigma + \tau'$ from (12.26) together with $(k-1)/k \le 2/3$ to get

$$(3-4\sigma)\frac{k-1}{k} + \frac{\tau'}{2} + \frac{5}{4}\frac{\rho}{k} \le \frac{13}{4} - \frac{31}{6}\sigma + \frac{7}{4}\tau' \le \frac{2(45-44\sigma)}{2\sigma+15}\tau',$$

where the last inequality is verified at $\tau' = 6/5$. Furthermore, using $\rho/k \le \min(1 - 2\sigma + \tau', 18/5 - 4\sigma)$ and $(k-1)/k \ge 1/2$ one has

$$\begin{aligned}\frac{(8-16\sigma)(k-1)/k + 4\tau' + 9\rho/k}{5} &\le \min\left(\frac{13}{5} - \frac{26}{5}\sigma + \frac{13}{5}\tau', \frac{182}{25} - \frac{44}{5}\sigma + \frac{4}{5}\tau'\right) \\ &\le \max\left(\frac{5(18-19\sigma)}{2(5\sigma+3)}, \frac{2(45-44\sigma)}{2\sigma+15}\right)\tau'\end{aligned}$$

for $1 \le \tau' \le 6/5$. Therefore,

$$\frac{\rho^*}{k} \le \max\left(\frac{5(18-19\sigma)}{2(5\sigma+3)}\tau', \frac{2(45-44\sigma)}{2\sigma+15}\tau', (1-2\sigma)\frac{k-1}{k} + \frac{\tau'}{4} + \frac{21}{8}\frac{\rho}{k}\right).$$

If $1 \le \tau' \le 1 + 2\sigma/15$, we use the bound $\rho/k \le 1 - 2\sigma + \tau'$ to get

$$(1-2\sigma)\frac{k-1}{k} + \frac{\tau'}{4} + \frac{21}{8}\frac{\rho}{k} \le (1-2\sigma)\frac{k-1}{k} + \frac{\tau'}{4} + \frac{21}{8}(1-2\sigma+\tau') \le \frac{2(45-44\sigma)}{2\sigma+15}\tau'$$

where by (12.23) it suffices to check the inequality at the upper limit $\tau' = 1+2\sigma/15$ (where we have equality if $k = 2$). On the other hand if $1+2\sigma/15 \le \tau' \le 6/5$, we use $\rho/k \le 1-2\sigma+\tau'$ to get

$$4 - 4\sigma + \frac{\rho}{k} \le 5 - 6\sigma + \tau' \le \frac{2(45-44\sigma)}{2\sigma+15}\tau'$$

where by (12.23) it suffices to check the last inequality at $\tau' = 1 + 2\sigma/15$ (where we have equality). Therefore, one has

$$\frac{\rho^*}{k} \leq \max\left(\frac{5(18-19\sigma)}{2(5\sigma+3)}, \frac{2(45-44\sigma)}{2\sigma+15}\right)\tau'$$

in this case too.

□

Theorem 12.9. *For $3/4 \leq \sigma \leq 4/5$, one has*

$$\mathrm{A}^*(\sigma) \leq \max\left(\frac{197-220\sigma}{8(5\sigma-1)(1-\sigma)}, \frac{3(29-30\sigma)}{5(5\sigma-1)(1-\sigma)}, \frac{4(10-9\sigma)}{5(4\sigma-1)(1-\sigma)}\right)$$

Derived in `derived.py` *as:*
`prove_zero_density_energy_3()`

Proof. Throughout assume that $3/4 \leq \sigma \leq 4/5$ and take $\tau_0 := 8\sigma - 4$ in Corollary 12.4. It suffices to show that

$$\rho^* \leq \max\left(\frac{197-220\sigma}{8(5\sigma-1)}, \frac{3(29-30\sigma)}{5(5\sigma-1)}\right)\tau \tag{12.36}$$

for all $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ satisfying $\tau_0 \leq \tau \leq 2\tau_0$, and

$$\rho^* \leq \frac{4(10-9\sigma)}{5(4\sigma-1)}\tau \tag{12.37}$$

for all $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}_\zeta$ such that $2 \leq \tau \leq \tau_0$. In the proof of Theorem 12.7 we have already shown that (12.37) holds in the large range $65/86 \leq \sigma \leq 5/6$, so it remains to prove (12.36). Given σ, τ, let $k \geq 2$ be the integer for which

$$k \leq \frac{\tau}{4\sigma-2} < k+1 \tag{12.38}$$

so that $k = 2, 3$ for $\tau_0 \leq \tau < 2\tau_0$, and as before write $\tau' := \tau/k$.
By Theorem 10.27 and Lemma 7.8 one has

$$\rho/k \leq \max(18/5-4\sigma, 12/5-4\sigma+\tau') = \begin{cases} 18/5-4\sigma, & \tau' \leq 6/5 \\ 12/5-4\sigma+\tau', & \tau' > 6/5 \end{cases}$$

and from Theorem 7.12 and Lemma 7.8, for any integer ℓ,

$$\rho/\ell \leq \max(2-2\sigma, 4-6\sigma+\tau/\ell) = \begin{cases} 2-2\sigma, & \tau/\ell \leq 4\sigma-2, \\ 4-6\sigma+\tau/\ell, & \tau/\ell > 4\sigma-2. \end{cases}$$

so that in particular, taking $\ell = k+1$ and noting that $\tau/(k+1) \leq 4\sigma - 2$ by (12.38),

$$\rho/k = \frac{k+1}{k}\frac{\rho}{k+1} \leq \frac{k+1}{k}\max(2-2\sigma, 4-6\sigma+\frac{\tau}{k+1}) = (2-2\sigma)\frac{k+1}{k} \leq 3-3\sigma.$$

Combining everything, one obtains (for $k \geq 2$)

$$\rho/k \leq \begin{cases} 4-6\sigma+\tau', & 4\sigma-2 \leq \tau' \leq 2\sigma-2/5, \\ 18/5-4\sigma, & 2\sigma-2/5 \leq \tau' \leq 6/5, \\ 12/5-4\sigma+\tau', & 6/5 \leq \tau' \leq \sigma+3/5, \\ 3-3\sigma, & \sigma+3/5 \leq \tau' \leq (4\sigma-2)(k+1)/k. \end{cases} \tag{12.39}$$

First, suppose that $6/5 \le \tau' \le (4\sigma - 2)(k+1)/k$. By Lemma 10.12, $(\sigma, \tau', \rho', \rho^*/k, s') \in \mathcal{E}$ for some $\rho' \le \rho/k$ and $s' \le s/k$. Applying Theorem 10.20, and noting that $\rho/k + 1 \ge 2\rho/k$ since $\rho/k \le 1$ by (12.39),

$$\frac{\rho^*}{k} \le 1 - 2\sigma + \frac{1}{2}\max\left(\frac{\rho}{k} + 1, \frac{5\rho}{4k} + \frac{\tau'}{2}\right) + \frac{1}{2}\max\left(\frac{\rho^*}{k} + 1, \frac{4\rho}{k}, \frac{3\rho^*}{4k} + \frac{\rho}{k} + \frac{\tau'}{2}\right).$$

If $\rho/k + 1 \ge 5\rho/(4k) + \tau'/2$, then

$$\frac{\rho^*}{k} \le \frac{3}{2} - 2\sigma + \frac{\rho}{2k} + \frac{1}{2}\max\left(\frac{\rho^*}{k} + 1, \frac{4\rho}{k}, \frac{3\rho^*}{4k} + \frac{\rho}{k} + \frac{\tau'}{2}\right).$$

Solving for ρ^*/k gives

$$\frac{\rho^*}{k} \le \max\left(4 - 4\sigma + \frac{\rho}{k}, \frac{3 - 4\sigma + 5\rho/k}{2}, \frac{2}{5}(6 - 8\sigma + \tau' + \frac{4\rho}{k})\right).$$

Applying $\rho/k \le \min(12/5 - 4\sigma + \tau', 3 - 3\sigma)$ to the RHS, one may ultimately verify that

$$\frac{\rho^*}{k} \le \max\left(\frac{197 - 220\sigma}{8(5\sigma - 1)}, \frac{3(29 - 30\sigma)}{5(5\sigma - 1)}\right)\tau'.$$

If $\rho/k + 1 \le 5\rho/(4k) + \tau'/2$ one has

$$\rho^*/k \le 1 - 2\sigma + \frac{5\rho}{8k} + \frac{\tau'}{4} + \frac{1}{2}\max(\frac{\rho^*}{k} + 1, \frac{4\rho}{k}, \frac{3}{4}\frac{\rho^*}{k} + \frac{\rho}{k} + \frac{\tau'}{2})$$

and solving for ρ^*/k gives

$$\rho^*/k \le \max(3 - 4\sigma + \frac{\tau'}{2} + \frac{5}{4}\frac{\rho}{k}, 1 - 2\sigma + \frac{\tau'}{4} + \frac{21}{8}\frac{\rho}{k}, \frac{1}{5}(8 - 16\sigma + 4\tau' + 9\frac{\rho}{k})).$$

Once again applying $\rho/k \le \min(12/5 - 4\sigma + \tau', 3 - 3\sigma)$, one again ultimately obtains

$$\rho^*/k \le \frac{197 - 220\sigma}{8(5\sigma - 1)}\tau'.$$

Now suppose that $4\sigma - 2 \le \tau' \le 6/5$. By Lemma 10.12, for any integer $k \ge 2$ one has $(\sigma, \tau/(k-1), \rho', \rho^*/(k-1), s') \in \mathcal{E}$ for some $\rho' \le \rho/(k-1)$ and $s' \le s/(k-1)$. Applying Theorem 10.20 to this tuple, followed by $\rho' \le \rho/(k-1)$ and rearranging, one obtains

$$\begin{aligned}\rho^* \le (1 - 2\sigma)(k-1) &+ \frac{1}{2}\max(\rho + k - 1, 2\rho, \frac{5\rho}{4} + \frac{\tau}{2}) \\ &+ \frac{1}{2}\max(\rho^* + k - 1, 4\rho, \frac{3\rho^*}{4} + \rho + \frac{\tau}{2}).\end{aligned} \tag{12.40}$$

Consider the first maximum of (12.40). If $\rho + k - 1$ is maximal, then

$$\rho^* \le (\frac{3}{2} - 2\sigma)(k-1) + \frac{\rho}{2} + \frac{1}{2}\max(\rho^* + k - 1, 4\rho, \frac{3\rho^*}{4} + \rho + \frac{\tau}{2}).$$

Solving for ρ^* and dividing by k gives

$$\frac{\rho^*}{k} \le \max((4 - 4\sigma)\frac{k-1}{k} + \frac{\rho}{k}, \frac{(3 - 4\sigma)(k-1)/k + 5\rho/k}{2}, \frac{2}{5}((6 - 8\sigma)(k-1)/k + \tau' + \frac{4\rho}{k})).$$

Since $3/4 \le \sigma \le 1$ and $1/2 \le (k-1)/k \le 2/3$, we have

$$\frac{\rho^*}{k} \le \max\left(\frac{8}{3}(1-\sigma)+\frac{\rho}{k}, \frac{3}{4}-\sigma+\frac{5}{2}\frac{\rho}{k}, \frac{2}{5}(3-4\sigma+\tau'+\frac{4\rho}{k})\right).$$

Bounding the RHS with $\rho/k \le \min(4-6\sigma+\tau', 18/5-4\sigma)$, one ultimately obtains

$$\rho^*/k \le \max\left(\frac{197-220\sigma}{8(5\sigma-1)}, \frac{3(29-30\sigma)}{5(5\sigma-1)}\right)\tau'.$$

Now suppose $5\rho/4+\tau'/2$ is maximal in (12.40). Then

$$\rho^* \le (1-2\sigma)(k-1)+\frac{5}{8}\rho+\frac{\tau}{4}+\frac{1}{2}\max(\rho^*+k-1, 4\rho, \frac{3\rho^*}{4}+\rho+\frac{\tau}{2}).$$

Solving for ρ^* and dividing by k gives

$$\frac{\rho^*}{k} \le \max((3-4\sigma)\frac{k-1}{k}+\frac{\tau'}{2}+\frac{5}{4}\frac{\rho}{k}, (1-2\sigma)\frac{k-1}{k}+\frac{\tau'}{4}+\frac{21}{8}\frac{\rho}{k}, \frac{8}{5}((1-2\sigma)\frac{k-1}{k}+\frac{\tau'}{2}+\frac{9}{8}\frac{\rho}{k})).$$

As before, since $3/4 \le \sigma \le 1$ and $1/2 \le (k-1)/k \le 2/3$, one has

$$\frac{\rho^*}{k} \le \max(\frac{3-4\sigma}{2}+\frac{\tau'}{2}+\frac{5}{4}\frac{\rho}{k}, \frac{1}{2}-\sigma+\frac{\tau'}{4}+\frac{21}{8}\frac{\rho}{k}, \frac{8}{5}(\frac{1}{2}-\sigma+\frac{\tau'}{2}+\frac{9}{8}\frac{\rho}{k})).$$

Once again we apply $\rho/k \le \min(4-6\sigma+\tau', 18/5-4\sigma)$ to ultimately obtain

$$\rho^*/k \le \max\left(\frac{197-220\sigma}{8(5\sigma-1)}, \frac{3(29-30\sigma)}{5(5\sigma-1)}\right)\tau'.$$

Lastly, if 2ρ is maximal in (12.40), then

$$\rho^* \le (1-2\sigma)(k-1)+\rho+\frac{1}{2}\max(\rho^*+k-1, 4\rho, \frac{3\rho^*}{4}+\rho+\frac{\tau}{2}).$$

Solving for ρ^* and dividing by k gives

$$\rho^*/k \le \max((3-4\sigma)\frac{k-1}{k}+\frac{2\rho}{k}, (1-2\sigma)\frac{k-1}{k}+\frac{3\rho}{k}, \frac{8}{5}((1-2\sigma)\frac{k-1}{k}+\frac{\tau'}{4}+\frac{3}{2}\frac{\rho}{k})).$$

Once again we apply $\rho/k \le \min(4-6\sigma+\tau', 18/5-4\sigma)$ to ultimately obtain

$$\rho^*/k \le \max\left(\frac{197-220\sigma}{8(5\sigma-1)}, \frac{3(29-30\sigma)}{5(5\sigma-1)}\right)\tau'$$

in this case too. □

Modest improvements are possible by incorporating more large value estimates; these are recorded in the next few theorems.

Theorem 12.10. *For* $664/877 \le \sigma \le 31/40$*, one has*

$$\mathrm{A}^*(\sigma) \le \max\left(\frac{72-91\sigma}{7(11\sigma-8)(1-\sigma)}, \frac{5(18-19\sigma)}{2(5\sigma+3)(1-\sigma)}\right).$$

Derived in `derived.py` *as:*
`prove_zero_density_energy_4()`

Proof. Fix $664/877 \leq \sigma \leq 31/40$ and take $\tau_0 = 2$. It suffices to show that

$$\rho^* \leq \max\left(\frac{72-91\sigma}{7(11\sigma-8)}, \frac{5(18-19\sigma)}{2(5\sigma+3)}\right)\tau$$

for all $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ satisfying $2 \leq \tau \leq 4$.
Let $k = 2$ if $2 \leq \tau < 3$ and $k = 3$ otherwise, and as usual let $\tau' = \tau/k$. By Lemma 10.12, $(\sigma, \tau', \rho', \rho^*/k, s') \in \mathcal{E}$ for some $\rho' \leq \rho/k$ and $s' \leq s/k$. Applying Theorem 10.20, and noting that $\rho/k + 1 \geq 2\rho/k$ since $\rho/k \leq 1$ by (12.39),

$$\frac{\rho^*}{k} \leq 1 - 2\sigma + \frac{1}{2}\max(\frac{\rho}{k}+1, \frac{5\rho}{4k}+\frac{\tau'}{2}) + \frac{1}{2}\max(\frac{\rho^*}{k}+1, \frac{4\rho}{k}, \frac{3\rho^*}{4k}+\frac{\rho}{k}+\frac{\tau'}{2}).$$

Rearranging the inequality and solving for ρ^*/k, one must either have

$$\frac{\rho^*}{k} \leq \max(4-4\sigma+\frac{\rho}{k}, \frac{3-4\sigma+5\rho/k}{2}, \frac{2}{5}(6-8\sigma+\tau'+\frac{4\rho}{k})). \tag{12.41}$$

or

$$\frac{\rho^*}{k} \leq \max(3-4\sigma+\frac{\tau'}{2}+\frac{5}{4}\frac{\rho}{k}, 1-2\sigma+\frac{\tau'}{4}+\frac{21}{8}\frac{\rho}{k}, \frac{1}{5}(8-16\sigma+4\tau'+9\frac{\rho}{k})). \tag{12.42}$$

Here we will divide our argument into several cases. Suppose first that

$$\tau' \geq \frac{9\sigma-1}{5}.$$

By Theorem 10.27, one has

$$\rho/k \leq \max(18/5-4\sigma, 12/5-4\sigma+\tau')$$

and by Theorem 7.9, one has

$$\rho/(k+1) \leq \max(2-2\sigma, 1-2\sigma+\tau/(k+1)).$$

Combining the two inequalities, we have

$$\rho/k \leq \begin{cases} 18/5-4\sigma, & (9\sigma-1)/5 \leq \tau' < 6/5, \\ 12/5-4\sigma+\tau', & 6/5 \leq \tau' < 2\sigma-2/5+2(1-\sigma)/k, \\ (2-2\sigma)(k+1)/k, & 2\sigma-2/5+2(1-\sigma)/k \leq \tau' \leq (k+1)/k. \end{cases} \tag{12.43}$$

Substituting (12.43) into (12.41) and (12.42), one may verify that

$$\frac{\rho^*}{k} \leq \frac{5(18-19\sigma)}{2(5\sigma+3)}\tau'$$

for all $(9\sigma-1)/5 \leq \tau' \leq 1+1/k$ (with equality occurring at $\tau' = 2\sigma-2/5+2(1-\sigma)/k$ and $k = 2$).
Now consider the case where

$$1 \leq \tau' \leq \frac{9\sigma-1}{5}.$$

Taking $k = 10$ in Theorem 7.16, one has

$$\rho/k \leq \max(2 - 2\sigma, 19/5 - 29\sigma/5 + \tau', 60 - 80\sigma + \tau'). \tag{12.44}$$

Suppose first that $\sigma \geq 281/371$. Then, this reduces to

$$\rho/k \leq \max(2 - 2\sigma, 19/5 - 29\sigma/5 + \tau'). \tag{12.45}$$

By Lemma 10.12, for any integer $k \geq 2$ one has $(\sigma, \tau/(k-1), \rho', \rho^*/(k-1), s') \in \mathcal{E}$ for some $\rho' \leq \rho/(k-1)$ and $s' \leq s/(k-1)$. Applying Theorem 10.20 to this tuple, followed by $\rho' \leq \rho/(k-1)$ and rearranging, one obtains

$$\begin{aligned}\rho^* \leq (1-2\sigma)(k-1) + \frac{1}{2}\max(\rho + k - 1, 2\rho, \frac{5\rho}{4} + \frac{\tau}{2})\\ + \frac{1}{2}\max(\rho^* + k - 1, 4\rho, \frac{3\rho^*}{4} + \rho + \frac{\tau}{2}).\end{aligned} \tag{12.46}$$

By considering each case of the two maximums individually, and solving for ρ^*/k, one obtains

$$\begin{aligned}\frac{\rho^*}{k} \leq \max\Bigg(&(4-4\sigma)\frac{k-1}{k} + \frac{\rho}{k}, \frac{(3-4\sigma)(k-1)/k + 5\rho/k}{2},\\ &\frac{2}{5}((6-8\sigma)\frac{k-1}{k} + \tau' + \frac{4\rho}{k}), (3-4\sigma)\frac{k-1}{k} + 2\frac{\rho}{k},\\ &(1-2\sigma)\frac{k-1}{k} + 3\frac{\rho}{k}, \frac{2}{5}((4-8\sigma)\frac{k-1}{k} + \tau' + 6\frac{\rho}{k}),\\ &(3-4\sigma)\frac{k-1}{k} + \frac{\tau'}{2} + \frac{5}{4}\frac{\rho}{k}, (1-2\sigma)\frac{k-1}{k} + \frac{\tau'}{4} + \frac{21}{8}\frac{\rho}{k},\\ &\frac{(8-16\sigma)(k-1)/k + 4\tau' + 9\rho/k}{5}\Bigg).\end{aligned} \tag{12.47}$$

Bounding each term on the RHS using (12.45), one may verify that in each case

$$\frac{\rho^*}{k} < \frac{5(18-19\sigma)}{2(5\sigma+3)}\tau'$$

for $1 \leq \tau' \leq (9\sigma - 1)/5$ and $k = 2, 3$.
Suppose now that $\sigma < 281/371$ so that (12.44) reduces to

$$\frac{\rho}{k} \leq \max(2 - 2\sigma, 60 - 80\sigma + \tau'). \tag{12.48}$$

If $1 \leq \tau' < 77\sigma/2 - 28$ then substituting $\rho/k \leq \max(2 - 2\sigma, 60 - 80\sigma + \tau')$ into (12.47), one may ultimately verify in each case that

$$\frac{\rho^*}{k} \leq \max\left(\frac{72-91\sigma}{7(11\sigma-8)}, \frac{5(18-19\sigma)}{2(5\sigma+3)}\right)\tau'$$

for $k = 2, 3$, with equality when $\tau' = 77\sigma/2 - 28$ and $k = 2$ (here we make use of the assumption $\sigma \geq 664/877 = 0.7571...$).
Lastly, if $77\sigma/2 - 28 \leq \tau' \leq (9\sigma - 1)/5$ then (12.48) simplifies to $\rho/k \leq 60 - 80\sigma + \tau'$. Substituting this into (12.41) and (12.42), one obtains in either case that

$$\frac{\rho^*}{k} \leq \max\left(\frac{72-91\sigma}{7(11\sigma-8)}, \frac{5(18-19\sigma)}{2(5\sigma+3)}\right)\tau'$$

with equality when $\tau' = 77\sigma/2 - 28$ and $k = 2$. □

Theorem 12.11. *For* $42/55 \le \sigma \le 79/103$*, one has*

$$\mathrm{A}^*(\sigma) \le \max\left(\frac{18-19\sigma}{6(15\sigma-11)(1-\sigma)}, \frac{3(18-19\sigma)}{4(4\sigma-1)(1-\sigma)}\right).$$

Derived in **`derived.py`** *as:*
`prove_zero_density_energy_5()`

Proof. Fix $42/55 \le \sigma \le 79/103$ and take $\tau_0 = 2$. It suffices to show that

$$\rho^* \le \max\left(\frac{18-19\sigma}{6(15\sigma-11)}, \frac{3(18-19\sigma)}{4(4\sigma-1)}\right)\tau$$

for all $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ for which $2 \le \tau \le 4$. As before let

$$k := \begin{cases} 2, & 2 \le \tau < 3, \\ 3, & 3 \le \tau \le 4, \end{cases} \qquad \tau' := \tau/k,$$

so that in particular $1 \le \tau' \le (k+1)/k$.
By Lemma 10.12, for $k = 2, 3$,

$$(\sigma, \tau', \rho', \rho^*/k, s'), \quad (\sigma, \tau/(k+1), \rho'', \rho^*/(k+1), s'') \in \mathcal{E} \tag{12.49}$$

for some $\rho' \le \rho/k$, $s' \le s/k$, $\rho'' \le \rho/(k+1)$ and $s'' \le s/(k+1)$. Applying Theorem 7.16 with $k = 6$ to each tuple, one has

$$\begin{aligned} \rho/k &\le \max(2-2\sigma, 11/3 - 17\sigma/3 + \tau', 36 - 48\sigma + \tau'), \\ \rho/(k+1) &\le \max(2-2\sigma, 11/3 - 17\sigma/3 + \tau/(k+1), 36 - 48\sigma + \tau/(k+1)). \end{aligned} \tag{12.50}$$

First suppose $\sigma \ge 97/127$, in which case the above simplifies to

$$\begin{aligned} \rho/k &\le \max(2-2\sigma, 11/3 - 17\sigma/3 + \tau'), \\ \rho/(k+1) &\le \max(2-2\sigma, 11/3 - 17\sigma/3 + \tau/(k+1)). \end{aligned}$$

This combines to give

$$\rho/k \le \begin{cases} 2-2\sigma, & 1 \le \tau' < (11\sigma-5)/3, \\ 11/3 - 17\sigma/3 + \tau', & (11\sigma-5)/3 \le \tau' < (11\sigma-5)/3 + 2(1-\sigma)/k, \\ (2-2\sigma)(k+1)/k, & (11\sigma-5)/3 + 2(1-\sigma)/k \le \tau' \le 1 + 1/k. \end{cases} \tag{12.51}$$

First suppose that $\tau' \ge (11\sigma-5)/3$. Applying Theorem 10.20 to the first tuple of (12.49), and noting that $\rho/k + 1 \ge 2\rho/k$ since $\rho/k \le 1$,

$$\frac{\rho^*}{k} \le 1 - 2\sigma + \frac{1}{2}\max(\frac{\rho}{k} + 1, \frac{5\rho}{4k} + \frac{\tau'}{2}) + \frac{1}{2}\max(\frac{\rho^*}{k} + 1, \frac{4\rho}{k}, \frac{3\rho^*}{4k} + \frac{\rho}{k} + \frac{\tau'}{2}).$$

Rearranging the inequality and solving for ρ^*/k, one must either have

$$\frac{\rho^*}{k} \le \max(4 - 4\sigma + \frac{\rho}{k}, \frac{3 - 4\sigma + 5\rho/k}{2}, \frac{2}{5}(6 - 8\sigma + \tau' + \frac{4\rho}{k})) \tag{12.52}$$

or

$$\frac{\rho^*}{k} \leq \max(3-4\sigma+\frac{\tau'}{2}+\frac{5}{4}\frac{\rho}{k}, 1-2\sigma+\frac{\tau'}{4}+\frac{21}{8}\frac{\rho}{k}, \frac{1}{5}(8-16\sigma+4\tau'+9\frac{\rho}{k})). \qquad (12.53)$$

In either case, upon substituting the last two cases of (12.51) one obtains

$$\rho^*/k \leq \frac{3(18-19\sigma)}{4(4\sigma-1)}\tau', \qquad (\frac{11\sigma-5}{3} \leq \tau' \leq 1+\frac{1}{k}). \qquad (12.54)$$

Now suppose that $\tau' < (11\sigma-5)/3$. Then, note that for $k=2,3$ one has

$$(\sigma, \tau/(k-1), \rho''', \rho^*/(k-1), s''') \in \mathcal{E}$$

for some $\rho''' \leq \rho/(k-1)$ and $s''' \leq s/(k-1)$. Applying Theorem 10.20 to this tuple, followed by $\rho''' \leq \rho/(k-1)$ and rearranging, one obtains

$$\begin{aligned}\rho^* \leq (1-2\sigma)(k-1) &+ \frac{1}{2}\max(\rho+k-1, 2\rho, \frac{5\rho}{4}+\frac{\tau}{2}) \\ &+ \frac{1}{2}\max(\rho^*+k-1, 4\rho, \frac{3\rho^*}{4}+\rho+\frac{\tau}{2}).\end{aligned} \qquad (12.55)$$

By considering each case of the two maximums individually, and solving for ρ^*/k, one obtains

$$\begin{aligned}\frac{\rho^*}{k} \leq \max\Bigg(&(4-4\sigma)\frac{k-1}{k}+\frac{\rho}{k}, \frac{(3-4\sigma)(k-1)/k+5\rho/k}{2}, \\ &\frac{2}{5}((6-8\sigma)\frac{k-1}{k}+\tau'+\frac{4\rho}{k}), (3-4\sigma)\frac{k-1}{k}+2\frac{\rho}{k}, \\ &(1-2\sigma)\frac{k-1}{k}+3\frac{\rho}{k}, \frac{2}{5}((4-8\sigma)\frac{k-1}{k}+\tau'+6\frac{\rho}{k}), \\ &(3-4\sigma)\frac{k-1}{k}+\frac{\tau'}{2}+\frac{5}{4}\frac{\rho}{k}, (1-2\sigma)\frac{k-1}{k}+\frac{\tau'}{4}+\frac{21}{8}\frac{\rho}{k}, \\ &\frac{(8-16\sigma)(k-1)/k+4\tau'+9\rho/k}{5}\Bigg).\end{aligned} \qquad (12.56)$$

Note that for all $1 \leq \tau' < (11\sigma-5)/3$, one has by (12.51) that $\rho/k \leq 2-2\sigma$. Substituting this into (12.56), one obtains

$$\rho^*/k \leq \frac{3(18-19\sigma)}{4(4\sigma-1)}\tau'$$

in this case too. Combined with (12.54), we have shown that

$$\rho^* \leq \frac{3(18-19\sigma)}{4(4\sigma-1)}\tau$$

for all $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$ for $97/127 \leq \sigma \leq 79/103$ and $2 \leq \tau \leq 4$, as required.
The proof in the range $42/55 \leq \sigma \leq 97/127$ is similar. Here (12.50) reduces to

$$\rho/k \leq \max(2-2\sigma, 36-48\sigma+\tau'),$$
$$\rho/(k+1) \leq \max(2-2\sigma, 36-48\sigma+\tau/(k+1))$$

so that

$$\rho/k \leq \begin{cases} 2-2\sigma, & 1 \leq \tau' < 46\sigma-34, \\ 36-48\sigma+\tau', & 46\sigma-34 \leq \tau' < 46\sigma-34+2(1-\sigma)/k, \\ (2-2\sigma)(k+1)/k, & 46\sigma-34+2(1-\sigma)/k \leq \tau' \leq 1+1/k. \end{cases} \qquad (12.57)$$

If $\tau' \geq 46\sigma - 34$, we substitute the last two cases of this bound into (12.52) and (12.53), one may verify that

$$\frac{\rho^*}{k} \leq \frac{18 - 19\sigma}{6(15\sigma - 11)}\tau' \tag{12.58}$$

with equality when $\tau' = 46\sigma - 34 + 2(1-\sigma)/k$ and $k = 2$.
On the other hand if $1 \leq \tau' < 46\sigma - 34$ then substituting $\rho/k \leq 2 - 2\sigma$ into (12.56) gives

$$\frac{\rho^*}{k} \leq \frac{18 - 19\sigma}{6(15\sigma - 11)}\tau', \qquad (1 \leq \tau' \leq 46\sigma - 34)$$

in each case. Combined with (12.58), the desired result follows for $42/55 \leq \sigma \leq 97/127$. ☐

Theorem 12.12. *For* $79/103 \leq \sigma \leq 84/109$*, one has*

$$\mathrm{A}^*(\sigma) \leq \max\left(\frac{18 - 19\sigma}{2(37\sigma - 27)(1-\sigma)}, \frac{5(18 - 19\sigma)}{2(13\sigma - 3)(1-\sigma)}\right).$$

Derived in ***derived.py*** *as:*
`prove_zero_density_energy_6()`

Proof. Fix $79/103 \leq \sigma \leq 84/109$ and take

$$\tau_0 = \begin{cases} (36\sigma - 16)/5, & 79/103 \leq \sigma < 33/43, \\ 38\sigma - 28, & 33/43 \leq \sigma \leq 84/109. \end{cases}$$

Let

$$k := \begin{cases} 2, & \tau_0 \leq \tau < 3\tau_0/2 \\ 3, & 3\tau_0/2 \leq \tau \leq 2\tau_0 \end{cases}, \qquad \tau' := \tau/k.$$

Suppose that $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}$. We will first show

$$\rho^* \leq \max\left(\frac{18 - 19\sigma}{2(37\sigma - 27)}, \frac{5(18 - 19\sigma)}{2(13\sigma - 3)}\right)\tau, \qquad (\tau_0 \leq \tau \leq 2\tau_0). \tag{12.59}$$

By Lemma 10.12, one has that

$$(\sigma, \tau', \rho', \rho^*/k, s'), \quad (\sigma, \tau/(k+1), \rho'', \rho^*/(k+1), s'') \in \mathcal{E} \tag{12.60}$$

for some $\rho' \leq \rho/k$, $s' \leq s/k$, $\rho'' \leq \rho/(k+1)$ and $s'' \leq s/(k+1)$. First suppose that $\sigma \geq 33/43$. In this range, Theorem 7.16 with $k = 5$ gives

$$\rho/k \leq \max(2 - 2\sigma, 18/5 - 28\sigma/5 + \tau'),$$

$$\rho/(k+1) \leq \max(2 - 2\sigma, 18/5 - 28\sigma/5 + \tau/(k+1)).$$

For $\tau \geq \tau_0$, the first inequality reduces to $\rho/k \leq 18/5 - 28\sigma/5 + \tau'$, while for $\tau \leq 2\tau_0$ the second inequality reduces to

$$\rho/k = \frac{k+1}{k}\frac{\rho}{k+1} \leq \frac{k+1}{k}(2 - 2\sigma) \leq 3 - 3\sigma.$$

Combining these two inequalities gives

$$\rho/k \leq \min\left(18/5 - 28\sigma/5 + \tau', 3 - 3\sigma\right), \qquad (\tau_0 \leq \tau \leq 2\tau_0).$$

In particular this implies $\rho/k \le 1$. Applying Theorem 10.20 to the first tuple of (12.60),

$$\frac{\rho^*}{k} \le 1 - 2\sigma + \frac{1}{2}\max(\frac{\rho}{k} + 1, \frac{5\rho}{4k} + \frac{\tau'}{2}) + \frac{1}{2}\max(\frac{\rho^*}{k} + 1, \frac{4\rho}{k}, \frac{3\rho^*}{4k} + \frac{\rho}{k} + \frac{\tau'}{2}).$$

By considering each case of the first maximum and solving for ρ^*/k, one must either have

$$\frac{\rho^*}{k} \le \max(4 - 4\sigma + \frac{\rho}{k}, \frac{3 - 4\sigma + 5\rho/k}{2}, \frac{2}{5}(6 - 8\sigma + \tau' + \frac{4\rho}{k})) \tag{12.61}$$

or

$$\frac{\rho^*}{k} \le \max(3 - 4\sigma + \frac{\tau'}{2} + \frac{5}{4}\frac{\rho}{k}, 1 - 2\sigma + \frac{\tau'}{4} + \frac{21}{8}\frac{\rho}{k}, \frac{1}{5}(8 - 16\sigma + 4\tau' + 9\frac{\rho}{k})). \tag{12.62}$$

However one may verify in both cases that

$$\frac{\rho^*}{k} \le \frac{5(18 - 19\sigma)}{2(13\sigma - 3)}\tau' \tag{12.63}$$

for $33/43 \le \sigma \le 84/109$ and $\tau_0 \le \tau \le 2\tau_0$ (with equality at $\tau = 2(13\sigma - 3)/5$).
Now consider $\sigma \le 33/43$. In this range of σ, Theorem 7.16 with $k = 5$ gives

$$\rho/k \le \max(2 - 2\sigma, 30 - 40\sigma + \tau'),$$

$$\rho/(k+1) \le \max(2 - 2\sigma, 30 - 40\sigma + \tau/(k+1))$$

which gives

$$\rho/k \le \min(30 - 40\sigma + \tau', 3 - 3\sigma), \qquad (\tau_0 \le \tau \le 2\tau_0).$$

Substituting this bound into (12.61) and (12.62), one obtains

$$\frac{\rho^*}{k} \le \frac{18 - 19\sigma}{2(37\sigma - 27)}\tau'$$

for $79/103 \le \sigma \le 33/43$ and $\tau_0 \le \tau \le 2\tau_0$ (with equality at $\tau = 74\sigma - 54$). Combined with (12.63), one obtains (12.59).
It remains to show that

$$\rho^* \le \max\left(\frac{18 - 19\sigma}{2(37\sigma - 27)}, \frac{5(18 - 19\sigma)}{2(13\sigma - 3)}\right)\tau \tag{12.64}$$

for all $(\sigma, \tau, \rho, \rho^*, s) \in \mathcal{E}_\zeta$ for which $79/103 \le \sigma \le 84/109$ and $2 \le \tau \le \tau_0$. This follows from substituting $\rho/k \le 6 - 12\sigma + 2\tau'$ (Theorem 9.7) into (12.61) and (12.62). □

Theorem 12.13. *For* $84/109 \le \sigma \le 5/6$, *one has*

$$\mathrm{A}^*(\sigma) \le \max\left(\frac{18 - 19\sigma}{9(3\sigma - 2)(1 - \sigma)}, \frac{4(10 - 9\sigma)}{5(4\sigma - 1)(1 - \sigma)}\right).$$

Derived in ***derived.py*** *as:*
`prove_zero_density_energy_7()`

Theorem 12.14. *For* $165/226 \le \sigma \le 42/55$ *one has*

$$\mathrm{A}^*(\sigma) \le \max\left(\frac{457 - 546\sigma}{2(61 - 58\sigma)(1 - \sigma)}, \frac{5(18 - 19\sigma)}{2(5\sigma + 3)(1 - \sigma)}\right).$$

Derived in `derived.py` *as:*
`prove_zero_density_energy_12()`

Table 12.1 records the sharpest known unconditional upper bounds on $A^*(\sigma)$ for $1/2 \le \sigma \le 1$ (except when close to $\sigma = 1$, when sharper estimates are available by applying Lemma 12.2 with known zero-density bounds).

Table 12.1: Current best upper bound on $A^*(\sigma)$

$A^*(\sigma)$ bound	Range	Reference
$\dfrac{10-11\sigma}{(2-\sigma)(1-\sigma)}$	$\dfrac{1}{2} \le \sigma \le \dfrac{2}{3} = 0.6666\ldots$	Theorem 12.6
$\dfrac{18-19\sigma}{(4-2\sigma)(1-\sigma)}$	$\dfrac{2}{3} \le \sigma \le \dfrac{7}{10} = 0.7$	Theorem 12.6
$\dfrac{5(18-19\sigma)}{2(5\sigma+3)(1-\sigma)}$	$\dfrac{7}{10} \le \sigma \le \dfrac{539-\sqrt{42121}}{460} = 0.7255\ldots$	Theorem 12.8
$\dfrac{2(45-44\sigma)}{(2\sigma+15)(1-\sigma)}$	$\dfrac{539-\sqrt{42121}}{460} \le \sigma \le \dfrac{165}{226} = 0.7300\ldots$	Theorem 12.8
$\dfrac{457-546\sigma}{2(61-58\sigma)(1-\sigma)}$	$\dfrac{165}{226} \le \sigma \le \dfrac{5831+\sqrt{60001}}{8240} = 0.7373\ldots$	Theorem 12.14
$\dfrac{5(18-19\sigma)}{2(5\sigma+3)(1-\sigma)}$	$\dfrac{5831+\sqrt{60001}}{8240} \le \sigma \le \dfrac{42}{55} = 0.7636\ldots$	Theorem 12.14, 12.10
$\dfrac{18-19\sigma}{6(15\sigma-11)(1-\sigma)}$	$\dfrac{42}{55} \le \sigma \le \dfrac{97}{127} = 0.7637\ldots$	Theorem 12.11
$\dfrac{3(18-19\sigma)}{4(4\sigma-1)(1-\sigma)}$	$\dfrac{97}{127} \le \sigma \le \dfrac{79}{103} = 0.7669\ldots$	Theorem 12.11
$\dfrac{18-19\sigma}{2(37\sigma-27)(1-\sigma)}$	$\dfrac{79}{103} \le \sigma \le \dfrac{33}{43} = 0.7674\ldots$	Theorem 12.12
$\dfrac{5(18-19\sigma)}{2(13\sigma-3)(1-\sigma)}$	$\dfrac{33}{43} \le \sigma \le \dfrac{84}{109} = 0.7706\ldots$	Theorem 12.12
$\dfrac{18-19\sigma}{9(3\sigma-2)(1-\sigma)}$	$\dfrac{84}{109} \le \sigma \le \dfrac{1273-\sqrt{128689}}{1184} = 0.7721\ldots$	Theorem 12.13
$\dfrac{4(10-9\sigma)}{5(4\sigma-1)(1-\sigma)}$	$\dfrac{1273-\sqrt{128689}}{1184} \le \sigma \le \dfrac{5}{6} = 0.8333\ldots$	Theorem 12.7, 12.9, 12.13
$\dfrac{12}{4\sigma-1}$	$\dfrac{5}{6} \le \sigma \le 1$	Theorem 12.6

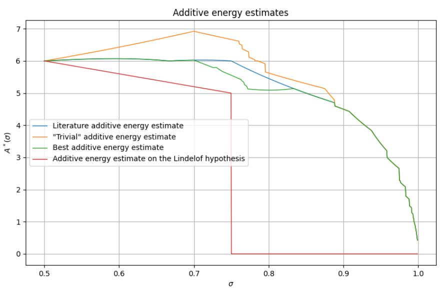

Figure 12.1: Comparison of bounds on $\mathrm{A}^*(\sigma)$ under various assumptions.

Chapter 13

Zero free regions

A zero of the Riemann zeta function is a complex number $\rho = \beta + i\gamma$ for which $\zeta(\rho) = 0$.

- There are infinitely many "trivial" zeros of the form $\rho = -2n$ for integer $n \geq 1$; these zeros are well understood.
- There are a countably infinite number of "non-trivial zeros lying inside the critical strip $0 < \Re z < 1$.

Definition 13.1 (Zero free region of $\zeta(s)$)**.** *A zero-free region of the Riemann zeta function is a set $D \subset \mathbb{C}$ for which $\zeta(s) \neq 0$ for all $s \in D$.*

Lemma 13.2 (Basic properties of zero free regions)**.** *The following properties hold:*

(i) (Symmetry about the real axis) If $\zeta(\sigma + it) \neq 0$ then $\zeta(\sigma - it) \neq 0$.

(ii) (Symmetry about the critical line $\Re s = 1/2$) For $0 \leq \sigma \leq 1$, if $\zeta(\sigma + it) \neq 0$ then $\zeta(1 - \sigma + it) \neq 0$.

(iii) (Non vanishing for $\Re s > 1$) If $\Re s > 1$ then $\zeta(s) \neq 0$.

Proof. Claim (i) follows directly from the property $\overline{\zeta(s)} = \zeta(\overline{s})$. Claim (ii) follows from the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s)$$

and claim (iii) follows from the Euler product formula

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \qquad (\Re s > 1).$$

□

A well-known conjecture regarding the non-trivial zeroes of $\zeta(s)$ is the Riemann hypothesis.

Conjecture 13.3 (Riemann hypothesis)**.** *If ρ is a non-trivial zero of the Riemann zeta function, then $\Re \rho = 1/2$.*

In light of Lemma 13.2, for the rest of the chapter we will focus on the quadrant

$$D \subseteq \{z \in \mathbb{C} : \Re z > 1/2, \Im z > 0\}.$$

The first non-trivial zero-free region was due to de la Vallée Poussin [56] and Hadamard [89], who showed independently that:

Theorem 13.4 (Non-vanishing on the 1-line)**.** *One has $\zeta(1+it) \neq 0$ for any real t.*

Proof. For $\Re s > 1$, one has

$$\Re \log \zeta(s) = \sum_p \sum_{m=1}^{\infty} \frac{\cos(t \log p^m)}{m p^{m\sigma}}$$

where the outer sum runs through all primes. Applying this formula at $s = \sigma, \sigma + it$ and $\sigma + 2it$ $(t \neq 0)$, and since $3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2 \geq 0$, one has

$$3\Re \log \zeta(\sigma) + 4\Re \log \zeta(\sigma+it) + \Re \log \zeta(\sigma+2it) = \sum_p \sum_{m=1}^{\infty} \frac{3 + \cos(t \log p^m) + \cos(2t \log p^m)}{m p^{m\sigma}} \geq 0.$$

It follows that $|\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+2it)| \geq 1$. Now as $\sigma \to 1$ from above (and t remains fixed), one has

$$\zeta(\sigma) \ll \frac{1}{\sigma - 1}, \qquad \zeta(\sigma + 2it) \ll 1,$$

since ζ has a simple pole at $s = 1$ and no pole at $\sigma + 2it$. If $\zeta(1+it) = 0$, then $\zeta(\sigma+it) \ll \sigma - 1$ so that $|\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+2it)| \ll \sigma - 1$, a contradiction. □

This was used to prove the prime number theorem $\pi(x) \sim x/\log x$ as $x \to \infty$ (in fact the two statements are equivalent).

13.1 Relation to growth rates of zeta

Using estimates of $\zeta(\sigma + it)$ close to the line $\sigma = 1$, one can extend the zero free region slightly inside the critical strip.

Lemma 13.5 (Relation to growth exponents of zeta)**.** *Suppose and $0 < g(t) \leq 1 < f(t)$ are real-valued functions for $t \geq 0$, with $f(t)$ non-decreasing and tending to infinity with t, and $g(t)$ non-increasing. Suppose further that $e^{f(t)}/g(t) = o(f(t))$. If*

$$\zeta(\sigma + it) \ll f(t) \qquad (1 - g(t) \leq \sigma \leq 2, t \geq 0)$$

then $\zeta(\sigma + it) \neq 0$ for

$$\sigma \geq 1 - A \frac{g(2t+1)}{\log f(2t+1)}$$

where $A > 0$ is an absolute constant.

Proof. See [277, Theorem 3.10]. □

Theorem 13.6 (Classical zero free region)**.** *One has $\zeta(\sigma + it) \neq 0$ if*

$$\sigma \geq 1 - \frac{A}{\log t}.$$

for an absolute constant $A > 0$ and t sufficiently large.

Proof. Thanks to the convexity bound $\mu(\sigma) \leq (1-\sigma)/2$, one may take $g(t) = 1/2$, $f(t) = t^{1/4+o(1)}$ in Lemma 13.5. The result follows. □

This classical result has been improved in a number of works, most of which make crucial use of non-trivial estimates of certain types of exponential sums.

Theorem 13.7 (Littlewood zero free region)**.** *One has* $\zeta(\sigma + it) \neq 0$ *if*

$$\sigma \geq 1 - \frac{A \log\log t}{\log t}$$

for an absolute constant $A > 0$ *and* t *sufficiently large.*

Proof. Follows from the zeta bound corresponding to

$$\mu\left(1 - \frac{k}{2^k - 2}\right) \leq \frac{1}{2^k - 2}$$

for integer $k \geq 3$, which is generated by the van der Corput exponent pair $A^{k-2}B(0,1) = (\frac{1}{2^k-2}, 1 - \frac{k-1}{2^k-2})$. However, one needs to make explicit the $o(1)$ term in the bound $\zeta(\sigma+it) \ll t^{\mu(\sigma)+o(1)}$. In particular, by [277, Theorem 5.14], one has

$$\zeta(1 - \frac{k}{2^k - 2} + it) \ll t^{1/(2^k-2)} \log t.$$

Taking

$$k = \left\lfloor \frac{1}{\log 2} \log\left(\frac{\log t}{\log\log t}\right)\right\rfloor$$

and using the Phragmén Lindelöf principle, one has

$$\zeta(\sigma + it) \ll (\log t)^5, \qquad (\sigma \geq 1 - \frac{(\log\log t)^2}{\log t}),$$

so we may take $f(t) = (\log t)^5$ and $g(t) = (\log\log t)^2 / \log t$ in Lemma 13.5. ∎

Theorem 13.8 (Chudakov zero free region)**.** *One has* $\zeta(\sigma + it) \neq 0$ *if*

$$\sigma \geq 1 - \frac{1}{(\log t)^{3/4+o(1)}}$$

for t *sufficiently large.*

Theorem 13.9 (Korobov-Vinogradov zero free region)**.** *One has* $\zeta(\sigma + it) \neq 0$ *if*

$$\sigma \geq 1 - \frac{A}{(\log t)^{2/3}(\log\log t)^{1/3}}$$

for an absolute constant $A > 0$ *and* t *sufficiently large.*

Proof. Via estimates of Vinogradov's integral, one may obtain an estimate of the form (see e.g. Richert [248])

$$\zeta(\sigma + it) \ll t^{B(1-\sigma)^{3/2}} (\log t)^{O(1)} \qquad (1/2 \leq \sigma \leq 1)$$

where $B > 0$ is a constant and t sufficiently large. Take

$$g(t) = \left(\frac{\log\log t}{\log t}\right)^{2/3}$$

so that

$$\zeta(\sigma + it) \ll f(t) = (\log t)^{O(1)} \qquad (\sigma \geq 1 - g(t)).$$

The result follows from applying Lemma 13.5. ∎

Chapter 14

Distribution of primes: long ranges

Let $\Lambda(n)$ denote the von Mangoldt function, i.e. $\Lambda(n) = \log p$ if $n = p^m$ where p is prime and m is a positive integer, and $\Lambda(n) = 0$ otherwise.

Definition 14.1. *For all $x \geq 1$ define the Chebyshev prime counting functions $\psi(x)$, $\theta(x)$ and $\pi(x)$ as*

$$\psi(x) := \sum_{n \leq x} \Lambda(n), \qquad \theta(x) := \sum_{p \leq x} \log p, \qquad \pi(x) := \sum_{p \leq x} 1$$

where the first sum is over positive integers n and the last two sums are over primes p.

These functions, particularly $\pi(x)$, are central to number theory because they measure the distribution of prime numbers among the integers. A well-known result is the prime number theorem.

Theorem 14.2 (Prime number theorem)**.** *As $x \to \infty$,*

$$\pi(x) \sim \frac{x}{\log x} \sim \mathrm{li}(x) := \int_2^\infty \frac{dt}{\log t}.$$

The following are equivalent formulations of the prime number theorem.

Theorem 14.3 (Prime number theorem, alternative formulations)**.** *As $x \to \infty$, one has $\psi(x) \sim x$ and $\theta(x) \sim x$.*

14.1 Error bounds for prime counting functions

In addition to their asymptotic behaviour, various bounds on the deviation from their respective asymptotics are known. The current best-known error bounds are derived from zero-free regions of the Riemann zeta function $\zeta(s)$. The relation between zeroes of $\zeta(s)$ and error bounds for prime counting functions are illustrated through von Mangoldt's explicit formula: for all non-integer $x > 0$, one has

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log(1 - x^{-2}),$$

where ρ runs through all non-trivial zeroes of $\zeta(s)$.

Theorem 14.4 (Korobov–Vinogradov estimate)**.** *There exists a positive constant A, such that*

$$\psi(x) - x,\ \theta(x) - x,\ \pi(x) - \mathrm{li}(x) \ll x \exp\left(-A\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right).$$

Table 14.1 lists the historical progression on estimates of $\pi(x)$.

Table 14.1: Historical estimates of $\pi(x)$, for x sufficiently large.

Reference	Estimate of $\pi(x)$
Chebyshev	$c_1\frac{x}{\log x} \le \pi(x) \le c_2\frac{x}{\log x}$ for some constants $0 < c_1 < 1 < c_2$, i.e. $\pi(x$
de la Vallée Poussin [56], Hadamard [89]	$\pi(x) = \frac{x}{\log x}(1+o(1))$ i.e. $\pi(x) \sim \frac{x}{\log x}$
de la Vallée Poussin [57]	$\pi(x) = \mathrm{li}(x) + O(x\exp(-A\sqrt{\log x}))$ for some $A > 0$
Littlewood [193]	$\pi(x) = \mathrm{li}(x) + O(x\exp(-A\sqrt{\log x \log\log x}))$ for some $A > 0$
Korobov, Vinogradov [285]	$\pi(x) = \mathrm{li}(x) + O\left(x\exp\left(-\frac{A(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right)$ for some $A > 0$

Under the Riemann hypothesis, stronger error bounds are known.

Theorem 14.5 ([168])**.** *If the Riemann hypothesis is true, then*

$$\psi(x) - x,\ \theta(x) - x \ll x^{1/2}(\log x)^2, \qquad \pi(x) - \mathrm{li}(x) \ll x^{1/2}\log x.$$

Slightly sharper estimates are possible if one assumes even stronger hypotheses.

Theorem 14.6 (Heath-Brown [109])**.** *Assume that the Riemann hypothesis is true. Furthermore, assume that*

$$F_T(X) := \sum_{0<\gamma_1,\gamma_2\le T} \frac{e((\gamma_1-\gamma_2)X)}{1+(\gamma_1-\gamma_2)^2/4} = o(T^2(\log T)^2)$$

where the sum is over the imaginary parts of all pairs of non-trivial zeroes of $\zeta(s)$. Then

$$\psi(x) = x + o(x^{1/2}(\log x)^2).$$

The same result was previously proved (assuming stronger hypotheses) by Gallagher–Mueller [80] and later by Mueller.

14.2 Relation to zero free region of zeta

Lemma 14.7 (Relation to zero free regions)**.** *[139] Suppose $\zeta(\sigma+it) \ne 0$ for $\sigma \ge 1-\eta(t)$ where $\eta(t)$ is a positive and decreasing function. Then*

$$\psi(x) - x \ll x\exp(-A\omega(x)) \qquad (x \to \infty)$$

for an absolute constant $A > 0$, where

$$\omega(x) := \inf_{t \geq 1}(\eta(t) \log x + \log t).$$

Applying Lemma 14.7, one obtains the error term estimates in the prime number theorem given in Table 14.2.

Table 14.2: Zero free regions for $\zeta(s)$, along with the bound on $\psi(x) - x$ that they imply. Here A represents an absolute, positive constant, which may be different at each occurrence.

Reference	Zero free region	Bound on $(\psi(x) - x)/x$
Theorem 13.6	$\sigma \geq 1 - \dfrac{A}{\log t}$	$\exp(-A(\log x)^{1/2})$
Theorem 13.7	$\sigma \geq 1 - \dfrac{A \log\log t}{\log t}$	$\exp(-A(\log x \log\log x)^{1/2})$
Theorem 13.8	$\sigma \geq 1 - \dfrac{A}{(\log t)^{3/4+o(1)}}$	$\exp(-A(\log x)^{4/7+o(1)})$
Theorem 13.9	$\sigma \geq 1 - \dfrac{A}{(\log t)^{2/3}(\log\log t)^{1/3}}$	$\exp\left(-A\dfrac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)$

The following type of converse statement is also known.

Theorem 14.8 ([281] Theorem 40.1)**.** *If for some $0 < \alpha \leq 1$ one has*

$$\psi(x) - x \ll x \exp(-A(\log x)^{1/(1+\alpha)}) \qquad (x \to \infty)$$

then $\zeta(\sigma + it) \neq 0$ for t sufficiently large and

$$\sigma > 1 - \frac{A}{(\log t)^{\alpha}}.$$

Here A denotes an absolute positive constant, not necessarily the same at each occurrence.

14.3 Omega results

In the opposite direction, it is known that

Theorem 14.9 (Schmidt [260])**.** *As $x \to \infty$,*

$$\psi(x) = x + \Omega(x^{1/2}).$$

This can be improved slightly conditioned on the Riemann hypothesis.

Theorem 14.10 (Littlewood [192])**.** *If the Riemann hypothesis is true, then as $x \to \infty$,*

$$|\pi(x) - \mathrm{li}(x)| = \Omega\left(x^{1/2}\frac{\log\log\log x}{\log x}\right).$$

Furthermore it is also known that

Theorem 14.11 (Grosswald [87]). *If*

$$\theta = \sup_{\rho:\zeta(\rho)=0} \Re\rho > 1/2$$

then as $x \to \infty$,

$$\psi(x) = x + \Omega(x^{\theta}).$$

Chapter 15

Distribution of primes: short ranges

Recall that Λ is the von Mangoldt function, and that the prime number theorem asserts that

$$\sum_{n \leq x} \Lambda(n) = x + o(x)$$

for unbounded x. If p_n denotes the n^{th} prime, the prime number theorem is also equivalent to

$$p_n = (1 + o(1)) n \log n$$

for unbounded n.
We now consider local versions of the prime number theorem.

Definition 15.1 (Prime number theorem in short interval exponents). *(i) We let θ_{PNT} denote the least exponent with the following property: if $\varepsilon > 0$ is fixed, and x is unbounded, then*

$$\sum_{x \leq n < x+y} \Lambda(n) = y + o(y)$$

whenever $x^{\theta_{\mathrm{PNT}}+\varepsilon} \leq y \leq x^{1-\varepsilon}$.

(ii) We let $\theta_{\mathrm{PNT-AA}}$ denote the least exponent with the following property: if $\varepsilon > 0$ is fixed, and X is unbounded, then we have

$$\int_X^{2X} \Big| \sum_{x \leq n < x+y} \Lambda(n) - y \Big| \, dx = o(Xy)$$

whenever $X^{\theta_{\mathrm{PNT-AA}}+\varepsilon} \leq y \leq X^{1-\varepsilon}$.

(iii) We let θ_{gap} denote the least exponent such that, if p_n denotes the n^{th} prime, that

$$p_{n+1} - p_n \ll n^{\theta_{\mathrm{gap}}+o(1)} = p_n^{\theta_{\mathrm{gap}}+o(1)}$$

as $n \to \infty$.

(iv) We let $\theta_{\mathrm{gap},2}$ denote the least exponent such that

$$\sum_{p_n \le x} (p_{n+1} - p_n)^2 \ll x^{\theta_{\mathrm{gap},2}+o(1)}$$

as $x \to \infty$.

(v) We let $\theta_{\mathrm{gap-AA}}$ denote the least exponent such that for every $\varepsilon > 0$, the intervals $[n, n^{\theta_{\mathrm{gap-AA}}+\varepsilon}]$ contain a prime for a density 1 set of natural numbers n.

Lemma 15.2 (Trivial bounds)**.** *We have*

$$0 \le \theta_{\mathrm{gap-AA}} \le \theta_{\mathrm{PNT-AA}}, \theta_{\mathrm{gap}} \le \theta_{\mathrm{PNT}} \le 1$$

and $1 \le \theta_{\mathrm{gap},2} \le 1 + \theta_{\mathrm{gap}}$.

Proof. These are all immediate, after noting from the prime number theorem that $\sum_{p_n \le x} p_{n+1} - p_n = x^{1+o(1)}$. □

The Cramér random model [48] predicts

Conjecture 15.3 (Prime gap conjecture)**.** *$\theta_{\mathrm{PNT}} = 0$, and hence (by Lemma 15.2) $\theta_{\mathrm{gap-AA}} = \theta_{\mathrm{PNT-AA}} = \theta_{\mathrm{gap}} = 0$ and $\theta_{\mathrm{gap},2} = 1$.*

We note that the results of Maier [203] show that there is some deviation from the prime number theorem at very small scales (of order $\log^{O(1)} x$), but this does not directly affect the exponents discussed here due to the epsilons in our definitions.
A basic connection with zero density exponents is

Proposition 15.4 (Zero density theorems and prime gaps)**.** *Let*

$$\|\mathrm{A}\|_\infty := \sup_{1/2 \le \sigma \le 1} A(\sigma). \tag{15.1}$$

Then

$$\theta_{\mathrm{PNT}} \le 1 - \frac{1}{\|\mathrm{A}\|_\infty}$$

and

$$\theta_{\mathrm{PNT-AA}} \le 1 - \frac{2}{\|\mathrm{A}\|_\infty}.$$

Proof. See for instance [88, §13.2]. □

Corollary 15.5 (Ingham-Huxley bound)**.** *We have $\theta_{\mathrm{PNT}} \le \frac{7}{12}$ and $\theta_{\mathrm{PNT-AA}} \le \frac{1}{6}$.*

Proof. From Theorem 11.14 and Theorem 11.15 one as $\|\mathrm{A}\|_\infty \le 12/5$, and the claim now follows from Proposition 15.4. □

Corollary 15.6 (Ingham-Guth-Maynard bound)**.** *[88] We have $\theta_{\mathrm{PNT}} \le \frac{17}{30}$ and $\theta_{\mathrm{PNT-AA}} \le \frac{2}{15}$.*

These are currently the best known upper bounds on θ_{PNT} and $\theta_{\mathrm{PNT-AA}}$.

Proof. From Theorem 11.14 and Theorem 11.16 one as $\|\mathrm{A}\|_\infty \le 30/13$, and the claim now follows from Proposition 15.4. □

Table 15.1: Historical upper bounds on θ_{gap}.

Reference	Upper bound
Hoheisel (1930) [118]	$1 - \frac{1}{33000} = 0.999\ldots$
Heilbronn (1933) [116]	$1 - \frac{1}{250} = 0.996$
Ingham (1937) [137]	$\frac{5}{8} = 0.625$
Montgomery (1969) [217]	$\frac{3}{5} = 0.6$
Huxley (1972) [122]	$\frac{7}{12} = 0.5833\ldots$
Iwaniec–Jutila (1979)[148]	$\frac{13}{23} = 0.5652\ldots$
Heath-Brown–Iwaniec (1979) [115], Lou–Yao (1993) [201]	$\frac{11}{20} = 0.55$
Pintz (1981) [233]	$\frac{17}{31} = 0.5483\ldots$
Iwaniec–Pintz (1984) [151]	$\frac{23}{42} = 0.5476\ldots$
Mozzochi (1986) [222]	$\frac{1051}{1920} = 0.5473\ldots$
Lou–Yao (1984) [196]	$\frac{35}{64} = 0.5469\ldots$
Lou–Yao (1992) [200]	$\frac{6}{11} = 0.5454\ldots$
Baker-Harman (1996) [5]	$\frac{107}{200} = 0.535$
Baker–Harman–Pintz (2001) [8]	$\frac{21}{40} = 0.525$
R. Li (2025) [183]	$\frac{13}{25} = 0.52$

Corollary 15.7. *The density hypothesis implies that* $\theta_{\mathrm{PNT}} \leq 1/2$ *and* $\theta_{\mathrm{PNT-AA}} = 0$.

The current unconditional best bound on θ_{gap} is

Theorem 15.8. *[183] We have* $\theta_{\mathrm{gap}} \leq 13/25 = 0.52$.

Historical bounds on θ_{gap} are summarized in the following table:
Bounds on $\theta_{\mathrm{gap-AA}}$ are recorded in Table 15.
Historical bounds on $\theta_{\mathrm{gap},2}$ are recorded in Table 15.
The following general bound on $\theta_{\mathrm{gap},2}$ is known:

Proposition 15.9. *We have*

$$\theta_{\mathrm{gap},2} \leq \max\left(2 - \frac{2}{\|\mathrm{A}\|_\infty}, \sup_{1/2 \leq \sigma \leq 1} \max(\alpha(\sigma), \beta(\sigma))\right)$$

where

$$\alpha(\sigma) := 4\sigma - 2 + 2\frac{B(\sigma)(1-\sigma) - 1}{B(\sigma) - A(\sigma)}$$

and

$$\beta(\sigma) := 4\sigma - 2 + \frac{B(\sigma)(1-\sigma) - 1}{A(\sigma)}$$

where $A(\sigma), B(\sigma)$ *are any upper bounds for* $\mathrm{A}(\sigma), \mathrm{A}^*(\sigma)$ *respectively.*

Proof. See [104, Lemma 2]. We remark that this lemma allows σ to range over $0 \leq \sigma \leq 1$ rather than $1/2 \leq \sigma \leq 1$, but it is easy to see that the contributions of the $0 \leq \sigma < 1/2$ cases are dominated by the $\sigma = 1/2$ case. □

Table 15.2: Historical upper bounds on $\theta_{\mathrm{gap-AA}}$.

Reference	Upper bound
Selberg (1943) [263]	$\frac{19}{77} = 0.2467\ldots$
Montgomery (1971) [218]	$\frac{1}{5} = 0.2$
Huxley (1972) [122]	$\frac{1}{6} = 0.1666\ldots$
Harman (1982) [99]	$\frac{1}{10} = 0.1$
Harman (1983) [100], Heath-Brown (1983) [110]	$\frac{1}{12} = 0.0833\ldots$
Jia (1995) [154]	$\frac{1}{13} = 0.0769\ldots$
Lou–Yao (1985) [197]	$\frac{17}{227} = 0.0748\ldots$
H. Li (1995) [178]	$\frac{2}{27} = 0.0740\ldots$
Jia (1995) [153], Watt (1995) [294]	$\frac{1}{14} = 0.0714\ldots$
H. Li (1997) [179]	$\frac{1}{15} = 0.0666\ldots$
Baker–Harman–Pintz (1997) [7]	$\frac{1}{16} = 0.625$
Wong (1996) [298], Jia (1996) [156], Harman (2007) [101]	$\frac{1}{18} = 0.0555\ldots$
Jia (1996) [155]	$\frac{1}{20} = 0.05$
R. Li (2024) [181]	$\frac{2}{43} = 0.0465\ldots$
R. Li (2025) [185]	$\frac{1}{22} = 0.0455\ldots$

`compute_gap2()`

This proposition can be used to recover the following bounds on $\theta_{\mathrm{gap},2}$:

Corollary 15.10.

(i) *Assuming the Riemann hypothesis,* $\theta_{\mathrm{gap},2} = 1$. *(Selberg, 1943 [263])*

(ii) *Assuming the Lindelof hypothesis,* $\theta_{\mathrm{gap},2} \leq 7/6$. *(Heath-Brown, 1979 [104])*

(iii) *Unconditionally,* $\theta_{\mathrm{gap},2} \leq 23/18$. *(Heath-Brown, 1979 [105]).*

Proof. For (i), we observe that $\|A\|_\infty = 2$ and that one can take $A(\sigma) = B(\sigma) = \varepsilon$ for any $\sigma > 1/2$ and $\varepsilon > 0$, and $A(\sigma) = 2$, $B(\sigma) = 6$ for $\sigma = 1/2$, and then the claim follows from Proposition 15.9.
For (ii), from Theorem 11.12 we may take $A(\sigma) = 2$ for $\sigma \leq 3/4$ and $A(\sigma) = \varepsilon$ for $3/4 < \sigma \leq 1$ and any $\varepsilon > 0$, while from Theorem 12.5 one can take $B(\sigma) = 8-4\sigma$ for $\sigma \leq 3/4$ and $B(\sigma) = \varepsilon$ for $3/4 < \sigma \leq 1$. The claim now follows from Proposition 15.9 and a routine calculation.
Part (iii) follows from applying Proposition 15.9 using the bounds from Theorem 12.6, together and various bounds on $\mathrm{A}(\sigma)$; see [144, Theorem 12.14] for details. □

Two variants of $\theta_{\mathrm{gap},2}$ are $\theta_{\mathrm{gap},>}$ and $\theta_{\mathrm{gap},\geq}$, defined respectively as the least exponent for which

$$\sum_{p_n \leq x : p_{n+1} - p_n \geq x^{1/2+\varepsilon}} (p_{n+1} - p_n) \ll x^{\theta_{\mathrm{gap},>} + o(1)}$$

(for any fixed $\varepsilon > 0$ for unbounded $x \geq 1$) and

$$\sum_{p_n \leq x : p_{n+1} - p_n \geq x^{1/2}} (p_{n+1} - p_n) \ll x^{\theta_{\mathrm{gap},\geq} + o(1)}$$

Table 15.3: Historical upper bounds on $\theta_{\mathrm{gap},2}$.

Reference	Upper bound
Selberg (1943) [263]	1 (on RH)
Heath-Brown (1978) [102]	$\frac{4}{3} = 1.3333\dots$
Heath-Brown (1979) [104]	$\frac{7}{6} = 1.1666\dots$ (on LH)
Heath-Brown (1979) [104]	$\frac{1413}{1067} = 1.3242\dots$
Heath-Brown (1979) [105]	$\frac{23}{18} = 1.2777\dots$
Yu (1996) [312]	1 (on LH)
Peck (1996) [230], Maynard (2012) [207]	$\frac{5}{4} = 1.25$
Stadlmann (2022) [268]	$\frac{123}{100} = 1.23$

(for unbounded $x \geq 1$). The trivial bounds are

Proposition 15.11 (Trivial bounds on large gaps)**.** *One has* $\theta_{\mathrm{gap},>} \leq \theta_{\mathrm{gap},\geq}$*. If* $\theta_{\mathrm{gap}} < 1/2$*, then* $\theta_{\mathrm{gap},>} = -\infty$*. In general, we have*

$$\max(1/2, \theta_{\mathrm{gap}}) \leq \max(1/2, \theta_{\mathrm{gap},>})$$

and $\theta_{\mathrm{gap},>} \leq 1$*. Also* $\theta_{\mathrm{gap},>} \leq \theta_{\mathrm{gap},2} - 1/2$*.*

The proofs are routine and are omitted. Historical bounds on $\theta_{\mathrm{gap},>}$ are recorded in Table 15.

Table 15.4: Historical upper bounds on $\theta_{\mathrm{gap},>}$ and $\theta_{\mathrm{gap},\geq}$.

Reference	Upper bound on $\theta_{\mathrm{gap},>}$	Upper bound on $\theta_{\mathrm{gap},\geq}$
Selberg (1943) [263]	$\frac{1}{2} = 0.5$ (on RH)	
Wolke (1975) [297]		$\frac{29}{30} = 0.966\dots$
Cook (1979) [44]	$\frac{85}{98} = 0.8673\dots$	
Huxley (1980) [126]	$\frac{1759}{2134} = 0.8242\dots$	
Huxley (1980) [126]	$\frac{3}{4} = 0.75$ (on LH)	
Ivíc (1979) [140]	$\frac{215}{266} = 0.8082\dots$	
Heath-Brown (1979) [105]		$\frac{3}{4} = 0.75$
Heath-Brown (1979) [104]	$\frac{5}{8} = 0.625$	
Peck (1998) [231]		$\frac{25}{36} = 0.6944\dots$
Matomäki (2007) [205]		$\frac{2}{3} = 0.6666\dots$
Heath-Brown (2020) [114]		$\frac{3}{5} = 0.6$
Järviniemi (2022) [152]		$\frac{57}{100} = 0.57$

For any $0 < \theta < 1$, let $\mu_{\mathrm{PNT}}(\theta)$ denote the least exponent μ such that for all unbounded X, one has $\sum_{x \leq n < x + x^\theta} \Lambda(n) = (1 + o(1))x^\theta$ for all $x \in [X, 2X]$ outside of an exceptional set of measure $O(X^{\mu + o(1)})$. Thus for instance $\mu_{\mathrm{PNT}}(\theta) = -\infty$ for $\theta > \theta_{\mathrm{PNT}}$ (and $\mu_{\mathrm{PNT}}(\theta) \geq 0$ for $\theta < \theta_{\mathrm{PNT}}$), and $\mu_{\mathrm{PNT}}(\theta) < 1$ implies $\theta \geq \theta_{\mathrm{PNT-AA}}$. The quantity $\mu_{\mathrm{PNT}}(\theta)$ is clearly non-decreasing in θ.

The following bounds are known:

Lemma 15.12 (Bounds on μ)**.**

(i) *[12, Theorem 2(i)] For sufficiently small* $\Delta > 0$*, we have* $\mu_{\mathrm{PNT}}(1/6+\Delta) \le 1 - c\Delta$ *and* $\mu_{\mathrm{PNT}}(7/12-\Delta) \le \frac{5}{8} + \frac{7}{4}\Delta + O(\Delta^2)$*.*

(ii) *[12, Theorem 2(ii)] Assuming RH, we have* $\mu_{\mathrm{PNT}}(\theta) \le 1-\theta$ *for* $0 < \theta \le 1/2$*.*

(iii) *[11, Lemma 1] We have*

$$\mu_{\mathrm{PNT}}(\theta) \le \begin{cases} \frac{3(1-\theta)}{2} & \frac{1}{2} < \theta \le \frac{11}{21} \\ \frac{47-42\theta}{35} & \frac{11}{21} < \theta \le \frac{23}{42} \\ \frac{36\theta^2-96\theta+55}{39-36\theta} & \frac{23}{42} < \theta \le \frac{7}{12} \end{cases}$$

Some further bounds were claimed in the region $1/6 < \theta \le 1/2$*, but unfortunately the arguments provided are incomplete (the claim (13) of that paper is not justified for* $\theta \le 1/2$*).*

(iv) *[78] For any* $0 < \theta < 1$*, one has*

$$\mu_{\mathrm{PNT}}(\theta) \le \inf_{\varepsilon>0} \sup_{\substack{0\le\sigma<1 \\ \mathrm{A}(\sigma)\ge\frac{1}{1-\theta}-\varepsilon}} \min(\mu_{2,\sigma}(\theta), \mu_{4,\sigma}(\theta))$$

where

$$\mu_{2,\sigma}(\theta) := (1-\theta)(1-\sigma)\mathrm{A}(\sigma) + 2\sigma - 1$$

and

$$\mu_{4,\sigma}(\theta) := (1-\theta)(1-\sigma)\mathrm{A}^*(\sigma) + 4\sigma - 3.$$

(v) *[114, Theorem 2]* $\mu_{\mathrm{PNT}}(1/2) \le 3/5$*.*

`prime_excep()`

In 2004, under the assumption of the existence of exceptional Dirichlet characters, Friedlander and Iwaniec [77] proved the following result:

Theorem 15.13. *[77] Let* $\chi = \chi_D$ *denotes the real primitive character of conductor* D*,* $x \geqslant D^r$ *with* $r = 18290$*. Then we have*

$$\pi(x) - \pi\left(x - x^{\frac{39}{79}}\right) = \frac{x^{\frac{39}{79}}}{\log x}\left(1 + O\left(L(1,\chi)(\log x)^{r^r}\right)\right).$$

Moreover, if we have

$$L(1,\chi) \ll (\log x)^{-r^r-1},$$

then there is always a prime number in the interval $\left[x - x^{\frac{39}{79}}, x\right]$ *for any* $D^r \leqslant x \leqslant \exp\left(L(1,\chi)^{-\frac{1}{r^r+1}}\right)$*.*

Note that $\frac{39}{79} = 0.4936\dots$. In 2024, Li [180] improved the exponent $\frac{39}{79}$ to 0.4923 with $r = 433433$.

15.1 Extremal values of prime gaps

Consider now the problem of determining upper bounds on

$$H_1 := \liminf_{n\to\infty}(p_{n+1} - p_n) \tag{15.2}$$

as well as lower bounds on

$$G(X) := \max_{p_{n+1}\le X}(p_{n+1} - p_n). \tag{15.3}$$

From the prime number theorem one expects $p_{n+1} - p_n$ to be of size $\log p_n$ on average, so that

Theorem 15.14 (Consequences of the prime number theorem)**.** *One has*

$$\liminf_{n\to\infty}\frac{p_{n+1} - p_n}{\log p_n} \le 1, \qquad G(X) \ge (1 + o(1))\log X \quad (X \to \infty).$$

However, $p_{n+1} - p_n$ can be sometimes be much smaller or much larger than its average size. The following is a classical conjecture regarding small prime gaps.

Conjecture 15.15 (Twin prime conjecture)**.** *One has*

$$\liminf_{n\to\infty}(p_{n+1} - p_n) = 2.$$

Since all sufficiently large primes are odd, the twin prime conjecture states that prime gaps achieve the smallest possible size, infinitely often. In the other direction, it is conjectured that

Conjecture 15.16 (Cramér [48])**.** *One has*

$$\limsup_{X\to\infty}\frac{G(X)}{(\log X)^2} = 1.$$

Note that by Theorem 15.8 it is known that $G(X) \ll X^{0.52}$.
The current best known result concerning (15.2) is

Theorem 15.17 (Polymath 8b [240])**.** *One has*

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \le 246.$$

Sharper conditional bounds are also known.

Theorem 15.18 (Maynard [209])**.** *Assuming the Elliott-Halberstam conjecture (EH), one has*

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \le 12.$$

Theorem 15.19 (Polymath 8b [240])**.** *Assuming the Generalized Elliott-Halberstam conjecture (GEH), one has*

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \le 6.$$

Historical progress towards this problem is recorded in Section 15.1.
The current best known lower bound on $G(X)$ is

Theorem 15.20 (Ford–Green–Konyagin–Maynard–Tao (2017) [68])**.** *For unbounded X, one has*

$$G(X) \gg \frac{\log X \log\log X \log\log\log\log X}{\log\log\log X}$$

Table 15.5: Historical progression of bounds related to (15.2).

Reference	Unconditional result	Assuming EH
Goldston–Pintz–Yıldırım (2009) [82]	$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = 0$	$H_1 \leq 16$
Goldston–Pintz–Yıldırım (2010) [83]	$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{(\log p_n)^{1/2}(\log\log p_n)^2} < \infty$	
Pintz (2013) [235]	$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{(\log p_n)^{3/7}(\log\log p_n)^{4/7}} < \infty$	
Zhang (2014) [315]	$H_1 < 7 \cdot 10^7$	
Polymath 8a (2014) [30]	$H_1 \leq 4680$	
Maynard (2015) [209]	$H_1 \leq 600$	$H_1 \leq 12$
Polymath 8b (2015) [240]	$H_1 \leq 246$	

Table 15.6: Historical progression of bounds related to (15.3).

Reference	Lower bound on $G(X)$ (for X sufficient
Westzynthius (1931) [295]	$G(X) \gg \log X \frac{\log\log\log X}{\log\log\log\log X}$
Erdős (1935) [65]	$G(X) \gg \log X \frac{\log\log X}{(\log\log\log X)^2}$
Rankin (1938) [244]	$G(X) > (c_0 + o(1)) \log X \frac{\log\log X \log\log\log\log}{(\log\log\log X)^2}$
Schönhage (1963) [262]	$c_0 = \frac{1}{2}e^{\gamma}$
Rankin (1963) [246]	$c_0 = e^{\gamma}$
Maier–Pomerance (1990) [204]	$c_0 = 1.31256e^{\gamma}$
Pintz (1997) [234]	$c_0 = 2e^{\gamma}$
Ford–Green–Konyagin–Tao (2016) [69], Maynard (2016) [210]	$G(X) \gg f(X) \log X \frac{\log\log X \log\log\log\log X}{(\log\log\log X)^2}$ for
Ford–Green–Konyagin–Maynard–Tao (2017) [68]	$G(X) \gg \frac{\log X \log\log X \log\log\log \mathrm{lo}}{\log\log\log X}$

Chapter 16

The generalized Dirichlet divisor problem

For any fixed integer $k \geq 1$, let

$$d_k(n) := \sum_{n_1 \cdots n_k = n} 1$$

denote the number of ways a positive integer n may be written as a product of exactly k positive integers. The divisor sum

$$D_k(x) := \sum_{n \leq x} d_k(n)$$

is known to satisfy the asymptotic formula

$$D_k(x) = x P_{k-1}(\log x) + \Delta_k(x)$$

where P_{k-1} is an explicit polynomial of degree $k-1$ and $\Delta_k(x) = o(x)$ is an error term. The (generalized) Dirichlet divisor problem concerns bounding the growth rate of $\Delta_k(x)$ as $x \to \infty$.

Definition 16.1 (Divisor sum exponents)**.** *Let $k \geq 1$ be a fixed integer. Then, α_k is the least (fixed) exponent for which*

$$\Delta_k(x) \ll x^{\alpha_k + o(1)}$$

for unbounded $x > 0$. Furthermore, β_k is the least (fixed) exponent for which

$$\left(\frac{1}{x}\int_1^x (\Delta_k(t))^2\, dt\right)^{1/2} \ll x^{\beta_k + o(1)}$$

for unbounded $x > 0$ (in both definitions, the implied constant may depend on k).

One can also give a non-asymptotic definition: α_k, β_k are respectively the least exponent such that for all $\varepsilon > 0$, there exists $C = C(\varepsilon, k) > 0$ for which

$$|\Delta_k(x)| \leq C x^{\alpha_k + \varepsilon}, \qquad (x \geq C)$$

and

$$\left|\frac{1}{x}\int_1^x (\Delta_k(t))^2 \mathrm{d}t\right|^{1/2} \leq C x^{\beta_k + \varepsilon}, \qquad (x \geq C).$$

In the case $k = 1$, the problem is trivial. In particular:

Lemma 16.2 (d_1 exponent)**.** *One has* $\alpha_1 = \beta_1 = 0$.

Proof. Follows from $\sum_{n\le x} 1 = x + O(1)$. □

However, the value of α_k is not known for $k \ge 2$. On the other hand, the values of β_2 and β_3 are known.

Theorem 16.3 (Hardy [94])**.** *One has* $\beta_2 = 1/4$.

Theorem 16.4 (Cramér [47])**.** *One has* $\beta_3 = 1/3$.

Nevertheless, the value of β_k is not known for $k \ge 4$. Unconditionally, the following lower-bounds are known to hold.

Lemma 16.5 (Lower bound on α_k and β_k)**.** *For all* $k \ge 1$*, one has*

$$\alpha_k \ge \beta_k \ge \frac{1}{2} - \frac{1}{2k}.$$

Proof. The first inequality follows from inserting the bound $\Delta_k(x) \ll x^{\alpha_k + o(1)}$ into the definition of β_k. The second inequality is due to Titchmarsh [275]. Note also that the weaker inequality $\alpha_k \ge 1/2 - 1/(2k)$ was first proved by Hardy [93]. □

It is conjectured that this lower bound on α_k and β_k is in fact an equality [277, p. 320]. Amongst other consequences, this conjecture implies the Lindelöf hypothesis [277, Chapter XII].

Conjecture 16.6 (Generalised Dirichlet divisor problem conjecture)**.** *For all* $k \ge 1$*, one has*

$$\alpha_k = \beta_k = \frac{1}{2} - \frac{1}{2k}.$$

The remainder of this chapter focuses on upper bounds on α_k and β_k.

16.1 Known pointwise bounds on divisor sum exponents

Currently the sharpest known upper bound on α_2 is:

Theorem 16.7. *[186, Theorem 1.2] One has* $\alpha_2 \le \alpha^* = 0.314483\ldots$*, where* α^* *is the solution to the equation*

$$\frac{8}{25}\alpha - \frac{(\sqrt{2(1+14\alpha)} - 5\sqrt{-1+8\alpha})^2}{200} + \frac{51}{200} = \alpha$$

on the interval $\alpha \in [0.3, 0.35]$.

Table 16.1 records the historical progression of upper bounds on α_2.
Currently, the sharpest known bound on α_3 is:

Theorem 16.8. *[171] One has* $\alpha_3 \le 43/96$.

Table 16.2 records the historical progression of upper bounds on α_3.
For larger k, estimates typically make use of the following relationship with zeta-moments.

Lemma 16.9. *Let* $k \ge 2$ *be an integer. If* $M(\sigma, k) = 1$ *then* $\alpha_k \le \sigma$.

Proof. See [144, §13.3]. □

Table 16.1: Historical bounds on α_2

Reference	Upper bound on α_2
Dirichlet (1849) [?], Piltz [?]	$1/2 = 0.5$
Voronoi (1903) [286]	$1/3 = 0.3333\dots$
van der Corput (1922) [45]	$33/100 = 0.33$
van der Corput (1928) [46]	$27/82 = 0.3292\dots$
Chih (1950) [39], Richert (1953) [247]	$15/46 = 0.3260\dots$
Kolesnik (1969) [169]	$12/37 = 0.3243\dots$
Kolesnik (1973) [170]	$346/1067 = 0.3242\dots$
Kolesnik (1982) [172]	$35/108 = 0.3240\dots$
Kolesnik (1985) [173, p. 118]	$139/429 = 0.3240\dots$
Iwaniec–Mozzochi (1988) [150]	$7/22 = 0.3181\dots$
Huxley (1993) [128]	$23/73 = 0.3150\dots$
Huxley (2003) [130]	$131/416 = 0.3149\dots$
Li–Yang (2023) [186]	$0.314483\dots$

For completeness we record the historical progression in bounds for α_k.

Lemma 16.10 (Piltz bound)**.** *For $k \geq 2$, one has*

$$\alpha_k \leq 1 - \frac{1}{k}.$$

Lemma 16.11 (Voronoi, Landau bound)**.** *For $k \geq 2$, one has*

$$\alpha_k \leq 1 - \frac{2}{k+1}.$$

Proof. See Voronoi [286] for $k = 2$ and Landau [176] for $k \geq 3$. □

Lemma 16.12 (Hardy–Littlewood bound for $k \geq 4$)**.** *For $k \geq 4$, one has*

$$\alpha_k \leq 1 - \frac{3}{k+2}.$$

Proof. See [96]. The original proof relied on the assumption that $\mu(1/2) \leq 1/6$ which was published later. □

Lemma 16.13 (Tong bound for $4 \leq k \leq 11$)**.** *One has*

$$\alpha_4 \leq 1/2, \quad \alpha_5 \leq 4/7, \quad \alpha_6 \leq 5/8, \quad \alpha_7 \leq 71/107$$
$$\alpha_8 \leq 41/59, \quad \alpha_9 \leq 31/43, \quad \alpha_{10} \leq 26/35, \quad \alpha_{11} \leq 19/25$$

Proof. See Tong [278]. □

Theorem 16.14. *[108] For $4 \leq k \leq 8$, one has*

$$\alpha_k \leq \frac{3k-4}{4k}.$$

Table 16.2: Historical bounds on α_3

Reference	Upper bound on α_3
Walfisz (1926) [288]	$43/87 = 0.4942\ldots$
Atkinson (1941) [1]	$37/75 = 0.4933\ldots$
Rankin (1955) [245]	$0.4931466\ldots$
Yue (1958) [313]	$14/29 = 0.4827\ldots$
Yin (1959) [308]	$25/52 = 0.4807\ldots$
Yin (1959) [309]	$10/21 = 0.4761\ldots$
Yue–Wu (1962) [314]	$8/17 = 0.4705\ldots$
Chen (1965) [33]	$5/11 = 0.4545\ldots$
Yin (1964) [310]	$34/75 = 0.4533\ldots$
Yin–Li (1981) [311], Zheng (1988) [317]	$127/282 = 0.4503\ldots$
Kolesnik (1981) [171]	$43/96 = 0.4479\ldots$

Theorem 16.15 (Ivić–Ouellet bound for large k)**.** *[145] One has*

$$\begin{aligned} &\alpha_{10} \le 27/40, \quad \alpha_{11} \le 0.6957, \quad \alpha_{12} \le 0.7130, \quad \alpha_{13} \le 0.7306, \\ &\alpha_{14} \le 0.7461, \quad \alpha_{15} \le 0.75851, \quad \alpha_{16} \le 0.7691, \quad \alpha_{17} \le 0.7785, \\ &\alpha_{18} \le 0.7868, \quad \alpha_{19} \le 0.7942, \quad \alpha_{20} \le 0.8009. \end{aligned}$$

Theorem 16.16. *[144, Theorem 13.12] One can bound α_k by*

$$\begin{gathered} (3k-4)/4k \text{ for } 4 \le k \le 8 \\ 35/54 \text{ for } k = 9 \\ 41/60 \text{ for } k = 10 \\ 7/10 \text{ for } k = 11 \\ (k-2)/(k+2) \text{ for } 12 \le k \le 25 \\ (k-1)/(k+4) \text{ for } 26 \le k \le 50 \\ (31k-98)/32k \text{ for } 51 \le k \le 57 \\ (7k-34)/7k \text{ for } k \ge 58. \end{gathered}$$

Lemma 16.17 (Heath-Brown bound for large k)**.** *For any $k \ge 2$, one has*

$$\alpha_k \le 1 - 0.849k^{-2/3}.$$

Proof. See Heath-Brown [113]. □

Theorem 16.18 ([15])**.** *For integer $k \ge 30$, one has*

$$\alpha_k \le 1 - 1.421(k-1.18)^{-2/3}.$$

Moreover, $\alpha_k \le 1 - 1.889k^{-2/3}$ for sufficiently large k.

Theorem 16.19 (Trudgian–Yang bound for large k)**.** *[[279], Theorem 2.9]One has*

$$\alpha_9 \le 0.64720, \quad \alpha_{10} \le 0.67173, \quad \alpha_{11} \le 0.69156, \quad \alpha_{12} \le 0.70818,$$
$$\alpha_{13} \le 0.72350, \quad \alpha_{14} \le 0.73696, \quad \alpha_{15} \le 0.74886, \quad \alpha_{16} \le 0.75952,$$
$$\alpha_{17} \le 0.76920, \quad \alpha_{18} \le 0.77792, \quad \alpha_{19} \le 0.78581, \quad \alpha_{20} \le 0.79297,$$
$$\alpha_{21} \le 0.79951.$$

Theorem 16.20 (Li bound for large k)**.** *[[184], Theorem 2]One has*

$$\alpha_9 \le 0.638889, \quad \alpha_{10} \le 0.663329, \quad \alpha_{11} \le 0.684349, \quad \alpha_{12} \le 0.701768,$$
$$\alpha_{13} \le 0.717523, \quad \alpha_{14} \le 0.731898, \quad \alpha_{15} \le 0.744898, \quad \alpha_{16} \le 0.75638,$$
$$\alpha_{17} \le 0.766588, \quad \alpha_{18} \le 0.775721, \quad \alpha_{19} \le 0.783939, \quad \alpha_{20} \le 0.791374.$$

Chapter 17

The number of Pythagorean triples

Definition 17.1 (Pythagorean triple exponent)**.** *Let θ_{Pythag} be the least exponent for which one has*

$$P(N) = cN^{1/2} - c'N^{1/3} + N^{\theta_{\mathrm{Pythag}}+o(1)}$$

for unbounded N and some fixed c, c', where $P(N)$ is the number of primitive Pythagorean triples of area no greater than N.

Lemma 17.2. *One has $\theta_{\mathrm{Pythag}} \leq 1/4$.*

Proof. See [296, 62]. The previous bound $\theta_{\mathrm{Pythag}} \leq 1/3$ was obtained in [175]. □

Lemma 17.3. *If (k, ℓ) is an exponent pair, and RH holds, then*

$$\theta_{\mathrm{Pythag}} \leq \max\left(\frac{1}{3} - \frac{5}{6}\frac{k+\ell-3/2}{4(k+\ell)-7}, \frac{1}{2} - \frac{3}{2}\frac{k+\ell-3/2}{4(k+\ell)-7}\right)$$

Proof. See [214] and and [279, Section 5.10]. □

Lemma 17.4. *Assuming RH, one has $\theta_{\mathrm{Pythag}} \leq 71/316$.*

Proof. See [279, Section 5.10]. □

Chapter 18

The de Bruijn–Newman constant

A survey on this topic may be found at [224].
Let $H_0\colon \mathbf{C} \to \mathbf{C}$ denote the function

$$H_0(z) := \frac{1}{8}\xi\left(\frac{1}{2} + \frac{iz}{2}\right), \tag{18.1}$$

where ξ denotes the Riemann xi function

$$\xi(s) := \frac{s(s-1)}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{18.2}$$

and ζ is the Riemann zeta function. Then H_0 is an entire even function with functional equation $H_0(\overline{z}) = \overline{H_0(z)}$, and the Riemann hypothesis is equivalent to the assertion that all the zeroes of H_0 are real.
It is a classical fact (see [277, p. 255]) that H_0 has the Fourier representation

$$H_0(z) = \int_0^\infty \Phi(u)\cos(zu)\ du$$

where Φ is the super-exponentially decaying function

$$\Phi(u) := \sum_{n=1}^{\infty}(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u})\exp(-\pi n^2 e^{4u}). \tag{18.3}$$

The sum defining $\Phi(u)$ converges absolutely for negative u also. From Poisson summation one can verify that Φ satisfies the functional equation $\Phi(u) = \Phi(-u)$ (i.e., Φ is even).
De Bruijn [55] introduced the more general family of functions $H_t\colon \mathbf{C} \to \mathbf{C}$ for $t \in \mathbf{R}$ by the formula

$$H_t(z) := \int_0^\infty e^{tu^2}\Phi(u)\cos(zu)\ du. \tag{18.4}$$

As noted in [52, p.114], one can view H_t as the evolution of H_0 under the backwards heat equation $\partial_t H_t(z) = -\partial_{zz}H_t(z)$. As with H_0, each of the H_t are entire even functions with functional equation $H_t(\overline{z}) = \overline{H_t(z)}$. From results of Pólya [239] it is known that if H_t has

purely real zeroes for some t then $H_{t'}$ has purely real zeroes for all $t' > t$. De Bruijn showed that the zeroes of H_t are purely real for $t \geq 1/2$. Strengthening these results, Newman [223] showed that there is an absolute constant $-\infty < \Lambda \leq 1/2$, now known as the *De Bruijn-Newman constant*, with the property that H_t has purely real zeroes if and only if $t \geq \Lambda$. The Riemann hypothesis is then clearly equivalent to the upper bound $\Lambda \leq 0$. Newman conjectured the complementary lower bound $\Lambda \geq 0$, and noted that this conjecture asserts that if the Riemann hypothesis is true, it is only "barely so".
Known lower bounds on Λ are listed in the tables below.

Table 18.1: Lower bounds on Λ.

Lower bound on Λ	Reference
$> -\infty$	Newman 1976 [223]
> -50	Csordas–Norfolk–Varga 1988 [49]
> -5	te Riele 1991 [271]
> -0.385	Norfolk–Ruttan–Varga 1992 [226]
> -0.0991	Csordas–Ruttan–Varga 1991 [51]
$> -4.379 \times 10^{-6}$	Csordas–Smith–Varga 1994 [52]
$> -5.895 \times 10^{-9}$	Csordas–Odlyzko–Smith–Varga 1993 [50]
$> -2.63 \times 10^{-9}$	Odlyzko 2000 [227]
$> -1.15 \times 10^{-11}$	Saouter–Gourdon–Demichel 2011 [257]
≥ 0	Rodgers–Tao 2020 [255]
≥ 0	Dobner 2021 [60]

The argument of Dobner applies more generally to the Selberg class.
For upper bounds, we have

Table 18.2: Upper bounds on Λ.

Upper bound on Λ	Reference
$\leq 1/2$	Newman 1976 [223]
$< 1/2$	Ki–Kim–Lee 2009 [166]
≤ 0.22	Polymath 2019 [241]
≤ 0.2	Platt–Trudgian 2021 [238]

Chapter 19

Brun-Titchmarsh type theorems

Definition 19.1 (Prime counting function on arithmetic progressions)**.** *Suppose* $a, q \in \mathbb{Z}$ *with* $\gcd(a, q) = 1$*. For each* $x \geq 0$*, define*

$$\pi(x; q, a) = \sum_{\substack{p \leq x \\ p \text{ prime} \\ q \mid (p-a)}} 1.$$

The ordinary prime counting function can be recovered by $\pi(x) = \pi(x; 1, 1)$*.*

The Prime Number Theorem (PNT) shows that $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$. On the other hand, known results on the asymptotic behavior of $\pi(x; q, a)$ depend greatly on how q and x are sent to ∞. Heuristically, it is expected that for "most" sequences $q_n, x_n \to \infty$ with $q_n < x_n$, $\pi(x_n; q_n, a) \sim \frac{x_n}{\varphi(q_n)\log(x_n)}$ as $n \to \infty$. Brun-Titchmarsh type theorems make this precise by provide asymptotic upper or lower bounds on $\pi(x; q, a)$ in terms of $\frac{x}{\varphi(q)\log x}$ or related quantities, presupposing constraints between q and x.

Definition 19.2 (Logarithmic integral function)**.** *Define the offset logarithmic integral function for* $x \geq 2$ *by* $\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}u}{\log u}$*. Note that* $\mathrm{Li}(x) \sim \frac{x}{\log x}$*.*

We first record two early results which recover the correct asymptotic under stringent assumptions.

Theorem 19.3 (Brun-Titchmarsh theorem under GRH (1929) [272])**.** *Under the Generalized Riemann Hypothesis (GRH), if* $q < x$*, then*

$$\pi(x; q, a) = \frac{\mathrm{Li}(x)}{\varphi(q)} + O(x^{1/2}\log x).$$

Theorem 19.4 (Walfisz (1936) [290])**.** *Fix* $B \geq 0$ *and suppose* $q \leq (\log x)^B$*. Then there exists* $A = A(B) > 0$ *such that*

$$\pi(x; q, a) = \frac{\mathrm{Li}(x)}{\varphi(q)} + O(x\exp(-A\sqrt{\log x})).$$

19.1 Upper bounds

Titchmarsh's original theorem establishes a coarse asymptotic upper bound.

Theorem 19.5 (Brun-Titchmarsh theorem (1929) [272])**.** *If* $0 < \theta < 1$ *and* $q \le x^\theta$*, then*

$$\pi(x;q,a) = O\left(\frac{x}{\varphi(q)\log x}\right) + O(x^{6(1-\theta)/7}).$$

Later bounds more generally bound the number of prime numbers equivalent to $a \pmod q$ in the interval $[x, x+y]$. Observe that setting $x = 0$ indeed yields an improvement on previous results.

Theorem 19.6 (Lint, Richert (1965) [283])**.** *If* $y > q$*, then*

$$\pi(x+y;q,a) - \pi(x;q,a) < \frac{2y}{\varphi(q)\log(y/q)} \min\left(\frac{3}{2}, 1 + \frac{6}{\log(y/q)}\right).$$

Theorem 19.7 (Montgomery, Vaughan (1973) [219])**.** *If* $y > q$*, then*

$$\pi(x+y;q,a) - \pi(x;q,a) < \frac{2y}{\varphi(q)\log(y/q)}.$$

On the other hand, various bounds improve on this result under polynomial relationships of the form $q \le x^\theta$. To state these, we need the following definition.

Definition 19.8 (θ and C_θ)**.** *Suppose* $x > 0$ *and* $q \in \mathbb{Z}$*. Define* $\theta := \frac{\log q}{\log x}$*, and let* $C_\theta > 0$ *be the smallest constant such that*

$$\max_{a:\gcd(a,q)=1} \pi(x;q,a) \le \frac{(C_\theta + o(1))x}{\varphi(q)\log(x)}$$

as $x \to \infty$*.*

Here is the historical progression of bounds on C_θ, where

$$\begin{aligned} C_1(\theta) = &-\frac{66}{33-16\theta}\int_2^4 \frac{\log(t-1)}{t}dt \\ &+ \frac{8}{4-\left(3+\frac{7}{64}\right)\theta}\int_{\frac{8-\left(7+\frac{7}{32}\right)\theta}{4\theta}}^{\frac{165\left(4-\left(3+\frac{7}{64}\right)\theta\right)}{4(33-16\theta)}-\frac{1}{4}} \frac{\log(t-1)}{t}dt \\ &+ \frac{8}{4-\left(1+\frac{7}{64}\right)\theta}\int_{\max\left(\frac{4-\left(1+\frac{7}{64}\right)\theta}{2(2-3\theta)}-\frac{5}{4},2\right)}^{\frac{8-\left(7+\frac{7}{32}\right)\theta}{4\theta}} \frac{\log(t-1)}{t}dt \end{aligned}$$

and

$$\begin{aligned} C_2(\theta) = &-\frac{66}{33-16\theta}\int_2^4 \frac{\log(t-1)}{t}dt \\ &+ \frac{16}{8-7\theta}\int_{\frac{8-7\theta}{4\theta}}^{\frac{165(8-7\theta)}{8(33-16\theta)}} \frac{\log(t-1)}{t}dt \\ &+ \frac{16}{8-3\theta}\int_{\max\left(\frac{9\theta}{4(2-3\theta)},2\right)}^{\frac{8-7\theta}{4\theta}} \frac{\log(t-1)}{t}dt. \end{aligned}$$

Let $\theta_{\rm char}$ be the least constant such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that one has a character sum bound of the form

$$\sum_{l \leq L} \chi(l) \ll L q^{-\delta}$$

whenever χ is a non-principal character mod q and $L \geq q^{\theta_{\rm char}} + \varepsilon$. The Burgess bound [27, 28] shows that $\theta_{\rm char} \leq 3/8$, which can be improved to $\theta_{\rm char} \leq 1/4$ for cube-free q. The extended Lindelöf hypothesis implies that $\theta_{\rm char} = 0$.

In [147, Theorem 3] it was shown that

$$C_\theta \leq \max(\frac{2}{1 - \theta\theta_{\rm char}}, \frac{2}{2 - 12\theta/5}).$$

This was improved in [198] to

$$C_\theta \leq \max(\frac{2}{1 - \theta\theta_{\rm char}}, \frac{2}{8/7 - 24\theta/35}).$$

A further (complicated) bound on C_θ in the range $3/7 \leq \theta < 9/20$ may be found in [3, Theorem 2].

In [301], the bound $C_\theta \leq 16/(8 - (3 + 2\theta_{\rm RP})\theta)$ for $9/20 < \theta < 1/2$ was established, where $\theta_{\rm RP}$ is the exponent for the Ramanujan–Petersson conjecture for $GL_2(\mathbf{Q})$. By the work of Kim and Sarnak [167] one has $\theta_{\rm RP} \leq 7/64$. One can also convert exponent pairs to bounds on C_θ:

Theorem 19.9 (From exponent pairs to Brun–Titchmarsh)**.** *[301, Theorem 1.4] If (k, ℓ) is an exponent pair, then*

$$C_\theta \leq \frac{4}{(3 + k - \ell) - (3 + 3k - \ell)\theta}$$

whenever

$$\frac{1 + k - \ell}{2 + 2k - 2\ell} \leq \theta \leq \frac{1 + k - \ell}{1 + 2k - \ell}.$$

Averaged versions of the Brun–Titchmarsh inequality were proven in [119], [120], [147], [59], [72], [73] [215], [4], [6], [74] and [182].

For any θ, let C'_θ denote the best constant for which one has an upper bound

$$\pi(x + x^\theta) - \pi(x) \leq (C'_\theta + o(1))\frac{x^\theta}{\log x}$$

for unbounded x. The following bounds on C'_θ are known:

19.2 Lower bounds

The most basic lower bound is Dirichlet's theorem, stating that $\lim_{x\to\infty} \pi(x; q, a) = \infty$; we shall not record it here. Until relatively recently, good lower bounds were not known on $\pi(x; q, a)$ other than Theorem 19.4 for small q, but there are many known estimates for the smallest value of x for which $\pi(x; q, a) > 0$.

Definition 19.10 (Linnik's constant L)**.** *Define L to be the infimum over all $L' > 0$ where there exists $q_0(L') > 0$ such that for all $q \geq q_0(L')$ and $x > q^{L'}$, $\min_{\gcd(a,q)=1} \pi(x; q, a) > 0$.*

Here is the historical progression of L.
Recent work by Maynard [208] establishes asymptotic lower bounds for $\pi(x; q, a)$.

Theorem 19.11 (Maynard (2013) [208])**.** *For sufficiently large q and $x > q^8$, we have*

$$\frac{\log q}{q^{1/2}} \left(\frac{x}{\varphi(q) \log x} \right) \ll \pi(x; q, a).$$

Theorem 19.12 (Maynard (2013) [208])**.** *Let $\epsilon > 0$. There exists $q_0(\epsilon) > 0$ such that for all $q \geq q_0(\epsilon)$,*

$$\frac{q^{-\epsilon} x}{\varphi(q) \log x} \ll \pi(x; q, a).$$

Table 19.1: Historical bounds on C_θ

Reference	Range of θ	Upper bound on C_θ
Titchmarsh (1930)	$(0,1)$	Finite
van Lint & Richert (1965) [283] Montgomery & Vaughan (1973) [219] Selberg (1991) [265]	$(0,1)$	$2/(1-\theta)$
Motohashi (1973) [220]	$(0,1/3)$	$16/(8-3\theta)$
Motohashi (1974) [221]	$(0,1/3]$	2 (on LH)
Motohashi (1973) [220]	$(2/5,1/2]$	$2/(2-3\theta)$
Motohashi (1974) [221]	$[1/3,2/5]$	$4/(2-\theta)$
Motohashi (1974) [221]	$[1/3,2/5]$	$2/(2-3\theta)$ (on LH)
Goldfeld (1975) [81]	$(0,24/71)$	$16/(8-3\theta)$
Iwaniec (1982) [147]	$(0,9/20)$	$16/(8-3\theta)$
Iwaniec (1982) [147]	$(0,9/20)$	$8/(4-2\theta)$ (if q cube-free)
Iwaniec (1982) [147]	$[9/20,2/3]$	$8/(6-7\theta)$
Baker (1996) [3]	$(9/20,1/2)$	$4/(2-\theta)$
Friedlander & Iwaniec (1997) [75]	$[6/11,1)$	$(2-((1-\theta)/4)^6)/(1-\theta)$
Maynard (2013) [208]	$(0,1/8]$	2
Bourgain & Garaev (2014) [25]	$[1-\delta,1)$	$(2-c_0(1-\theta)^2)/(1-\theta)$
Xi & Zheng (2024) [301]	$(9/20,1/2)$	$16/(8-(3+7/32)\theta)$
Xi & Zheng (2024) [301]	$(9/20,1/2)$	$16/(8-3\theta)$ (if q prime)
Xi & Zheng (2024) [301]	$[1/2,12/23)$	$8/(5-5\theta)$ (if q prime)
Xi & Zheng (2024) [301]	$[12/23,32/61)$	$32/(32-43\theta)$ (if q prime)
Xi & Zheng (2024) [301]	$[32/61,8/15)$	$24/(16-17\theta)$ (if q prime)
Xi & Zheng (2024) [301]	$[8/15,7/13)$	$48/(40-49\theta)$ (if q prime)
Xi & Zheng (2024) [301]	$[7/13,6/11)$	$16/(11-12\theta)$ (if q prime)
Xi & Zheng (2024) [301]	$[6/11,4/7)$	$32/(28-35\theta)$ (if q prime)
Xi & Zheng (2024) [301]	$[9/51,9/11]$	$160/(89-91\theta)$ (if q smooth squar
Xi & Zheng (2024) [301]	$[1/8,5/12)$	2 (if q smooth square-free)
Xi & Zheng (2024) [301]	$[5/12,9/20)$	$5/(5-6\theta)$ (if q smooth square-f
Xi & Zheng (2025) [302]	$[9/20,1/2)$	$66/(33-16\theta)-C_1(\theta)$
Xi & Zheng (2025) [302]	$[9/20,1/2)$	$66/(33-16\theta)-C_2(\theta)$ (if q prin
Xi & Zheng (2025) [302]	$[3/10,3/4]$	$24/(15-16\theta)$ (if q smooth square
Xi & Zheng (2025) [303]	$[1/2,34/67]$	$240/(184-217\theta)$ (if q prime
Xi & Zheng (2025) [303]	$[1/2,(\nu(2\nu+1))/(4\nu^2+\nu+4)]$	$\frac{8}{6-7\theta+\frac{2\nu-(3\nu+4)\theta}{\nu(2\nu-1)}}$ for every integer $\nu\geq 5$ (

Table 19.2: Historical bounds on C'_θ

Reference	Range of θ	Upper bound on C'_θ
Montgomery & Vaughan (1973) [219]	$(0,1)$	$2/\theta$
Iwaniec (1982) [147]	$(1/3,1)$	$18/(15\theta-2)$
Iwaniec (1982) [147]	$(1/2,1)$	$4/(1+\theta)$
Lou & Yao (1989) [199]	$(6/11,11/20]$	$22/(100\theta-45)$
Lou & Yao (1992) [200]	$(6/11,1]$	1.031
Baker, Harman, & Pintz (1997) [7]	$(0.55,1)$	1.0001
R. Li (2025) [183]	$(0.52,0.521]$	2.874
	$(0.521,0.522]$	2.700
	$(0.522,0.523]$	2.583
	$(0.523,0.524]$	2.536
	$(0.524,0.525]$	2.437
	$(0.525,0.535]$	2.347

Table 19.3: Historical bounds on L

Reference	Upper bound on L
Linnik (1994) [190]	$< \infty$
Pan (1957) [228]	10000
Pan (1958) [229]	5448
Chen (1965) [34]	777
Jutila (1970) [158]	630
Jutila (1970) [157]	550
Chen (1977) [35]	168
Jutila (1977) [159]	80
Graham (1977) [84]	36
Graham (1981) [86]	20
Chen (1979) [36]	17
Wang (1986) [291]	16
Chen & Liu (1989) [37]	13.5
Chen & Liu (1990) [38]	11.5
Wang (1991) [292]	8
Heath-Brown (1992) [112]	5.5
Meng (2000) [211]	4.5 (if q prime)
Meng (2001, 2010) [212] [213]	4.5 (if q has bounded cubic part)
Xylouris (2009) [304]	5.2
Xylouris (2011) [305]	5.18
Xylouris (2011, 2018) [306] [307]	5
Montgomery (1971) [218]	$\frac{5}{2} = 2.5$ (if q is a power of a fixed prime)
Forti & Viola (1973) [70]	$\frac{45}{20-\sqrt{3}} = 2.4633\dots$ (if q is a power of a fixed prime)
Jutila (1972) [162]	$\frac{3(9+\sqrt{17})}{16} = 2.4606\dots$ (if q is a power of a fixed prime)
Huxley (1975) [125]	$\frac{12}{5} = 2.4$ (if q is a power of a fixed prime)
B. Chen (2025) [31]	$\frac{7}{3} = 2.3333\dots$ (if q is a power of a fixed prime)
Banks & Shparlinski (2019) [10]	$\frac{1}{0.4736} = 2.1115\dots$ (if q is a power of a fixed prime)
R. Li (2025) [183]	$\frac{1}{0.4752} = 2.1044\dots$ (if q is a power of a fixed prime)
Heath-Brown (1990) [111]	$3 + \varepsilon$ (under the existence and some conditions of the exceptional character $2 + \varepsilon$ (under the existence and some conditions of the exceptional character,
Friedlander & Iwaniec (2003) [76]	$\frac{117}{59} = 1.983\dots$ (under the existence and some conditions of the exceptional

Chapter 20

Waring and Goldbach type problems, and Schnirelman's constant

20.1 Waring Problem

Definition 20.1. *Let $A \subset \mathbf{N}$ be such that there exists k for which*

$$\underbrace{A + A + \cdots + A}_{k \text{ times}} = \mathbf{N} \tag{20.1}$$

Then A is called an additive basis of $\mathbf{N}$. The minimum k for which (20.1) *holds is the order of A.*

Definition 20.2. *For any $k \geq 1$ let $A_k = \{n^k : n \in \mathbf{N} \cup \{0\}\}$. Let $g(k)$ be the order of A_k when it exists. That is, $g(k)$ is the minimum number of k powers needed to write any natural number as the sum of (not necessarily unique) $g(k)$ many k powers including 0.*

Definition 20.3. *For any $k \geq 1$, let $G(k)$ be the minimum m such that there exists $N \geq 1$ for which*

$$\underbrace{A_k + \cdots + A_k}_{m \text{ times}} = \mathbf{N} \setminus J_N.$$

where $J_N = \{1, \ldots, N\}$. That is, $G(k)$ is the minimum number of k powers such that every sufficiently large integer may be written as the sum of (not necessarily unique) $G(k)$ many k powers including 0.

20.1.1 Known values of g(k)

Theorem 20.4 (Lagrange's Four Square Theorem)**.** *We have $g(2) = 4$; that is every natural number may be written as the sum of* 4 *perfect squares.*

Theorem 20.5 (Linnik [188])**.** *$g(k)$ exists for all $k \geq 1$.*

Linnik's proof relied on the notion of Schnirelmann density, which will be discussed later.

In fact, the exact value of $g(k)$ is known for almost all $k \geq 1$. We have

$$g(k) = 2^k + \left[\frac{3}{2}\right]^k - 2 \quad \text{if} \quad 2^k \left\{\frac{3}{2}\right\}^k + \left[\frac{3}{2}\right]^k \leq 2^k$$

and, otherwise,

$$g(k) = 2^k + \left[\frac{3}{2}\right]^k + \left[\frac{4}{3}\right]^k - \xi$$

where $\xi = 2$ if

$$\left[\frac{3}{2}\right]^k + \left[\frac{4}{3}\right]^k + \left[\frac{3}{2}\right]^k \left[\frac{4}{3}\right]^k \geq 2^k$$

and 3 otherwise. Note that $[x]$ is the greatest integer less than x and $\{x\} = x - [x]$. It has been shown that there at at most finitely many exceptions [202]. To complete the proof, it suffices to show

$$\left\{\left(\frac{3}{2}\right)^k\right\} \leq 1 - \left(\frac{3}{4}\right)^{k-1}$$

It has been shown for all $k > 5000$, $\{(3/2)^k\} \leq 1 - a^k$ where $a = 2^{-0.9} \approx 0.53$, and for sufficiently large k, $\{(3/2)^k\} \leq 1 - (0.5769\ldots)^{-k}$ [61] [16].

20.1.2 Known values of $G(k)$

Only 2 values of $G(k)$ are known definitively: $G(2) = 4$ as shown by Lagrange and $G(4) = 16$ as shown by Davenport [53].

Definition 20.6. *Let $G_1(k)$ be the smallest number m such that*

$$d(\underbrace{A_k + \cdots + A_k}_{m \text{ times}}) = 1$$

where $d(A)$ represents the natural density of A:

$$d(A) = \lim_{N \to \infty} \frac{\#(A \cap J_N)}{N}$$

$G_1(k)$ has been determined for 5 values:

Davenport [54]	$G_1(3) = 4$
Hardy and Littlewood [98]	$G_1(4) = 15$
Vaughan [284]	$G_1(8) = 32$
Wooley [299]	$G_1(16) = 64$
Wooley [299]	$G_1(32) = 128$

Table 20.1: Known values of $G_1(k)$

	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Best Bound	31	39	47	55	63	72	81	89	97	105	113	121	129	137
Conjectured	8	32	13	12	12	16	14	15	16	64	18	27	20	25

Table 20.2: Conjectured and best upper bounds for $G(k)$ for $7 \leq k \leq 20$

20.1.3 General bounds for $G(k)$

Theorem 20.7 (Brudern and Wooley 2022 [26]). *For all $k \geq 1$,*

$$G(k) < k(\log k + C_1) + C_2$$

Furthermore,

$$G(k) \leq \lceil k(\log k + 4.20032) \rceil$$

This bound is the sharpest to date and was a significant improvement over the previous bound by Wooley [299]: for sufficiently large k,

$$G(k) \leq k(\log k + \log \log k + 2 + O(\log \log k / \log k))$$

20.1.4 Bounds for special cases for $G(k)$

$k = 3$

Lemma 20.8. $G(3) \geq 4$

Proof. Note cubes are congruent $1, -1, 0$ modulo 9. Thus, numbers congruent $4, 5$ modulo 9 may not be expressed as the sum of 3 cubes. ☐

Theorem 20.9 (Linnik [189]). $G(3) \leq 7$

The exact value of $G(3)$ is conjectured to be 4, but has not been proven.

Conjectured $G(k)$ for small k

Table 20.2 summarizes the best upper bounds for $G(k)$ and the conjectured values of $G(k)$ for $7 \leq k \leq 20$.
The upper bounds for $k \leq 13$ were deduced from Wooley [300] and the bounds for $14 \leq k \leq 20$ are from Theorem 20.7.

20.1.5 Generalized Waring problem and connections to the Generalized Riemann Hypothesis

Waring's Problem concerns the solvability of equations of the form

$$x_1^k + x_2^k + \cdots + x_n^k = m \tag{20.2}$$

for $m, n, k \in \mathbf{N}$, and Theorem 20.5 states that for any fixed $k \geq 1$, there exists $n \in \mathbf{N}$ such that (20.2) is solvable for all $m \in \mathbf{N}$. A more generalized problem arises when k is not fixed. Given any $\mathbf{k} = (k_1, \dots, k_n) \in \mathbf{N}^n$, the generalized Waring problem concerns the solvability of the equations of the form

$$x_1^{k_1} + x_2^{k_2} + \cdots + x_n^{k_n} = m \tag{20.3}$$

The following theorem due to Erich Kamke provides a partial result.

Theorem 20.10 (Kamke)**.** *Let $f(x)$ be an integer valued polynomial such that there does not exist $d \in \mathbf{N}$ such that $d | f(n)$ for all $n \in \mathbf{N}$. Then for sufficiently large k,*

$$f(x_1) + f(x_2) + \cdots + f(x_k) = m$$

is solvable for all large enough m.

Assuming the Generalized Riemann Hypothesis (GRH), The solvability of (20.3) can be guaranteed for specific $\mathbf{k}$. For example:

Theorem 20.11 (Wooley)**.** *Assuming GRH, then*

$$x_1^2 + x_2^2 + x_3^3 + x_4^3 + x_5^6 + x_6^6 = n \tag{20.4}$$

is solvable for sufficiently high n. Furthermore, (20.4) is not solvable for at most $O((\log N)^{3+\epsilon})$ integers between 1 and N.

20.2 Goldbach-Style Problems

Goldbach's original conjecture stated that every positive integer could be written as the sum of 3 primes. In light of Waring's problem, a natural extension of Goldbach's problem asks when

$$p_1^k + p_2^k + \cdots + p_m^k = n \tag{20.5}$$

is solvable for all $n \in \mathbf{N}$ for $p_1, \ldots p_k$ prime and $k \in \mathbf{N}$. It is conjectured when $m \geq k+1$ and for sufficiently large n satisfying local conditions, which will be made more explicit for specific values of k, (20.5) is solvable.

20.2.1 When $k = 2$

It is conjectured that

$$n = p_1^2 + p_2^2 + p_3^2 + p_4^2 \tag{20.6}$$

is solvable whenever $n \equiv 4 \pmod{24}$. The following theorem gives the closest solution.

Theorem 20.12 (Liu, Wooley, Yu [195])**.** *Let $E(N)$ be the number of integers $n \equiv 4 \pmod{24}$ for which (20.6) is not solvable. For any $\epsilon > 0$,*

$$E(N) \ll O(N^{\frac{3}{8}+\epsilon})$$

20.2.2 When $k = 4, 5$

Following Kawada and Wooley, we define the following quantities to give the relevant local conditions for the cases $k = 4$ and $k = 5$. Let $\theta = \theta(p, k)$ be the greatest power of p dividing k; that is $p^\theta | k$ but $p^{\theta+1} \nmid k$. Then, let

$$\gamma(k,p) = \begin{Bmatrix} \theta + 2 & \text{when } p = 2, \theta > 0 \\ \theta + 1 & \text{otherwise} \end{Bmatrix}$$

and

$$K(k) = \prod_{(p-1)|k} p^{\gamma(k,p)}$$

In particular, $K(4) = 240$ and $K(5) = 2$.

Definition 20.13. *For $k \in \mathbf{N}$, let $H(k)$ be the minimum integer s such that*

$$p_1^k + p_2^k + \cdots + p_s^k = n$$

is solvable for sufficiently large n whenever $n \equiv s \pmod{K(k)}$.

Finding the value of $H(k)$ is the main focus of the modern Waring-Goldbach problem.

Theorem 20.14 (Wooley, Kawada 2001 [164])**.** *We have*

- $H(4) \leq 14$
- *For any positive A,*

  $$p_1^4 + p_2^4 + \cdots + p_7^4 = n$$

  has at most $O(N(\log N)^{-A})$ exceptions for $n \equiv 7 \pmod{240}$ and $1 \leq n \leq N$.
- $H(5) \leq 21$
- *For any positive A,*

  $$p_1^5 + p_2^5 + \cdots + p_{11}^5 = n$$

  has at most $O(N(\log N)^{-A})$ exceptions for n odd and $1 \leq n \leq N$.

In 2014, Zhao improved the bound for $k = 4$ and showed $H(4) \leq 13$ [316]. He also showed $H(6) \leq 32$ in the same paper.

20.2.3 When $k \geq 7$

Theorem 20.15 (Kumchev, Wooley 2016 [174])**.** *For large values of k,*

$$H(k) \leq (4k-2)\log k - (2\log 2 - 1)k - 3 \tag{20.7}$$

Table 20.3 summarizes the best bounds on $H(k)$ for $7 \leq k \leq 20$.

7	8	9	10	11	12	13	14	15	16	17	18	19	20
45	57	69	81	93	107	121	134	149	163	177	193	207	223

Table 20.3: Upper bounds for $H(k)$

20.3 Schnirelmann Density

20.3.1 Existence of Additive Basis

Definition 20.16. *Define the Schnirelmann density of $A \subset \mathbf{N}$ as*

$$\sigma A = \inf_{n \geq 1} \frac{\#(A \cap J_n)}{n}$$

Definition 20.17. *Define the lower asymptotic density of $A \subset \mathbf{N}$ as*

$$\delta A = \liminf_{n \to \infty} \frac{\#(A \cap J_n)}{n}$$

Theorem 20.18 (Schnirelmann [261])**.** *Suppose $\sigma A > 0$. Then A is an additive basis for* $\mathbf{N}$.

With Theorem 20.18, it is possible to prove many sets form additive bases. For example:

Theorem 20.19 (Schnirelmann [261])**.** *Let $\mathbb{P}$ denote the set of primes. Then, $\delta(\mathbb{P} + \mathbb{P}) > 0$. Therefore, $\mathbb{P}$ is an additive basis for $\mathbf{N}$. The order of $\mathbb{P}$ is denoted C and called Schnirelmann's constant.*

Schnirelmann originally bounded $C < 80000$ and Helfgott showed in 2013 that $C \leq 4$ [117]. Goldbach's conjecture claims that $C = 3$.

Theorem 20.20 (Romanoff [242])**.** *Let $\mathfrak{S}_a = \{p + a^k : p \in \mathbb{P}, k \in \mathbf{N}\}$. Then, $\sigma \mathfrak{S}_a > 0$ for all $a \in \mathbf{N}$. Thus, each integer n may be written as the sum of at most C_a primes and C_a powers of a, where C_a is a constant depending only on a.*

20.3.2 Essential Components

Definition 20.21. *$B \subset \mathbf{N}$ is called an essential component if $\sigma(A + B) > \sigma(A)$ for any $A \subset \mathbf{N}$ with $0 < \sigma A < 1$.*

Linnik showed in 1933 gave the first example of an essential component that is not a basis [187]. Erdos showed in 1936 that every basis is also an essential component [64]. The minimum possible size of an essential component remained an open problem until Ruzsa showed in 1984[256] that for any $\epsilon > 0$, there exists an essential component H such that

$$\#(H \cap J_n) \ll (\log n)^{1+\epsilon}$$

but there does not exist an essential component such that

$$\#(H \cap J_n) \ll (\log n)^{1+o(1)}$$

Chapter 21

The Gauss circle problem and its generalizations

This chapter is not yet integrated into the main blueprint.
For any fixed integer $k \geq 2$ and unbounded R, consider the problem of estimating the number of integer lattice points contained in $B_k(R)$, a k-dimensional ball of radius R:

$$S_k(R) := \#\mathbb{Z}^k \cap B_k(R) = \#\{x \in \mathbb{Z}^k : |x| \leq R\}.$$

Equivalently, $S_k(R)$ may be written as the partial sum

$$S_k(R) = \sum_{n \leq R^2} r_k(n)$$

where $r_k(n)$ counts the number of integer solutions to the equation $x_1^2 + \cdots + x_k^2 = n$.
By considering the volume of a k-dimensional ball of radius R, one has the asymptotic

$$S_k(R) \sim \mathrm{Vol}(B_k(R)) = \frac{\pi^{k/2}}{\Gamma(k/2+1)} R^k.$$

The generalized Gauss circle problem concerns estimating the error term in this approximation.

Definition 21.1. *For fixed integer $k \geq 2$, define $\theta_k^{\mathrm{Gauss}}$ as the least (fixed) exponent for which*

$$S_k(R) - \mathrm{Vol}(B_k(R)) \ll R^{\theta_k^{\mathrm{Gauss}} + o(1)}.$$

Figure 21.1 and Figure 21.2 plots the magnitude of this error term for $k = 2$ and $k = 3$ respectively (for $0 < R \leq 1000$).
It is conjectured that

Conjecture 21.2. *One has*

$$\theta_k^{\mathrm{Gauss}} = \begin{cases} 1/2, & k = 2, \\ k-2, & k \geq 3. \end{cases}$$

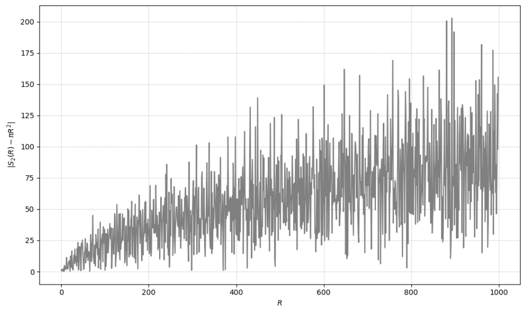

Figure 21.1: $|S_k(R) - \mathrm{Vol}(B_k(R))|$ for $k = 2$ and $0 < R \leq 1000$

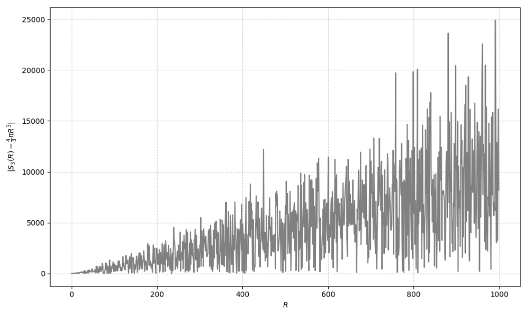

Figure 21.2: $|S_k(R) - \mathrm{Vol}(B_k(R))|$ for $k = 3$ and $0 < R \leq 1000$

21.1 Known upper and lower bounds

?? 21.2 is known to hold for $k \geq 4$, i.e.

Theorem 21.3. *For integer $k \geq 4$, one has $\theta_k^{\mathrm{Gauss}} = k - 2$.*

The remaining open cases are $k = 2, 3$. For such cases the following lower-bounds on $\theta_k^{\mathrm{Gauss}}$ are known:

Theorem 21.4. *One has $\theta_2^{\mathrm{Gauss}} \geq 1/2$ and $\theta_3^{\mathrm{Gauss}} \geq 1$.*

In light of **??** 21.2 and Theorem 21.4, in the rest of this section we shall focus on upper bounds on $\theta_k^{\mathrm{Gauss}}$ for $k = 2, 3$.
The case $k = 2$ is known classically as Gauss's circle problem. The current sharpest known bound on $\theta_2^{\mathrm{Gauss}}$ is

Theorem 21.5 (Li–Yang (2023) [186])**.** *One has $\theta_2^{\mathrm{Gauss}} \leq 2\alpha$, where $\alpha = 0.31448\ldots$ is the solution to the equation*

$$\frac{8}{25}\alpha - \frac{(\sqrt{2(1+14\alpha)} - 5\sqrt{-1+8\alpha})^2}{200} + \frac{51}{200} = \alpha$$

on the interval $[0.3, 0.35]$.

Remark 21.6. *The value of α is the same as that appearing in Theorem 16.7. Historically, methods used to make progress in the α_2 exponent in the Dirichlet divisor problem have led to corresponding improvements in $\theta_2^{\mathrm{Gauss}}$ (and vice versa). This may be unsurprising given that both problems reduce to counting the number of lattice points contained in a curved region with a smooth boundary (with the region being the hyperbola $\{(m, n) \in [0, \infty)^2 : mn \leq x\}$ in the case of the Dirichlet divisor problem).*

The historical progression of upper bounds on $\theta_2^{\mathrm{Gauss}}$ is recorded in Table 21.1 and Figure 21.3.

Table 21.1: Historical upper bounds on $\theta_2^{\mathrm{Gauss}}$

Reference	Bound on $\theta_2^{\mathrm{Gauss}}$
Gauss (1834)	1
Sierpiński (1906) [266]	$2/3 = 0.6666\ldots$
van der Corput (1923) [282]	$2/3 - \delta$ for some $\delta > 0$
Littlewood–Walfisz (1924) [194]	$37/56 = 0.6607\ldots$
Walfisz (1927) [289]	$163/247 = 0.6599\ldots$
Nieland (1928) [225]	$27/41 = 0.6585\ldots$
Titchmarsh (1935) [274]	$15/23 = 0.6521\ldots$
Hua (1942) [121]	$13/20 = 0.65$
Iwaniec–Mozzochi (1988) [150]	$7/11 = 0.6363\ldots$
Huxley (1993) [127]	$46/73 = 0.6301\ldots$
Huxley (2003) [130]	$131/208 = 0.6298\ldots$
Li–Yang (2023) [186]	$2\alpha* = 0.6289\ldots$

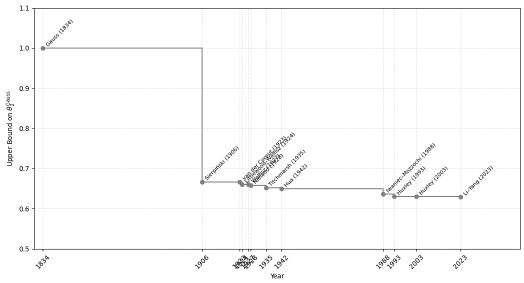

Figure 21.3: Historical upper bounds on θ_2^{Gauss}

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