
Reconciling Predictive and Statistical Parity: A Causal Approach

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Abstract

Since the rise of fair machine learning as a critical field of inquiry, many different notions on how to quantify and measure discrimination have been proposed in the literature. Some of these notions, however, were shown to be mutually incompatible. Such findings make it appear that numerous different kinds of fairness exist, thereby making a consensus on the appropriate measure of fairness harder to reach, hindering the applications of these tools in practice. In this paper, we investigate one of these key impossibility results that relates the notions of statistical and predictive parity. Specifically, we derive a new causal decomposition formula for the fairness measures associated with predictive parity, and obtain a novel insight into how this criterion is related to statistical parity through the legal doctrines of disparate treatment, disparate impact, and the notion of business necessity. Our results show that through a more careful causal analysis, the notions of statistical and predictive parity are not really mutually exclusive, but complementary and spanning a spectrum of fairness notions through the concept of business necessity. Finally, we demonstrate the importance of our findings on a real-world example.

1 Introduction

As society increasingly relies on AI-based tools, an ever larger number of decisions that were once made by humans are now delegated to automated systems, and this trend is likely to only accelerate in the coming years. Such automated systems may exhibit discrimination based on gender, race, religion, or other sensitive attributes, as witnessed by various examples in criminal justice [1], facial recognition [13, 5], targeted advertising [12], and medical treatment allocation [19], to name a few.

In light of these challenges, a large amount of effort has been invested in attempts to detect and quantify undesired discrimination based on society’s current ethical standards, and then design learning methods capable of removing possible unfairness from future predictions and decisions. During this process, many different notions on how to quantify discrimination have been proposed. In fact, the current literature is abundant with different fairness metrics, some of which are mutually incompatible [9]. The incompatibility of these measures can create a serious obstacle for practitioners since choosing among them, even for the system designer, is usually a non-trivial task.

In the real world, issues of discrimination and unfairness are analyzed through two major legal doctrines. The first one is *disparate treatment*, which enforces the equality of treatment of different groups and prohibits the use of the protected attribute (e.g., race) during the decision process. One of the legal formulations for showing disparate treatment is that “a similarly situated person who is not a member of the protected class would not have suffered the same fate” [3]. Disparate treatment is commonly associated with the notion of direct effects in the causal literature. The second doctrine is known as *disparate impact* and focuses on *outcome fairness*, namely, the equality of outcomes among protected groups. Discrimination through disparate impact occurs if a facially neutral practice

has an adverse impact on members of the protected group, including cases where discrimination is unintended or implicit. In practice, the law may not necessarily prohibit the usage of all characteristics correlated with the protected attribute due to their relevance to the business itself, which is legally known as “business necessity” (labelled BN from now on) or “job-relatedness”. Therefore, some of the variables may be used to distinguish between individuals, even if they are associated with the protected attribute [14]. From a causal perspective, disparate impact is realized through indirect forms of discrimination, and taking into account BN considerations is the essence of this doctrine [3].

We now note how BN requirements span a range of fairness notions between statistical (SP) and predictive parity (PP). Consider a set of causal pathways between the attribute X and the predictor $\hat{Y}$, labeled $\mathcal{C}_1, \dots, \mathcal{C}_k$. For example, these pathways could represent the direct, indirect, and spurious effects of X on $\hat{Y}$, or more generally any set of path-specific effects. For each $\mathcal{C}_i$, the system designer needs to decide whether the causal pathway in question is considered as discriminatory (i.e., *not* in the BN set), or if it is considered non-discriminatory (i.e., in the BN set), as shown in Fig. 1 left. If $\mathcal{C}_i$ is not in the BN set, then the causal effect transmitted along this pathway should equal zero, written $\mathcal{C}_i(X, \hat{Y}) = 0$ (which is the main focus of previous works on path-specific counterfactual fairness [22, 7]). However, if $\mathcal{C}_i$ is in the BN set, then the transmitted causal effect does not need to equal 0, i.e., $\mathcal{C}_i(X, \hat{Y}) \neq 0$ may be allowed. Interestingly, the transmitted causal effect should not take an arbitrary value in this case. Rather, it should equal the *transmitted effect from X to the original outcome Y (observed in the real world) along the same pathway*, written as $\mathcal{C}_i(X, \hat{Y}) = \mathcal{C}_i(X, Y)$. Therefore, considerations of BN can be summarized as a 0/1 vector of length k , where each pathway $\mathcal{C}_i$ is represented by an entry. As we demonstrate both intuitively and formally throughout the paper, the choice of the BN set being empty, written $(0, \dots, 0)$, will ensure the notion of statistical parity. The choice of the BN set that includes all pathways, written $(1, \dots, 1)$, will lead to predictive parity. Crucially, various intermediate fairness notions between the two ends of the spectrum are possible, depending on what is allowed or not; for an illustration, see Fig. 1 (right side).

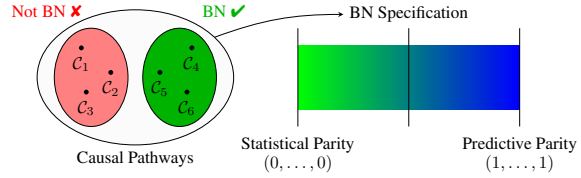


Figure 1: BN specification (left) and the spectrum between statistical and predictive parity (right).

The unification of the principles behind statistical and predictive parity through the concept of BN has the potential to bridge the gap between the two notions and improve the current state-of-the-art, by providing an argument against the seemingly discouraging impossibility result between the two notions. The practitioner is no longer faced with a false dichotomy of choosing between statistical or predictive parity but rather faces a spectrum of different fairness notions determined by the choice of the business necessity set, which is usually fixed through societal consensus and legal requirements.

1.1 Organization & Contributions

In Sec. 2, we introduce important preliminary notions in causal inference, and the formal tools for specifying causal pathways $\mathcal{C}_1, \dots, \mathcal{C}_k$ described above. Further, we discuss the notions of statistical and predictive parity, together with the impossibility result that separates them. In Sec. 2.2 we discuss how different causal variations with the statistical parity measure can be disentangled using an additive decomposition [23]. In Sec. 3, we develop a novel decomposition of the predictive parity measure in terms of the underlying causal mechanisms and discuss how this notion is in fact complementary to statistical parity through the concept of business necessity. In Sec. 3.1, we unify the theoretical findings by introducing a formal procedure that shows how to assess the legal doctrines of discrimination by leveraging both the concepts of causal predictive parity and causal statistical parity. In Sec. 4, we apply our approach in the context of criminal justice using the COMPAS dataset [1], and demonstrate empirically the trade-off between SP and PP. Our key formal contributions are the following:

- We develop the first non-parametric decomposition of the predictive parity measure in terms of the underlying causal mechanisms (Thm. 1).
- Building on the previous result, we define a natural notion of causal predictive parity (Def. 5). We then develop a procedure (Alg. 1) for evaluating if a classifier satisfies the desired notions

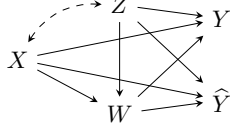


Figure 2: Standard Fairness Model.

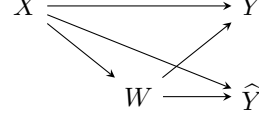


Figure 3: Standard Fairness Model with $Z = \emptyset$ from Thm. 1, extended with the predictor $\hat{Y}$.

of causal statistical parity and causal predictive parity, which provides a unified framework for incorporating desiderata from both predictive and statistical parity and sheds light on the impossibility theorem that relates the two notions.

2 Background

We use the language of structural causal models (SCMs) as our basic semantical framework [15]. A structural causal model (SCM) is a tuple $\mathcal{M} := \langle V, U, \mathcal{F}, P(u) \rangle$, where V, U are sets of endogenous (observables) and exogenous (latent) variables, respectively, $\mathcal{F}$ is a set of functions f_{V_i} , one for each $V_i \in V$, where $V_i \leftarrow f_{V_i}(\text{pa}(V_i), U_{V_i})$ for some $\text{pa}(V_i) \subseteq V$ and $U_{V_i} \subseteq U$. $P(u)$ is a strictly positive probability measure over U . Each SCM $\mathcal{M}$ is associated to a causal diagram $\mathcal{G}$ [15] over the node set V where $V_i \rightarrow V_j$ if V_i is an argument of f_{V_j} , and $V_i \leftrightarrow V_j$ if the corresponding U_{V_i}, U_{V_j} are not independent. An instantiation of the exogenous variables $U = u$ is called a *unit*. By $Y_x(u)$ we denote the potential response of Y when setting $X = x$ for the unit u , which is the solution for $Y(u)$ to the set of equations obtained by evaluating the unit u in the submodel $\mathcal{M}_x$, in which all equations in $\mathcal{F}$ associated with X are replaced by $X = x$. Building on the notion of a potential response, one can further define the notions of counterfactual and factual contrasts, given by:

Definition 1 (Contrasts [16]). *Given an SCM $\mathcal{M}$, a contrast $\mathcal{C}$ is any quantity of the form*

$$\mathcal{C}(C_0, C_1, E_0, E_1) = \mathbb{E}[y_{C_1} \mid E_1] - \mathbb{E}[y_{C_0} \mid E_0], \quad (1)$$

where E_0, E_1 are observed (factual) clauses and C_0, C_1 are counterfactual clauses to which the outcome Y responds. Furthermore, whenever

- (a) $E_0 = E_1$, the contrast $\mathcal{C}$ is said to be counterfactual;
- (b) $C_0 = C_1$, the contrast $\mathcal{C}$ is said to be factual.

For instance, the contrast $(C_0 = \{x_0\}, C_1 = \{x_1\}, E_0 = \emptyset, E_1 = \emptyset)$ corresponds to the *average treatment effect (ATE)* $\mathbb{E}[y_{x_1} - y_{x_0}]$. Similarly, the contrast $(C_0 = \{x_0\}, C_1 = \{x_1\}, E_0 = \{x_0\}, E_1 = \{x_0\})$ corresponds to the *effect of treatment on the treated (ETT)* $\mathbb{E}[y_{x_1} - y_{x_0} \mid x_0]$. Many other important causal quantities can be represented as contrasts, as exemplified later on.

Throughout this manuscript, we assume a specific cluster causal diagram $\mathcal{G}_{\text{SFM}}$ known as the standard fairness model (SFM) [16] over endogenous variables $\{X, Z, W, Y, \hat{Y}\}$ shown in Fig. 2. The SFM consists of the following: *protected attribute*, labeled X (e.g., gender, race, religion), assumed to be binary; the set of *confounding* variables Z , which are not causally influenced by the attribute X (e.g., demographic information, zip code); the set of *mediator* variables W that are possibly causally influenced by the attribute (e.g., educational level or other job-related information); the *outcome* variable Y (e.g., GPA, salary); the *predictor* of the outcome $\hat{Y}$ (e.g., predicted GPA, predicted salary). The SFM also encodes the assumptions typically used in the causal inference literature about the lack of hidden confounding¹. We next introduce the key notions and results from the fair ML literature needed for our discussion.

2.1 Statistical & Predictive Parity Notions

The notions of statistical parity and predictive parity are defined as follows²:

¹Partial identification techniques for bounding effects can be used for relaxing these assumptions [24].

²The paper [8] introduces predictive parity for a binary predictor $\hat{Y}$, and calibration for a continuous classification score $\hat{Y}$. In this paper we are agnostic to this distinction, and thus use predictive parity for both.

Definition 2 (Statistical [11] and Predictive [8] Parity). *Let X be the protected attribute, Y the true outcome, and $\hat{Y}$ the outcome predictor. The predictor $\hat{Y}$ satisfies statistical parity (SP) with respect to X if $X \perp\!\!\!\perp \hat{Y}$, or equivalently if the measure*

$$SPM_{x_0, x_1}(\hat{y}) := P(\hat{y} | x_1) - P(\hat{y} | x_0) = 0. \quad (2)$$

Further, $\hat{Y}$ satisfies predictive parity (PP) with respect to X, Y if $Y \perp\!\!\!\perp X | \hat{Y}$, or equivalently if

$$PPM_{x_0, x_1}(y | \hat{y}) := P(y | x_1, \hat{y}) - P(y | x_0, \hat{y}) = 0 \quad \forall \hat{y}. \quad (3)$$

Statistical parity ($\hat{Y} \perp\!\!\!\perp X$) requires that the predictor $\hat{Y}$ contains no information about the protected attribute X . In contrast to this, the notion of predictive parity ($Y \perp\!\!\!\perp X | \hat{Y}$), requires that $\hat{Y}$ should “exhaust”, or capture all the variations of X coming into the outcome Y in the current real world. An important known result that further corroborates this point is the following (see proof in Appendix C):

Proposition 1 (PP and Efficient Learning). *Let $\mathcal{M}$ be an SCM compatible with the Standard Fairness Model (SFM). Suppose that the predictor $\hat{Y}$ is based on the features X, Z, W . Suppose also that $\hat{Y}$ is an efficient learner, meaning that:*

$$\hat{Y}(x, z, w) = P(y | x, z, w). \quad (4)$$

Then, it follows that $\hat{Y}$ satisfies predictive parity w.r.t. X and Y .

Prop. 1 shows that PP is expected to hold whenever we learn the true conditional distribution $P(y | x, z, w)$. This is in stark contrast with the SP notion. For the SP to hold, the predictor $\hat{Y}$ should not be associated at all with the attribute X , whereas for PP to hold, $\hat{Y}$ should contain all variations coming from X . Perhaps unsurprisingly after this discussion, the following result holds:

Proposition 2 (SP and PP impossibility [4]). *Statistical parity ($\hat{Y} \perp\!\!\!\perp X$) and predictive parity ($Y \perp\!\!\!\perp X | \hat{Y}$) are mutually exclusive except in degenerate cases when $Y \perp\!\!\!\perp X$.*

The above theorem might lead the reader to believe that SP and PP criteria come from two different realities, and bear no relation to each other. After all, the theorem states that it is not possible for the predictor $\hat{Y}$ to include *all variations* of X coming into Y and simultaneously include *no variations* of X coming into Y . This realization is the starting point of our discussion in the rest of the manuscript.

2.2 Statistical Parity Decomposition

We continue by analyzing the decomposition of the statistical parity measure (also known as parity gap, or total variation), which is commonly used to determine if statistical parity is satisfied. The causal decomposition of the SPM requires the usage of causal measures known as counterfactual direct, indirect, and spurious effects, which are defined as:

Definition 3 (Counterfactual Causal Measures). *The counterfactual- $\{direct, indirect, spurious\}$ effects of X on $\hat{Y}$ are defined as follows*

$$Ctf-DE_{x_0, x_1}(\hat{y} | x) = P(\hat{y}_{x_1, W_{x_0}} | x) - P(\hat{y}_{x_0} | x) \quad (5)$$

$$Ctf-IE_{x_1, x_0}(\hat{y} | x) = P(\hat{y}_{x_1, W_{x_0}} | x) - P(\hat{y}_{x_1} | x) \quad (6)$$

$$Ctf-SE_{x_1, x_0}(\hat{y}) = P(\hat{y}_{x_1} | x_0) - P(\hat{y}_{x_1} | x_1). \quad (7)$$

The measures capture the variations from X to $\hat{Y}$ going through: (i) the direct mechanism $X \rightarrow \hat{Y}$; (ii) the indirect mechanism $X \rightarrow W \rightarrow \hat{Y}$; (iii) the spurious mechanisms $X \leftrightarrow Z \rightarrow \hat{Y}$, and $X \leftrightarrow Z \rightarrow W \rightarrow \hat{Y}$, respectively (see Fig. 2). Based on the defined measures, $SPM_{x_0, x_1}(\hat{y})$ admits an additive decomposition, first obtained in [23], given in the following theorem³:

Proposition 3 (Causal Decomposition of Statistical Parity [23]). *The statistical parity measure admits a decomposition into its direct, indirect, and spurious variations:*

$$SPM_{x_0, x_1}(\hat{y}) = Ctf-DE_{x_0, x_1}(\hat{y} | x_0) - Ctf-IE_{x_1, x_0}(\hat{y} | x_0) - Ctf-SE_{x_1, x_0}(\hat{y}). \quad (8)$$

³We note that the same decomposition can also be applied for the true outcome Y instead of the predictor $\hat{Y}$, as will be relevant in later sections.

This result shows how we can disentangle direct, indirect, and spurious variations within the SPM. We emphasize the importance of this result in the context of assessing the legal doctrines of discrimination. If a causal pathway (direct, indirect, or spurious) does not lie in the business necessity set, then the corresponding counterfactual measure (Ctf-DE, IE, or SE) needs to equal 0. To formalize this notion, we can now introduce the criterion of *causal statistical parity*:

Definition 4 (Causal Statistical Parity). *We say that $\hat{Y}$ satisfies causal statistical parity with respect to the protected attribute X if*

$$\text{Ctf-DE}_{x_0, x_1}(\hat{y} \mid x_0) = \text{Ctf-IE}_{x_1, x_0}(\hat{y} \mid x_0) = \text{Ctf-SE}_{x_1, x_0}(\hat{y}) = 0. \quad (9)$$

In practice, causal statistical parity can be a strong requirement, but the notion can be easily relaxed to include only a subset of the Ctf- $\{\text{DE}, \text{IE}, \text{or SE}\}$ measures, under BN requirements.

3 Predictive Parity Decomposition

After discussing the decomposition of the SPM, our aim is to obtain a causal understanding of the predictive parity criterion, $Y \perp\!\!\!\perp X \mid \hat{Y}$. To do so, we derive a formal decomposition result of the PP measure that involves both Y and $\hat{Y}$, shown in the following theorem:

Theorem 1 (Causal Decomposition of Predictive Parity). *Let $\mathcal{M}$ be an SCM compatible with the causal graph in Fig. 3 (i.e., SFM with $Z = \emptyset$). Then, it follows that the $\text{PPM}_{x_0, x_1}(y \mid \hat{y}) = P(y \mid x_1, \hat{y}) - P(y \mid x_0, \hat{y})$ can be decomposed into its causal and spurious anti-causal variations as:*

$$\text{PPM}_{x_0, x_1}(y \mid \hat{y}) = P(y_{x_1} \mid x_1, \hat{y}) - P(y_{x_0} \mid x_1, \hat{y}) + P(y_{x_0} \mid \hat{y}_{x_1}) - P(y_{x_0} \mid \hat{y}_{x_0}). \quad (10)$$

Thm. 1 offers a non-parametric decomposition result of the predictive parity measure that can be applied to any SCM compatible with the graph in Fig. 3. In Appendix A.1 we provide a proof of the theorem (together with the proof of Cor. 1 stated below), and in Appendix A.2 we perform an empirical study to verify the decomposition result of the theorem. For the additional special case of linear SCMs, the terms appearing in the decomposition in Eq. 10 can be computed explicitly:

Corollary 1 (Causal Decomposition of Predictive Parity in the Linear Case). *Under the additional assumptions that (i) the SCM $\mathcal{M}$ is linear and Y is continuous; (ii) the learner $\hat{Y}$ is efficient, then*

$$\mathbb{E}(y_{x_1} \mid x_1, \hat{y}) - \mathbb{E}(y_{x_0} \mid x_1, \hat{y}) = \alpha_{XW}\alpha_{WY} + \alpha_{XY} \quad (11)$$

$$\mathbb{E}(y_{x_0} \mid x_1, \hat{y}_{x_1}) - \mathbb{E}(y_{x_0} \mid x_1, \hat{y}_{x_0}) = -(\alpha_{XW}\alpha_{WY} + \alpha_{XY}), \quad (12)$$

where $\alpha_{V_i V_j}$ is the linear coefficient between variables V_i, V_j .

We now carefully unpack the key insight from Thm. 1. In particular, we showed that in the case of an SFM with $Z = \emptyset$ ⁴ the predictive parity measure can be written as:

$$\text{PPM} = \underbrace{P(y_{x_1} \mid x_1, \hat{y}) - P(y_{x_0} \mid x_1, \hat{y})}_{\text{Term (I) causal}} + \underbrace{P(y_{x_0} \mid \hat{y}_{x_1}) - P(y_{x_0} \mid \hat{y}_{x_0})}_{\text{Term (II) reverse-causal spurious}}. \quad (13)$$

The first, causal term can be expanded as

$$P(y_{x_1} \mid x_1, \hat{y}) - P(y_{x_0} \mid x_1, \hat{y}) = \sum_u \underbrace{[y_{x_1}(u) - y_{x_0}(u)]}_{\text{unit-level difference}} \underbrace{P(u \mid x_1, \hat{y})}_{\text{posterior}}. \quad (14)$$

The expansion shows us that the term is a weighted average of unit-level differences $y_{x_1}(u) - y_{x_0}(u)$. Each unit-level difference measures the causal effect of a transition $x_0 \rightarrow x_1$ on Y . The associated weights are given by the posterior $P(u \mid x_1, \hat{y})$, which determines the probability mass corresponding to a unit u within the set of all units compatible with $x_1, \hat{y}$. Therefore, the term represents an average causal effect of X on Y for a specific group of units. Interestingly, for any set of units selected by $x_1, \hat{y}$, the effect in Term (I) *does not depend on the constructed predictor $\hat{Y}$* , but only on the underlying system, i.e., it is not under the control of the predictor designer. The additional linear

⁴We remark that the essence of the argument is unchanged in the case with $Z \neq \emptyset$, but handling this case limits the clarity of presentation.

Algorithm 1 Business Necessity Cookbook

```
1: Input: data  $\mathcal{D}$ , BN-Set  $\text{BN} \subseteq \{\text{DE}, \text{IE}, \text{SE}\}$ 
2: for  $\text{CE} \in \{\text{DE}, \text{IE}, \text{SE}\}$  do
3:   if  $\text{CE} \in \text{BN}$  then
4:     Compute the effects  $\text{Ctf-CE}(y), \text{Ctf-CE}(\hat{y})$ 
5:     Assert that  $\text{Ctf-CE}(y) = \text{Ctf-CE}(\hat{y})$ , otherwise FAIL
6:   else
7:     Compute the effect  $\text{Ctf-CE}(\hat{y})$ 
8:     Assert that  $\text{Ctf-CE}(\hat{y}) = 0$ , otherwise FAIL
9:   end if
10: end for
11: if not FAIL then
12:   return SUCCESS
13: end if
14: Output: SUCCESS or FAIL of ensuring that disparate impact and treatment hold under BN.
```

result in Cor. 1 may also help the reader ground this idea, since it shows that Term (I) indeed captures the causal effect, which, in the linear case, can be obtained using the path-analysis of [21].

To achieve the criterion $\text{PPM} = 0$, the second term needs to be exactly the reverse of the causal effect, captured by the spurious variations induced by changing $\hat{y}_{x_0} \rightarrow \hat{y}_{x_1}$ in the selection of units. The second term, which is in the control of the predictor $\hat{Y}$ designer, needs to cancel out the causal effect measured by the first term for PPM to vanish. Therefore, we see that achieving predictive parity is about constructing $\hat{Y}$ in a way that reflects the causal effect of X on Y , across various groups of units. This key observation motivates a novel definition that we call *causal predictive parity*:

Definition 5 (Causal Predictive Parity). *Let $\hat{Y}$ be a predictor of the outcome Y , and let X be the protected attribute. Then, $\hat{Y}$ is said to satisfy causal predictive parity (CPP) with respect to a counterfactual contrast (C_0, C_1, E, E) if*

$$\mathbb{E}[y_{C_1} \mid E] - \mathbb{E}[y_{C_0} \mid E] = \mathbb{E}[\hat{y}_{C_1} \mid E] - \mathbb{E}[\hat{y}_{C_0} \mid E]. \quad (15)$$

Furthermore, $\hat{Y}$ is said to satisfy CPP with respect to a factual contrast (C, C, E_0, E_1) if

$$\mathbb{E}[y_C \mid E_1] - \mathbb{E}[y_C \mid E_0] = \mathbb{E}[\hat{y}_C \mid E_1] - \mathbb{E}[\hat{y}_C \mid E_0]. \quad (16)$$

The intuition behind the notion of causal predictive parity captures the intuition behind predictive parity. If a contrast $\mathcal{C}$ describes some amount of variation in the outcome Y , then it should describe the same amount of variation in the predicted outcome $\hat{Y}$. For any of the contrasts $\text{Ctf-}\{\text{DE}, \text{IE}, \text{SE}\}$ corresponding to a causal pathway, causal predictive parity would require that $\mathcal{C}(X, \hat{Y}) = \mathcal{C}(X, Y)$.

3.1 Reconciling Statistical and Predictive Parity

We now tie the notions of statistical and predictive parity through the concept of *business necessity*. In particular, if a contrast $\mathcal{C}$ is associated with variations that are not in the business necessity set, then the value of this contrast should be $\mathcal{C}(X, \hat{Y}) = 0$, following the intuition of causal statistical parity from Def. 4. However, if the variations associated with the contrast *are* in the business necessity set, then the value of that contrast should be equal for the predictor to the value for the true outcome

$$\mathcal{C}(X, \hat{Y}) = \mathcal{C}(X, Y), \quad (17)$$

following the intuition of causal predictive parity. Combining these two notions through business necessity results in Alg. 1. The algorithm requires the user to compute the measures

$$\text{Ctf-}\{\text{DE}, \text{IE}, \text{SE}\}(y), \text{Ctf-}\{\text{DE}, \text{IE}, \text{SE}\}(\hat{y}). \quad (18)$$

Importantly, under the SFM, these measures are *identifiable* from observational data:

Proposition 4. *Under the assumptions of the standard fairness model in Fig. 2, the causal measures of fairness $\text{Ctf-}\{\text{DE}, \text{IE}, \text{SE}\}(y), \text{Ctf-}\{\text{DE}, \text{IE}, \text{SE}\}(\hat{y})$ are identifiable from observational data, that is, they can be computed uniquely from the observational distribution $P(V)$.*

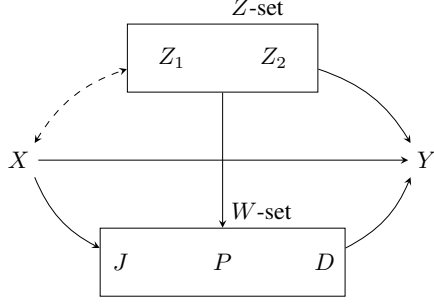


Figure 4: SFM for the COMPAS dataset.

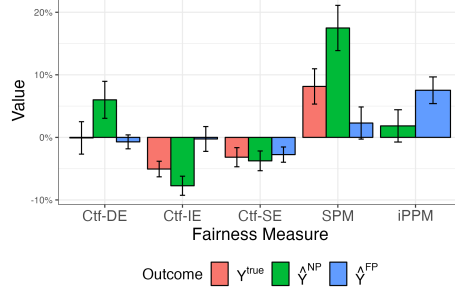


Figure 5: Fairness measures on COMPAS.

The explicit identification expression for each of the measures is given in Appendix B. The above result guarantees that the procedure of Alg. 1 is applicable to practical data analysis, in a fully non-parametric nature.

Further, for each of the DE, IE, and SE effects, the user needs to determine whether the causal effect (CE) in question falls into the business necessity set. If yes, then the algorithm asserts that

$$\text{Ctf-CE}(y) = \text{Ctf-CE}(\hat{y}). \quad (19)$$

In the other case, when the causal effect is not in the business necessity set, the algorithm asserts that

$$\text{Ctf-CE}(\hat{y}) = 0. \quad (20)$$

We remark that Alg. 1 is written in its population level version, in which the causal effects are estimated perfectly with no uncertainty. In the finite sample case, one needs to perform hypothesis testing to see whether the effects differ. If one is also interested in constructing a new fair predictor before using Alg. 1 (instead of testing an existing one), one may use tools for causally removing discrimination, such as [7] or [17, 18].

4 Experiments

We now apply Alg. 1 to the COMPAS dataset [1], as described in the following example.

Courts in Broward County, Florida use machine learning algorithms, developed by Northpointe, to predict whether individuals released on parole are at high risk of re-offending within 2 years (Y). The algorithm is based on the demographic information Z (Z_1 for gender, Z_2 for age), race X (x_0 denoting White, x_1 Non-White), juvenile offense counts J , prior offense count P , and degree of charge D .

We construct the standard fairness model (SFM) for this example, which is shown in Fig. 4. The bidirected arrow between X and $\{Z_1, Z_2\}$ indicates possible co-variations of race with age and sex, which may not be causal in nature⁵. Furthermore, $\{Z_1, Z_2\}$ are the confounders, not causally affected by race X . The set of mediators $\{J, P, D\}$, however, may be affected by race, due to an existing societal bias in policing and criminal justice. Finally, all of the above mentioned variables may influence the outcome Y .

Having access to data from Broward County, and equipped with Alg. 1, we want to prove that the recidivism predictions produced by Northpointe (labeled $\hat{Y}^{NP}$) violate legal doctrines of anti-discrimination. Suppose that in an initial hearing, the Broward County district court determines that the direct and indirect effects are not in the business necessity set, while the spurious effect is. In words, gender (Z_1) and age (Z_2) are allowed to be used to distinguish between the minority and majority groups when predicting recidivism, while other variables are not. If Northpointe’s predictions are found to be discriminatory, we are required by the court to produce better, non-discriminatory predictions.

In light of this information, we proceed as follows (see source code). We first obtain a causally fair predictor $\hat{Y}^{FP}$ using the `fairadapt` package, which sequentially performs optimal transport

⁵The causal model is non-committal regarding the complex historical/social processes that lead to such co-variations.

to match conditional distributions between groups for each variable in the causal diagram. Then, we compute the counterfactual causal measures of fairness for the true outcome Y , Northpointe’s predictions $\hat{Y}^{NP}$, and the fair predictions $\hat{Y}^{FP}$ (see Fig. 5). For the direct effect, we have:

$$\text{Ctf-DE}_{x_0, x_1}(y \mid x_0) = -0.08\% \pm 2.59\%, \quad (21)$$

$$\text{Ctf-DE}_{x_0, x_1}(\hat{y}^{NP} \mid x_0) = 6\% \pm 2.96\%, \quad (22)$$

$$\text{Ctf-DE}_{x_0, x_1}(\hat{y}^{FP} \mid x_0) = -0.72\% \pm 1.11\%. \quad (23)$$

The indicated 95% confidence intervals are computed using repeated bootstrap repetitions of the dataset. Since the direct effect is not in the business necessity set, Northpointe’s predictions clearly violate the disparate treatment doctrine (green bar for the Ctf-DE measure in Fig. 5). Our predictions, however, do not exhibit a statistically significant direct effect of race on the outcome, so they do not violate the criterion (blue bar). Next, for the indirect effects, we obtain:

$$\text{Ctf-IE}_{x_1, x_0}(y \mid x_0) = -5.06\% \pm 1.24\%, \quad (24)$$

$$\text{Ctf-IE}_{x_1, x_0}(\hat{y}^{NP} \mid x_0) = -7.73\% \pm 1.53\%, \quad (25)$$

$$\text{Ctf-IE}_{x_1, x_0}(\hat{y}^{FP} \mid x_0) = -0.25\% \pm 1.98\%. \quad (26)$$

Once again, the indirect effect, which is not in the business necessity set, is different from 0 for the Northpointe’s predictions (violating disparate impact, see green bar for Ctf-IE in Fig. 5), but not statistically different from 0 for our predictions (blue bar). Interestingly, the indirect effect is different from 0 for the true outcome (red bar), indicating a bias in the current real world. Finally, for the spurious effects, we obtain

$$\text{Ctf-SE}_{x_1, x_0}(y) = -3.17\% \pm 1.53\%, \quad (27)$$

$$\text{Ctf-SE}_{x_1, x_0}(\hat{y}^{NP}) = -3.75\% \pm 1.58\%, \quad (28)$$

$$\text{Ctf-SE}_{x_1, x_0}(\hat{y}^{FP}) = -2.75\% \pm 1.22\%. \quad (29)$$

Since the spurious effect is in the business necessity set and each confidence interval contains all three point estimates, no violations with respect to spurious effects are found. The estimated effects for the outcome Y , $\hat{Y}^{NP}$, and $\hat{Y}^{FP}$ are also shown graphically in Fig. 5. We conclude that Northpointe’s predictions $\hat{Y}^{NP}$ violate the legal doctrines of fairness, while our predictions $\hat{Y}^{FP}$ do not. Importantly, tying back to the original discussion that motivated our approach, Northpointe’s predictions $\hat{Y}^{NP}$ are further away from statistical parity than our predictions $\hat{Y}^{FP}$ according to the SPM (see SPM column in Fig. 5), while at the same time better calibrated according to the integrated PP measure (iPPM) that averages the PP measures from Eq. 3 across different values of $\hat{y}$ (see iPPM column in the figure). This observation demonstrates the trade-off between statistical and predictive parity through business necessity.

The choice of the business necessity set in the example above, $\text{BN} = \{W\}$, was arbitrary. In general, the BN set can be any of $\emptyset, Z, W, \{Z, W\}$, which is the setting we explore next.

Statistical vs. Predictive Parity Pareto Frontier. We now investigate other possible choices of the BN set, namely BN sets $\emptyset, Z, W, \{Z, W\}$. Based on the theoretical analysis of Sec. 3, we expect that different choices of the BN set will lead to different trade-offs between SP and PP.

In particular, with the BN set being empty, any variation from X to Y would be considered discriminatory, so one would expect to be close to statistical parity. In contrast, if the BN set is $\{Z, W\}$, then all variations (apart from the direct effect) are considered as non-discriminatory, so one would expect to be closer to satisfying predictive parity. The remaining options, $\text{BN} = W$, and $\text{BN} = Z$ should interpolate between these two extremes.

Based on the above intuition, we proceed as follows. For each choice of the business necessity set,

$$\text{BN} \in \{\emptyset, Z, W, \{Z, W\}\}, \quad (30)$$

we compute the adjusted version of the data again using the `fairadapt` package, with the causal effects in the complement of the BN set being removed. That is, if $\text{BN} = W$, then our procedure removes the spurious effect, but keeps the indirect effect intact (other choices of the BN set are similar). Therefore, for each BN set, we obtain an appropriately adjusted predictor $\hat{Y}_{\text{BN}}^{FP}$, and in

particular compute the predictors $\hat{Y}_\emptyset^{FP}$, $\hat{Y}_Z^{FP}$, $\hat{Y}_W^{FP}$, and $\hat{Y}_{\{Z,W\}}^{FP}$. We note that the fair predictor $\hat{Y}_\emptyset^{FP}$ from the first part of the example is the predictor with the BN set equal to Z , i.e., it corresponds to $\hat{Y}_Z^{FP}$. For each $\hat{Y}_{BN}^{FP}$, we in particular compute the SPM and iPPM measures, namely:

$$\text{SPM}_{x_0,x_1}(\hat{y}_{BN}^{FP}), \text{iPPM}_{x_0,x_1}(\hat{y}_{BN}^{FP}), \quad (31)$$

across 10 different repetitions of the adjustment procedure that yields $\hat{Y}_{BN}^{FP}$. For each business necessity set, this allows us to compute the SPM (measuring statistical parity), and iPPM (measuring predictive parity).

The results of the experiment are shown in Fig. 6, where the error bars indicate the standard deviation over different repetitions. As predicted by our theoretical analysis, the choice of $BN = \emptyset$ yields the lowest SPM, but the largest iPPM. Conversely, $BN = \{Z, W\}$ yields the lowest iPPM, but the largest SPM. The BN sets Z, W interpolate between the two notions, but the data indicates the spurious effect explained by Z does not have a major contribution. Fig. 6, therefore, shows a trade-off between statistical and predictive parity described through different business necessity options, and gives an empirical validation of the hypothesized spectrum of fairness notions in Fig. 1.

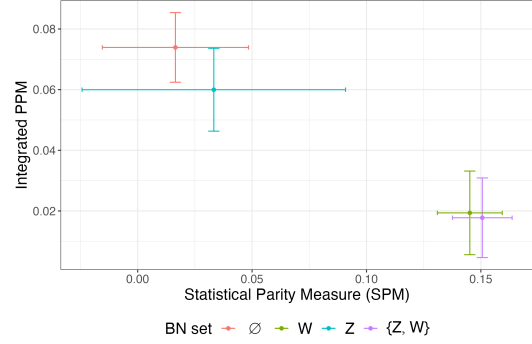


Figure 6: SP vs. PP Pareto frontier on COMPAS.

5 Conclusions

The literature in fair ML is abundant with fairness measures [9], many of which are mutually incompatible. Nonetheless, it is doubtful that each of these measures corresponds to a fundamentally different ethical conception of fairness. The multitude of possible approaches to quantifying discrimination makes the consensus on an appropriate notion of fairness unattainable. Further, the impossibility results between different measures may be discouraging to data scientists who wish to quantify and remove discrimination, but are immediately faced with a choice of which measure they wish to subscribe to.

In this work, we attempt to remedy a part of this issue by focusing on the impossibility of simultaneously achieving SP and PP. As our discussion shows, the guiding idea behind SP is that variations transmitted along causal pathways from the protected attribute to the predictor should equal 0, i.e., the decision should not depend on the protected attribute through the causal pathway in question (Def. 4). Complementary to this notion, and based on Thm. 1, the guiding principle behind PP is that variations transmitted along a causal pathway should be equal for the predictor as they are for the outcome *in the real world* (Def. 5). SP will therefore be satisfied when the BN set includes all variations coming from X to Y , while PP will be satisfied when the BN set is empty. The choice of the BN set interpolates between SP and PP forming a spectrum of fairness notions (see Fig. 1), in a way that can be formally assessed based on Alg. 1.

Therefore, our work complements the previous literature by reconciling the impossibility result between SP and PP [4]. Furthermore, it complements the existing literature on path-specific notions of fairness [14, 22, 7], which does not consider the true outcome Y and the predictor $\hat{Y}$ simultaneously, and does not explicitly specify which levels of discrimination are deemed acceptable along causal pathways in the BN set. Finally, we also mention the work on counterfactual predictive parity (Ctf-PP) [10] that is similar in name to our notion of causal predictive parity, but is in fact a very different notion. Ctf-PP deals with the setting of decision-making and considers counterfactuals of the outcome Y with respect to a treatment decision D that precedes it, while our work considers counterfactuals of the outcome Y and the predictor $\hat{Y}$ with respect to the protected attribute X , in the context of fair predictions, and thus offers a different line of reasoning.

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Supplementary Material for *Reconciling Predictive and Statistical Parity: A Causal Approach*

The source code for reproducing all the experiments can be found in the anonymized repository. The code is also included with the supplementary materials, in the folder `source-code`.

A Thm. 1 Proof and Evaluation

A.1 Proof of Thm. 1

Proof We prove the theorem and the corollary jointly. Note that $\mathbb{E}(y \mid x_1, \hat{y}) - \mathbb{E}(y \mid x_0, \hat{y})$ equals

$$\mathbb{E}(y_{x_1} \mid x_1, \hat{y}_{x_1}) - \mathbb{E}(y_{x_0} \mid x_0, \hat{y}_{x_0}) = \underbrace{\mathbb{E}(y_{x_1} \mid x_1, \hat{y}_{x_1}) - \mathbb{E}(y_{x_0} \mid x_1, \hat{y}_{x_1})}_{\text{Term (I)}} \quad (32)$$

$$+ \underbrace{\mathbb{E}(y_{x_0} \mid x_1, \hat{y}_{x_1}) - \mathbb{E}(y_{x_0} \mid x_1, \hat{y}_{x_0})}_{\text{Term (II)}} \quad (33)$$

$$+ \underbrace{\mathbb{E}(y_{x_0} \mid x_1, \hat{y}_{x_0}) - \mathbb{E}(y_{x_0} \mid x_0, \hat{y}_{x_0})}_{\text{Term (III)}}. \quad (34)$$

Since there are no backdoor paths between X and $Y, \hat{Y}$, Term (III) vanishes. By noting that $\mathbb{E}(y_x \mid x_1, \hat{y}_{x_1}) = \mathbb{E}(y_x \mid x_1, \hat{y}) \ \forall x$ by consistency axiom [15, Ch.7] (and applying it to Term (I)), and also that $Y_x \perp\!\!\!\perp X$ (and applying it to Term (II)) gives us the theorem. For the corollary, we further assume that the SCM is linear, and that the predictor $\hat{Y}$ is efficient, i.e., $\hat{Y}(x, w) = \mathbb{E}[Y \mid x, w]$. In this case, the efficiency simply translates to the fact that

$$\alpha_{W\hat{Y}} = \alpha_{WY}, \alpha_{X\hat{Y}} = \alpha_{XY}. \quad (35)$$

Due to linearity, for every unit u , we have that

$$y_{x_1}(u) - y_{x_0}(u) = \alpha_{XW}\alpha_{WY} + \alpha_{XY}, \quad (36)$$

and since Term (I) can be written as $\sum_u [y_{x_1}(u) - y_{x_0}(u)]P(u \mid x_1, \hat{y})$ using the unit-level expansion of counterfactual distributions [20, 2], Eq. 11 follows. Similarly, Term (II) can be expanded as

$$\sum_u \hat{y}_{x_0}(u)[P(u \mid \hat{y}_{x_1}) - P(u \mid \hat{y}_{x_0})]. \quad (37)$$

We now look at units u which are compatible with $\hat{Y}_{x_1}(u) = \hat{y}$ and $\hat{Y}_{x_0}(u) = \hat{y}$. We can expand $\hat{Y}_{x_1}(u)$ as

$$\hat{Y}_{x_1}(u) = \alpha_{X\hat{Y}} + \alpha_{XW}\alpha_{W\hat{Y}} + \alpha_{W\hat{Y}}u_W. \quad (38)$$

Thus, we have that

$$\hat{Y}_{x_1}(u) = \hat{y} \implies \alpha_{W\hat{Y}}u_W = \hat{y} - \alpha_{X\hat{Y}} + \alpha_{XW}\alpha_{W\hat{Y}}. \quad (39)$$

Similarly, we also obtain that

$$\hat{Y}_{x_0}(u) = \hat{y} \implies \alpha_{W\hat{Y}}u_W = \hat{y}. \quad (40)$$

Due to the efficiency of learning which implies that $\alpha_{W\hat{Y}} = \alpha_{WY}$ and $\alpha_{X\hat{Y}} = \alpha_{XY}$, Eq. 39 and 40 imply

$$y_{x_0}(u) = \hat{y} - (\alpha_{XY} + \alpha_{XW}\alpha_{WY}) \ \forall u \text{ s.t. } \hat{Y}_{x_1}(u) = \hat{y}, \quad (41)$$

$$y_{x_0}(u) = \hat{y} \ \forall u \text{ s.t. } \hat{Y}_{x_0}(u) = \hat{y}, \quad (42)$$

which in turn shows that

$$\mathbb{E}(y_{x_0} \mid \hat{y}_{x_1}) - \mathbb{E}(y_{x_0} \mid \hat{y}_{x_0}) = -\alpha_{XY} - \alpha_{XW}\alpha_{WY}. \quad (43)$$

■

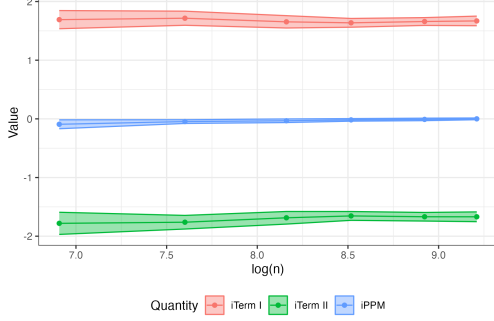


Figure 7: iTerm I, iTerm II, iPPM estimated values.

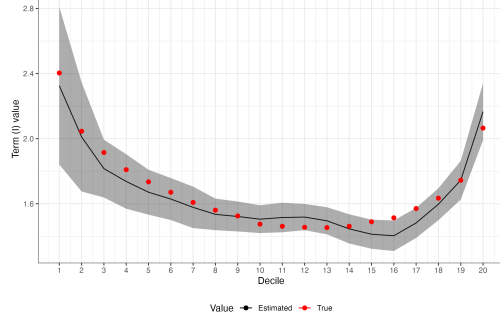


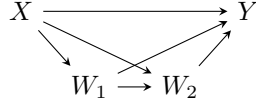
Figure 8: iTerm I and $\widehat{\text{Term I}}_i$ values.

A.2 Empirical Evaluation of Thm. 1

In this section, we empirically demonstrate the validity of Thm. 1 through an example (see source code for reproducing the results). In particular, we consider the following structural causal model:

$$\mathcal{F}^*, P^*(U) : \begin{cases} X \leftarrow U_X & (44) \\ W_1 \leftarrow X + U_1 & (45) \\ W_2 \leftarrow \frac{1}{4}W_1^2 - \frac{1}{3}X + U_2 & (46) \\ Y \leftarrow \frac{1}{6}W_1W_2 + W_1 + \frac{1}{2}X + U_Y. & (47) \\ U_X \in \{0, 1\}, P(U_X = 1) = 0.5, & (48) \\ U_1, U_2, U_Y \sim N(0, 1), & (49) \end{cases}$$

with the following causal diagram:



We proceed as follows. We investigate the decomposition in Thm. 1 with respect to a varying number of samples available, with $n \in \{1000, 2000, 3500, 5000, 7500, 10000\}$. For each value, we generate n samples from the SCM in Eqs. 44-49. Then, based on the samples, we construct a predictor $\hat{Y}$ using `xgboost` [6] (with a fixed $\eta = 0.1$ and number of rounds chosen using cross-validation with 10 folds). Then, we bin the samples into 20 intervals corresponding to the quantiles of the predictor $\hat{Y}$:

$$b_1 = [0\%-\text{quant}(\hat{Y}), 5\%-\text{quant}(\hat{Y})], \dots, b_{20} = [95\%-\text{quant}(\hat{Y}), 100\%-\text{quant}(\hat{Y})].$$

Further, we estimate the potential outcome Y_{x_0} for each sample using the `fairadapt` package [18], and the estimator is labeled $Y_{x_0}^*$ (also note that $Y_{x_0}(u) = Y(u)$ for all u s.t. $X(u) = x_0$). Then, within each bin b_i , we compute

$$\widehat{\text{Term I}}_i = \widehat{\mathbb{E}}[Y_{x_1} \mid x_1, \hat{y} \in b_i] - \widehat{\mathbb{E}}[Y_{x_0}^* \mid x_1, \hat{y} \in b_i], \quad (50)$$

$$\widehat{\text{Term II}}_i = \widehat{\mathbb{E}}[Y_{x_0}^* \mid x_1, \hat{y} \in b_i] - \widehat{\mathbb{E}}[Y_{x_0} \mid x_0, \hat{y} \in b_i], \quad (51)$$

$$\widehat{\text{iPPM}}_i = \widehat{\mathbb{E}}[Y_{x_1} \mid x_1, \hat{y} \in b_i] - \widehat{\mathbb{E}}[Y_{x_0} \mid x_0, \hat{y} \in b_i], \quad (52)$$

where $\widehat{\mathbb{E}}$ denotes the empirical expectation. We then compute $\widehat{\text{iTerm I}} = \frac{1}{20} \sum_{i=1}^{20} \widehat{\text{Term I}}_i$, where iTerm I is the “integrated” value of Term I across all 20 bins. We similarly estimate $\widehat{\text{iTerm II}}$ and $\widehat{\text{iPPM}}$. For each dataset size n , the experiment is repeated 20 times. In Fig. 7, we plot the obtained estimates of iTerm I, iTerm II, and the iPPM, where the shaded region represents the standard deviation of the estimates across 20 repetitions. The figure illustrates that as the number of samples increases, the predictor $\hat{Y}$ is closer to calibration (iPPM is closer to 0), as predicted by Prop. 1. Furthermore, we

can see that iTerm I and iTerm II are different from 0, and cancel each other out, as predicted by Thm. 1.

Furthermore, we verify if the estimates $\widehat{\text{Term I}}_i$ obtained in Eq. 50 correspond to the ground truth. To this end, instead of estimating the potential outcome Y_{x_0} , we obtain the true potential outcome values Y_{x_0} from the SCM. Then, we compute

$$\text{Term I}_i = \mathbb{E}[Y_{x_1} \mid x_1, \hat{y} \in b_i] - \mathbb{E}[Y_{x_0} \mid x_1, \hat{y} \in b_i]. \quad (53)$$

We take 50 repetitions of $n = 5000$ samples and obtain the values of $\widehat{\text{Term I}}_i$ and Term I_i . The obtained results are shown in Fig. 8. In red, we can see the ground truth values for each Term I_i , and in black we see the estimated values of $\widehat{\text{Term I}}_i$, where the shaded area indicates the standard deviation of the estimate across the 50 repetitions. Once again, we see that the terms Term I_i are estimated correctly, which in combination with Fig. 7 demonstrates empirically the theoretical result predicted by Thm. 1.

B Identification Expressions of Prop. 4

We provide the identification expressions for the counterfactual measures $\text{Ctf-DE}_{x_0, x_1}(y \mid x)$, $\text{Ctf-IE}_{x_1, x_0}(y \mid x)$, and $\text{Ctf-SE}_{x_1, x_0}(y)$. For the measures related measures of $\hat{y}$, the expressions are analogous, with y replaced by $\hat{y}$ throughout:

$$\text{Ctf-DE}_{x_0, x_1}(y \mid x) = \sum_{z, w} [P(y \mid x_1, z, w) - P(y \mid x_0, z, w)] P(w \mid x_0, z) P(z \mid x), \quad (54)$$

$$\text{Ctf-IE}_{x_1, x_0}(y \mid x) = \sum_{z, w} P(y \mid x_1, z, w) [P(w \mid x_0, z) - P(w \mid x_1, z)] P(z \mid x), \quad (55)$$

$$\text{Ctf-SE}_{x_1, x_0}(y) = \sum_z P(y \mid x_1, z) [P(z \mid x_1) - P(z \mid x_0)]. \quad (56)$$

C Proof of Prop. 1

Proof Notice that for any $X = x$

$$P(y \mid x, \hat{y}) = \sum_{z, w: \hat{Y}(x, z, w) = \hat{y}} P(y \mid x, z, w, \hat{y}) P(z, w \mid x, \hat{y}) \quad (57)$$

$$= \sum_{z, w: \hat{Y}(x, z, w) = \hat{y}} P(y \mid x, z, w) P(z, w \mid x, \hat{y}) \quad (58)$$

$$= \hat{y} * \sum_{z, w: \hat{Y}(x, z, w) = \hat{y}} P(z, w \mid x, \hat{y}) = \hat{y}. \quad (59)$$

The first step (Eq. 57) follows from the law of total probability, the second (Eq. 58) from noting that $\hat{Y} \perp\!\!\!\perp Y \mid X, Z, W$, and the third (Eq. 59) from the efficiency of the learner (Eq. 4). Therefore, it follows that $P(y \mid x_1, \hat{y}) = P(y \mid x_0, \hat{y})$, meaning that $\hat{Y}$ satisfies PP. ■