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Tannaka Duality for Finite Groups

Bachelor's thesis, July 11, 2011

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CONTENTS

1 INTRODUCTION

The reader of this thesis is assumed to know what a category, a functor, and a natural transformation is. For an introduction to Category Theory we refer to (Stevenhagen 2011, §27) (Dutch) or to the first chapters of (MacLane 1971).

Let k be a field and G a group. A k-linear group action of G on a k-vector space V is called a representation of G over k. Such a representation is a pair (V, ρ) where V is a k-vector space and ρ is a group homomorphism $G \to \mathrm{Aut}(V)$. The dimension of a representation (V, ρ) is defined as $\dim V$.

The finite dimensional k-vector spaces form a category, denoted with $\mathbf{Vec}_k$. The finite dimensional representations of G over k form a category as well, which we denote with $\mathbf{Rep}_k(G)$. Now assume G is finite. In this thesis we 'reconstruct' G from the category $\mathbf{Rep}_k(G)$. Strictly speaking, this is not exactly what we do.

To reconstruct the group we need more data. The tensor product on vector spaces induces a tensor structure on $\mathbf{Rep}_k(G)$ and $\mathbf{Vec}_k$ in a natural way. Let $F\colon \mathbf{Rep}_k(G) \to \mathbf{Vec}_k$ be the forgetful functor defined by $(V, \rho) \mapsto V$. Now consider the subgroup $\mathrm{Aut}^{\otimes}(F) \subset \mathrm{Aut}(F)$ consisting of those natural isomorphisms that respect the tensor structure.

We construct an isomorphism $G \xrightarrow{\sim} \mathrm{Aut}^{\otimes}(F)$, hence recovering G from $\mathbf{Rep}_k(G)$, the tensor structure, and the forgetful functor F. This is the main theorem of this thesis, and is called Tannaka duality for finite groups. The precise statement is in Theorem 4.3.

Tannaka duality is a more general theorem by Grothendieck about reconstructing an affine group scheme from its category of finite dimensional representations. Finite groups can be viewed as a particular kind of affine group schemes. For more about Tannaka duality we refer to (Rivano 1972, §III; Deligne and Milne 1982, §2)

It is also natural to try to generalize the statement to infinite groups. In §5 we take $G = \mathbb{Z}$ and k an algebraic extension of a finite field. We then construct a canonical isomorphism $\hat{\mathbb{Z}} \to \mathrm{Aut}^{\otimes}(F)$. In particular $\mathrm{Aut}^{\otimes}(F)$ and $\mathbb{Z}$ are not isomorphic. Probably it can be shown in the case where k is not algebraic over a finite field, that $\mathrm{Aut}^{\otimes}(F)$ is not isomorphic to $\mathbb{Z}$, nor to $\hat{\mathbb{Z}}$, using the theory of Tannakian categories.

2 REPRESENTATIONS AND TENSOR PRODUCTS

2.1 REPRESENTATIONS

Given a field k and a finite-dimensional vector space V over k, we can look at the automorphism group $\mathrm{Aut}(V)$ of V and consider group homomorphisms of other groups to $\mathrm{Aut}(V)$. This is the basic idea of representations.

Definition 2.1. Let G be a group, and k a field. A *representation of G over k* is a pair (V, ρ) where V is a vector space over k, and ρ a homomorphism $\rho\colon G \to \mathrm{Aut}(V)$. «

We define the *dimension* of a representation to be the dimension of the underlying vector space.

Example 2.2. Let k be a field and G a group. Consider k^G, the vector space of k-valued functions on G. Define

$$\begin{aligned} \tau\colon G &\to \mathrm{Aut}(k^G) \\ s &\mapsto (f \mapsto f \circ r_s), \end{aligned}$$

where r_s is the right multiplication with s on G.

Note that τ is a homomorphism, since for all $s, t \in G$ and $f \in k^G$ we have

$$\big(\tau(s) \circ \tau(t)\big)(f) = \tau(s)(f \circ r_t) = f \circ r_t \circ r_s = f \circ r_{st} = \tau(st)(f).$$

So (k^G, τ) is a representation. «

Definition 2.3. Let (V_1, ρ_1) and (V_2, ρ_2) be two representations of G over k. A *morphism* of (V_1, ρ_1) to (V_2, ρ_2) is a k-linear map $\phi\colon V_1 \to V_2$ such that for all $s \in G$ the diagram

$$\begin{tikzcd} V_1 \arrow[r, "\phi"] \arrow[d, "\rho_1(s)"'] & V_2 \arrow[d, "\rho_2(s)"] \\ V_1 \arrow[r, "\phi"] & V_2 \end{tikzcd}$$

commutes. «

Later on we will see that the finite-dimensional representations form a category. First we will introduce the tensor product on vector spaces, which will induce the tensor product on representations.

Definition 2.4. Let V and W be k-vector spaces. The *tensor product* $V \otimes W$ of V and W is defined as the free k-vector space on $\{(v, w) : v \in V, w \in W\}$ modulo the equivalence relations generated by

$$\begin{aligned} (v, (w_1 + w_2)) &\sim (v, w_1) + (v, w_2) \\ ((v_1 + v_2), w) &\sim (v_1, w) + (v_2, w) \\ \lambda(v, w) &\sim (\lambda v, w) \sim (v, \lambda w), \end{aligned}$$

for all $v, v_1, v_2 \in V$ and $w, w_1, w_2 \in W$ and $\lambda \in k$. «

For $(v, w) \in V \times W$ the equivalence class of (v, w) in $V \otimes W$ is written as $v \otimes w$, and is called a *pure tensor*. Note that the pure tensors generate $V \otimes W$ as k-vector space.

We will now state some properties of the tensor product.

Lemma 2.5. *Let V and W be k-vector spaces. Then for any k-vector space Z and any k-bilinear map $\phi\colon V \times W \to Z$ there exists a unique k-linear map $\psi\colon V \otimes W \to Z$, with the property that the diagram*

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "\phi"'] & V \otimes W \arrow[d, dotted, "!\psi"] \\ & Z \end{tikzcd}$$

commutes. This property is called the universal property *of the tensor product. The map $V \times W \to V \otimes W$ is called the* universal bilinear map. «

Proof. Observe that there indeed exists a k-linear map $\psi\colon V \otimes W \to Z$, since we can define ψ on the pure tensors

$$\psi\colon v \otimes w \mapsto \phi(v, w).$$

This is independent of the representative for $v \otimes w$ precisely because ϕ is bilinear. By linearity this induces the entire map, hence proving existence.

On the other hand, it is clear from the diagram that there is no other definition possible on the pure tensors, which proves uniqueness. □

Lemma 2.6. *Let V and W be two finite-dimensional k-vector spaces. Then* $\dim(V \otimes W) = \dim V \cdot \dim W$. «

Proof. Note that $k \otimes k \cong k$ holds, since any pure tensor $\lambda \otimes \mu \in k \otimes k$ is equivalent to $\lambda\mu \cdot 1 \otimes 1$.

Let V_1 and V_2 be k-vector spaces such that $V = V_1 \times V_2$. We claim that $(V_1 \times V_2) \otimes W \cong (V_1 \otimes W) \times (V_2 \otimes W)$.

Define the bilinear map

$$\begin{aligned} \otimes'\colon (V_1 \times V_2) \times W &\to (V_1 \otimes W) \times (V_2 \otimes W) \\ ((v_1, v_2), w) &\mapsto (v_1 \otimes w, v_2 \otimes w). \end{aligned}$$

Using the universal property of the tensor product on $V_1 \times V_2$ and W, we see that there exists a unique k-linear map α such that

$$\begin{tikzcd}
(V_1 \times V_2) \times W \arrow[r, "\otimes"] \arrow[dr, "\otimes'"] & (V_1 \times V_2) \otimes W \arrow[d, "!\alpha"] \\
& (V_1 \otimes W) \times (V_2 \otimes W)
\end{tikzcd}$$

commutes. We will now show that $\otimes'$ is the universal bilinear map from $(V_1 \times V_2) \times W$ to $(V_1 \otimes W) \times (V_2 \otimes W)$ and deduce that α is an isomorphism.

Let Z be any k-vector space, and $\phi\colon (V_1 \times V_2) \times W \to Z$ a bilinear map. Then $\phi = (\phi_1, \phi_2)$ where $\phi_1\colon V_1 \times W \to Z$ and $\phi_2\colon V_2 \times W \to Z$ are bilinear maps.

The universal property now states that ϕ_1 factors through $V_1 \otimes W$ inducing a map ψ_1. Analogously we obtain a map ψ_2. Define $\psi = (\psi_1, \psi_2)$ and observe that $\phi = \psi \circ \otimes'$ holds.

Finally, ψ is unique, since ϕ determines its image on all 'pure' elements $(v_1 \otimes w, v_2 \otimes w)$. Hence $\otimes'$ satisfies the universal property of the tensor product and it follows that α is an isomorphism.

Analogously to the claim above, $V \otimes (W_1 \times W_2) \cong (V \otimes W_1) \times (V \otimes W_2)$ holds. Combining the proven claim, and the fact $k \otimes k \cong k$ it is immediate that $\dim(V \otimes W) = \dim V \cdot \dim W$. $\square$

Definition 2.7. Let V_1, W_1, V_2, and W_2 be k-vector spaces. Using the universal property of $\otimes\colon V_1 \times W_1 \to V_1 \otimes W_1$, we define the tensor product of two k-linear maps $\phi\colon V_1 \to V_2$ and $\psi\colon W_1 \to W_2$ to be the unique k-linear map $\phi \otimes \psi$ such that

$$\begin{tikzcd}
V_1 \times W_1 \arrow[r, "\otimes"] \arrow[dr, "\otimes \circ (\phi, \psi)"'] & V_1 \otimes W_1 \arrow[d, "!\phi \otimes \psi"] \\
& V_2 \otimes W_2
\end{tikzcd}$$

commutes. «

Definition 2.8. Let k be a field and G a group. If (V_1, ρ_1) and (V_2, ρ_2) are two representations of G over k, then we define the *tensor product of representations* as

$$(V_1, \rho_1) \otimes (V_2, \rho_2) = (V_1 \otimes V_2, \rho_1 \otimes \rho_2),$$

where $\rho_1 \otimes \rho_2$ is the homomorphism

$$\begin{aligned} \rho_1 \otimes \rho_2 \colon G &\to \mathrm{Aut}(V_1 \otimes W_1) \\ s &\mapsto \rho_1(s) \otimes \rho_2(s). \end{aligned}$$ «

3 THE CATEGORIES $\mathbf{Vec}_k$ AND $\mathbf{Rep}_k(G)$

3.1 TENSOR CATEGORIES

Let k be a field and G a group.

Definition 3.1. The category $\mathbf{Vec}_k$ is defined as the category with

- as objects the finite-dimensional vector spaces over k;
- as morphisms the k-linear maps. «

Definition 3.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *product category* $\mathcal{C} \times \mathcal{D}$ is defined as the category with

- as objects the pairs (X, Y) with $X \in \operatorname{ob} \mathcal{C}$ and $Y \in \operatorname{ob} \mathcal{D}$;
- as morphisms
$$\operatorname{mor}_{\mathcal{C}\times\mathcal{D}}\left((X_1, Y_1), (X_2, Y_2)\right) = \operatorname{mor}_{\mathcal{C}}(X_1, X_2) \times \operatorname{mor}_{\mathcal{D}}(Y_1, Y_2),$$
for all $X_1, X_2 \in \operatorname{ob} \mathcal{C}$ and $Y_1, Y_2 \in \operatorname{ob} \mathcal{D}$;
- and componentwise composition. «

Definition 3.3. A *tensor structure* on a category $\mathcal{C}$ is a functor

$$\begin{aligned} \otimes_{\mathcal{C}} \colon \mathcal{C} \times \mathcal{C} &\to \mathcal{C} \\ (X, Y) &\mapsto X \otimes_{\mathcal{C}} Y. \end{aligned}$$ «

Definition 3.4. A *tensor category* is a pair $(\mathcal{C}, \otimes_{\mathcal{C}})$ of a category $\mathcal{C}$ together with a tensor structure $\otimes_{\mathcal{C}}$ on $\mathcal{C}$. «

Lemma 3.5. $(\mathbf{Vec}_k, \otimes)$ *is a tensor category.* «

Proof. In Lemma 2.6 we proved that the tensor product of two finite-dimensional vector spaces is again finite-dimensional.

It is immediate that $\_ \otimes \_$ preserves identity and composition. □

As mentioned before, the finite-dimensional representations of a finite group G over a field k form a category. This category will play a very important role in this thesis.

Definition 3.6. The category $\mathbf{Rep}_k(G)$ is defined as the category with

- as objects the finite-dimensional representations of G over k;
- as morphisms the morphisms of representations (Definition 2.3).«

We will write $\mathbf{Vec}_k$ respectively $\mathbf{Rep}_k(G)$ for the tensor categories $(\mathbf{Vec}_k, \otimes)$ and $(\mathbf{Rep}_k(G), \otimes)$.

Definition 3.7. Let $(\mathcal{C}, \otimes_\mathcal{C})$ and $(\mathcal{D}, \otimes_\mathcal{D})$ be two tensor categories. A *tensor functor* is a pair (F, f), consisting of a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a natural isomorphism

$$f\colon \otimes_\mathcal{D} \circ (F, F) \to F \circ \otimes_\mathcal{C}.$$

«

Recall that a natural isomorphism, as appearing in the previous definition is an isomorphism

$$f_{X,Y}\colon F(X) \otimes_\mathcal{D} F(Y) \to F(X \otimes_\mathcal{C} Y),$$

for all $X \in \operatorname{ob} \mathcal{C}$ and $Y \in \operatorname{ob} \mathcal{D}$, such that for all $X_1 \to X_2 \in \operatorname{mor} \mathcal{C}$ and $Y_1 \to Y_2 \in \operatorname{mor} \mathcal{D}$ the diagram

$$\begin{tikzcd}
F(X_1) \otimes_\mathcal{D} F(Y_1) \arrow[r, "f_{X_1,Y_1}"] \arrow[d] & F(X_1 \otimes_\mathcal{C} Y_1) \arrow[d] \\
F(X_2) \otimes_\mathcal{D} F(Y_2) \arrow[r, "f_{X_2,Y_2}"] & F(X_2 \otimes_\mathcal{C} Y_2)
\end{tikzcd}$$

commutes.

We will often write F for the pair (F, f).

Now we define a particular functor $\mathbf{Rep}_k(G) \to \mathbf{Vec}_k$ forgetting all about a representation but its underlying vector space. This functor is called the *fibre functor* of $\mathbf{Rep}_k(G)$ to $\mathbf{Vec}_k$.

Definition 3.8. The *fibre functor* of $\mathbf{Rep}_k(G)$ to $\mathbf{Vec}_k$ is defined as

$$\begin{aligned} F\colon \mathbf{Rep}_k(G) &\to \mathbf{Vec}_k \\ (V, \rho) &\mapsto V \\ \phi &\mapsto \phi, \end{aligned}$$

where ϕ denotes the morphisms of $\mathbf{Rep}_k(G)$. «

Note that F is indeed a functor, since it preserves identities of morphisms and composition of morphisms by definition of the morphisms of $\mathbf{Rep}_k(G)$. Besides that, observe that for all finite-dimensional representations (V_1, ρ_1) and (V_2, ρ_2) it holds that $F\big((V_1, \rho_1) \otimes (V_2, \rho_2)\big)$ is *equal* to $F\big((V_1, \rho_1)\big) \otimes F\big((V_2, \rho_2)\big)$. Hence $F = (F, \mathrm{id})$ is a tensor functor.

3.3 THE TENSOR AUTOMORPHISM GROUP OF F

Definition 3.9. Let $(\mathcal{C}, \otimes_\mathcal{C})$ and $(\mathcal{D}, \otimes_\mathcal{D})$ be tensor categories, and let (F, f) and (G, g) be tensor functors $(\mathcal{C}, \otimes_\mathcal{C}) \to (\mathcal{D}, \otimes_\mathcal{D})$. A *tensor natural transformation* $\eta\colon (F, f) \to (G, g)$ is a natural transformation $F \to G$ such that

$$\begin{tikzcd}
F(X) \otimes_\mathcal{D} F(Y) \arrow[r, "\eta_X \otimes_\mathcal{D} \eta_Y"] \arrow[d, "f_{X,Y}"'] & G(X) \otimes_\mathcal{D} G(Y) \arrow[d, "g_{X,Y}"] \\
F(X \otimes_\mathcal{C} Y) \arrow[r, "\eta_{X \otimes_\mathcal{C} Y}"] & G(X \otimes_\mathcal{C} Y)
\end{tikzcd}$$

commutes, for all $X, Y \in \operatorname{ob} \mathcal{C}$. A tensor natural transformation that is a natural isomorphism is called a *tensor natural isomorphism*. «

Definition 3.10. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F\colon \mathcal{C} \to \mathcal{D}$ a functor. The *automorphism group of* F, written as $\operatorname{Aut}(F)$, is the group of natural isomorphisms $F \to F$. «

Definition 3.11. Let $(\mathcal{C}, \otimes_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}})$ be tensor categories and F a tensor functor $(\mathcal{C}, \otimes_{\mathcal{C}}) \to (\mathcal{D}, \otimes_{\mathcal{D}})$. The *tensor automorphism group of* F, written as $\operatorname{Aut}^{\otimes}(F)$, is the group of tensor natural isomorphisms $F \to F$. «

Note that for every tensor functor $F\colon (\mathcal{C}, \otimes_{\mathcal{C}}) \to (\mathcal{D}, \otimes_{\mathcal{D}})$ the tensor automorphism group $\operatorname{Aut}^{\otimes}(F)$ is a subgroup of $\operatorname{Aut}(F)$.

Example 3.12. In this example we compute the automorphism group of the identity functor on $\mathbf{Vec}_k$, $\mathrm{id}\colon \mathbf{Vec}_k \to \mathbf{Vec}_k$.

Let $\eta \in \operatorname{Aut}(\mathrm{id})$ be a natural isomorphism of id. Note that η_k is a k-linear isomorphism $k \to k$, hence multiplication with some $\lambda \in k^*$.

Let V be a k-vector space, and let $v \in V$. Define the k-linear map $\phi\colon k \to V$ by $x \mapsto x \cdot v$. By definition of natural isomorphism the diagram

$$\begin{tikzcd}
k \arrow[r, "\sim", "\eta_k"'] \arrow[d, "\phi"'] & k \arrow[d, "\phi"] \\
V \arrow[r, "\sim", "\eta_V"'] & V
\end{tikzcd}$$

commutes. Thus we have

$$\eta_V(v) = \eta_V \circ \phi(1) = \phi \circ \eta_k(1) = \phi(\lambda) = \lambda \cdot v,$$

and find $\operatorname{Aut}(\mathrm{id}) \subset k^*$. Since it is trivial that $k^* \subset \operatorname{Aut}(\mathrm{id})$ holds, we conclude $\operatorname{Aut}(\mathrm{id}) = k^*$. «

4

TANNAKA DUALITY FOR FINITE GROUPS

Tannaka duality for finite groups gives an isomorphism between a finite group G and the tensor automorphism group of the fibre functor from $\mathbf{Rep}_k(G)$ to $\mathbf{Vec}_k$. The construction of this isomorphism is the subject of this section. Therefore, let G be a finite group, k a field, and F the fibre functor from $\mathbf{Rep}_k(G)$ to $\mathbf{Vec}_k$.

4.1 STATEMENT OF THE THEOREM

We will first discuss the elements of $\mathrm{Aut}(F)$. Such an element is by definition a natural isomorphism $F \to F$. In other words, it is a collection of k-linear maps $\eta_{(V,\rho)}$, such that for all representations $(V_1, \rho_1), (V_2, \rho_2) \in \mathrm{ob}\,\mathbf{Rep}_k(G)$ and for all morphisms $\phi\colon (V_1, \rho_1) \to (V_2, \rho_2)$ the diagram

$$\begin{tikzcd} V_1 \arrow[r, "\sim", "\eta_{(V_1,\rho_1)}"'] \arrow[d, "\phi"'] & V_1 \arrow[d, "\phi"] \\ V_2 \arrow[r, "\sim", "\eta_{(V_2,\rho_2)}"'] & V_2 \end{tikzcd}$$

commutes.

We define the map

$$\begin{aligned} T\colon G &\to \mathrm{Aut}(F) \\ s &\mapsto \big(\rho(s)\big)_{(V,\rho)}. \end{aligned}$$

Lemma 4.1. *The map $T\colon G \to \mathrm{Aut}(F)$ is a homomorphism of groups.* «

Proof. Let $(V, \rho) \in \mathrm{ob}\,\mathbf{Rep}_k(G)$ be a finite-dimensional representation. By definition $T(s)_{(V,\rho)} = \rho(s)$ is an automorphism of V, for all $s \in G$. Moreover, we find

$$\begin{aligned} T(s)_{(V,\rho)} \circ T(t)_{(V,\rho)} &= \rho(s) \circ \rho(t) \\ &= \rho(st) \\ &= T(st)_{(V,\rho)}, \end{aligned}$$

for all $s, t \in G$, proving that T is indeed a homomorphism. □

The homomorphism T is often called the 'tautological' homomorphism. We will now prove that the image of T lies in $\mathrm{Aut}^{\otimes}(F)$. In other words that T factors as follows.

$$\begin{tikzcd} G \arrow[r, "T"] \arrow[dr, "\sim"'] & \mathrm{Aut}(F) \\ & \mathrm{Aut}^{\otimes}(F) \arrow[u, hook] \end{tikzcd}$$

Lemma 4.2. *The image of $T\colon G \to \mathrm{Aut}(F)$ lies in $\mathrm{Aut}^{\otimes}(F)$.* «

Proof. Fix $s \in G$. Observe that for all $(V_1, \rho_1), (V_2, \rho_2), \in \mathrm{ob}\,\mathbf{Rep}_k(G)$, by definition of the tensor product of representations, we have

$$\rho_1(s) \otimes \rho_2(s) = \rho_1 \otimes \rho_2(s),$$

and hence

$$T(s)_{(V_1,\rho_1)} \otimes T(s)_{(V_2,\rho_2)} = T(s)_{(V_1,\rho_1)\otimes(V_2,\rho_2)}.$$

Recall that $F = (F, \mathrm{id})$ to conclude that $T(s)$ is a tensor natural isomorphism of F. Hence $\operatorname{im} T \subset \mathrm{Aut}^{\otimes}(F)$ holds. □

We can now state the main theorem of this thesis, which we will prove in the next section.

Theorem 4.3. *The map $T\colon G \to \mathrm{Aut}^{\otimes}(F)$ is an isomorphism.* «

4.2 PROOF OF THE THEOREM

In this section we will prove that $T\colon G \to \mathrm{Aut}^{\otimes}(F)$ is an isomorphism of groups. First we will prove that T is injective, then we prove the surjectiveness of T. In the entire proof the representation (k^G, τ), which we introduced in Example 2.2, will play an important role.

But first we define the indicator functions on G. For all $s \in G$ we define

$$\begin{aligned} e_s\colon G &\to k \\ t &\mapsto \begin{cases} 1 & \text{if } t = s \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

Lemma 4.4. *The homomorphism $T\colon G \to \mathrm{Aut}^{\otimes}(F)$ is injective.* «

Proof. Consider the representation (k^G, τ), fix $s \in G$, and assume that $T(s)_{(k^G,\tau)} = \mathrm{id}\, k^G$. Then $f = \tau(s)(f) = f \circ r_s$ holds for all $f \in k^G$. In particular we have $e_{\mathrm{id}} = e_{\mathrm{id}} \circ r_s = e_{s^{-1}}$. Hence $s = \mathrm{id}$ and T is injective. □

To prove that the map T is also surjective, we state and prove several lemmata. First note that k^G is a k-algebra, since we can add and multiply functions pointwise. Next, we define the evaluation functions on k^G.

$$\begin{aligned} \pi_s\colon k^G &\to k \\ f &\mapsto f(s). \end{aligned}$$

Lemma 4.5. *Let $\phi\colon k^G \to k$ be a k-algebra homomorphism. Then there exists a unique $s \in G$ such that $\phi = \pi_s$.* «

Proof. Because ϕ is surjective, $k^G / \ker \phi \cong k$ holds, and hence $\ker \phi$ is a maximal ideal of k^G.

Because k^G is isomorphic to $k^{|G|}$ as k-algebra, the ideals of k^G are products of the ideals of k. Since k is a field, the only ideals of k are 0 and k. So the maximal ideal $\ker \phi$ is equal to $\{f \in k^G : f(s) = 0\}$ for a certain $s \in G$. Thus $\phi = \pi_s$. $\square$

Lemma 4.6. *Let $(\alpha_{(V,\rho)})_{(V,\rho)} \in \mathrm{Aut}^{\otimes}(F)$ be a tensor natural transformation. Then $\alpha_{(k^G,\tau)}$ is a k-algebra homomorphism.* «

Proof. Since multiplication on k^G is bilinear, the universal property of the tensor product states that it factors through $k^G \otimes k^G$. Call the induced k-linear map μ. Observe that μ is a morphism of representations, since for all $s \in G$ the diagram

$$\begin{tikzcd}
k^G \otimes k^G \arrow[r, "\mu"] \arrow[d, "\tau(s) \otimes \tau(s)"'] & k^G \arrow[d, "\tau(s)"] \\
k^G \otimes k^G \arrow[r, "\mu"] & k^G
\end{tikzcd}$$

commutes, i.e., because for all $f, g \in k^G$ we have $(f \circ r_s)(g \circ r_s) = fg \circ r_s$.

To keep notation concise, write $\alpha = \alpha_{(k^G,\tau)}$. By definition of tensor natural transformation we know that $\alpha_{(k^G,\tau)\otimes(k^G,\tau)} = \alpha \otimes \alpha$, and by definition of natural transformation the diagram

$$\begin{tikzcd}
k^G \otimes k^G \arrow[r, "\mu"] \arrow[d, "\alpha \otimes \alpha"'] & k^G \arrow[d, "\alpha"] \\
k^G \otimes k^G \arrow[r, "\mu"] & k^G
\end{tikzcd}$$

commutes. Hence for all $f, g \in k^G$ we have $\alpha(f)\alpha(g) = \alpha(fg)$. Since α is an isomorphism of vector spaces, it has an inverse α^{-1}. Thus $\alpha(1) = \alpha(1)\alpha(\alpha^{-1}(1)) = \alpha(1 \cdot \alpha^{-1}(1)) = 1$ holds, and we conclude that α is a k-algebra homomorphism. $\square$

Denote the left multiplication with s on G with $l_s\colon G \to G, t \mapsto st$. We define

$$\begin{aligned}
\tau'\colon G &\to \mathrm{Aut}(k^G) \\
s &\mapsto (f \mapsto f \circ l_s).
\end{aligned}$$

Lemma 4.7. *Let $(\alpha_{(V,\rho)})_{(V,\rho)} \in \mathrm{Aut}^{\otimes}(F)$ be a tensor natural transformation. Then there exists a unique $s \in G$ such that $\alpha_{(k^G,\tau)} = \tau(s)$.* «

Proof. Again write $\alpha = \alpha_{(k^G,\tau)}$. Note that r_s and l_t commute for all $s, t \in G$ and hence so do $\tau(s)$ and $\tau'(t)$. Hence we have $\tau'(t) \in \mathrm{mor}\big((k^G,\tau), (k^G,\tau)\big)$.

By definition of natural transformation α commutes with all automorphisms of (k^G, τ), and hence with $\tau'(t)$ for all $t \in G$.

By Lemma 4.6 the map α is a k-algebra homomorphism, hence so is $\pi_{\mathrm{id}} \circ \alpha$. Now Lemma 4.5 gives that there exists a unique $s \in G$ such that $\pi_{\mathrm{id}} \circ \alpha = \pi_s$. We claim $\alpha = \tau(s)$.

Using that α commutes with τ' we compute for $t, u \in G$,

$$\begin{aligned}
\alpha(e_u)(t) &= \alpha(e_u) \circ l_t(\mathrm{id}) \\
&= \alpha(e_u \circ l_t)(\mathrm{id}) \\
&= \pi_{\mathrm{id}} \circ \alpha(e_{t^{-1}u}) \\
&= e_{t^{-1}u}(s).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\alpha(f)(t) &= \alpha\left(\sum_{u \in G} f(u) e_u\right)(t) \\
&= \sum_{u \in G} f(u) \alpha(e_u)(t) \\
&= \sum_{u \in G} f(u) e_{t^{-1}u}(s) \\
&= f(ts) \\
&= \tau(s)(f)(t).
\end{aligned}$$

Thus we conclude $\alpha = \tau(s)$. □

Lemma 4.8. *Let $\alpha, \beta \in \mathrm{Aut}^{\otimes}(F)$ such that $\alpha_{(k^G,\tau)} = \beta_{(k^G,\tau)}$. Then $\alpha = \beta$.* «

Proof. Let (V, ρ) be a finite-dimensional representation and let $v \in V$. We define

$$\begin{aligned}
\phi \colon (k^G, \tau) &\to (V, \rho) \\
f &\mapsto \sum_{s \in G} f(s) \rho(s^{-1}) v,
\end{aligned} \tag{4.9}$$

so that $\phi(e_{\mathrm{id}}) = v$. Note that $\phi \in \mathrm{mor}\big((k^G, \tau), (V, \rho)\big)$, since for all $t \in G$ we have

$$\begin{aligned}
\phi \circ \tau(t)(f) &= \sum_{s \in G} (f \circ r_t)(s) \rho(s^{-1}) v \\
&= \sum_{st^{-1} \in G} f(s) \rho(ts^{-1}) v \\
&= \sum_{s \in G} f(s) \rho(t) \rho(s^{-1}) v \\
&= \rho(t) \circ \phi(f).
\end{aligned}$$

By definition of natural isomorphism, the following diagram commutes.

$$\begin{tikzcd}
k^G \arrow[r, "\phi"] \arrow[d, "\alpha_{(k^G,\tau)} = \beta_{(k^G,\tau)}"'] & V \arrow[d, bend right, "\beta_{(V,\rho)}"'] \arrow[d, bend left, "\alpha_{(V,\rho)}"] \\
k^G \arrow[r, "\phi"'] & V
\end{tikzcd}$$

Since $v \in \operatorname{im}\phi$ holds, $\alpha_{(V,\rho)}(v) = \beta_{(V,\rho)}(v)$ follows and we conclude $\alpha = \beta$. □

Remark 4.10. Note that we really use the finiteness of G here. If we redefine $\mathbf{Rep}_k(G)$ (resp. $\mathbf{Vec}_k$) to include infinite-dimensional representations (resp. vector spaces), the proof presented in this thesis would not be valid for Lemma 4.8. The map ϕ defined in (4.9) would not be well defined.

Allthough we could amend Lemma 4.8 by substituting $k^{(G)}$ for k^G in the entire thesis, the proof would then fail at Lemma 4.5. ($k^{(G)}$ is the k-subalgebra of k^G of all functions with finite support.)

I do not know whether the proof can be amended in another way such that a similar result can be proven for infinite groups and infinite-dimensional vector spaces and representations. «

Proposition 4.11. *The map $T\colon G \to \operatorname{Aut}^{\otimes}(F)$ is surjective.* «

Proof. Let $\alpha \in \operatorname{Aut}^{\otimes}(F)$. By Lemma 4.7 there exists a unique $s \in G$ such that $\alpha_{(k^G,\tau)} = \tau(s) = T(s)_{(k^G,\tau)}$. Finally Lemma 4.8 states that $\alpha = T(s)$. □

The injectiveness and surjectiveness imply our main theorem.

Proof (of Theorem 4.3). By Lemma 4.1 the map T is a homomorphism, by Lemma 4.4 it is injective and by Proposition 4.11 it is surjective. Hence T is an isomorphism of groups. □

5 An example with an infinite group

Let k be an algebraic extension of a finite field. Let F be the fibre functor $F\colon \mathbf{Rep}_k(\mathbb{Z}) \to \mathbf{Vec}_k$. In this section we show that $\mathrm{Aut}^{\otimes}(F)$ is not isomorphic to $\mathbb{Z}$.

Therefore we define a group, $\hat{\mathbb{Z}} \ncong \mathbb{Z}$, and construct a natural isomorphism $\hat{\mathbb{Z}} \cong \mathrm{Aut}^{\otimes}(F)$.

Definition 5.1. The group $\hat{\mathbb{Z}}$ consists of infinite sequences $(s_n)_{n\in\mathbb{Z}_{>0}}$ with $s_n \in \mathbb{Z}/n\mathbb{Z}$, such that $s_n \equiv s_m \mod m$ if $m|n$. The group law is the componentwise addition. «

Define the injection $i\colon \mathbb{Z} \to \hat{\mathbb{Z}}$ by $n \mapsto (n + m\mathbb{Z})_{m\in\mathbb{Z}_{>0}}$. For all $n \in \mathbb{Z}_{>0}$ define $\pi_n\colon \hat{\mathbb{Z}} \to \mathbb{Z}/n\mathbb{Z}$ by $(s_m)_{m\in\mathbb{Z}_{>0}} \mapsto s_n$.

Lemma 5.2. *Let (V,ρ) be a finite-dimensional representation of $\mathbb{Z}$ over k. Then there is a unique map $\hat{\rho}\colon \hat{\mathbb{Z}} \to \mathrm{Aut}(V)$ such that $\rho = \hat{\rho} \circ i$ holds.* «

Proof. Note that ρ is determined by $\rho(1)$. Choose a basis for V. Observe that there exists a finite subfield $l \subset k$ such that all entries of the matrix representing $\rho(1)$ are in l. Since V is finite-dimensional, and l is finite, $\rho(1)$ is of finite order n, for some $n \in \mathbb{Z}_{>0}$.

Hence ρ factors through $\mathbb{Z}/n\mathbb{Z}$, inducing a map $\overline{\rho}$. If we define $\hat{\rho}$ as $\overline{\rho} \circ \pi_n$ we have the following commutative diagram.

$$\begin{tikzcd}
 & \mathbb{Z}/n\mathbb{Z} \arrow[ddr, hook, "\overline{\rho}"] & \\
 & \hat{\mathbb{Z}} \arrow[u, two heads, "\pi_n"] \arrow[dr, "\hat{\rho}"] & \\
\mathbb{Z} \arrow[uur, two heads, "\bmod n"] \arrow[ur, hook, "i"] \arrow[rr, "\rho"'] & & \mathrm{Aut}(V)
\end{tikzcd}$$

□

Define

$$\begin{aligned}\phi\colon \hat{\mathbb{Z}} &\to \mathrm{Aut}^{\otimes}(F)\\ s &\mapsto \big(\hat{\rho}(s)\big)_{(V,\rho)},\end{aligned}$$

and observe that, analogous to Lemma 4.1 and Lemma 4.2, it is a group homomorphism.

Proposition 5.3. *The map $\phi\colon \hat{\mathbb{Z}} \to \mathrm{Aut}^{\otimes}(F)$ is an isomorphism.* «

Before we prove this proposition, we first state some lemmata analogous to the case for finite groups.

Consider $G = \mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{Z}_{>0}$ and observe that we can view (k^G, τ) as a representation of $\mathbb{Z}$ (with some abuse of notation).

Lemma 5.4. *The map ϕ is injective.* «

Proof. Assume $\phi(s) = \mathrm{id}$ and consider (k^G, τ), with $G = \mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{Z}_{>0}$. Then we have

$$\mathrm{id}_{(k^G,\tau)} = \phi(s)_{(k^G,\tau)} = \hat{\tau}(s) = \tau(s_n),$$

which implies $s_n = 0$. Since n was arbitrary, this implies $s = 0$. □

Lemma 5.5. *Let $\alpha, \beta \in \mathrm{Aut}^{\otimes}(F)$ such that $\alpha_{(k^{\mathbb{Z}/n\mathbb{Z}},\tau)} = \beta_{(k^{\mathbb{Z}/n\mathbb{Z}},\tau)}$ holds, for all $n \in \mathbb{Z}_{>0}$. Then $\alpha = \beta$.* «

Proof. Let $(V, \rho) \in \mathrm{ob}\,\mathbf{Rep}_k(\mathbb{Z})$ and $v \in V$. As observed above, ρ factors through $\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{Z}_{>0}$. Put $G = \mathbb{Z}/n\mathbb{Z}$ and analogous to Lemma 4.8 we deduce $\alpha = \beta$. □

Lemma 5.6. *The map ϕ is surjective.* «

Proof. Let $\alpha \in \mathrm{Aut}^{\otimes}(F)$ be a tensor automorphism of F.

Recall that we can view $(k^{\mathbb{Z}/n\mathbb{Z}}, \tau)$ as representation of $\mathbb{Z}$ over k, for all $n \in \mathbb{Z}_{>0}$. By Lemma 4.7, there exists an $s_n \in \mathbb{Z}/n\mathbb{Z}$ such that $\alpha_{(k^G,\tau)} = \tau(s_n)$. We claim that $(s_n)_n \in \hat{\mathbb{Z}}$, i.e., for all $m, n \in \mathbb{Z}_{>0}$ it holds that $s_n \equiv s_m \mod m$ if $m|n$.

Suppose $m|n$ holds for $n, m \in \mathbb{Z}_{>0}$. Define

$$\begin{aligned} f\colon k^{\mathbb{Z}/m\mathbb{Z}} &\to k^{\mathbb{Z}/n\mathbb{Z}} \\ g &\mapsto (a + n\mathbb{Z} \mapsto g(a + m\mathbb{Z})). \end{aligned}$$

Note that f is well defined, and a morphism of representations since

$$\begin{tikzcd} k^{\mathbb{Z}/m\mathbb{Z}} \arrow[r, "\tau(s)"] \arrow[d, "f"'] & k^{\mathbb{Z}/m\mathbb{Z}} \arrow[d, "f"] \\ k^{\mathbb{Z}/n\mathbb{Z}} \arrow[r, "\tau(s)"] & k^{\mathbb{Z}/n\mathbb{Z}} \end{tikzcd}$$

commutes for all $s \in \mathbb{Z}$.

By definition of automorphism of F, and the construction of s_n and s_m we then find that

$$\begin{tikzcd} k^{\mathbb{Z}/m\mathbb{Z}} \arrow[r, "\tau(s_m)"] \arrow[d, "f"'] & k^{\mathbb{Z}/m\mathbb{Z}} \arrow[d, "f"] \\ k^{\mathbb{Z}/n\mathbb{Z}} \arrow[r, "\tau(s_n)"] & k^{\mathbb{Z}/n\mathbb{Z}} \end{tikzcd}$$

commutes, which means $s_n \equiv s_m \mod m$. Hence $(s_n)_n \in \hat{\mathbb{Z}}$ holds.

Since α agrees with $\phi(s)$ on $(k^{\mathbb{Z}/n\mathbb{Z}}, \tau)$ for all $n \in \mathbb{Z}_{>0}$, Lemma 5.5 gives $\alpha = \phi(s)$, which proves surjectiveness of ϕ. □

Proof (of Proposition 5.3). By Lemma 5.4 the map ϕ is injective, and by Lemma 5.6 it is surjective. □

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