Nakahara Ch. 1 Problems

Matt DeCross

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Topics covered: Euler-Lagrange, Canonical quantization, Abelian gauge transformations, Higgs mechanism, Magnetic monopoles in electromagnetism.

In the problems below, we use metric convention $\eta^{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ and units where $c = \hbar = 1$. The Levi-Civita tensor ϵ_{ijk} is totally antisymmetric with $\epsilon_{123} = 1$.

1 Monopoles, Gauge Transformations, and Charge Quantization

Source: Exercises in Chapter 1 of Nakahara, a problem from an 8.06 problem set, notes by R.L. Jaffe, and my own work in my 8.06 paper, combined.

Suppose that magnetic monopoles exist. In this case, Gauss' law for magnetic charges reads (in CGS units):

$$\vec{\nabla} \cdot \vec{B} = 4\pi g \delta^3(\vec{r}) \tag{1}$$

Comparing to Gauss' law for electricity, the solution is the analogue of Coulomb's law:

$$\vec{B} = \frac{g}{r^2}\hat{r} \tag{2}$$

(a) Prove that such a magnetic field cannot come from the curl of a single vector potential.

(b) Define two vector potentials in spherical coordinates (r, θ, ϕ):

$$\vec{A}^N = \frac{g(1-\cos\theta)}{r\sin\theta}\hat{\phi} \tag{3}$$

$$\vec{A}^S = -\frac{g(1+\cos\theta)}{r\sin\theta}\hat{\phi} \tag{4}$$

Show that $\vec{A}^N$ suffices to define the $\vec{B}$ field everywhere on the positive z axis and vice versa for $\vec{A}^S$.

(c) Show that these two vector potentials are gauge equivalent. If $\vec{A}^N = \vec{A}^S + \vec{\nabla}\alpha$, find the gauge transformation function α.

Now we will examine the consequences of having to make this gauge transformation. First, let's consider the wavefunction of an electron outside a magnetic monopole. To simplify the problem, we will assume the electron is nonrelativistic and ignore its spin, so that we may use ordinary quantum mechanics. In order to do this, we need to know what the Hamiltonian of a particle in a magnetic field looks like.

(d) Consider the Lagrangian of a classical particle in a magnetic field with vector potential $\vec{A}$:

$$\mathcal{L} = \frac{1}{2}m\dot{\vec{x}}^2 + e\dot{\vec{x}} \cdot \vec{A} \tag{5}$$

Show that this Lagrangian reproduces the Lorentz force law for the magnetic field.

(e) Find the momentum conjugate to the coordinate x_i. Using this expression, perform a Legendre transformation to show that the Hamiltonian of a particle in a magnetic field is:

$$\mathcal{H} = \frac{1}{2m}(\vec{p} - e\vec{A})^2 \tag{6}$$

The time-dependent Schrödinger equation for an electron in a magnetic field with wavefunction $\psi(x,t)$ is therefore:

$$i\frac{\partial\psi}{\partial t} = \frac{1}{2m}(\vec{p} - e\vec{A})^2\psi \tag{7}$$

We are ignoring the electric field of the electron, since it turns out to be irrelevant to the computation.

Note that this equation still looks a lot like the free time-dependent Schrödinger equation:

$$i\partial_t\psi = \frac{1}{2m}\hat{p}^2\psi \tag{8}$$

Recalling that in quantum mechanics $\hat{p}_x = -i\partial_x$ (in one dimension), they are in fact equivalent if we send the derivative to a "gauge-covariant derivative":

$$\partial_x \to D_x = \partial_x - ieA_x \tag{9}$$

and define the momentum instead as $\hat{p}_x = -iD_x$.

(f) Under a gauge transformation, the vector potential transforms as

$$\vec{A} \to \vec{A} - \vec{\nabla}\alpha \tag{10}$$

Show that in order for the Schrödinger equation to remain gauge-invariant, the electron wavefunction must transform as:

$$\psi \to \exp(-ie\alpha)\psi \tag{11}$$

The electron wavefunction must transform by an overall phase factor in order for the Schrödinger equation to still hold after the gauge transformation. This is one reason that electromagnetism is called a $U(1)$ gauge theory: particles coupled to electromagnetism transform under the group $U(1)$ (unitary 1×1 matrices, that is, phases) in gauge transformations.

(g) Prove that charge is quantized using the fact that the electron wavefunction must be single-valued. Hint: consider the effect of making a gauge transformation from $\vec{A}^N$ to $\vec{A}^S$.

Since one vector potential does not suffice to describe the magnetic field of a magnetic monopole, one must specify such a gauge transformation from $\vec{A}^N$ to $\vec{A}^S$ in order to describe the field everywhere.

2 Gauge Theories and the Higgs Mechanism

Source: Section 1.8 in Nakahara combined with a problem from a QFT II problem set with my own modification and extension.

The Higgs field is an example of a massive scalar field. A simpler example of massive scalars is the massive, real, free scalar field which obeys the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2. \tag{12}$$

(a) Find the equation of motion of ϕ.

In the Lagrangian formulation of electrodynamics, the electric and magnetic fields can be arranged into an antisymmetric 2-tensor called the **Faraday tensor**, with the electric and magnetic potentials combining into a **four-potential**:

$$A_\mu = (\varphi, \vec{A}) \tag{13}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \implies F_{0i} = E_i, \quad F_{ij} = -\frac{1}{2}\epsilon_{ijk}F^{jk}. \tag{14}$$

Given these definitions, the electromagnetic field itself obeys the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{15}$$

(b) Show that the Euler-Lagrange equations for this Lagrangian yield the source-free Maxwell equations.

Electromagnetism is a special example of a **gauge theory**, specifically, a **Yang-Mills theory**. In a gauge theory, the four-potential takes values in a **Lie algebra**, which is a type of space that corresponds to a continuous **group**. This group is called the **gauge group** of the theory. For electromagnetism, the four-potential lives in the set of real numbers as opposed to some set of matrices, so the Lie algebra is $\mathbb{R}$. This Lie algebra corresponds to the continuous groups $U(1)$ under multiplication or $\mathbb{R}$ under addition – the only difference between the two is that $U(1)$ is periodic. In the first problem, we showed that the choice between $U(1)$ and $\mathbb{R}$ (periodicity or non-periodicity) corresponds to enforcing charge quantization or not when charges are included in the theory.

A priori, one might think that there's no reason the photon shouldn't have a very small but immeasurable mass. It turns out there are subtle problems related to the degrees of freedom and polarizations if we make the photon massive, but there are more obvious problems as well. Suppose we naïvely add a mass term for the photon to the E&M Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \tag{16}$$

(c) Suppose we make a gauge transformation $A_\mu \to A_\mu + \partial_\mu \alpha$ for some function α. This is just the statement from classical E&M that the vector potential can be changed by the gradient of some function without affecting the fields, since the curl of a gradient vanishes. Show that the Lagrangian *is not* invariant under this gauge transformation.

As seen in the first problem, in a gauge theory the partial derivative must be modified to the **gauge-covariant derivative** D_μ in order for the physical theory (i.e., the Lagrangian) to remain gauge-invariant when it is coupled to another field e.g. the electron:

$$\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu \tag{17}$$

Consider now coupling electromagnetism to a complex scalar field which we suggestively call H which has a kinetic term using the gauge-covariant derivative:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu H D^\mu H \tag{18}$$

We have not included an explicit mass term for this putative Higgs; such a mass term causes additional complications beyond the point of this problem. Suppose the H field is expanded into radial and angular parts $H = ve^{i\theta/v}$, where v is a fixed constant called the **vacuum expectation value** of the Higgs and θ is measured in units of v so that the Lagrangian is appropriately normalized.

(d) Show that there exists a gauge transformation law for θ such that this Lagrangian is still gauge-invariant.

(e) Show that picking one particular gauge leads to the Lagrangian of the massive photon (the **Proca Lagrangian**) by defining an appropriate mass in terms of e and v.

This is a toy example of the Higgs mechanism called the "Stueckelberg trick". It's not possible to just have a massive vector boson in the theory. However, by introducing a scalar field and breaking gauge symmetry by choosing a gauge, we have ended up with a theory containing a massive vector boson. In this problem, we were dealing with a $U(1)$ gauge theory and the massive vector boson was the photon – this is not a real physical theory, since we do not have a massive photon. For the actual Higgs mechanism, however, we have a $SU(2)$ gauge theory and the massive vector bosons are the $W^\pm$ and Z bosons, which are indeed massive. The scalar introduced in that case is the Higgs itself, although additional complications arise because the Higgs also has a mass of about 125 GeV.