A finite-dimensional model for affine, linear quantum lambda calculi with general recursion

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Abstract. We introduce a concrete domain model for the quantum lambda calculus λ_ρ° extended with a fixpoint operator. A distinctive feature of λ_ρ° is that it relies on density matrices for describing both quantum information and probabilistic distributions over computation states. It has been shown that there is a conservative translation from λ_ρ° to the quantum lambda calculus of Selinger and Valiron. In contrast to existing models for quantum lambda calculi featuring recursion with intuitionistic arrows, our model is finite-dimensional and does not need more than cones of positive matrices and affine arrows.

Keywords: Quantum lambda calculus · Denotational semantics · Fixpoint operator · Linear affine type system.

1 Introduction

Quantum computation is a model of computation where data is encoded on the state of particles governed by the laws of quantum physics. In the mathematical formalism, a piece of quantum data can be regarded as a complex linear combination of pieces of classical data: quantum states are represented with Hilbert spaces. To recover classical data from quantum information, the physical operation is called measurement: it is a *probabilistic* operation, modifying the global state of the system. The bottom line is that a quantum algorithm in general produces a *probabilistic distribution of pure quantum states*. The standard denotational semantics for quantum data consists in *density matrices*, i.e. *positive matrices of trace 1*. In this compact representation, eigenvectors correspond to classical outcomes while eigenvalues encompass the probability of getting each of them. In the historical interpretation [6], a quantum algorithm inputs a quantum state, operates on it and outputs the resulting modified state: this simple situation can be regarded as a *superoperator*, a trace-preserving linear map acting on positive matrices. The semantics of quantum algorithms in this approach

is finite-dimensional: a quantum algorithm manipulates a finite amount of information.

The last twenty years have seen the development of quantum programming languages and semantics thereof. In particular, the design of functional programming languages for quantum computation has roots in the seminal work of Selinger [10], introducing *quantum flow charts* (QFCs). QFCs are (possibly recursive) first-order programs: the trace of the output might be smaller than 1, allowing the program to possibly diverge. The denotational semantics of QFCs therefore extends superoperators to *non-increasing* linear maps acting on cones of positive matrices. This approach has been subsequently extended in [11] to accommodate for higher-order programs: The *quantum lambda calculus* consists in a simply-typed, *linear* lambda calculus without recursion. Its denotational semantics is still finite-dimensional. It consists in an extension of the semantics of QFC where the requirement for trace-preservation for morphisms is relaxed. Objects are still cones of positive matrices, but morphisms are the so-called *completely positive maps* (CPM). Thanks to the compact-closure of the corresponding category [10,12], this makes it possible to capture internal homs in this semantics.

If the CPM category can encode *linear* quantum higher-order computation, it is however limited in several ways. First, its finite dimensional aspect makes it impossible to account for duplicable data (as it would require the possibility to have non-linear functions) or inductive types. Then, although CPM-homsets can be endowed with a partial order consistent with the trace (the *Löwner order*), the lack of constant (non-zero) functions places the least fixed point of any function at 0, essentially saying that the fixpoint construction sending $A \to A$ to A is always diverging (as its probability is then 0).

The main problem that has been tackled in the literature [3,4,7,8] consists in developing a semantics extending CPM to support infinite dimensional objects: the focus has been placed on duplicability [7] and inductive types [8]. In both cases, the extensions encompasses affine functions, allowing one to rely on the least fixed point construction to model recursion.

Contributions.   In this work, we instead concentrate on the problem of designing a *finite dimensional* extension of the CPM denotational semantics supporting fixpoints. In particular, we do not need to account for duplicable objects nor inductive datatypes. To support our approach, we follow an operational approach by building up on a concrete quantum lambda-calculus featuring recursion while admitting a finite-dimensional model. Concretely, we extend the quantum lambda calculus λ_ρ° [5] with a fixpoint operator while forbidding duplicable elements. We remark that λ_ρ° has been shown equivalent [1] to the one proposed by Selinger&Valiron [11], but relies on a presentation in terms of density matrices, which we felt makes it easier to study syntactically, since it is closer to its semantics. The denotational semantics of our language follows the approach of Selinger [10]; consequently, we interpret basic types as positive matrices with trace less than or equal to 1. Following the usual intuition, a matrix

whose trace is 1 represents a terminating program, while a matrix whose trace is smaller than 1 represents a program that might not terminate.

We build upon the standard Choi representation [2] of a completely positive linear map f as the positive matrix, and extend it to the affine case. We show that this finite model is sound and adequate, and suffices to interpret the fixpoint operator as the least upper bound of a chain of approximations, i.e., as $\lim_{n\to\infty} \overline{\chi}_f^n(\mathbf{0})$ where $\overline{\chi}_f$ is the extended Choi representation of the affine function f, and $\mathbf{0}$ denotes the null matrix.

Our model is only focused on affine behavior, which makes it possible to stay within a finite-dimensional setting, unlike e.g. [7]. We therefore claim to have found a sweet spot for the denotation of quantum programs with fixpoints, when duplication is not required.

Plan of the paper. In Section 2 we introduce the calculus λ_ρ^μ, give some examples, and state its syntactical correctness. In Section 3.1 we give a version of λ_ρ^μ with an incremental fixpoint operator parameterised by a bound on the number of iterations. The semantics interpretation of λ_ρ^μ and of this intermediate language are given in Sections 3.2 to 3.4. The soundness for the intermediate language is proved in Section 4.1, which allows us to study the existence of the limit when the bound tends to infinity, which is proven in Section 4.2. In Section 5 we prove the following adequacy result: the probability of termination for a term of the basic type is equal to the trace of its interpretation (Theorem 5.9). We finally conclude in Section 6. Omitted technical material and proofs are provided in the appendices for the reviewers' convenience.

2 The calculus λ_ρ^μ

In this section, we introduce the calculus λ_ρ^μ, which extends λ_ρ° [5] with a fixpoint operator.

Since the quantum measurement is a probabilistic operation, we first define probability (sub)distributions as follows.

Definition 2.1. *A (discrete) probability subdistribution over a set Ω is a function $p : \Omega \to [0,1]$ such that $\sum_{\omega \in \Omega} p(\omega) \leq 1$ and for any $A \subseteq \Omega$, $p(A) = \sum_{\omega \in A} p(\omega)$; moreover, p is said to be a distribution if $\sum_{\omega \in \Omega} p(\omega) = 1$.*

We write $\{(p_i, \omega_i)\}_{i \in \{1,\dots,n\}}$ (also $\{(p_1, \omega_1), \cdots, (p_n, \omega_n)\}$) for the probability subdistribution p over $\Omega = \{\omega_1, \dots, \omega_n\}$ defined such that $p(\omega_i) = p_i$ for all $i = 1, \dots, n$. With a slight abuse of notation, we write $\{(p, \omega), (q, \omega)\}$ in lieu of $\{(p + q, \omega)\}$, with the latter being the canonical form. Also, we shall write $\left\{ \left(p_i, \{(q_j, t_{ij})\}_j\right) \right\}_i$ for $\{(p_i q_j, t_{ij})\}_{ij}$. Unless otherwise noted, distributions are considered in canonical form. Given two subdistributions $\{(p_i, \omega_i)\}_{i \in I}$ and $\{(p_j, \omega_j)\}_{j \in J}$, we write $\{(p_i, \omega_i)\}_{i \in I} \cup \{(p_j, \omega_j)\}_{j \in J}$ for the (sub)distribution $\{(p_k, \omega_k)\}_{k \in I \cup J}$ (if defined).

$$
\begin{aligned}
t ::= \quad & x \mid \lambda x.t \mid tt \mid \mu x.t && \textbf{(Standard lambda terms)} \\
& \mid \rho^n \mid U^n t \mid \pi^m t \mid t \otimes t && \textbf{(Quantum postulates)} \\
& \mid \mathsf{letcase}^\circ \; x = t \; \mathsf{in} \; \{t, \ldots, t\} && \textbf{(Control)} \\
& \mid \{(p_i, t_i)\}_{i \in \{1, \ldots, n\}} && \textbf{(Distributions)}
\end{aligned}
$$

where $n, m \in \mathbb{N}$ and $\rho \in \mathcal{D}_n = \left\{ \rho \mid \rho \in \mathbb{C}^{2^n \times 2^n} \text{ positive with } tr(\rho) = 1 \right\}$

Fig. 1: Syntax for the λ_ρ^μ calculus

2.1   Syntax and Operational Semantics

The syntax of the λ_ρ^μ calculus is given in Figure 1 where $\mathbb{C}$ denotes the set of complex numbers and $\mathbb{C}^{n \times m}$ the set of $(n \times m)$-dimensional complex matrices. Terms are divided in four categories:

– Standard terms of the lambda calculus with fixpoint, namely, a variable x, an abstraction $\lambda x.t$, an application tr, and the fixpoint $\mu x.t$ of the abstraction $\lambda x.t$.
– Quantum postulates, which include a quantum state ρ^n, where ρ is an n-dimensional semidefinite positive Hermitian matrix (from now on referred to as positive matrix) with trace equal to 1 (we shall also write σ and τ for quantum states); the application $U^n t$ of the unitary operator U to the first n qubits of t; the measurement $\pi^m t$ of the first m qubits of t in the computational basis; and the tensor product of states $t \otimes r$.
– The control operator construction, $\mathsf{letcase}^\circ \; x = r \; \mathsf{in} \; \{t_0, \ldots, t_n\}$, that expresses the combination of the programs $t_0, \ldots t_n$ according to a probability distribution given by the result of the measurement described by r.
– Probability distributions of terms.

Our syntax allows for terms such as $\{(p_i, t_i)\}_{i \in \{1, \ldots, n\}} \{(q_j, r_j)\}_{j \in \{1, \ldots, m\}}$, in which the distribution $\{(p_i, t_i)\}_{i \in \{1, \ldots, n\}}$ is applied to $\{(q_j, r_j)\}_{j \in \{1, \ldots, m\}}$. However, such term describes a probability distribution over the set

$$
\{t_i r_j \mid 1 \le i \le n, 1 \le j \le m\}
$$

which can be equivalently written as $\{(p_i q_j, t_i r_j)\}_{\substack{i \in \{1, \ldots, n\} \\ j \in \{1, \ldots, m\}}}$. Formally, we consider the first term —applying a distribution to another— to be just a notation for the second. We extend this convention all over constructors as follows.

$$
\begin{aligned}
\lambda x. \{(p_i, t_i)\}_{i \in I} &:= \{(p_i, \lambda x.t_i)\}_{i \in I} & \mu x. \{(p_i, t_i)\}_{i \in I} &:= \{(p_i, \mu x.t_i)\}_{i \in I} \\
t \{(p_i, r_i)\}_{i \in I} &:= \{(p_i, t r_i)\}_{i \in I} & \{(p_i, t_i)\}_{i \in I}\, r &:= \{(p_i, t_i r)\}_{i \in I} \\
U^n \{(p_i, t_i)\}_{i \in I} &:= \{(p_i, U^n t_i)\}_{i \in I} & \pi^m \{(p_i, t_i)\}_{i \in I} &:= \{(p_i, \pi^m t_i)\}_{i \in I} \\
\{(p_i, t_i)\}_{i \in I} \otimes r &:= \{(p_i, t_i \otimes r)\}_{i \in I} & r \otimes \{(p_i, t_i)\}_{i \in I} &:= \{(p_i, r \otimes t_i)\}_{i \in I}
\end{aligned}
$$

$\mathsf{letcase}^\circ \; x = \{(p_i, t_i)\}_{i \in I} \; \mathsf{in} \; \{r_1, \ldots, r_n\} := \{(p_i, \mathsf{letcase}^\circ \; x = t_i \; \mathsf{in} \; \{r_1, \ldots, r_n\})\}_{i \in I}$

The above notation states that the distribution operator commutes with all operators. Additionally, any term t can be regarded as a distribution that assigns probability 1 to t.

We let Val be the set of values, which is defined as follows.

$$\mathsf{Val} ::= \rho^n \mid \pi^m \rho^n \mid \{(p_i, \lambda x.t_i)\}_{i \in \{1,\dots,n\}}$$

The operational semantics of the calculus is given by the rules in Figure 2. Rules for standard lambda terms correspond to those of the weak call-by-name lambda calculus (i.e., there are no reductions under lambda).

The first rule concerning quantum postulates accounts for the fact that any mixed state given as a probability distribution $\{(p_i, \rho_i^n)\}_{i \in I}$ over density matrices $\{\rho_i\}_{i \in I}$ can be represented as a unique density matrix, which is obtained as the linear combination of the matrices $\{\rho_i\}_{i \in I}$, i.e., $\sum_{i \in I} p_i \rho_i$. The reduction rule for $U^m \rho^n$ corresponds to the application of the unitary operator U over the first m qubits of the density matrix ρ, which has dimension $n \geq m$. Since m and n may be different, the term $U^m \rho^n$ stands for the application of the unitary operator $\overline{U} = U \otimes I_{n-m}$ with I_{n-m} being the $n-m$ dimensional identity matrix. As usual, $\overline{U}^\dagger$ denotes the conjugate transpose of $\overline{U}$. The rule corresponding to the tensor operator states that the combination of two density matrices reduces to the tensor product of the matrices.

The evaluation of a term $\mathsf{letcase}^\circ\ x = \pi^m \rho^n$ in $\{t_0, \dots, t_{2^m-1}\}$ gives rise to the probabilistic distribution of the branches $t_0, \dots, t_{2^m-1}$, where the probability assigned to each branch corresponds to the probability p_i of each of the possible outcomes ρ_i of the measurement $\pi^m \rho^n$. The bound variable x, which may appear on every t_i, is substituted with the corresponding measurement result ρ_i. The expression $|i\rangle\langle i|$ in the side condition of this rule represents the projectors $|i\rangle\langle i| \otimes I_{n-m}$. It is worth remarking that the operator π^m only measures the first m qubits of a density matrix. Should other set of qubits be measured, then the state can be previously transformed with the application of a suitable unitary operator that swaps the qubits as required.

The typing system for the λ_ρ^μ calculus is defined in Figure 3. The set A of types includes (i) n, which is the type of the density matrices of n-qubit states; (ii) (m, n) (with $m \leq n$) which stands for measurements over the first m qubits of n-qubit states; and (iii) the arrow type $A \multimap B$ of the affine functions from A to B.

Lambda terms are typed as in the affine lambda calculus, while typing rules for quantum postulates are straightforward. The typing rules for control and distribution terms use the auxiliary function on types $\ell(A)$, dubbed *last type of* A, which is inductively defined by:

$$\ell(n) = n \qquad \ell((m, n)) = (m, n) \qquad \ell(A \multimap B) = \ell(B)$$

Its usage is analogous in both rules (i.e., $+$ and m_e), where the premise $\ell(A) \neq (m, n)$ prevents the probabilistic combination of measurements. Note that π^m is the constructor for measurements while $\mathsf{letcase}^\circ$ is its destructor. Indeed, terms of the form $\pi^m t$ are only used inside their destructors $\mathsf{letcase}^\circ$.

$$
\begin{array}{l}
\textbf{(Standard lambda terms)}\\[4pt]
(\lambda x.t)r \longrightarrow t[x := r] \qquad\qquad \mu x.t \longrightarrow t[x := \mu x.t]\\[8pt]
\textbf{(Quantum postulates)}\\[4pt]
\{(p_i, \rho_i^n)\}_{i \in I} \longrightarrow (\sum_{i \in I} p_i \rho_i)^n \qquad U^m \rho^n \longrightarrow (\overline{U} \rho \overline{U}^\dagger)^n \qquad \rho^n \otimes \sigma^m \longrightarrow (\rho \otimes \sigma)^{n \times m}\\[8pt]
\textbf{(Control)}\\[4pt]
\mathsf{letcase}^\circ \ x = \pi^m \rho^n \text{ in } \{t_0, \ldots, t_{2^m - 1}\} \longrightarrow \{(p_i, t_i[x := \rho_i^n])\}_{i/p_i \neq 0}\\[4pt]
\text{where} \qquad p_i = \mathsf{tr}\left(\overline{|i\rangle\langle i|} \rho \overline{|i\rangle\langle i|}^\dagger\right) \qquad \rho_i = \frac{\overline{|i\rangle\langle i|} \rho \overline{|i\rangle\langle i|}^\dagger}{p_i}
\end{array}
$$

(Contextual rules)

$$
\frac{t \longrightarrow r}{ts \longrightarrow rs} \qquad \frac{t \longrightarrow r}{U^m t \longrightarrow U^m r} \qquad \frac{t \longrightarrow r}{\pi^m t \longrightarrow \pi^m r} \qquad \frac{t \longrightarrow r}{t \otimes s \longrightarrow r \otimes s}
$$

$$
\frac{t \longrightarrow r \quad s \nrightarrow}{s \otimes t \longrightarrow s \otimes r} \qquad \frac{j \in J, t_j \longrightarrow r_j \quad i \in I, t_i \nrightarrow}{\{(p_i, t_i)\}_{i \in I \cup J} \longrightarrow \{(p_i, t_i)\}_{i \in I} \cup \{(p_j, r_j)\}_{j \in J}}
$$

$$
\frac{t \longrightarrow r}{\mathsf{letcase}^\circ \ x = t \text{ in } \{s_0, \ldots, s_{2^m - 1}\} \longrightarrow \mathsf{letcase}^\circ \ x = r \text{ in } \{s_0, \ldots, s_{2^m - 1}\}}
$$

Fig. 2: Rewrite system for the λ_ρ^μ calculus

Despite rule m_e allows x to be used in the different branches $t_0, \ldots, t_{2^m - 1}$, we remark that such duplication of variables does not violate the quantum no-cloning theorem because each branch corresponds to the continuation associated with a particular result of the measurement.

We now illustrate the main features of the calculus by encoding a few representative examples from the literature.

Example 2.2 (Teleportation protocol). The teleportation protocol (see [6, Section 1.3.7]) can be implemented in λ_ρ^μ as follows. We start by defining the term ρ^2 representing the first Bell state describing maximum entanglement between two qubits:

$$
\rho = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)
$$

Then, the teleportation protocol can be expressed as

$$
\mathsf{telep} := \lambda x.\mathsf{letcase}^\circ \ y = \pi^2 (H^1 \ CNOT^2 \ (x \otimes \rho^2)) \text{ in } \{y, Z_3^3 y, X_3^3 y, Z_3^3 X_3^3 y\}
$$

where capitalised terms are (the density matrices corresponding to the gates) below.

- H is the Hadamard gate;
- $CNOT$ is the Controlled-Not gate;
- Z is the Phase gate;

$$A ::= n \mid (m,n) \mid A \multimap A \quad \text{where } m \leq n \in \mathbb{N}$$

(Standard lambda terms)

$$\frac{}{\Gamma, x : A \vdash x : A} \ \mathsf{ax} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B} \ \multimap_i$$

$$\frac{\Gamma \vdash t : A \multimap B \quad \Delta \vdash r : A}{\Gamma, \Delta \vdash tr : B} \ \multimap_e \qquad \frac{\Gamma, f : A \vdash t : A}{\Gamma \vdash \mu f.t : A} \ \mu$$

(Quantum postulates)

$$\frac{}{\Gamma \vdash \rho^n : n} \ \mathsf{ax}_\rho \qquad \frac{\Gamma \vdash t : n \quad m \leq n}{\Gamma \vdash U^m t : n} \ \mathsf{u}_i$$

$$\frac{\Gamma \vdash t : n}{\Gamma \vdash \pi^m t : (m,n)} \ \mathsf{m}_i \qquad \frac{\Gamma \vdash t : n \quad \Delta \vdash r : m}{\Gamma, \Delta \vdash t \otimes r : n + m} \ \otimes$$

(Control)

$$\frac{i=0,\dots,2^m-1 \quad \Delta_i, x : n \vdash t_i : A \quad \Gamma \vdash r : (m,n) \quad \ell(A) \neq (m',n')}{\Delta_0, \dots, \Delta_{2^m-1}, \Gamma \vdash \mathsf{letcase}^\circ \ x = r \ \mathsf{in} \ \{t_0, \dots, t_{2^m-1}\} : A} \ \mathsf{m}_e$$

(Distributions)

$$\frac{i=1,\dots,n \quad \Gamma \vdash t_i : A \quad \sum_{i=1}^n p_i = 1 \quad \ell(A) \neq (m,n')}{\Gamma \vdash \{(p_i, t_i)\}_{i \in \{1,\dots,n\}} : A} \ +$$

Fig. 3: Typing system for the λ_ρ^μ calculus

- X is the Not gate;
- $Z_3 = I_2 \otimes Z$; and
- $X_3 = I_2 \otimes X$.

The argument x of the lambda abstraction in the definition of telep corresponds to the the unknown state τ to be teleported. The $CNOT$ operator is applied to the first two qubits of the state $\tau \otimes \rho$. We recall that a unitary operator U of size m can be applied to a density matrix ρ of size $n \geq m$. A size mismatch is accommodated by the operational semantics, since the term $U^m \rho^n$ reduces to the density matrix $(\overline{U} \rho \overline{U}^\dagger)^n$ with $\overline{U} = U \otimes I_{n-m}$. Therefore, the term $CNOT^2 (\tau^1 \otimes \rho^2)$ is a well-typed term even though $CNOT^2$ has size 2 and $(\tau^1 \otimes \rho^2)$ has size 3. The Hadamard operator H is then applied to the first qubit; this is needed because the successive measurement is in the computational basis instead of the Bell basis. According to the letcase constructor, the evaluation proceeds differently depending on the possible outcomes of the measurement, which in this case are the possible combinations of two bits, i.e., $\{00, 01, 10, 11\}$. Note that each of the branches in $\{y, Z_3^3 y, X_3^3 y, Z_3^3 X_3^3 y\}$ is associated with one of the possible measurement outcomes. Correspondingly, the variable y is bound to different values in each of the branches, i.e., y represents the state after measurement. For instance, it takes value $(|00\rangle\langle00|\otimes\tau)^3$ when evaluating the first branch, $(|01\rangle\langle01| \otimes Z^{-1}\tau)^3$ when evaluating the second branch, etc. Hence, each branch

will apply a different transformation to the third qubit of the corresponding resulting state: the identity, Z, X, or ZX.

Finally, note that the evaluation of the letcase construction produces a probability distribution over the values corresponding to each of the branches; where probabilities are those arising from measurement. By (omitted) mechanical calculation, it can be checked that measuring the state $(H^1 \, CNOT^2 \, (x \otimes \rho^2))$ in the computational basis produces a uniform distribution on $\{00, 01, 10, 11\}$. Hence, telep τ^1 reduces as follows

$$\text{telep } \tau^1 \longrightarrow^* \left\{ \left(\tfrac{1}{4}, (|ij\rangle\langle ij| \otimes \tau)^3\right) \right\}_{i,j \in \{0,1\}} \longrightarrow^* \left(\tfrac{1}{4} I_2 \otimes \tau\right)^3$$

where the third qubit of the state coincides with the qubit τ that was expected to be teleported.

Example 2.3 (Deutsch's algorithm). Deutsch's algorithm (see [6, Section 1.4.3]) can be used to determine whether a given function $f : \{0,1\} \to \{0,1\}$ is constant $(f(0) = f(1))$ or not. Let U_f be the unitary application $|x, y\rangle \mapsto |x, y \dot{+} f(x)\rangle$, where $\dot{+}$ is addition modulo 2. Deutsch's algorithm can be implemented by the following term

$$\text{deutsch}_f := \text{letcase}^\circ \ x = \pi^1 \left(H^1 U_f^2 (H \otimes H)^2 |01\rangle\langle 01|^2 \right) \text{ in } \{|0\rangle\langle 0|^1, |1\rangle\langle 1|^1\}$$

The algorithm starts by picking a two qubit system with state $|01\rangle\langle 01|$, and successively applies the Hadamard operator independently to each of the qubits, i.e., it applies the quantum operator $H \otimes H$. Then, the unitary operator U_f is applied to the system, and the Hadamard gate H is applied to the first qubit. Finally, the first qubit of the resulting state is measured. It can be checked by mechanical calculation that the result is 0 if and only if $f(0) = f(1)$ (regardless of the definition of f). Consequently, depending on f, the measurement will produce either 0 with probability 1 (when $f(0) = f(1)$) or 1 with probability 1 (when $f(0) \neq f(1)$). Therefore, we have

$$\text{deutsch}_f \longrightarrow^* \begin{cases} |0\rangle\langle 0|^1 & \text{iff } f(0) = f(1) \\ |1\rangle\langle 1|^1 & \text{iff } f(0) \neq f(1) \end{cases}$$

Example 2.4. We illustrate the definition of a recursive (non-terminating) term that converges to the state $|0\rangle\langle 0|$, which is defined as follows

$$\mathsf{F} := \mu x.\text{letcase}^\circ \ z = \pi^1 |+\rangle\langle +|^1 \text{ in } \{x, |0\rangle\langle 0|^1\}$$

The term F defined above corresponds to the fixpoint of the lambda abstraction $\mathsf{f} := \lambda x.\text{letcase}^\circ \ z = \pi^1 |+\rangle\langle +|^1 \text{ in } \{x, |0\rangle\langle 0|^1\}$, which receives a one-qubit state x and behaves differently depending on the result of the measurement (in the computational basis) of the state $|+\rangle\langle +|$. The measurement behaves analogous to a coin toss: it produces 0 or 1 with probability $\frac{1}{2}$ each. If the measured value is 0, then the function returns the argument x; otherwise, it returns the state $|0\rangle\langle 0|$. Hence, the application of the abstraction f on a term t produces

the probability distribution $\left\{ (\frac{1}{2}, t), (\frac{1}{2}, |0\rangle\langle 0|^1) \right\}$. The only fixpoint of f is clearly $|0\rangle\langle 0|^1$.

Note that the term F reduces as follows:

$$\mathsf{F} = \mu x.\mathsf{letcase}^\circ \ z = \pi^1|{+}\rangle\langle{+}|^1 \ \mathsf{in} \ \{x, |0\rangle\langle 0|^1\}$$
$$\longrightarrow \ \mathsf{letcase}^\circ \ z = \pi^1|{+}\rangle\langle{+}|^1 \ \mathsf{in} \ \{\mathsf{F}, |0\rangle\langle 0|^1\}$$
$$\longrightarrow \ \left\{ (\tfrac{1}{2}, \mathsf{F}), (\tfrac{1}{2}, |0\rangle\langle 0|^1) \right\}$$
$$\longrightarrow \ \left\{ (\tfrac{1}{2}, \mathsf{letcase}^\circ \ z = \pi^1|{+}\rangle\langle{+}|^1 \ \mathsf{in} \ \{\mathsf{F}, |0\rangle\langle 0|^1\}), (\tfrac{1}{2}, |0\rangle\langle 0|^1) \right\}$$
$$\longrightarrow \ \left\{ (\tfrac{1}{4}, \mathsf{F}), (\tfrac{3}{4}, |0\rangle\langle 0|^1) \right\}$$
$$\longrightarrow^* \ \left\{ (\tfrac{1}{2^n}, \mathsf{F}), (\tfrac{2^n-1}{2^n}, |0\rangle\langle 0|^1) \right\}$$
$$\longrightarrow \ \dots$$

which converges to $\left\{ (1, |0\rangle\langle 0|^1) \right\}$.

The evaluation of the term F introduced in Example 2.4 does not terminate, i.e., it proceeds indefinitely. However, the evaluation of F admits another interpretation: if we were to choose (with probability $\frac{1}{2}$) just one path to follow at each coin flip, then the global probability of no termination would be 0. Indeed, λ_ρ° internalises all the paths in which computation may evolve as a probability distribution, as a sort of "generalised mixed state". The original presentation of λ_ρ° [5] is accompanied by an alternative presentation, dubbed λ_ρ, in which reduction rules are probabilistic (in the sense discussed above). Moreover, it has been also shown that the semantics of both calculi are coincident. For technical convenience, we opted for the presentation style of λ_ρ°, in which probabilistic paths are internalised; and understand that the evaluation of the term F terminates with probability 1. As a matter of fact, the denotational semantics in Section 3 establishes $|0\rangle\langle 0|$ as the denotation of F, which is obtained as the limit of the distribution $\left\{ (\frac{1}{2^n}, \mathsf{F}), (\frac{2^n-1}{2^n}, |0\rangle\langle 0|^1) \right\}$ when the number n of iterations goes to infinity.

2.2   Correctness

In this section we report on expected properties of the calculus; namely, progress, subject reduction and strong normalisation of typed terms not containing μ. It has been already shown that λ_ρ° (i.e., the fragment of λ_ρ^μ without fixpoints) enjoys progress for typed closed terms (i.e., any typed closed term is either a value or it reduces), subject reduction, and strong normalisation [5,9]. The last two results are directly adapted to λ_ρ^μ, while the first one follows by induction.

Theorem 2.5 (Progress). *If $\vdash t : A$, then either t is a value or there exists a term r such that $t \longrightarrow r$.*

Proof. By a straightforward induction on t.   □

Theorem 2.6 (Subject reduction). *If $\Gamma \vdash t : A$, and $t \longrightarrow r$, then $\Gamma \vdash r : A$.*

Proof. Straightforward extension of the proof for λ_ρ° [5, Theorem 4.4].   □

Theorem 2.7 (Strong normalisation). *If $\Gamma \vdash t : A$ and t does not contain any μ, then t is strongly normalising.*

Proof. Straightforward adaptation of the proof for $\lambda_\rho^\circ 2$ [9, Theorem 4.3.8], which is a polymorphic extension of λ_ρ°.   □

3   Denotational semantics on positive matrices

In this section we develop a denotational semantics of λ_ρ^μ, in which terms are interpreted as density matrices, along the lines of [10, 12]. From a semantic viewpoint, matrices with trace strictly smaller than 1 represent programs with a positive probability of non-termination [10]. As customary, we rely on the *Löwner order* $\sqsubseteq$ over density matrices of dimension n defined such that $M \sqsubseteq N$ if and only if $M - N$ is a positive matrix. As shown in [10], density matrices of dimension n equipped with the Löwner order conforms a CPO that has the null matrix 0 as its least element. For functions, we adopt the CPM approach of [12].

However, our interpretation of functions allows us to accommodate affine maps, i.e., maps f such that $f(0) \neq 0$; this is achieved by representing each affine mapping as the composition of a linear transformation and a constant translation. This change is essential for the interpretation of terms $\mu x.t$ as the least fixed point of the denotation of $\lambda x.t$: if every abstraction were interpreted as a linear map, then its least fixed point would be also 0 (i.e., the bottom of the domain).

Technically, our definition of the denotational semantics of λ_ρ^μ is obtained indirectly from an intermediate calculus in which fixpoints are incremental, i.e., the fixpoint operator is parameterised by a natural number that bounds the possible iterations. The interpretation of fixpoints in λ_ρ^μ is obtained as the limit of the interpretation of the incremental fixpoint.

The remaining of this section is structured as follows. We start by extending λ_ρ^μ with incremental fixpoints in Section 3.1. In Section 3.2 we define domains and the interpretation of types. In Section 3.3, we give a canonical representation for affine maps as an extension of the classical Choi representation of CPMs. The interpretation of terms is presented in Section 3.4.

3.1   Incremental fixpoint

In order to account for incremental fixpoints, we modify the syntax of λ_ρ^μ to define a calculus called $\lambda_\rho^{[\mu]}$ as follows:

$$t ::= \quad x \mid \lambda x.t \mid tt \mid \mu_n x.t \mid \bot \qquad \textbf{(Standard lambda terms)}$$

$$| \; \rho^n \; | \; U^n t \; | \; \pi^m t \; | \; t \otimes t \qquad \textbf{(Quantum postulates)}$$
$$| \; \mathsf{letcase}^\circ \; x = t \; \mathsf{in} \; \{t, \ldots, t\} \qquad \textbf{(Control)}$$
$$| \; \{(p_i, t_i)\}_{i \in \{1, \ldots, n\}} \qquad \textbf{(Distributions)}$$

where fixpoint terms are labelled by a natural number n, and $\perp$ stands for the undefined value. The set of typing rules in Figure 3 is extended with the following axiom

$$\overline{\Gamma \vdash \perp : A}$$

while the typing rule should now consider decorated fixpoints.

The rewrite system for the $\lambda_\rho^{[\mu]}$ calculus is given by the relation $\rightsquigarrow$. In addition to the rules in Figure 2, we consider following ones:

$$\mu_0 x.t \rightsquigarrow \perp \qquad \mu_{n+1} x.t \rightsquigarrow t[x := \mu_n x.t]$$

The evaluation of a fixpoint expression reduces the decoration $n + 1$ at each iteration until it reaches 0, when it reduces to the undefined term $\perp$. In addition, we need the following instrumental rules to propagate $\perp$.

$$\perp t \rightsquigarrow \perp \qquad U^n \perp \rightsquigarrow \perp \qquad \pi^n \perp \rightsquigarrow \perp$$
$$\perp \otimes t \rightsquigarrow \perp \qquad t \otimes \perp \rightsquigarrow \perp \qquad \mathsf{letcase}^\circ \; x = \perp \; \mathsf{in} \; \{t_1, \ldots, t_n\} \rightsquigarrow \perp$$

Note that the incremental version of the term F in Example 2.4, i.e., $\mathsf{F}_n = \mu_n x.\mathsf{letcase}^\circ \; z = \pi^1 |+\rangle\langle+|^1 \; \mathsf{in} \; \{x, |0\rangle\langle0|^1\}$ rewrites to $\left\{ (\frac{1}{2^n}, \perp), (\frac{2^n-1}{2^n}, |0\rangle\langle0|^1) \right\}$ after n iterations, and converges to $\left\{ (1, |0\rangle\langle0|^1) \right\}$ when n approaches infinity.

3.2 Interpretation of Types

In order to account for non-termination, we will consider domains involving positive matrices with trace equal or lesser than 1. Let

$$\mathcal{D}_n^\leq = \left\{ \rho \mid \rho \in \mathbb{C}^{2^n \times 2^n} \; \text{positive with} \; tr(\rho) \leq 1 \right\}$$

and note that $\mathcal{D}_n \subsetneq \mathcal{D}_n^\leq$. Then, types are interpreted as follows:

$$(\!| n |\!) = \mathcal{D}_n^\leq$$

$$(\!| (m, n) |\!) = \left\{ M \mid M \in \bigoplus_{i=1}^{2^m} \mathcal{D}_n^\leq \; \text{and} \; \mathsf{tr}\,(M) \leq 1 \right\}$$

$$(\!| A \multimap B |\!) = \{ f \mid f \; \text{positive in} \; (\!| A |\!) \otimes (\!| B |\!)) \oplus (\!| B |\!) \}$$

- The type n is interpreted as the set of density matrices of dimension 2^n, since it is associated with n-qubit systems.
- The type (m, n) is interpreted as the set of coproducts of 2^m density matrices of dimension 2^n, with global trace bounded by 1. Intuitively, the type (m, n) describes all the possible outcomes of measuring the first m qubits of a state of n qubits, i.e., the combination of 2^m possible states, each of them in $\mathcal{D}_n^\leq$. For example, $M = (\frac{1}{2}|0\rangle\langle0| \oplus \frac{1}{2}|1\rangle\langle1|) \in (\!| (1, 1) |\!)$ because $\frac{1}{2}|0\rangle\langle0| \in \mathcal{D}_1^\leq, \frac{1}{2}|1\rangle\langle1| \in \mathcal{D}_1^\leq$ and $\mathsf{tr}\,(M) = 1$.

- The type $A \multimap B$ is interpreted as the set of positive matrices in $((\!|A|\!) \otimes (\!|B|\!)) \oplus (\!|B|\!)$, where the linear part is represented in $(\!|A|\!) \otimes (\!|B|\!)$ via its action on the canonical basis of $(\!|A|\!)$, and the constant part is a matrix in $(\!|B|\!)$.

Definition 3.1 (Domains). *The set* Dom *of interpretation domains is* Dom $= \bigcup_{A \in \mathsf{Types}} (\!|A|\!)$.

Definition 3.2 (Dimension). *The dimension of a type is defined as the dimension of its representation space, that is* $\dim(A) = dim((\!|A|\!))$:

$$\dim(n) = 2^n$$
$$\dim((m, n)) = 2^m 2^n = 2^{n+m}$$
$$\dim(A \multimap B) = (\dim(A) + 1) \dim(B)$$

3.3 Extended Choi representation for affine functions

Affine functions consist of a linear transformation and a translation; consequently, we interpret a function $f : A \multimap B$ as a matrix $\overline{\chi}_{[f]} \in (\!|A \multimap B|\!)$ that combines two matrices, one that represents its linear part and one that represents its constant part, i.e.,

$$\overline{\chi}_{[f]} = \left(\begin{array}{c|c|c} f(E_{11}^A) - f(\mathbf{0}_{\dim(A)}) & \cdots & f(E_{1n}^A) - f(\mathbf{0}_{\dim(A)}) \\ \hline \vdots & \ddots & \vdots \\ \hline f(E_{n1}^A) - f(\mathbf{0}_{\dim(A)}) & \cdots & f(E_{nn}^A) - f(\mathbf{0}_{\dim(A)}) \end{array} \right) \oplus f(\mathbf{0}_{\dim(A)})$$

where $\{E_{ij}^A\}_{ij}$ are the elements of the canonical basis of $(\!|A|\!)$, and $\mathbf{0}_{\dim(A)}$ is the null matrix in $(\!|A|\!)$. The matrix on the left-hand-side of the coproduct represents the linear transformation on the canonical basis of $(\!|A|\!)$, and the matrix on the right-hand-side represents the translation.

We can also write this representation in terms of the characteristic matrix defined in [10, Section 6.7] for the linear function $f - f(\mathbf{0}_{\dim(A)})$. Let g be a linear function, its characteristic matrix is

$$\chi_{[g]} = \left(\begin{array}{c|c|c} g(E_{11}^A) & \cdots & g(E_{1n}^A) \\ \hline \vdots & \ddots & \vdots \\ \hline g(E_{n1}^A) & \cdots & g(E_{nn}^A) \end{array} \right)$$

Then $\overline{\chi}_{[f]}$ can equivalently be defined as $\overline{\chi}_{[f]} = \chi_{[f - f(\mathbf{0}_{\dim(A)})]} \oplus f(\mathbf{0}_{\dim(A)})$.

Then, the application of an affine map (represented by its characteristic matrix) to an element of its domain requires

1. decomposing the element in the canonical basis,
2. applicating the linear transformation to each individual component, and
3. accumulating all partial results and the translation.

Definition 3.3 (Projection). *Let $\{E_{ij}^n\}$ be the canonical basis of the space $\mathbb{C}^{n \times n}$, and $\overline{\chi} = (\sum_{ij}(E_{ij}^n \otimes M_{ij})) \oplus M_\perp \in \mathbb{C}^{nm \times nm} \oplus \mathbb{C}^{m \times m}$ an affine mapping. Then, the projection of $\overline{\chi}$ with respect to the indexes $1 \leq k, l \leq n$ is $P_{kl}(\overline{\chi}) = M_{kl}$. Moreover, $P_\perp(\overline{\chi}) = M_\perp$.*

Intuitively, the operator P_{kl} projects the submatrix of size $\mathbb{C}^{m \times m}$ of the linear component of $\overline{\chi}$ that corresponds to the basis E_{ij}^n, while $P_\perp$ projects the constant component of the mapping.

Definition 3.4 (Application). *Let $\overline{\chi} \in \mathbb{C}^{nm \times nm} \oplus \mathbb{C}^{m \times m}$ be a linear mapping. Then, the application of $\overline{\chi}$ to an element in $\mathbb{C}^{n \times n}$ is denoted by the operator $\#$, which is defined as follows:*

$$\overline{\chi} \# \left(\sum_{ij} m_{ij} E_{ij}^n \right) = \left(\sum_{ij} m_{ij} P_{ij}(\overline{\chi}) \right) + P_\perp(\overline{\chi})$$

We shall write $\overline{\chi} \#_n M$ for n applications of $\overline{\chi}$ to M, e.g., $\overline{\chi} \#_3 M = \overline{\chi} \# (\overline{\chi} \# (\overline{\chi} \# M))$.

Remark 3.5. The operator $\#$ can be defined in terms of the standard linear application @ of Choi matrices directly as

$$\overline{\chi}_{[f]} \# M = \left(\chi_{[f - f(\mathbf{0}_n)]} \oplus f(\mathbf{0}_n) \right) \# M = \chi_{[f - f(\mathbf{0}_n)]} @ M + f(\mathbf{0}_n)$$

The following two results state two useful properties about the application operator.

Lemma 3.6 (# is right affine). *Let $\overline{\chi} \in \mathbb{C}^{nm \times nm} \oplus \mathbb{C}^{m \times m}$ be an affine mapping and $M, N \in \mathbb{C}^{n \times n}$ two matrices. Then,*

$$\overline{\chi} \# (M + N) = \overline{\chi} \# M + \overline{\chi} \# N - P_\perp(\overline{\chi}) \qquad \square$$

Lemma 3.7 (# is left linear). *Let $\overline{\chi}_1, \overline{\chi}_2 \in \mathbb{C}^{nm \times nm} \oplus \mathbb{C}^{m \times m}$ be affine mappings and $M \in \mathbb{C}^{n \times n}$ a matrix. Then,*

$$(\overline{\chi}_1 + \overline{\chi}_2) \# M = \overline{\chi}_1 \# M + \overline{\chi}_2 \# M \qquad \square$$

3.4   Interpretation of Terms

Our interpretation of terms depends on a *valuation function*, i.e., a partial function $\theta : \mathsf{Var} \to \mathsf{Dom}$ that maps each variable to an element of some domain. Then, the interpretation of a (typed) term t with respect to a valuation θ is inductively defined by the equations in Figure 4. For the sake of simplicity, we left implicit the typing judgement and write $(\!|t|\!)_\theta$ in lieu of $(\!|\Gamma \vdash t : A|\!)_\theta$. This is irrelevant for most of the equations but for abstractions and fixpoints. For abstractions, we rely on the representation of affine mappings introduced in Section 3.3, in this case, we implicitly assume $\Gamma \vdash \lambda x.t : A \multimap B$ and $a \in (\!|A|\!)$. Analogously, in the case of fixpoints we write $\mathbf{0}_{\dim(A)}$ to refer to the null matrix of a suitable dimension.

(Standard lambda terms)

$$(\!|x|\!)_\theta = \theta(x)$$

$$(\!|\lambda x.t|\!)_\theta = \overline{\chi}_{[a \mapsto (\!|t|\!)_{\theta, x=a}]}$$

$$(\!|tr|\!)_\theta = (\!|t|\!)_\theta \,\#\, (\!|r|\!)_\theta$$

$$(\!|\mu x.t|\!)_\theta = \lim_{n \to \infty} \left((\!|\lambda x.t|\!)_\theta \,\#_n\, \mathbf{0}_{\dim(A)} \right)$$

(Quantum postulates)

$$(\!|\rho^n|\!)_\theta = \rho$$

$$(\!|U^m t|\!)_\theta = \overline{U} (\!|t|\!)_\theta \overline{U}^\dagger$$

$$(\!|\pi^m t|\!)_\theta = \bigoplus_{i=0}^{2^m - 1} \left(\overline{|i\rangle\langle i|} (\!|t|\!)_\theta \overline{|i\rangle\langle i|}^\dagger \right)$$

$$(\!|t \otimes r|\!)_\theta = (\!|t|\!)_\theta \otimes (\!|r|\!)_\theta$$

(Control)

$$(\!|\mathsf{letcase}^\circ \; x = r \text{ in } \{t_0, \ldots, t_{2^m-1}\}|\!)_\theta = \sum_{i=0}^{2^m - 1} \mathsf{tr}\,(\rho_i) \, (\!|t_i|\!)_{\theta, x=\rho'_i}$$

$$\text{with } (\!|r|\!)_\theta = \bigoplus_{i=0}^{2^m - 1} \rho_i \text{ and } \rho'_i = \begin{cases} \frac{\rho_i}{\mathsf{tr}(\rho_i)} & \text{if } \mathsf{tr}\,(\rho_i) \neq 0 \\ \rho_i & \text{if } \mathsf{tr}\,(\rho_i) = 0 \end{cases}$$

(Distributions)

$$(\!|\{(p_i, t_i)\}_i|\!)_\theta = \sum_i p_i (\!|t_i|\!)_\theta$$

The type A is the type of the interpreted term.

Fig. 4: Denotational semantics for the λ_ρ^μ calculus.

Example 3.8. Let $\lambda x.t$ where $t = \mathsf{letcase}^\circ \; y = \pi^1 |{+}\rangle\langle{+}|^1$ in $\{x, |0\rangle\langle 0|^1\}$. Note that $\vdash \lambda x.t : 1 \multimap 1$. Then, $(\!|\lambda x.t|\!)_\emptyset = \overline{\chi}_{[f]}$ with $f = a \mapsto (\!|t|\!)_{x=a}$ for $a \in (\!|1|\!)$. We first note that the measurement in t is independent of a, i.e., $(\!|\pi^1|{+}\rangle\langle{+}|^1|\!)_{x=a} = (\!|\pi^1|{+}\rangle\langle{+}|^1|\!)_\emptyset$. By the interpretation of a measurement, $(\!|\pi^1|{+}\rangle\langle{+}|^1|\!)_\emptyset = \frac{1}{2}|0\rangle\langle 0| \oplus \frac{1}{2}|1\rangle\langle 1|$. By the interpretation of $\mathsf{letcase}$, $(\!|t|\!)_{x=a} = \frac{1}{2}(\!|x|\!)_{x=a} + \frac{1}{2}|0\rangle\langle 0| = \frac{1}{2}a + \frac{1}{2}|0\rangle\langle 0|$. Hence, $f = a \mapsto \frac{1}{2}a + \frac{1}{2}|0\rangle\langle 0|$. Consequently, $\overline{\chi}_{[f]} = \chi_{[a \mapsto \frac{1}{2}a]} \oplus \frac{1}{2}|0\rangle\langle 0|$, i.e.,

$$\overline{\chi}_{[f]} = \begin{pmatrix} \frac{1}{2}|0\rangle\langle 0| & \frac{1}{2}|0\rangle\langle 1| \\ \frac{1}{2}|1\rangle\langle 0| & \frac{1}{2}|1\rangle\langle 1| \end{pmatrix} \oplus \frac{1}{2}|0\rangle\langle 0|$$

Consider now the application $(\lambda x.t)\rho^1$ with $\rho = |0\rangle\langle 1| + |1\rangle\langle 1|$. Then, $(\!|(\lambda x.t)\rho^1|\!) = \overline{\chi}_{[f]} \,\#\, \rho = \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 1|)$.

Example 3.9. Let $\lambda x.t$ with $t = \mathsf{letcase}^\circ \; y = x$ in $\{|0\rangle\langle 0|^1, |1\rangle\langle 1|^1\}$. Note that $\vdash \lambda x.t : (1,1) \multimap 1$. As in the previous example, $(\!|\lambda x.t|\!)_\emptyset = \overline{\chi}_{[g]}$ with $g = a \mapsto (\!|t|\!)_{x=a}$ for $a \in (\!|(1,1)|\!)$. The canonical basis of the domain $(\!|(1,1)|\!)$ has the following elements:

$|0\rangle\langle 0| \oplus \mathbf{0}_2$	$|1\rangle\langle 0| \oplus \mathbf{0}_2$	$\mathbf{0}_2 \oplus |0\rangle\langle 1|$	$\mathbf{0}_2 \oplus |1\rangle\langle 1|$
$|0\rangle\langle 1| \oplus \mathbf{0}_2$	$|1\rangle\langle 1| \oplus \mathbf{0}_2$	$\mathbf{0}_2 \oplus |0\rangle\langle 0|$	$\mathbf{0}_2 \oplus |1\rangle\langle 0|$

Then, $\overline{\mathcal{X}}_{[g]}$ is given by

$$\overline{\mathcal{X}}_{[g]} = \left(\begin{pmatrix} |0\rangle\langle 0| & \mathbf{0}_2 \\ \mathbf{0}_2 & |0\rangle\langle 0| \end{pmatrix} \oplus \begin{pmatrix} |1\rangle\langle 1| & \mathbf{0}_2 \\ \mathbf{0}_2 & |1\rangle\langle 1| \end{pmatrix} \right) \oplus \mathbf{0}_2$$

Example 3.10. Consider the λ_ρ^μ term in Example 2.4. Its interpretation is:

$$(\!|\mu x.\mathsf{letcase}^\circ \ z = \pi^1|+\rangle\langle+|^1 \ \mathsf{in} \ \{x, |0\rangle\langle 0|^1\}|\!)_\emptyset$$
$$= \lim_{n \to \infty} (\!|\lambda x.\mathsf{letcase}^\circ \ z = \pi^1|+\rangle\langle+|^1 \ \mathsf{in} \ \{x, |0\rangle\langle 0|^1\}|\!)_\emptyset \ \#_n \ \mathbf{0}_2$$

Note that $(\!|\lambda x.\mathsf{letcase}^\circ \ z = \pi^1|+\rangle\langle+|^1 \ \mathsf{in} \ \{x, |0\rangle\langle 0|^1\}|\!)_\emptyset$ is $\overline{\mathcal{X}}_{[f]}$ defined in Example 3.8. Hence,

$$\lim_{n \to \infty} \overline{\mathcal{X}}_{[f]} \ \#_n \ \mathbf{0}_2 = \lim_{n \to \infty} \tfrac{1}{2^n}\mathbf{0}_2 + \sum_{i=1}^n \tfrac{1}{2^i}|0\rangle\langle 0|$$
$$= \lim_{n \to \infty} \left(\frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} - 1 \right) |0\rangle\langle 0| = |0\rangle\langle 0|$$

Example 3.11. The identity function has infinite fixpoints, all elements in the domain. Its fixpoint is defined as the least element in the domain, i.e., the null matrix:

$$(\!|\mu x.x|\!)_\emptyset = \lim_{n \to \infty} (\!|\lambda x.x|\!)_\emptyset \ \#_n \ \mathbf{0}_2 = \mathbf{0}_2$$

3.5 Interpretation of $\lambda_\rho^{[\mu]}$

In order to show that the interpretation function is well-defined, in particular, the existence of the limits in interpretation of fixpoints, we provide the following interpretation for the incremental fixpoint in the $\lambda_\rho^{[\mu]}$ calculus

$$(\!|\mu_n x.t|\!)_\theta = (\!|\lambda x.t|\!)_\theta \ \#_n \ \mathbf{0}_{\dim(A)}$$
$$(\!|\bot|\!)_\theta = \mathbf{0}_{\dim(A)}$$

4   Soundness of the interpretation

In this section we prove that our definition of interpretation is sound, i.e., the interpretation function maps a term of type A to an element of the domain of the type A. We first focus on the intermediate language $\lambda_\rho^{[\mu]}$, which features incremental fixpoints (Section 4.1). In Section 4.2 we address the full calculus λ_ρ^μ and show that the interpretation function is well-defined, i.e., the limits in the interpretation of fixpoint terms exist. This is achieved by showing that the interpretation function maps well-typed terms into matrices the traces of which are bounded by the types of the terms. Such bounds allow us to show that the proposed domains equipped with the Löwner order are CPOs.

4.1   Soundness of the interpretation of incremental fixpoints

We start by introducing some auxiliary notions and results that are instrumental for establishing the soundness of the interpretation for the calculus with incremental fixpoints. For the reviewer's convenience, omitted proofs are provided in Appendix A. Firstly, we show that the interpretation function behaves well with respect to substitution, application and reduction. The following is the usual expected substitution lemma.

Lemma 4.1 (Substitution). $(\!|t[x := r]|\!)_\theta = (\!|t|\!)_{\theta, x = (\!|r|\!)_\theta}$.

For application and reduction, we rely on the following property establishing that mappings $a \mapsto (\!|t|\!)_{\theta, x = a}$ arising from the interpretation of terms $\lambda x.t$ are affine or, equivalently, that $a \mapsto (\!|t|\!)_{\theta, x = a} - (\!|t|\!)_{\theta, x = \mathbf{0}_{\dim(A)}}$ is linear. The key point here is that θ should map variables to values of the proper type. We say that a valuation θ and a typing context Γ are consistent, written $\theta \vDash \Gamma$, if and only if for every $x : A \in \Gamma$ we have $\theta(x) \in (\!|A|\!)$.

Lemma 4.2 (Linearity).   *If* $\Gamma, x : A \vdash t : B$ *and* $\theta \vDash \Gamma$, *then for all* $a \in (\!|A|\!)$ *the function* $a \mapsto (\!|t|\!)_{\theta, x = a} - (\!|t|\!)_{\theta, x = \mathbf{0}_{\dim(A)}}$ *is linear.*

By relying on linearity, we show that the application operator behaves as expected with respect to the valuation used for the interpretation of terms.

Lemma 4.3 (Application).   *If* $\Gamma, x : A \vdash t : B$ *and* $\theta \vDash \Gamma$, *then for all* $a \in \mathbb{C}^{\dim(A) \times \dim(A)}$ *we have* $(\!|\lambda x.t|\!)_\theta \,\#\, a = (\!|t|\!)_{\theta, x = a}$.

Also, we show that interpretation is stable with respect to the reduction relation.

Lemma 4.4 (Reduction correctness).   *If* $\Gamma \vdash t : A$, $\theta \vDash \Gamma$, *and* $t \rightsquigarrow r$ *then* $(\!|t|\!)_\theta = (\!|r|\!)_\theta$.

We now concentrate on the soundness of the interpretation of abstractions. We need to show that $\Gamma \vdash \lambda x.t : A \multimap B$ implies $(\!|\lambda x.t|\!)_\theta \in (\!|A \multimap B|\!)$ for all valuations θ consistent with Γ (Lemma 4.6). Recall that $(\!|\lambda x.t|\!)_\theta = \overline{\chi}_{[a \mapsto (\!|t|\!)_{\theta, x = a}]}$. We first show that the linear part of $a \mapsto (\!|t|\!)_{\theta, x = a}$, i.e., $a \mapsto (\!|t|\!)_{\theta, x = a} - (\!|t|\!)_{\theta, x = \mathbf{0}_{\dim(A)}}$, is a completely positive map (CPM), i.e., it produces positive matrices when applied to positive matrices.

Lemma 4.5.   *Let* $\Gamma, x : A \vdash t : B$, *and for all* $\theta \vDash \Gamma$ *and* $a \in (\!|A|\!)$, *let* $(\!|t|\!)_{\theta, x = a} \in (\!|B|\!)$. *Then, the map* $F_{\theta \vDash \Gamma}^{t,x} = a \mapsto (\!|t|\!)_{\theta, x = a} - (\!|t|\!)_{\theta, x = \mathbf{0}_{\dim(A)}}$ *is a CPM.*

By relying on the previous lemma, we have the following expected result about the soundness of the interpretation for abstractions.

Lemma 4.6 (Soundness for abstractions).   *Let* $\Gamma, x : A \vdash t : B$ *and* $\theta \vDash \Gamma$, *such that* $(\!|t|\!)_{\theta, x = a}, (\!|t|\!)_{\theta, x = \mathbf{0}_{\dim(A)}} \in (\!|B|\!)$. *Then* $\overline{\chi}_{[a \mapsto (\!|t|\!)_{\theta, x = a}]} \in (\!|A \multimap B|\!)$.

Next Lemma 4.7 extends the previous result for all arrow-typed terms.

Lemma 4.7 (Soundness for arrow-type terms). *Let $\Gamma \vdash t : A \multimap B$ and $\theta \vDash \Gamma$. One of the following holds:*

- *There exist $t_1, \ldots, t_n$ and $p_1, \ldots, p_n$ such that for all i, $x : A \vdash t_i : B$, $p_i > 0$, $\sum_{i=1}^{n} p_i \leq 1$ and $(\!|t|\!)_\theta = \sum_{i=1}^{n} p_i (\!|\lambda x.t_i|\!)_\emptyset$.*
- *$(\!|t|\!)_\theta = \mathbf{0}_{\dim(A \multimap B)}$*

The soundness for sums is stated below.

Lemma 4.8. *Let $(\!|t_i|\!)_\theta \in (\!|A|\!)$ for $i \in \{1, \ldots, n\}$. Then for any $p_1, \ldots, p_n$ such that $0 < p_i \leq 1$ with $\sum_{i=1}^{n} p_i \leq 1$, we have $\sum_{i=1}^{n} p_i (\!|t_i|\!)_\theta \in (\!|A|\!)$.*

Finally we state the soundness theorem.

Theorem 4.9 (Soundness). *Let $\Gamma \vdash t : A$ and $\theta \vDash \Gamma$, then $(\!|t|\!)_\theta \in (\!|A|\!)$.*

4.2   Existence of fixpoints

We now proceed to show that $\lim_{n \to \infty} ((\!|\lambda x.t|\!)_\theta \#_n \mathbf{0}_{\dim(A)})$ actually exists for well-typed terms (omitted proofs are in Appendix B). Firstly, we show that closed terms are interpreted as matrices with bounded traces. While the traces of the elements in the domains associated with the types n and (m, n) are bounded by 1 by definition, the traces of the elements in the domains associated with arrow types are not. Since arrows $A \multimap B$ are interpreted in $((\!|A|\!) \otimes (\!|B|\!)) \oplus (\!|B|\!)$, their traces can be greater than 1. For example, the identity function in $1 \multimap 1$ has as interpretation a matrix of trace 2, as illustrated below

$$(\!|\lambda x.x|\!)_\emptyset = \overline{\chi}_{[a \mapsto (\!|x|\!)_{x=a}]} = (|00\rangle\langle00| + |01\rangle\langle10| + |10\rangle\langle01| + |11\rangle\langle11|) \oplus \mathbf{0}_2$$

However, we can associate a bound, which we call size, to each type.

Definition 4.10 (Size of a type). *Let A be a type, we write N_A for its size, which is inductively defined as follows:*

- *$N_n = 1$*
- *$N_{(m,n)} = 1$*
- *$N_{A \multimap B} = (\dim(A) + 1)N_B$*

Theorem 4.11. *Let $\vdash t : A$, then $\mathsf{tr}\,((\!|t|\!)_\emptyset) \leq N_A$.*

Next, we define the Löwner order (Definition 4.12) and show that abstractions terms in the calculus with incremental fixpoints that have identity type preserve this order (Lemma 4.13) and are continuous (Lemma 4.14).

Definition 4.12 (Löwner order). *Let M, N be positive matrices. Then $M \sqsubseteq N$ if and only if $N - M$ is positive.*

Lemma 4.13. *Let $\Gamma, x : A \vdash t : A$ and $\theta \vDash \Gamma$. Then for all $a, b \in (\!|A|\!)$, if $a \sqsubseteq b$ we have $(\!|\lambda x.t|\!)_\theta \# a \sqsubseteq (\!|\lambda x.t|\!)_\theta \# b$.*

Lemma 4.14. *Let $\overline{\chi} \in \mathbb{C}^{nm \times nm} \oplus \mathbb{C}^{m \times m}$ and let (P_n) be an increasing sequence of positive matrices in $\mathbb{C}^{n \times n}$ such that $\lim_{n \to \infty} P_n = P$. Then, $\overline{\chi}$ is monotone and $\lim_{n \to \infty} \overline{\chi} \ \# \ P_n = \overline{\chi} \ \# \ P$.*

We recall that the least upper bound of an increasing sequence of square complex matrices is equal to its limit (see [10, Remark 3.8]).

We have already shown that every term in the calculus with incremental fixpoint is interpreted as a positive matrix (Theorem 4.9) the trace of which is bounded by the size of its type (Theorem 4.11). Moreover, we have shown that terms of type $A \multimap A$ are interpreted as continuous functions. This allows us to show that the image of the interpretations form CPOs.

For any type A, we let $\mathfrak{D}_A = \{M \mid M \in (\!|A|\!) \text{ and } \mathsf{tr}\,(M) \le N_A\}$. By Theorem 4.11, the image of the interpretation of closed terms $\vdash t : A$ lies on $\mathfrak{D}_A$. The following lemma states that this set, with the Löwner order, forms a CPO.

Lemma 4.15. *For any type A, $(\mathfrak{D}_A, \sqsubseteq)$ is a complete partial order.*

Finally, the following result states that the denotation of the fixpoint in λ_ρ^μ is well defined.

Theorem 4.16. *If $\Gamma \vdash \lambda x.t : A \multimap A$ and $\theta \vDash \Gamma$, then*

$$\lim_{n \to \infty} (\!|\lambda x.t|\!)_\theta \ \#_n \ \mathbf{0}_{\dim(A)}) \in \mathfrak{D}_A$$

5   Adequacy

We now show that the denotational semantics of terms of the basic types n is adequate, i.e., that the probability of termination of the evaluation of a term coincides with the trace of its denotation. We start by establishing some useful properties about the reduction relations of the λ_ρ^μ calculus ($\longrightarrow$) and of the $\lambda_\rho^{[\mu]}$ calculus ($\rightsquigarrow$) which features incremental fixpoints (see Section 3.1). We shall write $\rightsquigarrow\!\!\!\rightarrow$ to denote either $\longrightarrow$ or $\rightsquigarrow$. In particular, we show that the probability of termination does not decrease with reduction. For this reason, we associate each term (seen as a distribution) with a probability of being a value, as defined below.

Definition 5.1 (Probability of being a value). *Let $t = \{(p_i, \rho_i^n)\}_i \cup \{(q_j, r_j)\}_j$ be a distribution such that $\vdash t : n$, and for all j either $r_j \rightsquigarrow\!\!\!\rightarrow r_j'$ or $r_j = \bot$. The subdistribution of values of t, written $\mathcal{V}(t)$, is $\{(p_i, \rho_i^n)\}_i$. Moreover, the probability of being a value, written $P(t)$, is $\sum_i p_i$.*

We first note that $P(t) = P(\mathcal{V}(t))$. Then, we show that the function P is not decreasing with respect to term rewriting.

Lemma 5.2. *Let $\vdash t : n$, if $t \rightsquigarrow\!\!\!\rightarrow t'$ then $P(t) \le P(t')$.*

Proof. We can write t as $\{(p_i, \rho_i^n)\}_{i \in I} \cup \{(q_j, r_j)\}_{j \in J}$ where for all $j \in J$, either $r_j \rightsquigarrow r_j'$ or $r_j = \bot$. Then by definition $P(t) = \sum_i p_i$. By hypothesis $t \rightsquigarrow t'$, and so there must be a $k \in J$ such that $r_k \rightsquigarrow \{(q_l', r_l')\}_{l \in L}$. Then we have $t' = \{(p_i, \rho_i^n)\}_{i \in I} \cup \{(q_l', r_l')\}_{l \in L} \cup \{(q_j, r_j)\}_{j \in J \setminus \{k\}}$. Since $\mathcal{V}(t) \subseteq \mathcal{V}(t')$, we have that $P(t) \leq P(t')$. $\square$

We define the probability of termination for the evaluation of a term as the least upper bound of the set of the probabilities of being a value of all its reductions (Definition 5.4).

Definition 5.3. *Let t be a term. The set of reductions of t is defined as $Red^*(t) = \{r \mid t \rightsquigarrow^* r\}$.*

We remark that $Red^*(t)$ with the order given by reduction is a directed set. With abuse of notation, we use P for its extension on sets, defined as follows,

$$P(\{t_1, t_2, ...\}) = \{P(t_1), P(t_2), ...\}$$

Definition 5.4 (Probability of termination). *The probability of termination of a term t, written $P_\infty(t)$, is the least upper bound of the set of the probabilities of being a value of its reductions, i.e., $P_\infty(t) = \bigvee P(Red^*(t))$.*

Notice that $P_\infty(t)$ is well defined because $P(Red^*(t))$ is a directed set with respect to $\leq$ in $\mathbb{R}_{\geq 0}$, and it is bounded by 1.

Finally, we prove that the that probability of termination coincides with the trace of the interpretation of the term (Theorem 5.9). To prove it we use Lemmas 5.7 and 5.8.

Definition 5.5. *Let t be a term in λ_ρ^μ such that $\vdash t : n$, and $N \in \mathbb{N}_0$. We write $[t]_N$ for the term obtained by substituting every occurrence of μ in t by μ_N. This is a term in the incremental fixpoint calculus $\lambda_\rho^{[\mu]}$.*

Proposition 5.6. *Let t be a term in λ_ρ^μ such that $\vdash t : n$. Then, for all $N \in \mathbb{N}_0$, $P_\infty([t]_N) \leq P_\infty([t]_{N+1})$ and $P_\infty([t]_N) \leq P_\infty(t)$.* $\square$

Lemma 5.7. *Let t be a term in λ_ρ^μ such that $\vdash t : n$. Then,*

$$\lim_{N \to \infty} P_\infty([t]_N) = P_\infty(t)$$

Proof. We have to see that for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $N' \geq N$, $\mid P_\infty([t]_{N'}) - P_\infty(t) \mid < \varepsilon$.

Because of the correspondence between $Red^*(t)$ and $P(Red^*(t))$, there is an ω-sequence $(t_i)_{i \in \mathbb{N}} \subseteq Red^*(t)$ starting from t such that

$$P_\infty(t) = \bigvee P(Red^*(t)) = \lim_{i \to \infty} P(t_i)$$

Then, for all $\varepsilon > 0$ there is a $j \in \mathbb{N}$ such that $t_j \in (t_i)_{i \in \mathbb{N}}$ and

$$\mid P(t_j) - P_\infty(t) \mid < \varepsilon \tag{1}$$

Hence, there is an $N \in \mathbb{N}$ and a $K \leq N$ such that

$$[t]_N \leadsto^* [t_j]_K \leadsto^* \{(q_i, i)\}_{i \in \mathsf{Val}_n} \cup \left\{ \left(1 - \sum_{i \in \mathsf{Val}_n} q_i, \bot\right)\right\}$$

From this we have

$$P(t_j) \leq \sum_{i \in \mathsf{Val}_n} q_i = P_\infty([t]_N) \leq P_\infty(t) \tag{2}$$

From Equations (1) and (2) we have

$$\mid P_\infty([t]_N) - P_\infty(t) \mid < \varepsilon \tag{3}$$

Let $N' \in \mathbb{N}$ such that $N' > N$. By Proposition 5.6, we have that $P_\infty([t]_N) \leq P_\infty([t]_{N'}) \leq P_\infty(t)$, and from Equation (3)

$$\mid P_\infty(t_{N'}) - P_\infty(t) \mid < \varepsilon \qquad\qquad \square$$

Lemma 5.8. *Let t be a term in λ_ρ^μ such that $\vdash t : n$. Then,*

$$\lim_{N \to \infty} \mathsf{tr}\left(([t]_N)_\emptyset\right) = \mathsf{tr}\left((t)_\emptyset\right)$$

Proof. By Lemmas 3.6 and 3.7, and Theorem 4.16 we have

$$\lim_{N \to \infty} ([t]_N)_\emptyset = (t)_\emptyset$$

By trace linearity we have the result. $\qquad\qquad \square$

Finally, we can show that the definition of the denotational semantics is adequate for terms of types n.

Theorem 5.9 (Adequacy). *Let t be a term in λ_ρ^μ such that $\vdash t : n$. Then,*

$$P_\infty(t) = \mathsf{tr}\left((t)_\emptyset\right)$$

Proof. Assume that $P_\infty(t) \neq \mathsf{tr}\left((t)_\emptyset\right)$. Then, there exists $\delta > 0$ such that

$$\mid P_\infty(t) - \mathsf{tr}\left((t)_\emptyset\right) \mid > \delta \tag{4}$$

Take $\varepsilon < \frac{1}{2}\delta$. By Lemma 5.7, there is $N_1 \in \mathbb{N}$ such that $\mid P_\infty([t]_{N_1}) - P_\infty(t) \mid < \varepsilon$
By Lemma 5.8, there is $N_2 \in \mathbb{N}$ such that $\mid \mathsf{tr}\left(([t]_{N_2})_\emptyset\right) - \mathsf{tr}\left((t)_\emptyset\right) \mid < \varepsilon$.
Let $N = max\{N_1, N_2\}$, so, the previous two inequations become

$$\mid P_\infty([t]_N) - P_\infty(t) \mid < \varepsilon \tag{5}$$
$$\mid \mathsf{tr}\left(([t]_N)_\emptyset\right) - \mathsf{tr}\left((t)_\emptyset\right) \mid < \varepsilon \tag{6}$$

Since $[t]_N \leadsto^* \{(p, \rho^n), (1 - p, \bot)\}$, by Lemma 4.4 we have

$$([t]_N)_\emptyset = (\{(p, \rho^n), (1 - p, \bot)\})_\emptyset = p.\rho$$

and so $\text{tr}\left(\langle\!\langle[t]_N\rangle\!\rangle_\emptyset\right) = \text{tr}\left(p.\rho\right) = p$. In addition, by Lemma 5.2, we have that $P_\infty([t]_N) = P(\{(p,\rho^n),(1-p,\bot)\}) = p$. Hence, we have

$$P_\infty([t]_N) = \text{tr}\left(\langle\!\langle[t]_N\rangle\!\rangle_\emptyset\right) \tag{7}$$

Then, from Equations (6) and (7) we have,

$$\mid P_\infty([t]_N) - \text{tr}\left(\langle\!\langle t\rangle\!\rangle_\emptyset\right)\mid \, < \varepsilon \tag{8}$$

Therefore, from Equations (5) and (8),

$$\mid P_\infty(t) - \text{tr}\left(\langle\!\langle t\rangle\!\rangle_\emptyset\right)\mid \, < 2\varepsilon < \delta$$

which contradicts Equation (4). Hence, $P_\infty(t) = \text{tr}\left(\langle\!\langle t\rangle\!\rangle_\emptyset\right)$. $\square$

We remark that the adequacy result concerns only basic types n. It does not hold in general because, e.g., $\text{tr}\left(\langle\!\langle\lambda x.x\rangle\!\rangle_\emptyset\right) = 2$.

6   Conclusion

In this paper, we presented a finite-dimensional semantics for quantum higher-order computation with recursion. On one hand, to stay finite-dimensional we only account for non-duplicable elements. On the other hand, to be able to represent general recursion we allow the discarding of variables: the model is thus affine. In particular, we show how to extend the Choi-construction to linear, affine maps. The model is justified by the λ-calculus λ_ρ°, providing a concrete operational account of what affine, linear higher-order computation represents.

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A Proofs of results in Section 4.1

This section is devoted to the proofs of results in Section 4.1.

A.1 Proof of Lemma 4.1

Lemma 4.1 (Substitution). $(\!|t[x := r]|\!)_\theta = (\!|t|\!)_{\theta, x=(\!|r|\!)_\theta}$.

Proof. A routine proof by structural induction on t. $\square$

A.2 Proof of Lemma 4.2

Lemma 4.2 (Linearity). *If* $\Gamma, x : A \vdash t : B$ *and* $\theta \vDash \Gamma$, *then for all* $a \in (\!|A|\!)$ *the function* $a \mapsto (\!|t|\!)_{\theta, x=a} - (\!|t|\!)_{\theta, x=\mathbf{0}_{\dim(A)}}$ *is linear.*

Proof. Let $n = \dim(A)$. Then, note that the function $a \mapsto (\!|t|\!)_{\theta, x=a} - (\!|t|\!)_{\theta, x=\mathbf{0}_n}$ is linear if the equality below holds.

$$(\!|t|\!)_{\theta, x=\alpha A + \beta B} - (\!|t|\!)_{\theta, x=\mathbf{0}_n} = \alpha((\!|t|\!)_{\theta, x=A} - (\!|t|\!)_{\theta, x=\mathbf{0}_n}) + \beta((\!|t|\!)_{\theta, x=B} - (\!|t|\!)_{\theta, x=\mathbf{0}_n})$$

which, by rearranging terms, can be rewritten as follows.

$$(\!|t|\!)_{\theta, x=\alpha A + \beta B} = \alpha(\!|t|\!)_{\theta, x=A} + \beta(\!|t|\!)_{\theta, x=B} - (\alpha + \beta - 1)(\!|t|\!)_{\theta, x=\mathbf{0}_n} \qquad (9)$$

We show that Equation (9) holds by induction on the structure of t. Few interesting cases are shown below.

– Let $t = rs$. Hence, Equation (9) boils down to

$$(\!|rs|\!)_{\theta, x=\alpha A + \beta B} = \alpha(\!|rs|\!)_{\theta, x=A} + \beta(\!|rs|\!)_{\theta, x=B} - (\alpha + \beta - 1)(\!|rs|\!)_{\theta, x=\mathbf{0}_n} \qquad (10)$$

Since the type system is affine, either $x \in \mathsf{FV}(r)$ or $x \in \mathsf{FV}(s)$.

- Case $x \in \mathsf{FV}(r)$: By applying the definition of $(\!|\_|\!)_\_$ and by noting that $x \notin \mathsf{FV}(s)$ implies $(\!|s|\!)_{\theta, x=\alpha A + \beta B} = (\!|s|\!)_{\theta, x=\mathbf{0}_n} = (\!|s|\!)_\theta$, we rewrite Equation (10) as follows:

$$
\begin{aligned}
(\!|r|\!)_{\theta, x=\alpha A + \beta B} \mathbin{\#} (\!|s|\!)_\theta &= \alpha((\!|r|\!)_{\theta, x=A} \mathbin{\#} (\!|s|\!)_\theta) + \beta((\!|r|\!)_{\theta, x=B} \mathbin{\#} (\!|s|\!)_\theta) \qquad (11)\\
&\quad - (\alpha + \beta - 1)((\!|r|\!)_{\theta, x=\mathbf{0}_n} \mathbin{\#} (\!|s|\!)_\theta)
\end{aligned}
$$

that holds as follows.

$$
\begin{aligned}
&(\!|r|\!)_{\theta, x=\alpha A + \beta B} \mathbin{\#} (\!|s|\!)_\theta \\
&\overset{IH}{=} (\alpha(\!|r|\!)_{\theta, x=A} + \beta(\!|r|\!)_{\theta, x=B} - (\alpha + \beta - 1)(\!|r|\!)_{\theta, x=\mathbf{0}_n}) \mathbin{\#} (\!|s|\!)_\theta \\
&\overset{(3.7)}{=} \alpha((\!|r|\!)_{\theta, x=A} \mathbin{\#} (\!|s|\!)_\theta) + \beta((\!|r|\!)_{\theta, x=B} \mathbin{\#} (\!|s|\!)_\theta) \\
&\qquad - (\alpha + \beta - 1)((\!|r|\!)_{\theta, x=\mathbf{0}_n} \mathbin{\#} (\!|s|\!)_\theta)
\end{aligned}
$$

- Case $x \in \mathsf{FV}(s)$ follows analogously but relies on Lemma 3.6 instead of Lemma 3.7.

- Let $t = \{(p_i, t_i)\}_i$, with $\sum_i p_i = 1$ and $0 < p_i \le 1$ for all i. Hence, Equation (9) amounts to:

$$(\!|\{(p_i,t_i)\}_i|\!)_{\theta,x=\alpha A+\beta B} = \alpha (\!|\{(p_i,t_i)\}_i|\!)_{\theta,x=A} + \beta (\!|\{(p_i,t_i)\}_i|\!)_{\theta,x=B}$$
$$- (\alpha+\beta-1)(\!|\{(p_i,t_i)\}_i|\!)_{\theta,x=\mathbf{0}_n}$$

Rewriting the left-hand side:

$$(\!|\{(p_i,t_i)\}_i|\!)_{\theta,x=\alpha A+\beta B}$$
$$= \sum_i p_i (\!|t_i|\!)_{\theta,x=\alpha A+\beta B}$$
$$\overset{IH}{=} \sum_i p_i(\alpha(\!|t_i|\!)_{\theta,x=A} + \beta(\!|t_i|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|t_i|\!)_{\theta,x=\mathbf{0}_n})$$
$$= \alpha \sum_i p_i (\!|t_i|\!)_{\theta,x=A} + \beta \sum_i p_i (\!|t_i|\!)_{\theta,x=B} - (\alpha+\beta-1)\sum_i p_i(\!|t_i|\!)_{\theta,x=\mathbf{0}_n}$$
$$= \alpha(\!|\{(p_i,t_i)\}_i|\!)_{\theta,x=A} + \beta(\!|\{(p_i,t_i)\}_i|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|\{(p_i,t_i)\}_i|\!)_{\theta,x=\mathbf{0}_n}$$

- Let $t = \mathsf{letcase}^{\circ}\ y = r\ \mathsf{in}\ \{t_1, \ldots, t_m\}$. By definition of $(\!|\_|\!)_\_$, Equation (9) boils down to:

$$\sum_{i=0}^{m} \mathsf{tr}\left(\rho_{\alpha A+\beta B}^i\right)(\!|t_i|\!)_{\theta,x=\alpha A+\beta B,y=\widetilde{\rho_{\alpha A+\beta B}^i}}$$
$$= \alpha \sum_{i=0}^{m} \mathsf{tr}\left(\rho_A^i\right)(\!|t_i|\!)_{\theta,x=A,y=\widetilde{\rho_A^i}}$$
$$+ \beta \sum_{i=0}^{m} \mathsf{tr}\left(\rho_B^i\right)(\!|t_i|\!)_{\theta,x=B,y=\widetilde{\rho_B^i}}$$
$$- (\alpha+\beta-1)\sum_{i=0}^{m} \mathsf{tr}\left(\rho_0^i\right)(\!|t_i|\!)_{\theta,x=\mathbf{0}_n,y=\widetilde{\rho_0^i}} \qquad (12)$$

where

$$(\!|r|\!)_{\theta,x=\alpha A+\beta B} = \bigoplus_{i=1}^{m} \rho_{\alpha A+\beta B}^i \qquad (\!|r|\!)_{\theta,x=A} = \bigoplus_{i=1}^{m} \rho_A^i$$
$$(\!|r|\!)_{\theta,x=B} = \bigoplus_{i=1}^{m} \rho_B^i \qquad (\!|r|\!)_{\theta,x=\mathbf{0}_n} = \bigoplus_{i=1}^{m} \rho_0^i$$

$$\widetilde{\phi} = \begin{cases} \mathbf{0}_{n'} & \text{if } \mathsf{tr}\,(\phi) = 0 \\ \frac{\phi}{\mathsf{tr}(\phi)} & \text{otherwise} \end{cases} \qquad (13)$$

Since our type system is affine, if $x \in \mathsf{FV}(t)$ then either $x \in \mathsf{FV}(r)$ or $x \in \mathsf{FV}(t_i)$ for some i.

- Case $x \in \mathsf{FV}(r)$:
  For the next part we need the following equalities (14-17).
  By the induction hypothesis on r we have:

$$(\!|r|\!)_{\theta,x=\alpha A+\beta B} = \alpha(\!|r|\!)_{\theta,x=A} + \beta(\!|r|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|r|\!)_{\theta,x=\mathbf{0}_n}$$

Thus, for all i we have:

$$\rho^i_{\alpha A+\beta B} = \alpha\rho^i_A + \beta\rho^i_B - (\alpha+\beta-1)\rho^i_0 \tag{14}$$

By applying the trace to both sides of the equation:

$$\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) = \alpha\mathsf{tr}\left(\rho^i_A\right) + \beta\mathsf{tr}\left(\rho^i_B\right) - (\alpha+\beta-1)\mathsf{tr}\left(\rho^i_0\right) \tag{15}$$

In general for all ρ, by the induction hypothesis on t_i (with $\alpha = \frac{1}{\mathsf{tr}(\rho)}$, $A = \rho$, $\beta = 0$, and n' the correct dimension) we have that:

$$(\!|t_i|\!)_{\theta,y=\frac{\rho}{\mathsf{tr}(\rho)}} = \frac{1}{\mathsf{tr}\left(\rho\right)}(\!|t_i|\!)_{\theta,y=\rho} - (\frac{1}{\mathsf{tr}\left(\rho\right)}+0-1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}$$

Therefore,

$$(\!|t_i|\!)_{\theta,y=\frac{\rho}{\mathsf{tr}(\rho)}} = \frac{1}{\mathsf{tr}\left(\rho\right)}(\!|t_i|\!)_{\theta,y=\rho} + (1-\frac{1}{\mathsf{tr}\left(\rho\right)})(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}$$

Multiplying both sides by $\mathsf{tr}\left(\rho\right)$ we have:

$$\mathsf{tr}\left(\rho\right)(\!|t_i|\!)_{\theta,y=\frac{\rho}{\mathsf{tr}(\rho)}} = (\!|t_i|\!)_{\theta,y=\rho} + (\mathsf{tr}\left(\rho\right)-1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}} \tag{16}$$

From Equation (14) we have that:

$$(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}} = (\!|t_i|\!)_{\theta,y=\alpha\rho^i_A+\beta\rho^i_B-(\alpha+\beta-1)\rho^i_0}$$

By the induction hypothesis on the term in the right-hand-side, with $\alpha' = \alpha$, $A' = \rho^i_A$, $\beta' = 1$, $B' = \beta\rho^i_B - (\alpha+\beta-1)\rho^i_0$, we have that:

$$(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}} = \alpha(\!|t_i|\!)_{\theta,y=\rho^i_A} + (\!|t_i|\!)_{\theta,y=\beta\rho^i_B-(\alpha+\beta-1)\rho^i_0} - \alpha(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}$$

By the induction hypothesis on the second summand on the right-hand side with $\alpha' = \beta$, $A' = \rho^i_B$, $\beta' = -(\alpha+\beta-1)$ y $B' = \rho^i_0$ we have:

$$(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}} = \alpha(\!|t_i|\!)_{\theta,y=\rho^i_A} + \beta(\!|t_i|\!)_{\theta,y=\rho^i_B} - (\alpha+\beta-1)(\!|t_i|\!)_{\theta,y=\rho^i_0} \tag{17}$$

By definition, and since $x \in \mathsf{FV}(r)$ but $x \notin \mathsf{FV}(t_i)$ for all i, to show that Equation (12) holds we want to show that:

$$\sum_{i=1}^{n}\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)(\!|t_i|\!)_{\theta,y=\widetilde{\rho^i_{\alpha A+\beta B}}} =\alpha\sum_{i=1}^{n}\mathsf{tr}\left(\rho^i_A\right)(\!|t_i|\!)_{\theta,y=\widetilde{\rho^i_A}}$$

$$+ \beta \sum_{i=1}^{n} \text{tr}\left(\rho_B^i\right) (\!|t_i|\!)_{\theta, y = \widetilde{\rho_B^i}}$$

$$- (\alpha + \beta - 1) \sum_{i=1}^{n} \text{tr}\left(\rho_0^i\right) (\!|t_i|\!)_{\theta, y = \widetilde{\rho_0^i}}$$

We are going to show equality of the summation term by term, meaning that for all i we have:

$$\text{tr}\left(\rho_{\alpha A + \beta B}^i\right) (\!|t_i|\!)_{\theta, y = \widetilde{\rho_{\alpha A + \beta B}^i}} = \alpha \text{tr}\left(\rho_A^i\right) (\!|t_i|\!)_{\theta, y = \widetilde{\rho_A^i}} \tag{18}$$

$$+ \beta \text{tr}\left(\rho_B^i\right) (\!|t_i|\!)_{\theta, y = \widetilde{\rho_B^i}}$$

$$- (\alpha + \beta - 1) \text{tr}\left(\rho_0^i\right) (\!|t_i|\!)_{\theta, y = \widetilde{\rho_0^i}}$$

We show that Equation (18) holds by case analysis on whether the traces $\text{tr}\left(\rho_{\alpha A + \beta B}^i\right)$, $\text{tr}\left(\rho_A^i\right)$, $\text{tr}\left(\rho_B^i\right)$, and $\text{tr}\left(\rho_0^i\right)$ are equal to or different from 0. We illustrate the most interesting case since the remaining ones follow analogously.

1. Case $\text{tr}\left(\rho_{\alpha A + \beta B}^i\right) \neq 0$, $\text{tr}\left(\rho_A^i\right) \neq 0$, $\text{tr}\left(\rho_B^i\right) \neq 0$, $\text{tr}\left(\rho_0^i\right) \neq 0$. Then, Equation (18) boils down to:

$$\text{tr}\left(\rho_{\alpha A + \beta B}^i\right) (\!|t_i|\!)_{\theta, y = \frac{\rho_{\alpha A + \beta B}^i}{\text{tr}\left(\rho_{\alpha A + \beta B}^i\right)}} = \alpha \text{tr}\left(\rho_A^i\right) (\!|t_i|\!)_{\theta, y = \frac{\rho_A^i}{\text{tr}\left(\rho_A^i\right)}}$$

$$+ \beta \text{tr}\left(\rho_B^i\right) (\!|t_i|\!)_{\theta, y = \frac{\rho_B^i}{\text{tr}\left(\rho_B^i\right)}}$$

$$- (\alpha + \beta - 1) \text{tr}\left(\rho_0^i\right) (\!|t_i|\!)_{\theta, y = \frac{\rho_0^i}{\text{tr}\left(\rho_0^i\right)}}$$

By Equation (16),

$$(\!|t_i|\!)_{\theta, y = \rho_{\alpha A + \beta B}^i} + (\text{tr}\left(\rho_{\alpha A + \beta B}^i\right) - 1)(\!|t_i|\!)_{\theta, y = \mathbf{0}_{n'}}$$
$$= \alpha((\!|t_i|\!)_{\theta, y = \rho_A^i} + (\text{tr}\left(\rho_A^i\right) - 1)(\!|t_i|\!)_{\theta, y = \mathbf{0}_{n'}})$$
$$+ \beta((\!|t_i|\!)_{\theta, y = \rho_B^i} + (\text{tr}\left(\rho_B^i\right) - 1)(\!|t_i|\!)_{\theta, y = \mathbf{0}_{n'}})$$
$$- (\alpha + \beta - 1)((\!|t_i|\!)_{\theta, y = \rho_0^i} + (\text{tr}\left(\rho_0^i\right) - 1)(\!|t_i|\!)_{\theta, y = \mathbf{0}_{n'}})$$

By reordering terms,

$$(\!|t_i|\!)_{\theta, y = \rho_{\alpha A + \beta B}^i}$$
$$= \alpha (\!|t_i|\!)_{\theta, y = \rho_A^i} + \beta (\!|t_i|\!)_{\theta, y = \rho_B^i} - (\alpha + \beta - 1)(\!|t_i|\!)_{\theta, y = \rho_0^i}$$
$$+ (-\text{tr}\left(\rho_{\alpha A + \beta B}^i\right) + \alpha \text{tr}\left(\rho_A^i\right) + \beta \text{tr}\left(\rho_B^i\right)$$
$$- (\alpha + \beta - 1) \text{tr}\left(\rho_0^i\right))(\!|t_i|\!)_{\theta, y = \mathbf{0}_{n'}}$$

By Equation (15), the last term is null. Hence, Equation (17) holds.

$$(\!|t_i|\!)_{\theta, y = \rho_{\alpha A + \beta B}^i} = \alpha (\!|t_i|\!)_{\theta, y = \rho_A^i} + \beta (\!|t_i|\!)_{\theta, y = \rho_B^i} - (\alpha + \beta - 1)(\!|t_i|\!)_{\theta, y = \rho_0^i}$$

- Case $x \notin \mathsf{FV}(r)$. Assume

$$\left(r \right)_\theta = \bigoplus_{i=1}^{n} \rho^i$$

From Equation (12), in this case we want to show that:

$$\sum_{i=1}^{n} \mathsf{tr}\left(\rho^i\right) \left(t_i \right)_{\theta, x=\alpha A+\beta B, y=\widetilde{\rho^i}} = \alpha \sum_{i=1}^{n} \mathsf{tr}\left(\rho^i\right) \left(t_i \right)_{\theta, x=A, y=\widetilde{\rho^i}}$$

$$+ \beta \sum_{i=1}^{n} \mathsf{tr}\left(\rho^i\right) \left(t_i \right)_{\theta, x=B, y=\widetilde{\rho^i}}$$

$$- (\alpha + \beta - 1) \sum_{i=1}^{n} \mathsf{tr}\left(\rho^i\right) \left(t_i \right)_{\theta, x=\mathbf{0}_n, y=\widetilde{\rho^i}}$$

Let $\widetilde{}$ as defined in Equation (13), we will show that for all i the following holds:

$$\mathsf{tr}\left(\rho^i\right) \left(t_i \right)_{\theta, x=\alpha A+\beta B, y=\widetilde{\rho^i}} = \alpha \mathsf{tr}\left(\rho^i\right) \left(t_i \right)_{\theta, x=A, y=\widetilde{\rho^i}} + \beta \mathsf{tr}\left(\rho^i\right) \left(t_i \right)_{\theta, x=B, y=\widetilde{\rho^i}}$$

$$- (\alpha + \beta - 1)\mathsf{tr}\left(\rho^i\right) \left(t_i \right)_{\theta, x=\mathbf{0}_n, y=\widetilde{\rho^i}}$$

If $\mathsf{tr}\left(\rho^i\right) = 0$, it holds trivially. Otherwise it can be proven that for all i:

$$\left(t_i \right)_{\theta, x=\alpha A+\beta B, y=\frac{\rho^i}{\mathsf{tr}(\rho^i)}} = \alpha \left(t_i \right)_{\theta, x=A, y=\frac{\rho^i}{\mathsf{tr}(\rho^i)}} + \beta \left(t_i \right)_{\theta, x=B, y=\frac{\rho^i}{\mathsf{tr}(\rho^i)}}$$

$$- (\alpha + \beta - 1) \left(t_i \right)_{\theta, x=\mathbf{0}_n, y=\frac{\rho^i}{\mathsf{tr}(\rho^i)}}$$

Let $\theta' = \theta \cup \{y = \frac{\rho^i}{\mathsf{tr}(\rho^i)}\}$, then the previous equation is the same as:

$$\left(t_i \right)_{\theta', x=\alpha A+\beta B} = \alpha \left(t_i \right)_{\theta', x=A} + \beta \left(t_i \right)_{\theta', x=B} - (\alpha + \beta - 1) \left(t_i \right)_{\theta', x=\mathbf{0}_n}$$

This holds by the induction hypothesis on t_i. $\square$

A.3   Proof of Lemma 4.3

Lemma 4.3 (Application). *If $\Gamma, x : A \vdash t : B$ and $\theta \vDash \Gamma$, then for all $a \in \mathbb{C}^{\dim(A) \times \dim(A)}$ we have $\left(\lambda x.t \right)_\theta \# a = \left(t \right)_{\theta, x=a}$.*

Proof. By definition we have:

$$\left(\lambda x.t \right)_\theta = \overline{\mathcal{X}}_{[a \mapsto \left(t \right)_{\theta, x=a}]}$$

$$= \begin{pmatrix} \left(t \right)_{\theta, x=E_{11}^A} - \left(t \right)_{\theta, x=\mathbf{0}_{\dim(A)}} & \cdots & \left(t \right)_{\theta, x=E_{1n}^A} - \left(t \right)_{\theta, x=\mathbf{0}_{\dim(A)}} \\ \vdots & \ddots & \vdots \\ \left(t \right)_{\theta, x=E_{n1}^A} - \left(t \right)_{\theta, x=\mathbf{0}_{\dim(A)}} & \cdots & \left(t \right)_{\theta, x=E_{nn}^A} - \left(t \right)_{\theta, x=\mathbf{0}_{\dim(A)}} \end{pmatrix} \oplus \left(t \right)_{\theta, x=\mathbf{0}_{\dim(A)}}$$

Let $\{E_{ij}^A\}$ be the canonical basis for $\mathbb{C}^{\dim(A)\times\dim(A)}$, decomposing a on this basis and applying $(\!|\lambda x.t|\!)_\theta$ through $\#$:

$$(\!|\lambda x.t|\!)_\theta \# a = (\!|\lambda x.t|\!)_\theta \# \left(\sum_{i=1}^n \sum_{j=1}^n a_{ij} E_{ij}^A\right)$$

$$= \sum_{i=1}^n \sum_{j=1}^n a_{ij}\left((\!|t|\!)_{\theta,x=E_{ij}^A} - (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}}\right) + (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}}$$

By Lemma 4.2, $(\!|t|\!)_{\theta,x=E_{ij}^A} - (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}}$ is linear on E_{ij}^A, then we have:

$$(\!|\lambda x.t|\!)_\theta \# a = (\!|t|\!)_{\theta,x=\sum_{ij} a_{ij} E_{ij}^A} - (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}} + (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}} = (\!|t|\!)_{\theta,x=a} \quad \square$$

A.4   Proof of Lemma 4.4

We first need an auxiliary lemma. Let θ be a valuation and Γ a typing context. $\theta \vDash_{\dim} \Gamma$ if and only if for every pair $(x, A) \in \Gamma$, we have $\dim(\theta(x)) = \dim(A)$.

Lemma A.1. *Let $\Gamma \vdash t : A$ and $\theta \vDash_{\dim} \Gamma$, then $\dim((\!|t|\!)_\theta) = \dim(A)$.*

Proof. By induction on the structure of t. We illustrate a few interesting cases

- Let $t = x$. Hence, $x : A \in \Gamma$. Since $\theta \vDash_{\dim} \Gamma$, we have that $\dim((\!|x|\!)_\theta) = \dim(\theta(x)) = \dim(A)$.
- Let $t = \lambda x.u$. In this case $A = B \multimap C$. By definition of $(\!|\_|\!)_{\_}$, we have $(\!|\lambda x.u|\!)_\theta = \overline{\chi}_{[a \mapsto (\!|u|\!)_{\theta,x=a}]}$. Let $\{E_{ij}^B\}$ be the canonical basis of $\mathbb{C}^{\dim(B)\times\dim(B)}$. By inversion, we have $\Gamma, x : B \vdash u : C$. Moreover, $\theta \vDash_{\dim} \Gamma$, $\dim(E_{ij}^B) = \dim(B)$ and $\dim(\mathbf{0}_{\dim(B)}) = \dim(B)$. Hence,

$$\theta \cup \{x = E_{ij}^B\} \vDash_{\dim} \Gamma, x : B$$

$$\theta \cup \{x = \mathbf{0}_{\dim(B)}\} \vDash_{\dim} \Gamma, x : B$$

  Then by the induction hypothesis on $\Gamma, x : B \vdash u : C$:

$$\dim((\!|u|\!)_{\theta,x=E_{ij}^B}) = \dim(C)$$

$$\dim((\!|u|\!)_{\theta,x=\mathbf{0}_{\dim(B)}}) = \dim(C)$$

  By the definition of $\overline{\chi}_{[a \mapsto (\!|u|\!)_{\theta,x=a}]}$, we have that

$$\dim((\!|\lambda x.u|\!)_\theta) = \dim(B)\dim(C) + \dim(C) = \dim(B \multimap C).$$

- Let $t = \mathsf{letcase}^\circ\ x = r$ in $\{t_0, \ldots, t_{2^m-1}\}$. Let $\Gamma_0, \ldots, \Gamma_{2^m-1}, \Gamma' = \Gamma$ such that $\Gamma_i, x : n \vdash t_i : A$ for all i and $\Gamma' \vdash r : (m, n)$. Therefore $\theta \vDash_{\dim} \Gamma'$. By applying the induction hypothesis, $\dim((\!|r|\!)_\theta) = \dim((m, n)) = 2^{n+m}$. For $0 \le i \le 2^m - 1$, let ρ_i be the i-th $2^n \times 2^n$ sub-matrix of $(\!|r|\!)_\theta$'s non-superposing block diagonal. Define ρ_i' as ρ_i if $\mathsf{tr}\,(\rho_i) = 0$, and $\frac{\rho_i}{\mathsf{tr}(\rho_i)}$ otherwise. Clearly, $\dim(\rho_i') = 2^n$ for all i. Hence, $\theta \cup \{x = \rho_i'\} \vDash_{\dim} \Gamma_i, x : n$ for all i. By applying the induction hypothesis, $\dim(t_i) = \dim(A)$. Therefore, $\dim((\!|\mathsf{letcase}^\circ\ x = r$ in $\{t_0, \ldots, t_{2^m-1}\}|\!)_\theta) = \dim(\sum_{i=0}^{2^m-1} \mathsf{tr}\,(\rho_i)\,(\!|t_i|\!)_{\theta,x=\rho_i'}) = \dim(A).$ $\quad\square$

Lemma 4.4 (Reduction correctness). *If $\Gamma \vdash t : A$, $\theta \vDash \Gamma$, and $t \rightsquigarrow r$ then $(\!|t|\!)_\theta = (\!|r|\!)_\theta$.*

Proof. By induction on the derivation of $\rightsquigarrow$. Most of the cases follows by routine induction. We report some representative cases below.

- $(\lambda x.t)r \rightsquigarrow t[x := r]$
  In this case we want to show that $(\!|(\lambda x.t)r|\!)_\theta = (\!|t[x := r]|\!)_\theta$. By definition we have that $(\!|(\lambda x.t)r|\!)_\theta = (\!|\lambda x.t|\!)_\theta \mathbin{\#} (\!|r|\!)_\theta$. By Lemma 4.3 this is equal to $(\!|t|\!)_{\theta, x=(\!|r|\!)_\theta}$. By Lemma 4.1 it holds that $(\!|t|\!)_{\theta, x=(\!|r|\!)_\theta} = (\!|t[x := r]|\!)_\theta$.

- $\mu_0 x.t \rightsquigarrow \bot$
  In this case we want to show that $(\!|\mu_0 x.t|\!)_\theta = (\!|\bot|\!)_\theta$. By definition, we have $(\!|\mu_0 x.t|\!)_\theta = (\!|\lambda x.t|\!)_\theta \mathbin{\#_0} \mathbf{0}_{\dim(A)}$. This is equal to $\mathbf{0}_{\dim(A)} = (\!|\bot|\!)_\theta$.

- $\mu_{n+1} x.t \rightsquigarrow t[x := \mu_n x.t]$
  In this case we want to show that $(\!|\mu_{n+1} x.t|\!)_\theta = (\!|t[x := \mu_n x.t]|\!)_\theta$. By definition we have $(\!|\mu_{n+1} x.t|\!)_\theta = (\!|\lambda x.t|\!)_\theta \mathbin{\#_{n+1}} \mathbf{0}_{\dim(A)}$. This is the same as:

$$(\!|\lambda x.t|\!)_\theta \mathbin{\#} ((\!|\lambda x.t|\!)_\theta \mathbin{\#_n} \mathbf{0}_{\dim(A)})$$

By the induction hypothesis, this is equal to $(\!|\lambda x.t|\!)_\theta \mathbin{\#} (\!|\mu_n x.t|\!)_\theta$. Since $\Gamma \vdash \mu_{n+1} x.t : A$, by inversion $\Gamma, x : A \vdash t : A$, and using rule μ we have $\Gamma \vdash \mu_n x.t : A$. Since $\theta \vDash \Gamma$ we also have $\theta \vDash_{\dim} \Gamma$ and using Lemma A.1, $\dim((\!|\mu_n x.t|\!)_\theta) = \dim(A)$. Using Lemma 4.3 we have $(\!|\lambda x.t|\!)_\theta \mathbin{\#} (\!|\mu_n x.t|\!)_\theta = (\!|t|\!)_{\theta, x=(\!|\mu_n x.t|\!)_\theta}$. Using Lemma 4.1, this is equal to $(\!|t[x := \mu_n x.t]|\!)_\theta$.

- $\mathsf{letcase}^\circ \ x = \pi^m \rho^n \text{ in } \{t_0, \dots, t_{2^m-1}\} \rightsquigarrow \{(p_i, t_i[x := \rho_i^n])\}_{i/p_i \neq 0}$ where:

$$p_i = \mathsf{tr}\left(\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger\right) \qquad\qquad \rho_i = \frac{\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger}{p_i}$$

In this case we want to show that:

$$(\!|\mathsf{letcase}^\circ \ x = \pi^m \rho^n \text{ in } \{t_0, \dots, t_{2^m-1}\}|\!)_\theta = (\!|\{(p_i, t_i[x := \rho_i^n])\}_{i/p_i \neq 0}|\!)_\theta$$

Since $(\!|\pi^m \rho^n|\!)_\theta = \bigoplus_{i=0}^{2^m-1} \overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger$, we have

$$(\!|\mathsf{letcase}^\circ \ x = \pi^m \rho^n \text{ in } \{t_0, \dots, t_{2^m-1}\}|\!)_\theta = \sum_{i=0}^{2^m-1} p_i (\!|t_i|\!)_{\theta, x=\rho_i}.$$

Moreover, $(\!|\{(p_i, t_i[x := \rho_i^n])\}_{i/p_i \neq 0}|\!)_\theta = \sum_i p_i (\!|t_i[x := \rho_i^n]|\!)_\theta$ by the definition of $(\!|\_|\!)\_$. By Lemma 4.1, $\sum_i p_i (\!|t_i[x := \rho_i^n]|\!)_\theta = \sum_i p_i (\!|t_i|\!)_{\theta, x=(\!|\rho_i^n|\!)_\theta}$. Since $(\!|\rho_i^n|\!)_\theta = \rho_i$ for every valuation θ, we have to $\sum_i p_i (\!|t_i|\!)_{\theta, x=(\!|\rho_i^n|\!)_\theta} = \sum_i p_i (\!|t_i|\!)_{\theta, x=\rho_i}$.

$\square$

A.5   Proof of Lemma 4.5

Lemma 4.5. *Let $\Gamma, x : A \vdash t : B$, and for all $\theta \vDash \Gamma$ and $a \in (\!|A|\!)$, let $(\!|t|\!)_{\theta, x=a} \in (\!|B|\!)$. Then, the map $F^{t,x}_{\theta \vDash \Gamma} = a \mapsto (\!|t|\!)_{\theta, x=a} - (\!|t|\!)_{\theta, x=\mathbf{0}_{\dim(A)}}$ is a CPM.*

Proof. We proceed by induction on the structure of t.

- Let $t = y \neq x$, then $\theta = \theta' \cup \{y = c\}$ and $F^{y,x}_{\theta \vDash \Gamma, y:C} = a \mapsto (c - c) = \mathbf{0}_n$, which is completely positive.
- Let $t = x$, then $F^{x,x}_{\theta \vDash \Gamma} = a \mapsto (a - \mathbf{0}_n) = I_n$, which is completely positive.
- Let $t = \lambda y.r$. Then, $B = C \multimap D$, and, by inversion , $\Gamma, x : A, y : C \vdash r : D$. Hence, by the induction hypothesis, we have that $F^{r,y}_{\tau \vDash \Gamma, x:A}$ and $F^{r,x}_{\delta \vDash \Gamma, y:C}$ are CPMs.

  We need to prove that $F^{\lambda y.r,x}_{\theta \vDash \Gamma}$ is a CPM. This map is defined by

  $$F^{\lambda y.r,x}_{\theta \vDash \Gamma}(a)$$

  $$= (\!|\lambda y.r|\!)_{\theta, x=a} - (\!|\lambda y.r|\!)_{\theta, x=\mathbf{0}_n}$$

  $$= \overline{X}_{[c \mapsto (\!|r|\!)_{\theta, x=a, y=c}]} - \overline{X}_{[c \mapsto (\!|r|\!)_{\theta, x=\mathbf{0}_n, y=c}]}$$

  $$= X_{[c \mapsto (\!|r|\!)_{\theta, x=a, y=c} - (\!|r|\!)_{\theta, x=a, y=\mathbf{0}_m}]} \oplus (\!|r|\!)_{\theta, x=a, y=\mathbf{0}_m} \quad (\text{with } m = \dim(C))$$

  $$- X_{[c \mapsto (\!|r|\!)_{\theta, x=\mathbf{0}_n, y=c} - (\!|r|\!)_{\theta, x=\mathbf{0}_n, y=\mathbf{0}_m}]} \oplus (\!|r|\!)_{\theta, x=\mathbf{0}_n, y=\mathbf{0}_m}$$

  $$= X_{[c \mapsto ((\!|r|\!)_{\theta, x=a, y=c} - (\!|r|\!)_{\theta, x=a, y=\mathbf{0}_m}) - ((\!|r|\!)_{\theta, x=\mathbf{0}_n, y=c} - (\!|r|\!)_{\theta, x=\mathbf{0}_n, y=\mathbf{0}_m})]}$$

  $$\oplus ((\!|r|\!)_{\theta, x=a, y=\mathbf{0}_m} - (\!|r|\!)_{\theta, x=\mathbf{0}_n, y=\mathbf{0}_m})$$

  $$= X_{[F^{r,y}_{\theta \cup \{x=a\} \vDash \Gamma, x:A} - F^{r,y}_{\theta \cup \{x=\mathbf{0}_n\} \vDash \Gamma, x:A}]} \oplus F^{r,x}_{\theta \cup \{y=\mathbf{0}_m\} \vDash \Gamma, y:C}(a)$$

  Let

  $$G(b) = F^{r,y}_{\theta \cup \{x=b\} \vDash \Gamma, x:A} = c \mapsto (\!|r|\!)_{\theta, x=b, y=c} - (\!|r|\!)_{\theta, x=b, y=\mathbf{0}_n}$$

  $$H(a) = G(a) - G(\mathbf{0}_n)$$

  Since G is a CPM, $H(a)$ is also a CPM. Then,

  $$X_{[F^{r,y}_{\theta \cup \{x=a\} \vDash \Gamma, x:A} - F^{r,y}_{\theta \cup \{x=\mathbf{0}_n\} \vDash \Gamma, x:A}]} = X_{[H(a)]}$$

  is a CPM. Finally, the sum of CPMs is a CPM.
- Let $t = rs$. Then, by inversion, there are two cases:
  - $\Gamma, x : A \vdash r : C \multimap D$ and $\Delta \vdash s : C$. Hence, by the induction hypothesis, we have that $F^{r,x}_{\gamma \vDash \Gamma}$ is a CPM.

    We need to prove that $F^{rs,x}_{\theta \vDash \Gamma, \Delta}$, where $\theta = \gamma, \delta$ with $\gamma \vDash \Gamma$ and $\delta \vDash \Delta$, is a CPM.

    $$F^{rs,x}_{\theta \vDash \Gamma, \Delta}(a) = (\!|rs|\!)_{\theta, x=a} - (\!|rs|\!)_{\theta, x=\mathbf{0}_n}$$

$$= (\!|r|\!)_{\theta,x=a} \; \# \; (\!|s|\!)_{\theta,x=a} - (\!|r|\!)_{\theta,x=\mathbf{0}_n} \; \# \; (\!|s|\!)_{\theta,x=\mathbf{0}_n}$$
$$= (\!|r|\!)_{\gamma,x=a} \; \# \; (\!|s|\!)_\delta - (\!|r|\!)_{\gamma,x=\mathbf{0}_n} \; \# \; (\!|s|\!)_\delta$$
$$\text{(Lemma 3.7)} = ((\!|r|\!)_{\gamma,x=a} - (\!|r|\!)_{\gamma,x=\mathbf{0}_n}) \; \# \; (\!|s|\!)_\delta$$
$$= F^{r,x}_{\gamma \vDash \Gamma}(a) \; \# \; (\!|s|\!)_\delta$$

Notice that since $F^{r,x}_{\gamma \vDash \Gamma}(a)$ is a Choi matrix, then the application $\#$ is just the standard application, and since $(\!|s|\!)_\delta \in (\!|B|\!)$, it is positive, so $F^{r,x}_{\gamma \vDash \Gamma}(a) \; \# \; (\!|s|\!)_\delta$ is a CPM.

- $\Gamma \vdash r : C \multimap D$ and $\Delta, x : A \vdash s : C$ Hence, by the induction hypothesis, we have that $F^{s,x}_{\delta \vDash \Delta}$ is a CPM.
  We need to prove that $F^{rs,x}_{\theta \vDash \Gamma, \Delta}$, where $\theta = \gamma, \delta$ with $\gamma \vDash \Gamma$ and $\delta \vDash \Delta$, is a CPM.

$$F^{rs,x}_{\theta \vDash \Gamma, \Delta}(a) = (\!|rs|\!)_{\theta,x=a} - (\!|rs|\!)_{\theta,x=\mathbf{0}_n}$$
$$= (\!|r|\!)_{\theta,x=a} \; \# \; (\!|s|\!)_{\theta,x=a} - (\!|r|\!)_{\theta,x=\mathbf{0}_n} \; \# \; (\!|s|\!)_{\theta,x=\mathbf{0}_n}$$
$$= (\!|r|\!)_\gamma \; \# \; (\!|s|\!)_{\delta,x=a} - (\!|r|\!)_\gamma \; \# \; (\!|s|\!)_{\delta,x=\mathbf{0}_n}$$
$$\text{(Lemma 3.6)} = (\!|r|\!)_\gamma \; \# \; ((\!|s|\!)_{\delta,x=a} - (\!|s|\!)_{\delta,x=\mathbf{0}_n}) - P_\perp((\!|r|\!)_\gamma)$$
$$= M \; \# \; F^{s,x}_{\delta \vDash \Delta}(a) - M_2$$
$$\text{(with } (\!|r|\!)_\gamma = M = M_1 \oplus M_2)$$

As $M \in ((\!|A|\!) \otimes (\!|B|\!)) \oplus (\!|B|\!)$ is a positive matrix by the induction hypothesis, we have that:
1. $M_1 \in (\!|A|\!) \otimes (\!|B|\!)$ is a positive matrix $\iff$ it is the characteristic matrix for some CPM $g : (\!|A|\!) \to (\!|B|\!)$.
2. $M_2 \in (\!|B|\!)$ is a positive matrix.
Then $M \; \# \; F^{s,x}_{\delta \vDash \Delta}(a) - M_2 = g(F^{s,x}_{\delta \vDash \Delta}(a)) + M_2 - M_2 = g(F^{s,x}_{\delta \vDash \Delta}(a))$ is a CPM by composition.

- Let $t = \mu_n y.r$. Then, by inversion, $\Gamma, y : B, x : A \vdash r : B$. Hence, by the induction hypothesis, we have that $F^{r,y}_{\theta,x=a \vDash \Gamma, x:A}$ and $F^{r,y}_{\theta,y=b \vDash \Gamma, y:B}$ are CPMs. We need to prove that $F^{\mu_n y.r,x}_{\theta \vDash \Gamma}$ is a CPM.

$$F^{\mu_n y.r,x}_{\theta \vDash \Gamma}(a) = (\!|\mu_n y.r|\!)_{\theta,x=a} - (\!|\mu_n y.r|\!)_{\theta,x=\mathbf{0}_m}$$
$$= ((\!|\lambda y.r|\!)_{\theta,x=a} \;\#_n\; \mathbf{0}_m) - ((\!|\lambda y.r|\!)_{\theta,x=\mathbf{0}_m} \;\#_n\; \mathbf{0}_m)$$
$$\text{(Lemma 3.7)} = ((\!|\lambda y.r|\!)_{\theta,x=a} - (\!|\lambda y.r|\!)_{\theta,x=\mathbf{0}_m}) \;\#_n\; \mathbf{0}_m$$
$$= F^{\lambda y.r,x}_{\theta \vDash \Gamma}(a) \;\#_n\; \mathbf{0}_m$$

With the same reasoning as in case $t = \lambda y.r$, we have that $F^{\lambda y.r,x}_{\theta \vDash \Gamma}(a)$ is a CPM. Therefore, $F^{\mu_n y.r,x}_{\theta \vDash \Gamma}$ is a CPM.

- Let $t = \perp$. Hence, we have to prove that $F^{\perp,x}_{\theta \vDash \Gamma}$ is a CPM.

$$F^{\perp,x}_{\theta \vDash \Gamma}(a) = (\!|\perp|\!)_{\theta,x=a} - (\!|\perp|\!)_{\theta,x=\mathbf{0}_n} = \mathbf{0}_m - \mathbf{0}_m = \mathbf{0}_m$$

Therefore, $F^{\perp,x}_{\theta \vDash \Gamma}$ is a CPM.

- Let $t = \rho^n$. Hence, we have to prove that $F_{\theta \vDash \Gamma}^{\rho,x}$ is a CPM.

$$F_{\theta \vDash \Gamma}^{\rho,x}(a) = (\!|\rho^n|\!)_{\theta,x=a} - (\!|\rho^n|\!)_{\theta,x=\mathbf{0}_n} = \rho - \rho = \mathbf{0}_{2^n}$$

Therefore, $F_{\theta \vDash \Gamma}^{\rho,x}$ is a CPM.
- Let $t = U^m r$. Then, by inversion, $B = n$ and $\Gamma, x : A \vdash r : n$. Hence, by the induction hypothesis $F_{\theta \vDash \Gamma}^{r,x}$ is a CPM.
  We need to prove that $F_{\theta \vDash \Gamma}^{Ur,x}$ is a CPM.

$$
\begin{aligned}
F_{\theta \vDash \Gamma}^{U^m r,x}(a) &= (\!|U^m r|\!)_{\theta,x=a} - (\!|U^m r|\!)_{\theta,x=\mathbf{0}_m} \\
&= \overline{U}(\!|r|\!)_{\theta,x=a}\overline{U}^\dagger - \overline{U}(\!|r|\!)_{\theta,x=\mathbf{0}_m}\overline{U}^\dagger \\
&= \overline{U}\left((\!|r|\!)_{\theta,x=a} - (\!|r|\!)_{\theta,x=\mathbf{0}_m}\right)\overline{U}^\dagger \\
&= \overline{U}F_{\theta \vDash \Gamma}^{r,x}(a)\overline{U}^\dagger
\end{aligned}
$$

which is a CPM since $F_{\theta \vDash \Gamma}^{r,x}(a)$ is a CPM.
- Let $t = \pi^m r$. Then, by inversion, $B = n$ and $\Gamma, x : A \vdash r : n$. Hence, by the induction hypothesis $F_{\theta \vDash \Gamma}^{r,x}$ is a CPM.

$$
\begin{aligned}
F_{\theta \vDash \Gamma}^{\pi^m r,x}(a) &= (\!|\pi^m r|\!)_{\theta,x=a} - (\!|\pi^m r|\!)_{\theta,x=\mathbf{0}_k} \\
&= \bigoplus_{i=0}^{2^m-1}\left(\overline{|i\rangle\langle i|}(\!|r|\!)_{\theta,x=a}\overline{|i\rangle\langle i|}^\dagger\right) - \bigoplus_{i=0}^{2^m-1}\left(\overline{|i\rangle\langle i|}(\!|r|\!)_{\theta,x=\mathbf{0}_k}\overline{|i\rangle\langle i|}^\dagger\right) \\
&= \bigoplus_{i=0}^{2^m-1}\left(\overline{|i\rangle\langle i|}(\!|r|\!)_{\theta,x=a}\overline{|i\rangle\langle i|}^\dagger - \overline{|i\rangle\langle i|}(\!|r|\!)_{\theta,x=\mathbf{0}_k}\overline{|i\rangle\langle i|}^\dagger\right) \\
&= \bigoplus_{i=0}^{2^m-1}\left(\overline{|i\rangle\langle i|}\left((\!|r|\!)_{\theta,x=a} - (\!|r|\!)_{\theta,x=\mathbf{0}_k}\right)\overline{|i\rangle\langle i|}^\dagger\right) \\
&= \bigoplus_{i=0}^{2^m-1}\left(\overline{|i\rangle\langle i|}F_{\theta \vDash \Gamma}^{r,x}(a)\overline{|i\rangle\langle i|}^\dagger\right)
\end{aligned}
$$

which is a CPM since $F_{\theta \vDash \Gamma}^{r,x}(a)$ is a CPM.
- Let $t = r \otimes s$. Then, by inversion $B = n + m$ and there are two cases:
  - $\Gamma_1, x : A \vdash r : n$ and $\Gamma_2 \vdash s : m$, with $\Gamma = \Gamma_1, \Gamma_2$. Hence, by the induction hypothesis, $F_{\theta_1 \vDash \Gamma_1}^{r,x}$ is a CPM.
    We need to prove that $F_{\theta \vDash \Gamma}^{r \otimes s,x}$, with $\theta = \theta_1, \theta_2$, is a CPM.

$$
\begin{aligned}
F_{\theta \vDash \Gamma}^{r \otimes s,x}(a) &= (\!|r \otimes s|\!)_{\theta,x=a} - (\!|r \otimes s|\!)_{\theta,x=\mathbf{0}_k} \\
&= (\!|r|\!)_{\theta,x=a} \otimes (\!|s|\!)_{\theta,x=a} - (\!|r|\!)_{\theta,x=\mathbf{0}_k} \otimes (\!|s|\!)_{\theta,x=\mathbf{0}_k} \\
&= \left((\!|r|\!)_{\theta_1,x=a} - (\!|r|\!)_{\theta_1,x=\mathbf{0}_k}\right) \otimes (\!|s|\!)_{\theta_2} \\
&= F_{\theta_1 \vDash \Gamma_1}^{r,x}(a) \otimes (\!|s|\!)_{\theta_2}
\end{aligned}
$$

A CPM tensor a positive maps is a CPM.

- $\Gamma_1 \vdash r : n$ and $\Gamma_2, x : A \vdash s : m$. This case is analogous to the previous case.

− Let $t = \mathsf{letcase}^\circ\ y = r$ in $\{t_0, \ldots t_{2^m-1}\}$. By inversion, for all i, $\Delta_i, y : n \vdash t_i : B$ and $\Xi \vdash r : (m, n)$, with $\Gamma, x : A = \Delta_0, \ldots, \Delta_{2^m-1}, \Xi$.
  Cases:
  - $\Xi = \Xi', x = a$. Hence, by the induction hypothesis, $F^{r,x}_{\xi \models \Xi} = (\!|r|\!)_{\xi, x=a} - (\!|r|\!)_{\xi, x=\mathbf{0}_n}$ is a CPM.
    We need to prove that if $\theta = \delta_1, \ldots, \delta_{2^m-1}, \xi$, $F^{\mathsf{letcase}^\circ\ y=r\ \text{in}\ \{t_0,\ldots,t_{2^m-1}\},x}_{\theta \models \Gamma}$ is a CPM.
    Let $(\!|r|\!)_{\theta, x=c} = \bigoplus_{i=0}^{2^m-1} \rho_{i(c)}$. Then,

    $$F^{\mathsf{letcase}^\circ\ y=r\ \text{in}\ \{t_0,\ldots,t_{2^m-1}\},x}_{\theta \models \Gamma}(a)$$
    $$= (\!|\mathsf{letcase}^\circ\ y = r\ \text{in}\ \{t_0, \ldots, t_{2^m-1}\}|\!)_{\theta, x=a}$$
    $$\quad - (\!|\mathsf{letcase}^\circ\ y = r\ \text{in}\ \{t_0, \ldots, t_{2^m-1}\}|\!)_{\theta, x=\mathbf{0}_n}$$
    $$= \left(\sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho_{i(a)} \right) (\!|t_i|\!)_{\theta, x=a, y=\rho_{i(a)}} \right)$$
    $$\quad - \left(\sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho_{i(\mathbf{0}_n)} \right) (\!|t_i|\!)_{\theta, x=\mathbf{0}_n, y=\rho_{i(\mathbf{0}_n)}} \right)$$
    $$= \sum_{i=0}^{2^m-1} \left(\mathsf{tr}\left(\rho_{i(a)} \right) (\!|t_i|\!)_{\delta_i, y=\rho_{i(a)}} - \mathsf{tr}\left(\rho_{i(\mathbf{0}_n)} \right) (\!|t_i|\!)_{\delta_i, y=\rho_{i(\mathbf{0}_n)}} \right)$$
    $$\sqsupseteq \sum_{i=0}^{2^m-1} \left(\mathsf{tr}\left(\rho_{i(a)} \right) \left((\!|t_i|\!)_{\delta_i, y=\rho_{i(a)}} - (\!|t_i|\!)_{\delta_i, y=\rho_{i(\mathbf{0}_n)}} \right) \right)$$
    $$\sqsupseteq \sum_{i=0}^{2^m-1} \left(\mathsf{tr}\left(\rho_{i(a)} \right) \left((\!|t_i|\!)_{\delta_i, y=\rho_{i(a)}} - (\!|t_i|\!)_{\delta_i, y=\mathbf{0}_n} \right) \right)$$
    $$= \sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho_{i(a)} \right) F^{t_i, y}_{\delta_i \models \Gamma}(a)$$

    Positive linear combination of CPMs is a CPM.
  - For some k, $\Delta_k = \Delta'_k, x = a$ Hence, by the induction hypothesis, $F^{t_k, x}_{\delta_k \models \Delta_k}$ is a CPM.
    We need to prove that if $\theta = \delta_1, \ldots, \delta_{2^m-1}, \xi$, $F^{\mathsf{letcase}^\circ\ y=r\ \text{in}\ \{t_0,\ldots,t_{2^m-1}\},x}_{\theta \models \Gamma}$ is a CPM.
    Let $(\!|r|\!)_\xi = \bigoplus_{i=0}^{2^m-1} \rho_i$. Then,

    $$F^{\mathsf{letcase}^\circ\ y=r\ \text{in}\ \{t_0,\ldots,t_{2^m-1}\},x}_{\theta \models \Gamma}(a)$$
    $$= (\!|\mathsf{letcase}^\circ\ y = r\ \text{in}\ \{t_0, \ldots, t_{2^m-1}\}|\!)_{\theta, x=a}$$
    $$\quad - (\!|\mathsf{letcase}^\circ\ y = r\ \text{in}\ \{t_0, \ldots, t_{2^m-1}\}|\!)_{\theta, x=\mathbf{0}_n}$$
    $$= \left(\sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho_i \right) (\!|t_i|\!)_{\theta, x=a, y=\rho_i} \right)$$

$$-\left(\sum_{i=0}^{2^m-1} \mathsf{tr}\,(\rho_i)\, (\!|t_i|\!)_{\theta,x=\mathbf{0}_n,y=\rho_i}\right)$$

$$= (\!|t_k|\!)_{\delta_k,y=\rho_k} - (\!|t_k|\!)_{\delta_k,y=\rho_k}$$

$$= F^{t_k,x}_{\delta_k\vDash\Delta_k}$$

– Let $t = \{(p_i, t_i)\}_i$. Then, by inversion, for all i, $\Gamma, x : A \vdash t_i : B$. Hence, by the induction hypothesis, $F^{t_i,x}_{\theta\vDash\Gamma}$ are CPMs.

  We need to prove that $F^{\{(p_i,t_i)\}_i,x}_{\theta\vDash\Gamma}$ is a CPM.

$$\begin{aligned}
F^{\{(p_i,t_i)\}_i,x}_{\theta\vDash\Gamma}(a) &= (\!|\{(p_i,t_i)\}_i|\!)_{\theta,x=a} - (\!|\{(p_i,t_i)\}_i|\!)_{\theta,x=\mathbf{0}_n} \\
&= \sum_i p_i (\!|t_i|\!)_{\theta,x=a} - \sum_i p_i (\!|t_i|\!)_{\theta,x=\mathbf{0}_n} \\
&= \sum_i p_i \left((\!|t_i|\!)_{\theta,x=a} - (\!|t_i|\!)_{\theta,x=\mathbf{0}_n}\right) \\
&= \sum_i p_i F^{t_i,x}_{\theta\vDash\Gamma}(a)
\end{aligned}$$

which is a CPM since each $F^{t_i,x}_{\theta\vDash\Gamma}(a)$ are CPMs. $\qquad\square$

A.6   Proof of Lemma 4.6

We first restate the following theorem from [10, Theorem 6.5].

Theorem A.2. *Let $F : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ be a linear operator, and let $\chi_F \in \mathbb{C}^{nm\times nm}$ be its characteristic matrix.*

(a) *F if of the form $F(A) = UAU^\dagger$, for some $U \in \mathbb{C}^{m\times n}$, if and only if χ_F is pure.*

(b) *The following are equivalent:*
  (i) *F is completely positive.*
  (ii) *χ_F is positive.*
  (iii) *F if of the form $F(A) = \sum_i U_i A U_i^\dagger$, for some finite sequence of matrices $U_1, \dots, U_k \in \mathbb{C}^{m\times n}$.* $\qquad\square$

Lemma 4.6 (Soundness for abstractions). *Let $\Gamma, x : A \vdash t : B$ and $\theta \vDash \Gamma$, such that $(\!|t|\!)_{\theta,x=a}, (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}} \in (\!|B|\!)$. Then $\overline{\chi}_{[a\mapsto(\!|t|\!)_{\theta,x=a}]} \in (\!|A \multimap B|\!)$.*

Proof. Using Lemma 4.2 we have that $a \mapsto (\!|t|\!)_{\theta,x=a} - (\!|t|\!)_{\theta,x=\perp}$ is a linear function. It also is completely positive by Lemma 4.5, therefore its characteristic matrix is positive by Theorem A.2.

Its characteristic matrix is the following:

$$M_t = \begin{pmatrix}
(\!|t|\!)_{\theta,x=E_{11}} - (\!|t|\!)_{\theta,x=\perp} & \cdots & (\!|t|\!)_{\theta,x=E_{1n}} - (\!|t|\!)_{\theta,x=\perp} \\
\vdots & \ddots & \vdots \\
(\!|t|\!)_{\theta,x=E_{n1}} - (\!|t|\!)_{\theta,x=\perp} & \cdots & (\!|t|\!)_{\theta,x=E_{nn}} - (\!|t|\!)_{\theta,x=\perp}
\end{pmatrix}$$

By Definition (see Section 3.3), we have that

$$\overline{\chi}_{[a \mapsto (\!|t|\!)_{\theta,x=a}]} = M_t \oplus (\!|t|\!)_{\theta,x=\perp}$$

Since $M_t \in (\!|A|\!) \otimes (\!|B|\!)$, we have $\overline{\chi}_{[a \mapsto (\!|t|\!)_{\theta,x=a}]} \in ((\!|A|\!) \otimes (\!|B|\!)) \oplus (\!|B|\!)$. As $(\!|t|\!)_{\theta,x=\perp}$ is in $(\!|B|\!)$, it is a positive matrix. Then $\overline{\chi}_{[a \mapsto (\!|t|\!)_{\theta,x=a}]}$ is a positive matrix because it is a coproduct between positive matrices, and since it is in $((\!|A|\!) \otimes (\!|B|\!)) \oplus (\!|B|\!)$, it belongs to $(\!|A \multimap B|\!)$. □

A.7   Proof of Lemma 4.7

We first define closure functions, substitutions that close the open terms allowing to rewrite them according to the valuation in respect to which we want to interpret it. By closing the arrow-type terms in a manner consistent with the typing context, we can rewrite them and using progress arrive to a value.

Definition A.3 (Closure function). *Let $f : \mathsf{Var} \to \mathsf{Val} \cup \{\perp\}$, we note $f(t)$ as the substitution that replaces free variables in t by values according to the mapping given by f. We call f a closure function.*

Closure functions are used on typed terms, therefore they need to be coherent with the typing contexts, assigning values with the same type as the variable they are substituting.

Definition A.4. *Let f be a closure function, let Γ be a typing context. f satisfies Γ (noted $f \vDash \Gamma$) if and only if for all $x : A$ in Γ we have $\vdash f(x) : A$.*

The following is a coherence definition between a valuation and a closure function.

Definition A.5. *Let f be a closure function and let θ be a valuation, f and θ are coherent (noted $f \leftrightarrow \theta$) if and only if*

$$f(x) = v \iff \theta(x) = (\!|v|\!)_\emptyset$$

Lemma 4.7 (Soundness for arrow-type terms). *Let $\Gamma \vdash t : A \multimap B$ and $\theta \vDash \Gamma$. One of the following holds:*

- *There exist $t_1, \ldots, t_n$ and $p_1, \ldots, p_n$ such that for all i, $x : A \vdash t_i : B$, $p_i > 0$, $\sum_{i=1}^n p_i \leq 1$ and $(\!|t|\!)_\theta = \sum_{i=1}^n p_i (\!|\lambda x.t_i|\!)_\emptyset$.*
- *$(\!|t|\!)_\theta = \mathbf{0}_{\dim(A \multimap B)}$*

Proof. Let f be a closure function such that $f \vDash \Gamma$ and $f \leftrightarrow \theta$. By substitution we have that $\vdash f(t) : A \multimap B$. Using Progress (trivial extension of Theorem 2.5 to $\lambda_\rho^{[\mu]}$) we have that either $f(t)$ is in $\mathsf{Val} \cup \{\perp\}$ or it rewrites, and by Lemma 4.1 we have that $(\!|t|\!)_\theta = (\!|f(t)|\!)_\emptyset$.

– If $f(t)$ is in $\mathsf{Val} \cup \{\bot\}$, by its type we have that either $f(t) = \bot$ or $f(t) = \{(p_i, \lambda x.t_i)\}_{i=\in\{1,\dots,n\}}$ for a variable x and terms t_i such that $x : A \vdash t_i : B$, $0 < p_i \leq 1$ and $\sum_{i=1}^{n} p_i \leq 1$.
In the first case we have:

$$(\! |t| \!)_\theta = (\! |f(t)| \!)_\emptyset = (\! |\bot| \!)_\emptyset = \mathbf{0}_{\dim(A \multimap B)}$$

In the second case we have:

$$(\! |t| \!)_\theta = (\! |f(t)| \!)_\emptyset = (\! |\{(p_i, \lambda x.t_i)\}_{i\in\{1,\dots,n\}}| \!)_\emptyset = \sum_{i}^{n} p_i (\! |\lambda x.t_i| \!)_\emptyset$$

– If $f(t) \rightsquigarrow r$, since $\vdash f(t) : A \multimap B$, by Subject reduction we have that $\vdash r : A \multimap B$. As $f(r) = r$ because r is a closed term, using Progress successively we have that there exist $r_1, \dots, r_{n-1}$ closed terms and r_n in $\mathsf{Val} \cup \{\bot\}$, with $\vdash r_i : A \multimap B$, such that

$$f(t) \rightsquigarrow r \rightsquigarrow r_1 \rightsquigarrow \dots \rightsquigarrow r_n$$

This holds because there are no infinite rewritings, by the strong normalisation property of the calculus. By Lemma 4.4 we have that:

$$(\! |t| \!)_\theta = (\! |f(t)| \!)_\emptyset = (\! |r| \!)_\emptyset = (\! |r_1| \!)_\emptyset = \dots = (\! |r_n| \!)_\emptyset$$

As $\vdash r_n : A \multimap B$ and $r_n \in \mathsf{Val} \cup \{\bot\}$, we have the same results as in the previous item. $\qquad\square$

A.8   Proof of Lemma 4.8

Lemma 4.8. *Let $(\! |t_i| \!)_\theta \in (\! |A| \!)$ for $i \in \{1, \dots, n\}$. Then for any $p_1, \dots, p_n$ such that $0 < p_i \leq 1$ with $\sum_{i=1}^{n} p_i \leq 1$, we have $\sum_{i=1}^{n} p_i (\! |t_i| \!)_\theta \in (\! |A| \!)$.*

Proof. We prove more generally the following result: Let A be a type. For i in $\{1, \dots, n\}$, let $a_i \in (\! |A| \!)$ and $0 < p_i \leq 1$ with $\sum_{i=1}^{n} p_i \leq 1$. Then $\sum_{i=1}^{n} p_i a_i \in (\! |A| \!)$.
   We proceed by induction on types.

– If $A = n$, we have $(\! |A| \!) = \mathcal{D}_n^{\leq}$.
Since $p_i > 0$ for all i and positive matrices form a vector space over $\mathbb{R}_{\geq 0}$, we have that $\sum_{i=1}^{n} p_i a_i$ is a positive matrix in $\mathbb{C}^{n \times n}$.
By trace linearity, we have that $\mathsf{tr}\left(\sum_{i=1}^{n} p_i a_i\right) = \sum_{i=1}^{n} p_i \mathsf{tr}(a_i)$. This is bounded by $\sum_{i=1}^{n} p_i \leq 1$ because by hypothesis $a_i \in (\! |n| \!) = \mathcal{D}_n^{\leq}$.
– If $A = (m, n)$, we have $(\! |A| \!) = \{p \mid p \in \bigoplus_{i=1}^{2^m - 1} \mathcal{D}_n^{\leq} \text{ y } \mathsf{tr}(p) \leq 1\}$.
By hypothesis $a_i \in (\! |(m,n)| \!)$, so we can rewrite the a_i as $a_i = \bigoplus_{j=0}^{2^m - 1} a_{ij}$, where $a_{ij} \in \mathcal{D}_n^{\leq}$. Then,

$$\sum_{i=1}^{n} p_i a_i = \sum_{i=1}^{n} p_i \left(\bigoplus_{j=0}^{2^m - 1} a_{ij} \right) = \bigoplus_{j=0}^{2^m - 1} \left(\sum_{i=1}^{n} p_i a_{ij} \right)$$

By case $A = n$ we have that $\sum_{i=1}^{n} p_i a_{ij} \in \mathcal{D}_n^{\leq}$ for all $j \in \{0, \dots, 2^m - 1\}$, therefore $\sum_{i=1}^{n} p_i a_i \in \bigoplus_{i=0}^{2^m-1} \mathcal{D}_n^{\leq}$.
By trace linearity, we have that $\mathsf{tr}\left(\sum_{i=1}^{n} p_i a_i\right) = \sum_{i=1}^{n} p_i \mathsf{tr}(a_i)$. By hypothesis $a_i \in (\!(m,n)\!)$, then this is bounded by $\sum_{i=1}^{n} p_i \leq 1$.

- If $A = B \multimap C$, we have $(\!|A|\!) = \{f \mid f \text{ positive in } ((\!|B|\!) \otimes (\!|C|\!)) \oplus (\!|C|\!)\}$. Since $\sum_{i=1}^{n} p_i a_i$ is a positive real combination of elements from $(\!|B \multimap C|\!)$, it is in $(\!|B \multimap C|\!)$. $\qquad\square$

A.9   Proof of Theorem 4.9

We first give a lemma stating that function application preserves positivity. This implies that function application interpretation stays inside the domain.

Lemma A.6 (# preserves positivity). *Let t be a term and θ be a valuation such that $(\!|\lambda x.t|\!)_\theta \in (\!|A \multimap B|\!)$, then for all a in $(\!|A|\!)$ we have that $(\!|t|\!)_{\theta, x=a} \in (\!|B|\!)$.*

Proof. By Lemma 4.3 we have $(\!|t|\!)_{\theta, x=a} = (\!|\lambda x.t|\!)_\theta \,\#\, a$. By hypothesis $(\!|\lambda x.t|\!)_\theta \in (\!|A \multimap B|\!)$, so we can write $(\!|\lambda x.t|\!)_\theta = M_1 \oplus M_2$ with $M_1 \in (\!|A|\!) \otimes (\!|B|\!)$ and $M_2 \in (\!|B|\!)$, both positive matrices. By definition of the $\#$ operator, $(M_1 \oplus M_2) \,\#\, a = M_1 @ a + M_2$. Since @ preserves positivity, therefore $M_1 @ a$ is in $(\!|B|\!)$ and so is the sum. $\qquad\square$

We also give two auxiliary lemmas and a corollary, concerning trace bounds after projection of states. There are used in the measurement case of the proof of soundness.

Lemma A.7. *Let ρ be a positive matrix in $\mathbb{C}^{2^n \times 2^n}$. Then for all $m \leq n$*

$$\mathsf{tr}\left(\bigoplus_{i=0}^{2^m-1} \left(\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger\right)\right) = \mathsf{tr}(\rho)$$

Proof. By trace linearity:

$$\mathsf{tr}\left(\bigoplus_{i=0}^{2^m-1} \left(\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger\right)\right) = \sum_{i=0}^{2^m-1} \mathsf{tr}\left(\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger\right)$$

By the trace cyclic property, this is equal to $\sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho\overline{|i\rangle\langle i|}^\dagger\overline{|i\rangle\langle i|}\right)$. Since $\overline{|i\rangle\langle i|}$ is Hermitian and is also a projector, we have that $\overline{|i\rangle\langle i|}^\dagger\overline{|i\rangle\langle i|} = \overline{|i\rangle\langle i|}$, and the term is equal to $\sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho\overline{|i\rangle\langle i|}\right)$. Since $\sum_{i=0}^{2^m-1} \overline{|i\rangle\langle i|} = \mathbf{1}_{2^n}$:

$$\sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho\overline{|i\rangle\langle i|}\right) = \mathsf{tr}\left(\sum_{i=0}^{2^m-1} \rho\overline{|i\rangle\langle i|}\right) = \mathsf{tr}\left(\rho \sum_{i=0}^{2^m-1} \overline{|i\rangle\langle i|}\right) = \mathsf{tr}(\rho) \qquad\square$$

Corollary A.8. *Let ρ be a positive matrix in $\mathbb{C}^{2^n \times 2^n}$. For all $m \leq n$ and $0 \leq i \leq 2^m - 1$ we have that $\mathsf{tr}\left(\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger\right) \leq \mathsf{tr}(\rho)$.*

Proof. By bounding every sum term separately.   □

Lemma A.9. *Let ρ be a positive matrix in $\mathbb{C}^{2^n \times 2^n}$ such that $\mathsf{tr}\,(\rho) \leq 1$. Then for all $m \leq n$, for all $0 \leq i \leq 2^m - 1$ we have that $\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger$ is positive and its trace is bounded by 1.*

Proof.   – $\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger$ is Hermitian: we have $(\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger)^\dagger = \overline{|i\rangle\langle i|}\rho^\dagger\overline{|i\rangle\langle i|}^\dagger$. This
   is equal to $\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger$ because ρ es is Hermitian by hypothesis.
   – $\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger$ is semidefinite positive: multiplying to both sides by a vector u
   in $\mathbb{C}^{2^n}$ we have that $u^\dagger\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger u = (u^\dagger\overline{|i\rangle\langle i|})\rho(\overline{|i\rangle\langle i|}^\dagger u)$. Let $v = \overline{|i\rangle\langle i|}^\dagger u \in$
   $\mathbb{C}^{2^n}$, we have that $v^\dagger = (\overline{|i\rangle\langle i|}^\dagger u)^\dagger = u^\dagger\overline{|i\rangle\langle i|}$. Therefore $(u^\dagger\overline{|i\rangle\langle i|})\rho(\overline{|i\rangle\langle i|}^\dagger u) =$
   $v^\dagger\rho v \geq 0$ because ρ is semidefinite positive.
   – The trace of $\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger$ is bounded by 1: by Corollary A.8 we have that
   $\mathsf{tr}\left(\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger\right) \leq \mathsf{tr}\,(\rho) \leq 1$.   □

Theorem 4.9 (Soundness). *Let $\Gamma \vdash t : A$ and $\theta \vDash \Gamma$, then $(\!|t|\!)_\theta \in (\!|A|\!)$.*

Proof. By induction on the typing rules.

– $\dfrac{}{\Gamma, x : A \vdash x : A}$ ax
   In this case we have $(\!|x|\!)_\theta = \theta(x)$ by definition. By hypothesis we have that
   $\theta \vDash \Gamma, x : A$, therefore $\theta(x) = (\!|x|\!)_\theta \in (\!|A|\!)$.
– $\dfrac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}$ $\multimap_i$
   By the induction hypothesis we have that for all θ' such that $\theta' \vDash \Gamma, x : A$,
   $(\!|t|\!)_{\theta'} \in (\!|B|\!)$. Let $a \in (\!|A|\!)$, then $\theta' = \theta \cup \{x = a\} \vDash \Gamma, x : A$, therefore
   $(\!|t|\!)_{\theta, x=a} \in (\!|B|\!)$. Also, $\mathbf{0}_{\dim(A)} \in (\!|A|\!)$, and then $(\!|t|\!)_{\theta, x=\perp} \in (\!|B|\!)$. By definition
   we have that $(\!|\lambda x.t|\!)_\theta = \overline{\mathcal{X}}_{[a \mapsto (\!|t|\!)_{\theta, x=a}]}$ and by Lemma 4.6 this is in $(\!|A \multimap B|\!)$.
– $\dfrac{\Gamma \vdash t : A \multimap B \quad \Delta \vdash r : A}{\Gamma, \Delta \vdash tr : B}$ $\multimap_e$
   Since $\theta \vDash \Gamma, \Delta$, we have that $\theta \vDash \Gamma$ and $\theta \vDash \Delta$. By the induction hypothesis,
   $(\!|t|\!)_\theta \in (\!|A \multimap B|\!)$ and $(\!|r|\!)_\theta \in (\!|A|\!)$. By Lemma 4.7 we have that either that for
   some $n \in \mathbb{N}$ there exist n terms $t_1, \ldots, t_n$ and n real numbers $p_1, \ldots, p_n$ such
   that $(\!|t|\!)_\theta = \sum_{i=1}^{n} p_i(\!|\lambda x.t_i|\!)_\emptyset$ (with $x : A \vdash t_i : B$, $0 < p_i \leq 1$ y $\sum_{i=1}^{n} p_i \leq 1$),
   or else $(\!|t|\!)_\theta = \mathbf{0}_{\dim(A \multimap B)}$.
   By definition, $(\!|tr|\!)_\theta = (\!|t|\!)_\theta \,\#\, (\!|r|\!)_\theta$.

   • In the first case we have:

$$(\!|tr|\!)_\theta = \left(\sum_{i=1}^{n} p_i(\!|\lambda x.t_i|\!)_\emptyset\right) \,\#\, (\!|r|\!)_\theta = \sum_{i=1}^{n} p_i((\!|\lambda x.t_i|\!)_\emptyset \,\#\, (\!|r|\!)_\theta)$$

   By Lemma 4.3 this is equal to $\sum_{i=1}^{n} p_i(\!|t_i|\!)_{x=(\!|r|\!)_\theta}$. By Lemma A.6, we
   have $(\!|t_i|\!)_{x=(\!|r|\!)_\theta} \in (\!|B|\!)$ for all i. By Lemma 4.8 this linear combination is
   in $(\!|B|\!)$.

- In the second case we have:

$$(tr)_\theta = \mathbf{0}_{\dim(A \multimap B)} \;\#\; (r)_\theta = \mathbf{0}_{\dim(B)} \in (B)$$

  This holds because according to the definition for the $\#$ operator, $\mathbf{0}_{\dim(A \multimap B)}$ is the constant function $a \mapsto \mathbf{0}_{\dim(B)}$.

$$- \quad \frac{\Gamma, f : A \vdash t : A}{\Gamma \vdash \mu_n f.t : A} \; \mu$$

  Let Γ be a typing context and let θ be a valuation such that $\Gamma \vdash \mu_n f.t : A$ and $\theta \vDash \Gamma$. Then by inversion we have that $\Gamma, f : A \vdash t : A$. Using rule $\multimap_i$ we have $\Gamma \vdash \lambda f.t : A \multimap A$. By the soundness case for rule $\multimap_i$, we have that $(\lambda f.t)_\theta \in (A \multimap A)$.
  We want to show that $(\mu_n f.t)_\theta = (\lambda f.t)_\theta \;\#_n\; \mathbf{0}_{\dim(A)}$ is in (A). By induction on n:

  - Base case: $(\lambda f.t)_\theta \;\#_0\; \mathbf{0}_{\dim(A)} = \mathbf{0}_{\dim(A)} \in (A)$ by definition.
  - $(\lambda f.t)_\theta \;\#_{n+1}\; \mathbf{0}_{\dim(A)} = (\lambda f.t)_\theta \;\#\; ((\lambda f.t)_\theta \;\#_n\; \mathbf{0}_{\dim(A)})$. By the induction hypothesis we have that $(\lambda f.t)_\theta \;\#_n\; \mathbf{0}_{\dim(A)} \in (A)$. Since $(\lambda f.t)_\theta$ is in $(A \multimap A)$ we have that $(\lambda f.t)_\theta \;\#_{n+1}\; \mathbf{0}_{\dim(A)}$ is in (A) by Lemma A.6.

$$- \quad \frac{}{\Gamma \vdash \bot : A} \; \bot$$

  By definition we have that $(\bot)_\theta = \mathbf{0}_{\dim(A)}$. The null matrix is Hermitian, positive semidefinite and its trace is bounded by 1, so $\mathbf{0}_{\dim(A)} \in (A)$ for all type A.

$$- \quad \frac{}{\Gamma \vdash \rho^n : n} \; \mathsf{ax}_\rho$$

  For all θ, in particular such that $\theta \vDash \Gamma$, we have $(\rho^n)_\theta = \rho \in \mathcal{D}_{\bar{n}}^{\leq} = (n)$.

$$- \quad \frac{\Gamma \vdash t : n}{\Gamma \vdash U^m t : n} \; \mathsf{u}_i$$

  By the induction hypothesis we have that for all θ' such that $\theta' \vDash \Gamma$, $(t)_{\theta'} \in (n) = \mathcal{D}_{\bar{n}}^{\leq}$.
  Since $\theta \vDash \Gamma$, then $(t)_\theta \in \mathcal{D}_{\bar{n}}^{\leq}$. By definition we have that $(U^m t)_\theta = \overline{U} (t)_\theta \overline{U}^\dagger$. $\overline{U}$ is a unitary matrix, and so this product is in $\mathcal{D}_{\bar{n}}^{\leq}$.

$$- \quad \frac{\Gamma \vdash t : n}{\Gamma \vdash \pi^m t : (m, n)} \; \mathsf{m}_i$$

  By the induction hypothesis we have that for all θ' such that $\theta' \vDash \Gamma$, $(t)_{\theta'} \in (n) = \mathcal{D}_{\bar{n}}^{\leq}$.
  Since $\theta \vDash \Gamma$, we have $(t)_\theta \in \mathcal{D}_{\bar{n}}^{\leq}$. By definition we have that $(\pi^m t)_\theta = \bigoplus_{i=0}^{2^m - 1} (\overline{|i\rangle\langle i|} (t)_\theta \overline{|i\rangle\langle i|}^\dagger)$.
  As $(t)_\theta$ is in $\mathcal{D}_{\bar{n}}^{\leq}$, by Lemma A.9 $\overline{|i\rangle\langle i|} (t)_\theta \overline{|i\rangle\langle i|}^\dagger$ it is in $\mathcal{D}_{\bar{n}}^{\leq}$. By Lemma A.7, we have that $\mathsf{tr}\,((\pi^m t)_\theta) = \mathsf{tr}\left(\bigoplus_{i=0}^{2^m - 1} \overline{|i\rangle\langle i|} (t)_\theta \overline{|i\rangle\langle i|}^\dagger \right) = \mathsf{tr}\,((t)_\theta)$, and this is bounded by 1 by definition of $\mathcal{D}_{\bar{n}}^{\leq}$.
  Therefore $(\pi^m r)_\theta \in \bigoplus_{i=0}^{2^m - 1} \mathcal{D}_{\bar{n}}^{\leq} = ((m, n))$.

$$- \quad \frac{\Gamma \vdash t : n \quad \Delta \vdash r : m}{\Gamma, \Delta \vdash t \otimes r : n + m} \; \otimes$$

  By the induction hypothesis we have that for all θ' such that $\theta' \vDash \Gamma$, $(t)_{\theta'} \in (n) = \mathcal{D}_{\bar{n}}^{\leq}$; and for all θ'' such that $\theta'' \vDash \Delta$, $(t)_{\theta''} \in (m) = \mathcal{D}_{\bar{m}}^{\leq}$.

By hypothesis we have that $\theta \vDash \Gamma, \Delta$, then $\theta \vDash \Gamma$ and $\theta \vDash \Delta$. Thus, $(\!|t|\!)_\theta \in \mathcal{D}_n^{\leq}$ and $(\!|r|\!)_\theta \in \mathcal{D}_m^{\leq}$.

Then we have that $(\!|r \otimes s|\!)_\theta = (\!|r|\!)_\theta \otimes (\!|s|\!)_\theta \in \mathcal{D}_{n+m} = (\!|n+m|\!)$ because tensor product arity is given by $\otimes \colon \mathcal{D}_n \times \mathcal{D}_m \to \mathcal{D}_{n+m}$.

$$\frac{(i=1,\dots,2^m-1) \quad \Delta_i, x : n \vdash t_i : A \quad \Gamma \vdash r : (m,n) \quad \ell(A) \neq (m', n')}{\Delta_0, \dots, \Delta_{2^m-1}, \Gamma \vdash \mathsf{letcase}^\circ \ x = r \text{ in } \{t_0, \dots, t_{2^m-1}\} : A} \ \mathsf{m}_e$$

By the induction hypothesis we have that for all θ' such that $\theta' \vDash \Gamma$ we have that $(\!|r|\!)_{\theta'} \in (\!|(m,n)|\!)$. $\theta \vDash \Delta_0, \dots, \Delta_{2^m-1}, \Gamma$, then $\theta \vDash \Gamma$ and so $(\!|r|\!)_\theta \in (\!|(m,n)|\!)$.

Also by the induction hypothesis, for all i in $\{0, \dots, 2^m - 1\}$, for all θ' such that $\theta' \vDash \Delta_i, x : n$ we have that $(\!|t_i|\!)_{\theta'} \in (\!|A|\!)$. Since $\theta \vDash \Delta_0, \dots, \Delta_{2^m-1}, \Gamma$, in particular $\theta \vDash \Delta_i$ for all i and $\theta \cup \{x = \rho\} \vDash \Delta_i, x : n$ for all $\rho \in \mathcal{D}_n^{\leq}$. Therefore $(\!|t_i|\!)_{\theta, x=\rho} \in (\!|A|\!)$ for all $\rho \in \mathcal{D}_n^{\leq}$ and all $i \in \{0, \dots, 2^m - 1\}$.

By definition we have:

$$(\!|\mathsf{letcase}^\circ \ x = r \text{ in } \{t_0, \dots, t_{2^m-1}\}|\!)_\theta = \sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho_i\right) (\!|t_i|\!)_{\theta, x=\rho_i'}$$

where $(\!|r|\!)_\theta = \bigoplus_{i=0}^{2^m-1} \rho_i \in (\!|(m,n)|\!)$ and

$$\rho_i' = \begin{cases} \dfrac{\rho_i}{\mathsf{tr}\left(\rho_i\right)} & \text{if } \mathsf{tr}\left(\rho_i\right) \neq 0 \\[2mm] \rho_i & \text{if } \mathsf{tr}\left(\rho_i\right) = 0 \end{cases}$$

$\rho_i' \in \mathcal{D}_n^{\leq}$ because $\rho_i \in \mathcal{D}_n^{\leq}$, therefore by the induction hypothesis $(\!|t_i|\!)_{\theta, x=\rho_i'} \in (\!|A|\!)$ for all i in $\{0, \dots, 2^m - 1\}$.

Since $\rho_i \in \mathcal{D}_n^{\leq}$ for all i, we have $0 \leq \mathsf{tr}\left(\rho_i\right) \leq 1$. Also, as $(\!|r|\!)_\theta \in (\!|(m,n)|\!)$, we have $\mathsf{tr}\left((\!|r|\!)_\theta\right) \leq 1$. Therefore,

$$\mathsf{tr}\left((\!|r|\!)_\theta\right) = \mathsf{tr}\left(\bigoplus_{i=0}^{2^m-1} \rho_i\right) = \sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho_i\right) \leq 1$$

By Lemma 4.8, $\sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho_i\right) (\!|t_i|\!)_{\theta, x=\rho_i'} \in (\!|A|\!)$.

$$\frac{i=1,\dots,n \quad \Gamma \vdash t_i : A \quad \sum_{i=1}^n p_i = 1 \quad \ell(A) \neq (m, n')}{\Gamma \vdash \{(p_i, t_i)\}_{i \in \{1,\dots,n\}} : A} \ +$$

By using the induction hypothesis on i in $\{1, \dots, n\}$, we have that for all θ' such that $\theta' \vDash \Gamma$, $(\!|t_i|\!)_{\theta'} \in (\!|A|\!)$ holds. By hypothesis, $\theta \vdash \Gamma$. Hence, $(\!|t_i|\!)_\theta \in (\!|A|\!)$ for all i in $\{1, \dots, n\}$. By definition of $(\!|\_|\!)\_$, we have that $(\!|\sum_{i=1}^n p_i t_i|\!)_\theta = \sum_{i=1}^n p_i (\!|t_i|\!)_\theta$, which belongs to $(\!|A|\!)$ by Lemma 4.8. $\qquad \square$

B   Proofs of Section 4.2

B.1   Proof of Theorem 4.11

Theorem 4.11. *Let* $\vdash t \colon A$, *then* $\mathsf{tr}\left((\!|t|\!)_\emptyset\right) \leq N_A$.

Proof. By induction on types.

By Theorem 4.9 we have that $(\!|t|\!)_\emptyset \in (\!|A|\!)$ for every type A.

- If $A = n$ or $A = (m,n)$, we have that $\mathsf{tr}\left((\!|t|\!)_\emptyset\right) \le 1$ by definition of $(\!|n|\!)$ and $(\!|(m,n)|\!)$.
- If $A = B \multimap C$, by Lemma 4.7 there are two possibilities:
  - For $n \in \mathbb{N}$, there are n terms $t_1, \ldots, t_n$ and n real numbers $p_1, \ldots, p_n$ such that $x : B \vdash t_i : C$, $0 < p_i \le 1$, $\sum_{i=1}^n p_i \le 1$ and $(\!|t|\!)_\emptyset = \sum_{i=1}^n p_i (\!|\lambda x.t_i|\!)_\emptyset$. By trace linearity:

$$\mathsf{tr}\left((\!|t|\!)_\emptyset\right)$$
$$= \mathsf{tr}\left(\sum_{i=1}^n p_i (\!|\lambda x.t_i|\!)_\emptyset\right)$$
$$= \sum_{i=1}^n p_i \mathsf{tr}\left((\!|\lambda x.t_i|\!)_\emptyset\right)$$
$$= \sum_{i=1}^n p_i \left(\sum_{j=1}^{\dim(B)} \mathsf{tr}\left((\!|t_i|\!)_{x=E_{ii}^B} - (\!|t_i|\!)_{x=\mathbf{0}_{\dim(B)}}\right) + \mathsf{tr}\left((\!|t_i|\!)_{x=\mathbf{0}_{\dim(B)}}\right)\right)$$
$$= \sum_{i=1}^n p_i \left(\sum_{j=1}^{\dim(B)} \mathsf{tr}\left((\!|t_i|\!)_{x=E_{ii}^B}\right)\right.$$
$$\left. - \sum_{j=1}^{\dim(B)} \mathsf{tr}\left((\!|t_i|\!)_{x=\mathbf{0}_{\dim(B)}}\right) + \mathsf{tr}\left((\!|t_i|\!)_{x=\mathbf{0}_{\dim(B)}}\right)\right)$$

By Lemma A.6, both $(\!|t_i|\!)_{x=E_{ii}^B}$ and $(\!|t_i|\!)_{x=\mathbf{0}_{\dim(B)}}$ are in $(\!|C|\!)$, for all i. Therefore, and by the induction hypothesis, all those terms are positive and bounded by N_C. Then,

$$\mathsf{tr}\left((\!|t|\!)_\emptyset\right) \le \sum_{i=1}^n p_i \left(\sum_{j=1}^{\dim(B)} N_C + N_C\right) = \sum_{i=1}^n p_i (N_{B \multimap C}) \le N_{B \multimap C}$$

The last inequality holds because of the bound on the probability sum.
  - $(\!|t|\!)_\emptyset = \mathbf{0}_{\dim(B \multimap C)}$
    This case is trivial: $\mathsf{tr}\left((\!|t|\!)_\emptyset\right) = \mathsf{tr}\left(\mathbf{0}_{\dim(B \multimap C)}\right) = 0 \le N_{B \multimap C}$ □

B.2   Proof of Lemma 4.13

Lemma 4.13. *Let $\Gamma, x : A \vdash t : A$ and $\theta \vDash \Gamma$. Then for all $a, b \in (\!|A|\!)$, if $a \sqsubseteq b$ we have $(\!|\lambda x.t|\!)_\theta \# a \sqsubseteq (\!|\lambda x.t|\!)_\theta \# b$.*

Proof. By Lemma 4.5 and Lemma 4.2, we have that the following function is linear and completely positive, where $n = \dim(A)$.

$$f(c) = (\!|t|\!)_{\theta, x=c} - (\!|t|\!)_{\theta, x=\mathbf{0}_n}$$

Since $a \sqsubseteq b$ by hypothesis, $b - a$ is a positive matrix and since f is completely positive, $f(b-a)$ is also a positive matrix. In addition f is linear, then $f(b-a) = f(b) - f(a)$ is a positive matrix.

$$f(b - a) = (\!|t|\!)_{\theta, x=b} - (\!|t|\!)_{\theta, x=\mathbf{0}_n} - ((\!|t|\!)_{\theta, x=a} - (\!|t|\!)_{\theta, x=\mathbf{0}_n})$$
$$= (\!|t|\!)_{\theta, x=b} - (\!|t|\!)_{\theta, x=a}$$

By Lemma 4.3 we have $(\!|t|\!)_{\theta, x=a} = (\!|\lambda x.t|\!)_\theta \ \# \ a$ and $(\!|t|\!)_{\theta, x=b} = (\!|\lambda x.t|\!)_\theta \ \# \ b$. Rewriting we have:

$$f(b - a) = (\!|\lambda x.t|\!)_\theta \ \# \ b - (\!|\lambda x.t|\!)_\theta \ \# \ a$$

Therefore $(\!|\lambda x.t|\!)_\theta \ \# \ b - (\!|\lambda x.t|\!)_\theta \ \# \ a$ is a positive matrix, and this implies

$$(\!|\lambda x.t|\!)_\theta \ \# \ a \sqsubseteq (\!|\lambda x.t|\!)_\theta \ \# \ b \qquad\qquad \square$$

B.3   Proof of Lemma 4.14

Lemma 4.14. *Let $\overline{\chi} \in \mathbb{C}^{nm \times nm} \oplus \mathbb{C}^{m \times m}$ and let (P_n) be an increasing sequence of positive matrices in $\mathbb{C}^{n \times n}$ such that $\lim_{n \to \infty} P_n = P$. Then, $\overline{\chi}$ is monotone and $\lim_{n \to \infty} \overline{\chi} \ \# \ P_n = \overline{\chi} \ \# \ P$.*

Proof. By Lemma 4.13 we have that interpretations of abstraction terms in this calculus are monotone with respect to the Löwner order.

Remark that for all $1 \le i, j \le n, \lim_{n \to \infty}(P_n)_{ij} = P_{ij}$, where $(P_n)_{ij}$ is a sequence in $\mathbb{C}$. Let L_{ij}, K in $\mathbb{C}^{m \times m}$ and $\{E_{ij}^n\}$ the canonical basis for $\mathbb{C}^{n \times n}$ such that:

$$\overline{\chi} = \left(\sum_{i=1}^n \sum_{j=1}^n E_{ij}^n \otimes L_{ij} \right) \oplus K$$

By Lemma 3.6, since the first term of this matrix acts linearly on the argument, it is linear in every matrix element. Therefore application is continuous on every matrix element:

$$\lim_{n \to \infty} \overline{\chi} \ \# \ P_n = \lim_{n \to \infty} \sum_{i=1}^n \sum_{j=1}^n (P_n)_{ij} L_{ij} + K = \sum_{i=1}^n \sum_{j=1}^n P_{ij} L_{ij} + K = \overline{\chi} \ \# \ P \quad \square$$

B.4   Proof of Lemma 4.15

We need first the following technical lemma.

Lemma B.1. *For all positive M in $\mathbb{C}^{n \times n}$ and u in $\mathbb{C}^n$ we have that $u^\dagger M u \le \mathsf{tr}(M) \|u\|^2$.*

Proof. Since M is positive, it is diagonalisable and can be written as $P^{-1}DP$ where $D \in \mathbb{C}^{n \times n}$ is diagonal and $P \in \mathbb{C}^{n \times n}$ is unitary.

Therefore we have that

$$u^\dagger M u = u^\dagger (P^{-1}DP)u = (u^\dagger P^{-1})D(Pu) = v^\dagger D v$$

Defining $v = Pu \in \mathbb{C}^n$. Let $v_i \in \mathbb{C}$ be v's elements:

$$u^\dagger M u = \sum_i v_i^2 d_{ii} \leq \left(\sum_i v_i^2 \right) \left(\sum_i d_{ii} \right) = \mathsf{tr}\,(D)\,\|v\|^2 = \mathsf{tr}\,(M)\,\|u\|^2$$

This inequality holds because $d_{ii} \geq 0$ for all i since these are M's eigenvalues, that is a positive matrix. $\qquad\square$

Lemma 4.15. *For any type A, $(\mathfrak{D}_A, \sqsubseteq)$ is a complete partial order.*

Proof. We follow the structure of the proof at [10, Proposition 3.6].

We want to prove that in this set, the increasing sequences with respect to the Löwner order have a least upper bound.

$\mathfrak{D}_A$ is a subset of positive matrices in $\mathbb{C}^{2^n \times 2^n}$. Let M_1 and M_2 be matrices in $\mathfrak{D}_A$.

By definition $M_1 \sqsubseteq M_2$ if and only if $M_2 - M_1$ is a positive matrix, and this happens if and only if $u^\dagger (M_2 - M_1)u \geq 0$ for all u in $\mathbb{C}^{2^n}$. Therefore $M_1 \sqsubseteq M_2$ if and only if $u^\dagger M_1 u \leq u^\dagger M_2 u$ for all u in $\mathbb{C}^{2^n}$.

Thus, for all increasing sequences in $\mathfrak{D}_A$:

$$M_1 \sqsubseteq M_2 \sqsubseteq \ldots \sqsubseteq M_n \sqsubseteq \ldots$$

there is a corresponding increasing sequence in $\mathbb{R}_{\geq 0}$ for all u in $\mathbb{C}^{2^n}$:

$$u^\dagger M_1 u \leq u^\dagger M_2 u \leq \cdots \leq u^\dagger M_n u \leq \ldots$$

By Lemma B.1 and Theorem 4.11 we have that every element of the increasing sequence $\{u^\dagger M_n u\}$ are bounded by $N_A \|u\|^2$, since M_n matrices are in $\mathfrak{D}_A$. Any bounded increasing sequence in $\mathbb{R}$ has a least upper bound. Therefore the corresponding sequence $\{M_n\}$ in $\mathfrak{D}_A$ also has a least upper bound, and, by trace continuity, it is bounded by N_A. Hence, it is in $\mathfrak{D}_A$. $\qquad\square$