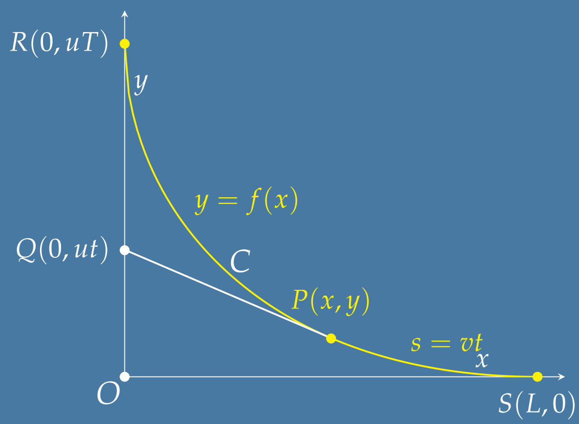
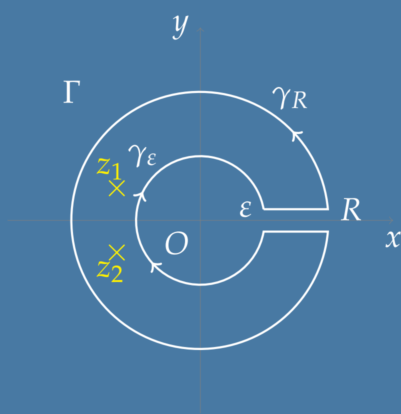
Some contributions to **Mathematics Stack Exchange**





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July 2020

Queluz

Contents

This is a small compilation of some answers and questions I posted to the site http://math.stackexchange.com.

My user profile page is http://math.stackexchange.com/users/752/americo-tavares.

Américo Tavares

(Last Updated on July 10, 2020)

Part I

Answers

Chap. 1

Elementary algebra

1.1 Equations

1.1.1 Inverse of $y = x^3 + x$?

Question by badatmath (http://math.stackexchange.com/users/9054/badatmath), Inverse of $y = x^3 + x$?, URL (version: 2011-10-06): http://math.stackexchange.com/q/60907

Can you help me find the inverse function for $y = x^3 + x$? (...)

Answer. URL (version: 2011-09-01): http://math.stackexchange.com/q/60913

Write $x^3 + x - y = 0$ and make $x = u + v$. You get

$$u^3 + v^3 - y + (u + v)(3uv + 1) = 0.$$

Then solve the system

$$\begin{cases} u^3 + v^3 = y \\ u^3 v^3 = -\frac{1}{27}, \end{cases}$$

which is equivalent to solving a quadratic equation in u^3 or v^3, because you know the sum of the two numbers u^3, v^3 and their product, e.g. the equation

$$\left(u^3\right)^2 - yu^3 - \frac{1}{27} = 0.$$

This technique is known as Cardano's method[1].

Reference: [6], pp. 215-216

[1] http://en.wikipedia.org/wiki/Cubic_function

1.1.2 Polynomials - Solutions

Question by Dek (http://math.stackexchange.com/users/83527/dek), Polynomials - Solutions, URL (version: 2013-06-27): http://math.stackexchange.com/q/428508

How I can find the exact solutions of this polynomial?

I can not get to the exact roots of the polynomial ... what methods occupy for this "problem"?

$$x^3 + 3x^2 - 7x + 1 = 0$$

(...)

Answer. URL (version: 2013-06-25): http://math.stackexchange.com/q/428545

> *How I can find the exact solutions of this polynomial?*

> *I can not get to the exact roots of the polynomial*

You have a real polynomial of the third degree with real coefficients. There are exact formulas to find the roots of any polynomial of this kind.
A general cubic equation of the form

$$ax^3 + bx^2 + cx + d = 0, \tag{1}$$

can be transformed by the substitution

$$x = t - \frac{b}{a} \tag{2}$$

into the reduced cubic equation

$$t^3 + pt + q = 0. \tag{3}$$

In the present case, we have

$$x^3 + 3x^2 - 7x + 1 = 0, \quad a = 1, b = 3, c = -7, d = 1. \tag{A}$$

For $x = t - 1$, we get the reduced equation

$$t^3 - 10t + 10 = 0, \qquad p = -10, q = 10. \tag{B}$$

It is known from the classical theory of the cubic equation that when the *discriminant*

$$\Delta = q^2 + \frac{4p^3}{27} = 10^2 + \frac{4\,(-10)^3}{27} < 0, \tag{C}$$

the three roots t_k of (3)*, with $k \in \{1,2,3\}$, are real and can be written in the following trigonometric form*[2]

$$t_k = 2\sqrt{-p/3}\cos\left(\frac{1}{3}\arccos\left(-\frac{q}{2}\sqrt{-27/p^3}\right) + \frac{(k-1)2\pi}{3}\right). \tag{4}$$

The roots of (1) are thus

$$x_k = t_k - \frac{b}{a}. \tag{5}$$

Consequently,

$$\begin{aligned}
x_1 &= 2\sqrt{10/3}\cos\left(\frac{1}{3}\arccos\left(-5\sqrt{27/10^3}\right)\right) - 1 \approx 1.4236,\\
x_2 &= 2\sqrt{10/3}\cos\left(\frac{1}{3}\arccos\left(-5\sqrt{27/10^3}\right) + \frac{2\pi}{3}\right) - 1 \approx -4.5771,\\
x_3 &= 2\sqrt{10/3}\cos\left(\frac{1}{3}\arccos\left(-5\sqrt{27/10^3}\right) + \frac{4\pi}{3}\right) - 1 \approx 0.15347.
\end{aligned} \tag{D}$$

1.1.3 Is there a general formula for solving 4th degree equations?

Question by John Gietzen (`http://math.stackexchange.com/users/38/john-gietzen`), Is there a general formula for solving 4th degree equations?, URL (version: 2012-07-30): `http://math.stackexchange.com/q/785`

[2] A deduction can be found in this post of mine `http://wp.me/p7Pyx-37s` in Portuguese.

There is a general formula for solving quadratic equations, namely the Quadratic Formula.

For third degree equations of the form $ax^3 + bx^2 + cx + d = 0$, there is a set of thee equations: one for each root.

Is there a general formula for solving equations of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ **?**

How about for higher degrees? If not, why not?

Answer. URL (version: 2011-08-16): http://math.stackexchange.com/q/57688

Yes. As an answer I will use a shorter version of this[3] Portuguese post of mine, where I deduce all the formulae. Suppose you have the *general* quartic equation (I changed the notation of the coefficients to Greek letters, for my convenience):

$$\alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \varepsilon = 0. \tag{1}$$

If you make the substitution $x = y - \frac{\beta}{4\alpha}$, you get a *reduced* equation of the form

$$y^4 + Ay^2 + By + C = 0, \tag{2}$$

with

$$A = \frac{\gamma}{\alpha} - \frac{3\beta^2}{8\alpha^2},$$

$$B = \frac{\delta}{\alpha} - \frac{\beta\gamma}{2\alpha^2} + \frac{\beta}{8\alpha},$$

$$C = \frac{\varepsilon}{\alpha} - \frac{\beta\delta}{4\alpha^2} + \frac{\beta^2 c}{16\alpha^3} - \frac{3\beta^4}{256\alpha^4}.$$

After adding and subtracting $2sy^2 + s^2$ to the LHS of (2) and rearranging terms, we obtain the equation

$$\underbrace{y^4 + 2sy^2 + s^2}_{(y^2+s)^2} - \left[(2s - A)\,y^2 - By + s^2 - C\right] = 0. \tag{2a}$$

[3]Resolução da equação do 4.º grau (ou quártica), http://wp.me/p7Pyx-39M

Then we factor the quadratic polynomial

$$(2s - A)\, y^2 - By + s^2 - C = (2s - A)\,(y - y_+)(y - y_-)$$

and make $y_+ = y_-$, which will impose a constraint on s (equation (4)). We will get:

$$\left(y^2 + s + \sqrt{2s - A}y - \frac{B}{2\sqrt{2s - A}}\right)\left(y^2 + s + \sqrt{2s - A}y + \frac{B}{2\sqrt{2s - A}}\right) = 0, \tag{3}$$

where s satisfies the *resolvent cubic equation*

$$8s^3 - 4As^2 - 8Cs + \left(4Ac - B^2\right) = 0. \tag{4}$$

The four solutions of (2) are the solutions of (3):

$$y_1 = -\frac{1}{2}\sqrt{2s - A} + \frac{1}{2}\sqrt{-2s - A + \frac{2B}{\sqrt{2s - A}}}, \tag{5}$$

$$y_2 = -\frac{1}{2}\sqrt{2s - A} - \frac{1}{2}\sqrt{-2s - A + \frac{2B}{\sqrt{2s - A}}}, \tag{6}$$

$$y_3 = -\frac{1}{2}\sqrt{2s - A} + \frac{1}{2}\sqrt{-2s - A - \frac{2B}{\sqrt{2s - A}}}, \tag{7}$$

$$y_4 = -\frac{1}{2}\sqrt{2s - A} - \frac{1}{2}\sqrt{-2s - A - \frac{2B}{\sqrt{2s - A}}}. \tag{8}$$

Thus, the original equation (1) *has the solutions*

$$x_k = y_k - \frac{\beta}{4\alpha}. \qquad k = 1, 2, 3, 4 \tag{9}$$

Example: $x^4 + 2x^3 + 3x^2 - 2x - 1 = 0$

$$y^4 + \frac{3}{2}y^2 - 4y + \frac{9}{16} = 0.$$

The resolvent cubic is

$$8s^3 - 6s^2 - \frac{9}{2}s - \frac{101}{8} = 0.$$

Making the substitution $s = t + \frac{1}{4}$, we get

$$t^3 - \frac{3}{4}t - \frac{7}{4} = 0.$$

One solution of the cubic is

$$s_1 = \left(-\frac{q}{2} + \frac{1}{2}\sqrt{q^2 + \frac{4p^3}{27}}\right)^{1/3} + \left(-\frac{q}{2} - \frac{1}{2}\sqrt{q^2 + \frac{4p^3}{27}}\right)^{1/3} - \frac{b}{3a},$$

where $a = 8, b = -6, c = -\frac{9}{2}, d = -\frac{101}{8}$ are the coefficients of the resolvent cubic and $p = -\frac{3}{4}, q = -\frac{7}{4}$ are the coefficients of the reduced cubic. Numerically, we have $s_1 \approx 1.6608$.
The four solutions are :

$$x_1 = -\frac{1}{2}\sqrt{2s_1 - A} + \frac{1}{2}\sqrt{-2s_1 - A + \frac{2B}{\sqrt{2s - A}}} - \frac{\beta}{4\alpha},$$

$$x_2 = -\frac{1}{2}\sqrt{2s_1 - A} - \frac{1}{2}\sqrt{-2s_1 - A + \frac{2B}{\sqrt{2s - A}}} - \frac{\beta}{4\alpha},$$

$$x_3 = -\frac{1}{2}\sqrt{2s_1 - A} + \frac{1}{2}\sqrt{-2s_1 - A - \frac{2B}{\sqrt{2s - A}}} - \frac{\beta}{4\alpha},$$

$$x_4 = -\frac{1}{2}\sqrt{2s_1 - A} - \frac{1}{2}\sqrt{-2s_1 - A - \frac{2B}{\sqrt{2s - A}}} - \frac{\beta}{4\alpha},$$

with $A = \frac{3}{2}, B = -4, C = \frac{9}{16}, D = \frac{9}{16}$. Numerically we have $x_1 \approx -1.1748 + 1.6393i$, $x_2 \approx -1.1748 - 1.6393i$, $x_3 \approx 0.70062$, $x_4 \approx -0.35095$.

Another method is to expand the LHS of the quartic into two quadratic polynomials, and find the zeroes of each polynomial. However, this method *sometimes fails*. Example: $x^4 - x - 1 = 0$. If we factor $x^4 - x - 1$ as $x^4 - x - 1 = \left(x^2 + bx + c\right)\left(x^2 + Bx + C\right)$ expand and equate coefficients we will get two equations, one of which is $-1/c - c^2\left(1+c^2\right)^2 + c = 0$. This is studied in Galois theory[4]

The general quintic is not solvable in terms of radicals, as well as equations of higher degrees.

1.2 Elementary Finite Sequences and Sums

1.2.1 Calculating the value of Annuities

Question by user78886 (http://math.stackexchange.com/users/78886/user78886), Calculating the value of Annuities, URL (version: 2014-01-13): http://math.stackexchange.com/q/636242

Instead of investing 3000 at the end of 5 years, and \$4000 at the end of 10 years, Steve wishes to make regular monthly payments that will amount to the same total after 10 years. Determine the monthly payment if interest is compounded monthly at an annual rate of 4%

Could anyone set up the annuity formula with the numbers from the question. Im not sure if what i did was correct:

$$4000 = R\frac{1-1.003^{120}}{0.004/12}$$

The answer when you isolate for R should be \$52.04.

Answer. URL (version: 2014-01-17): http://math.stackexchange.com/q/636431

You have some numerical errors and have considered the contribution of \$4000 only. Mathematically we can establish an equivalence between the investments (at the end of 5 years and at the end of 10 years) and a series of 120 monthly constant payments.

Since there are $m = 12$ compounding periods per year, the (nominal) annual interest rate $r = 4\% = 0.04$ indicates a monthly interest rate $i = \frac{r}{m} = \frac{4}{12}\% = \frac{0.04}{12}$. The hypothetical investment of 3000 at the end of 5 years (60 months) accumulates to the future value $F' = 3000\,(1+i)^{60} = 3000\,(1+0.04/12)^{60} \approx 3663.0$. Adding the second hypothetical investment $F'' = 4000$ yields the total future value $F = F' + F'' = 7663.0$ at the end of 10 years. Let A (the annuity) denote each monthly payment. The payment at the end of month k increases to a future

[4] http://en.wikipedia.org/wiki/Galois_theory.

value of $F_k = A(1+i)^{n-k}$ at the end of $n = 120$ months. Summing all these F_k the resulting geometric series (http://math.stackexchange.com/a/29035/752) of n payments, whose ratio is $c = 1 + i$, should be equal to F, as a consequence of the equivalence mentioned above. Applying the formula(http://math.stackexchange.com/a/29035/752) for such a sum, we get

$$F = \sum_{k=1}^{n} F_k = \sum_{k=1}^{n} A(1+i)^{n-k} = \sum_{j=1}^{n} Ac^{j-1} = A\frac{c^n - 1}{c-1} = A\frac{(1+i)^n - 1}{i}.$$

Numerically we obtain

$$A = F\frac{i}{(1+i)^n - 1} = 7663.0\frac{\frac{0.04}{12}}{(1+\frac{0.04}{12})^{120} - 1} \approx 52.04,$$

which agrees with the answer you indicate.

1.2.2 Proof that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$?

Question by Nathan Osman (http://math.stackexchange.com/users/4626/nathan-osman), Proof that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$?, URL (version: 2013-08-14): http://math.stackexchange.com/q/48080

Answer. URL (version: 2011-07-10): http://math.stackexchange.com/q/50631

Proof 1. *(Exercise 2.5.1 in Dias Agudo, Cândido da Silva, Matemáticas Gerais III).* Let $S := \sum_{k=1}^{n} k^2$. Consider $(1+a)^3 = 1 + 3a + 3a^2 + a^3$ and sum $(1+a)^3$ for $a = 1, 2, \ldots, n$:

$$\begin{aligned}
(1+1)^3 &= 1 + 3\cdot 1 + 3\cdot 1^2 + 1^3 \\
(1+2)^3 &= 1 + 3\cdot 2 + 3\cdot 2^2 + 2^3 \\
(1+3)^3 &= 1 + 3\cdot 3 + 3\cdot 3^2 + 3^3 \\
&\cdots \\
(1+n)^3 &= 1 + 3\cdot n + 3\cdot n^2 + n^3
\end{aligned}$$

The term $(1+1)^3$ on the LHs of the 1st sum cancels the term 2^3 on the RHS of the 2nd, $(1+2)^3$, the 3^3, $(1+3)^4$, the 4^3, ..., and $(1+n-1)^3$ cancels n^3. Hence

$$(1+n)^3 = n + 3\left(1+2+\ldots+n\right) + 3S + 1$$

and

$$S = \frac{n(n+1)(2n+1)}{6},$$

because $1+2+\ldots+n = \dfrac{n\left(n+1\right)}{2}$.

Proof 2. *(Exercise 1.42 in Balakrishnan, Combinatorics, Schaum's Outline of Combinatorics)*. From

$$\binom{k}{1} + 2\binom{k}{2} = k + 2\frac{k\left(k-1\right)}{2} = k^2,$$

we get

$$\begin{aligned} S &:= \sum_{k=1}^{n} k^2 = \sum_{k=0}^{n} \binom{k}{1} + 2\binom{k}{2} = \sum_{i=1}^{n} \binom{k}{1} + 2\sum_{k=1}^{n} \binom{k}{2} \\ &= \binom{n+1}{2} + 2\binom{n+1}{3} \\ &= \frac{n\left(n+1\right)\left(2n+1\right)}{6}. \end{aligned}$$

1.2.3 How to compute this sum

Question by Timo Willemsen (http://math.stackexchange.com/users/3605/timo-willemsen), How to compute this sum, URL (version: 2011-10-14): http://math.stackexchange.com/q/72618

Answer. URL (version: 2011-10-14): http://math.stackexchange.com/q/72621

Hints: expand $4(2k+1)^2 - 12k$ and use

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

You can find a proof of the second formula in the post of mine (in Portuguese) *Soma dos quadrados dos primeiros n números inteiros positivos*[5].

[5] http://wp.me/p7Pyx-3Yz

1.2.4 Adding powers of i

Question by jkottnauer (http://math.stackexchange.com/users/267/jkottnauer), Adding powers of i, URL (version: 2013-04-09): http://math.stackexchange.com/q/4062

> (...) But how do I get the result of $i^3 + i^4 + ... + i^{50}$? (...)

Answer. URL (version: 2011-10-14): http://math.stackexchange.com/q/72621

Observing that $i^3 + i^4 + \ldots + i^{50}$ is a *geometric progression* with ratio i, first term i^3 and $50 - 3 + 1 = 48$ terms, we have

$$\begin{aligned} i^3 + i^4 + \ldots + i^{50} &= i^3 \times \frac{1 - i^{50-3+1}}{1-i} = i^3 \times \frac{1 - i^{48}}{1-i} = i^2 i \times \frac{1-(i^2)^{24}}{1-i} \\ &= -i\frac{1-(-1)^{24}}{1-i} = -i\frac{1-1}{1-i} = 0 \end{aligned}$$

1.3 Inequalities

1.3.1 $\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} > \sqrt{a+b+c}$ is true for positive a,b,c

Question by mgamer (http://math.stackexchange.com/users/2562/mgamer), $\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} > \sqrt{a+b+c}$ is true for positive a,b,c, URL (version: 2010-10-16): http://math.stackexchange.com/q/6950

> How to prove that for all positive a,b,c this formula holds true:
>
> $$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} > \sqrt{a+b+c}$$
>
> (...)

Answer. URL (version: 2010-10-16): http://math.stackexchange.com/q/6954

Since

$$a+b+c = \sqrt{(a+b+c)^2} = \sqrt{a+b+c}\,\sqrt{a+b+c} \iff \frac{a+b+c}{\sqrt{a+b+c}} = \sqrt{a+b+c}$$

and

$$\frac{a}{\sqrt{a+b}} > \frac{a}{\sqrt{a+b+c}}$$
$$\frac{b}{\sqrt{b+c}} > \frac{b}{\sqrt{a+b+c}}$$
$$\frac{c}{\sqrt{c+a}} > \frac{c}{\sqrt{a+b+c}}$$

we have

$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} > \frac{a+b+c}{\sqrt{a+b+c}} = \sqrt{a+b+c}.$$

Chap. 2

Geometry and Trigonometry

2.1 Geometry

2.1.1 Orthogonal projection of a point onto a line

Question by jbennet (http://math.stackexchange.com/users/15269/jbennet), Orthogonal projection of a point onto a line, URL (version: 2011-09-08): http://math.stackexchange.com/q/62633

> (...) find an orthogonal projection of a point $\mathrm{T}(-4,5)$ onto a line $\frac{x}{3}+\frac{y}{-5}=1$

Answer. URL (version: 2011-09-08): http://math.stackexchange.com/q/62638

The slope of the line r, with equation $\frac{x}{3}+\frac{y}{-5}=1$, is $m_r=\frac{5}{3}$ (because $\frac{x}{3}+\frac{y}{-5}=1$ is equivalent to $y=\frac{5}{3}x-5$). The slope of the line s orthogonal to r is $m_s=-\frac{3}{5}$ (because $m_r m_s=-1$). Hence the equation of s is of the form

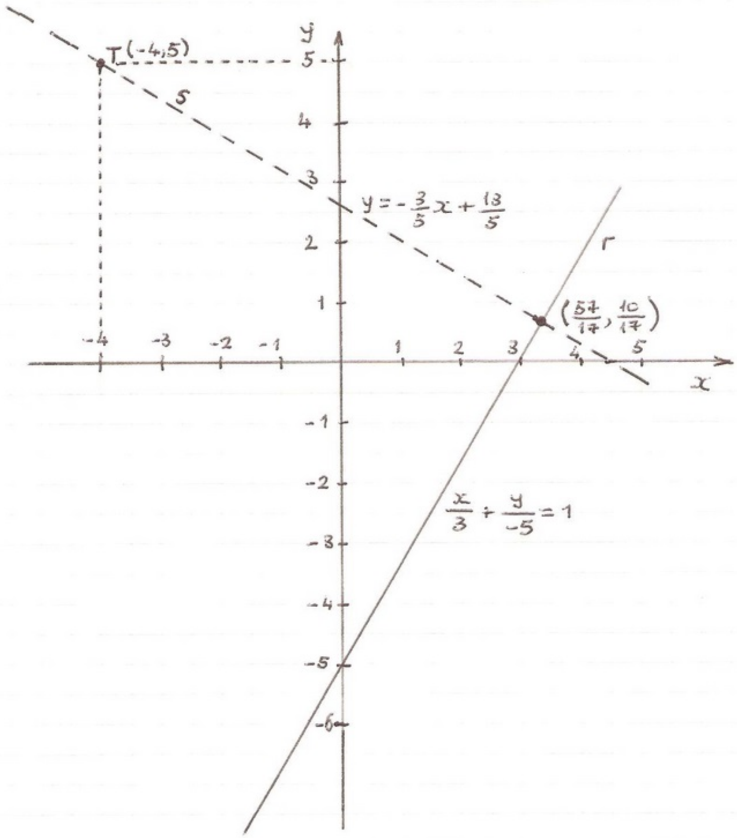
$$y=-\frac{3}{5}x+b_s.$$

Since $T(-4,5)$ is a point of s, we have

$$5=-\frac{3}{5}(-4)+b_s,$$

which means that $b_s=\frac{13}{5}$. So the equation of s is

$$y=-\frac{3}{5}x+\frac{13}{5}.$$



The coordinates of the orthogonal projection of T onto r are the solutions of the system

$$\begin{cases} y = \frac{5}{3}x - 5 \\ y = -\frac{3}{5}x + \frac{13}{5}, \end{cases}$$

which are $(x, y) = \left(\frac{57}{17}, \frac{10}{17}\right)$.

2.1.2 How to test any 2 line segments (3D) are collinear or not?

Question by niro (http://math.stackexchange.com/users/9275/niro), How to test any 2 line segments (3D) are collinear or not?, URL (version: 2012-01-27): http://math.stackexchange.com/q/103065

Answer. URL (version: 2012-01-27): http://math.stackexchange.com/q/103121

An alternative method. Assume PQ and RS are the line segments. Let the direction cosines[1] of the vectors $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{RS}$ be, respectively, $\alpha_u, \beta_u, \gamma_u$ and $\alpha_v, \beta_v, \gamma_v$. The angle ϕ between the line segments is such that[2]

$$\cos \phi = \alpha_u \alpha_v + \beta_u \beta_v + \gamma_u \gamma_v.$$

Hence the line segments are collinear if $\cos \phi = \pm 1$.

2.1.3 Rotation by 180° angle

Question by dato datuashvili (http://math.stackexchange.com/users/3196/dato-datuashvili), rotation by 180 angle, URL (version: 2011-07-23): http://math.stackexchange.com/q/53279

In general I know that if we rotate (x, y) about origin by the 180 degree we will get new image $(-x, -y)$, but suppose that we make rotation not about origin but some other point (a, b) does your result be rotation around origin + or - (a, b)? for suppose we have point $A(3, 27)$ and we want turn it by 180 around the point $(2, -1)$, if we rotate $(3, 27)$ about origin by 180 we get $(-3, -27)$ but how to connect $(2, -1)$ to this result?

Answer. URL (version: 2011-07-23): http://math.stackexchange.com/q/53282

You can make a translation of axes so that $(2, -1)$ becomes the new origin. The new axes are $X = x - 2, Y = y + 1$. Then you compute the new coordinates of $A(3 - 2, 27 + 1) = (1, 28)$. The symmetric point A' with respect to $(X, Y) = (0, 0)$ is thus $A'(-1, -28)$ in the $XY-$coordinate system or $A'(-1 + 2, -28 - 1) = (1, -29)$ in the original $xy-$coordinate system.

2.2 Trigonometry

2.2.1 Solve trigonometric equation: $1 = m \cos(\alpha) + \sin(\alpha)$

Question by ftiaronsem (http://math.stackexchange.com/users/3188/ftiaronsem), Solve trigonometric equation: $1 = m \cos(\alpha) + \sin(\alpha)$, URL (version: 2011-06-21): http://math.stackexchange.com/q/9138

Dealing with a physics Problem I get the following equation to solve for α

[1] http://en.wikipedia.org/wiki/Direction_cosine
[2] Formula 10.7 of [8]

$$1 = m\cos(\alpha) + \sin(\alpha)$$

Putting this in Mathematica gives the result:

$$a == 2\text{ArcTan}\left[\frac{1-m}{1+m}\right]$$

However I am unable to get this result myself. (...)

Answer. URL (version: 2011-07-09): `http://math.stackexchange.com/q/9185`

Added: As explained in the comments, certain trigonometric equations such as the linear equations in $\sin x$ and $\cos x$ can be solved by a resolvent quadratic equation. One method is to write the $\sin x$ and $\cos x$ functions in terms of the same trigonometric function. Since all [direct] trigonometric functions of the simple angle can be expressed rationally as a function of the tan of the half-angle, such a conversion is adequate for these equations [3].

Since

$$\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

and

$$\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

your equation

$$m\cos\alpha + \sin\alpha = 1$$

is equivalent to

$$m - m\tan^2\frac{\alpha}{2} + 2\tan\frac{\alpha}{2} = 1 + \tan^2\frac{\alpha}{2}.$$

One may set $x = \tan\frac{\alpha}{2}$ ($\alpha = 2\arctan x$), and thus get the quadratic equation

[3][5] states and proves: "All [direct] trigonometric functions of the double angle can be expressed rationally as a function of the simple angle tangent." (my translation). For instance $\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$ can be derived as follows

$$\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}} = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}.$$

$$(1+m)\,x^2 - 2x + 1 - m = 0.$$

Its solutions are: $x = \frac{1}{m+1}\,(-m+1)$ (if $m \neq -1$) or $x = 1$ (if $m = -1$), which gives

i) If $m \neq -1$,

$$\alpha = 2\arctan x = 2\arctan\frac{1-m}{m+1},$$

ii) If $m = -1$,

$$\alpha = 2\arctan 1 = \frac{\pi}{2}.$$

A different technique to solve a linear equation in $\sin\alpha$ and $\cos\alpha$ is to use an auxiliary angle φ. If you set $m = \tan\varphi$, your equation takes the form

$$\sin\alpha + \tan\varphi\cdot\cos\alpha = 1$$

or

$$\sin(\alpha+\varphi) = \cos\varphi = \frac{1}{\pm\sqrt{1+\tan^2\varphi}} = \pm\sqrt{\frac{1}{1+m^2}},$$

and obtain

$$\alpha = \pm\arcsin\sqrt{\frac{1}{1+m^2}} - \arctan m.$$

Detailed derivation: from $m\cos\alpha + \sin\alpha = 1$ and $m = \tan\varphi$,
we get

$$\begin{aligned}
&\sin\alpha + \tan\varphi\cdot\cos\alpha = 1 \iff \sin\alpha + \frac{\sin\varphi}{\cos\varphi}\cdot\cos\alpha = 1\\
&\iff \frac{\sin\alpha\cdot\cos\varphi + \sin\varphi\cdot\cos\alpha}{\cos\varphi} = 1 \iff \frac{\sin(\alpha+\varphi)}{\cos\varphi} = 1\\
&\iff \sin(\alpha+\varphi) = \cos\varphi.
\end{aligned}$$

The identity

$$\cos\varphi = \pm\sqrt{\frac{1}{1+\tan^2\varphi}}$$

can be obtained as follows

$$\sin^2\varphi+\cos^2\varphi=1 \iff \frac{\sin^2\varphi}{\cos^2\varphi}+1=\frac{1}{\cos^2\varphi}$$
$$\iff \tan^2\varphi+1=\frac{1}{\cos^2\varphi} \iff \cos^2\varphi=\frac{1}{1+\tan^2\varphi}.$$

Therefore

$$\sin(\alpha+\varphi)=\pm\sqrt{\frac{1}{1+\tan^2\varphi}} \iff \alpha+\varphi=\arcsin\left(\pm\sqrt{\frac{1}{1+\tan^2\varphi}}\right)$$
$$\iff \alpha+\arctan m=\arcsin\left(\pm\sqrt{\frac{1}{1+m^2}}\right) \qquad (m=\tan\varphi,\ \varphi=\arctan m)$$

and finally

$$\alpha=\arcsin\left(\pm\sqrt{\frac{1}{1+m^2}}\right)-\arctan m.$$

2.2.2 Is there a more efficient method of trig mastery than rote memorization?

Question by Joe Stavitsky (`http://math.stackexchange.com/users/16276/joe-stavitsky`), Is there a more efficient method of trig mastery than rote memorization?, URL (version: 2011-09-18): `http://math.stackexchange.com/q/65548`

Answer. URL (version: 2011-09-19): `http://math.stackexchange.com/q/65643`

I would *emphasize how to derive trigonometric identities from a few ones*. Learn:

- *how to derive* the relations between the direct functions of the same angle from the definition of the trigonometric functions and the Pythagorean formula;
- maxima, minima, zeroes and period of each function;
- if a function is odd or even;
- the trigonometric functions of 0°, 30°, 45°, 60° and 90°;

- the relations between functions of symmetric, complementary and supplementary angles;
- the relations between functions of angles whose difference is 180°;
- the relations between functions of angles whose sum is 360°;
- the inverse trigonometric functions;
- the addition formulas of sin and cos;
- *how to derive* the subtraction formulas of sin and cos;
- *how to derive* the addition and subtraction formulas of tan and cot;
- *how to derive* the double and half angle formulas;
- *how to derive* the sum to product formulas;
- how to solve some elementary trigonometric equations;
- (triangle) sin and cos laws;
- Heron's formula;
- derivatives of direct and inverse trigonometric functions.

Added. Examples. From

$$\sin(\alpha+\beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta, \tag{A}$$

if we set $\alpha = \beta = a$, we get

$$\sin 2a = 2\sin a \cdot \cos a. \tag{1}$$

And from

$$\cos(\alpha+\beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta \tag{B}$$

for $\alpha = \beta = a$, we have

$$\cos 2a = \cos^2 a - \sin^2 a. \tag{2}$$

Using the Pythagorean identity

$$\cos^2 a + \sin^2 a = 1, \tag{C}$$

if $\cos a \neq 0$, then

$$\begin{aligned}
\sin 2a = 2\sin a \cdot \cos a &= 2\frac{\sin a \cdot \cos a}{\cos^2 a + \sin^2 a} \\
&= \frac{2\dfrac{\sin a \cdot \cos a}{\cos^2 a}}{\dfrac{\cos^2 a + \sin^2 a}{\cos^2 a}} = \frac{2\dfrac{\sin a}{\cos a}}{1 + \dfrac{\sin^2 a}{\cos^2 a}} \\
&= \frac{2\tan a}{1 + \tan^2 a}.
\end{aligned} \tag{3}$$

Similarly

$$\begin{aligned}
\cos 2a = \cos^2 a - \sin^2 a &= \frac{\cos^2 a - \sin^2 a}{\cos^2 a + \sin^2 a} \\
&= \frac{\dfrac{\cos^2 a - \sin^2 a}{\cos^2 a}}{\dfrac{\cos^2 a + \sin^2 a}{\cos^2 a}} = \frac{1 - \dfrac{\sin^2 a}{\cos^2 a}}{1 + \dfrac{\sin^2 a}{\cos^2 a}} \\
&= \frac{1 - \tan^2 a}{1 + \tan^2 a}.
\end{aligned} \tag{4}$$

Then

$$\tan 2a = \frac{\sin 2a}{\cos 2a} = \frac{\dfrac{2\tan a}{1 + \tan^2 a}}{\dfrac{1 - \tan^2 a}{1 + \tan^2 a}} = \frac{2\tan a}{1 - \tan^2 a}. \tag{5}$$

Added 2. The linear equation in $\sin x$ and $\cos x$

$$A\sin x + B\cos x = C \tag{6}$$

can be solved by a resolvent quadratic equation in $\tan\frac{x}{2}$, by writting the $\sin x$ and the $\cos x$ functions in terms of $\tan\frac{x}{2}$ (set $x = 2a$ in (3) and (4)):

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \tag{7}$$

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}. \tag{9}$$

The equation (6) is equivalent to

$$\begin{aligned} A\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + B\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} &= C, \\ 2A\tan\frac{x}{2} + B - B\tan^2\frac{x}{2} &= C + C\tan^2\frac{x}{2}, \\ (B+C)\tan^2\frac{x}{2} - 2A\tan\frac{x}{2} + C - B &= 0. \end{aligned} \tag{10}$$

2.2.3 Why is it that when proving trig identities, one must work both sides independently?

Question by Ord (http://math.stackexchange.com/users/23387/ord), Why is it that when proving trig identities, one must work both sides independently?, URL (version: 2012-01-21): http://math.stackexchange.com/q/101053

Suppose that you have to prove the trig identity:

$$\frac{\sin\theta - \sin^3\theta}{\cos^2\theta} = \sin\theta$$

I have always been told that I should manipulate the left and right sides of the equation separately, until I have transformed them each into something identical. So I would do:

$$\begin{aligned} \frac{\sin\theta - \sin^3\theta}{\cos^2\theta} &= \frac{\sin\theta(1-\sin^2\theta)}{\cos^2\theta} \\ &= \frac{\sin\theta(\cos^2\theta)}{\cos^2\theta} \\ &= \sin\theta \end{aligned}$$

And then, since the left side equals the right side, I have proved the identity. My problem is: why can't I manipulate the entire equation? In this situation it probably won't make things any easier, but for certain identities, I can see ways to "prove" the identity by manipulating the entire equation, but cannot prove it by keeping both sides isolated.(...)

Answer. URL (version: 2014-05-03): http://math.stackexchange.com/q/101142

Why can't I manipulate the entire equation?

You can. The *analytical method* for proving an identity consists of starting with the identity you want to prove, in the present case

$$\frac{\sin\theta - \sin^3\theta}{\cos^2\theta} = \sin\theta, \qquad \cos\theta \neq 0 \tag{1}$$

and establish a sequence of identities so that each one is a consequence of the next one. For the identity (1) to be true is enough that the following holds

$$\sin\theta - \sin^3\theta = \sin\theta\cos^2\theta \tag{2}$$

or this equivalent one

$$\sin\theta\left(1 - \sin^2\theta\right) = \sin\theta\cos^2\theta \tag{3}$$

or finally this last one

$$\sin\theta\cos^2\theta = \sin\theta\cos^2\theta \tag{4}$$

Since (4) is true so is (1).

The book indicated below illustrates this method with the following identity

$$\frac{1+\sin a}{\cos a} = \frac{\cos a}{1-\sin a} \qquad a \neq (2k+1)\frac{\pi}{2}$$

It is enough that the following holds

$$(1+\sin a)(1-\sin a) = \cos a\cos a$$

or

$$1 - \sin^2 a = \cos^2 a,$$

which is true if

$$1 = \cos^2 a + \sin^2 a$$

is true. Since this was proven to be true, all the previous indentities hold, and so does the first identity.

References: [5]

2.2.4 Is it necessary to have θ in radians to obtain $\frac{\sin\theta}{\theta} \to 1$ as $\theta \to 0$?

Question by z_z (http://math.stackexchange.com/users/38294/z-z), Is it necessary to have θ in radians to obtain $\frac{\sin\theta}{\theta} \to 1$ as $\theta \to 0$?, URL (version: 2012-08-21): http://math.stackexchange.com/q/184593

Answer. URL (version: 2012-08-20): http://math.stackexchange.com/q/184668

The length L of an arc of a circle with radius r and subentending angle θ is

$$L = r\theta$$

if and only if θ is measured in radians[4]. Under this assumption and for $0 < \theta < \pi/2$ radians

$$\sin\theta < \frac{L}{r} = \theta < \tan\theta$$

and

[4]If θ is measured in degrees, then

$$L = \frac{\pi r}{180}\theta.$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta},$$

which implies that

$$\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1 \Leftrightarrow \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1,$$

because $\frac{\sin\theta}{\theta}$ is an even function.

Chap. 3

Calculus

3.1 Integration

3.1.1 Integral of $\dfrac{1}{(1+x^2)^2}$

Question by Virtuoso (http://math.stackexchange.com/users/7814/virtuoso), Integral of $\frac{1}{(1+x^2)^2}$, URL (version: 2011-04-30): http://math.stackexchange.com/q/35924

Answer. URL (version: 2011-04-30): http://math.stackexchange.com/q/35939

If we add and subtract x^2 in the numerator, we can integrate the first integral immediately

$$\begin{aligned}\int \frac{1}{\left(1+x^2\right)^2}dx &= \int \frac{1}{1+x^2}dx - \int \frac{x^2}{\left(1+x^2\right)^2}dx \\ &= \arctan x - \int x\frac{x}{\left(1+x^2\right)^2}dx\end{aligned}$$

and the second integral by parts:

$$\begin{aligned}\int x\frac{x}{\left(1+x^2\right)^2}dx &= x\left(-\frac{1}{2\left(1+x^2\right)}\right) + \int \frac{1}{2\left(1+x^2\right)}dx \\ &= -\frac{x}{2\left(1+x^2\right)} + \frac{1}{2}\arctan x.\end{aligned}$$

By applying this method $n-1$ times, we can reduce the integration of the function $f(x) = \dfrac{1}{(1+x^2)^n}$ to the integration of $\dfrac{1}{1+x^2}$.

3.1.2 Calculation of $\int \sqrt{\tan x + 2} dx$

Question by juantheron (http://math.stackexchange.com/users/14311/juantheron), Calculation of $\int \sqrt{\tan x + 2} dx$, URL (version: 2014-01-04): http://math.stackexchange.com/q/626942

Answer. URL (version: 2014-01-05): http://math.stackexchange.com/q/627511

Using your substitution

$$t = \sqrt{\tan x + 2}$$

we need to integrate

$$\frac{I}{2} = \frac{1}{2}\int \sqrt{\tan x + 2}\, dx = \int \frac{t^2}{\left(t^2-2\right)^2+1}\, dt + C,$$

as you show in your edited question. We can reduce it to a table integral if we factorize the denominator

$$t^4 - 4t^2 + 5 = \left(t^2 + \sqrt{4+\sqrt{20}}t + \sqrt{5}\right)\left(t^2 - \sqrt{4+\sqrt{20}}t + \sqrt{5}\right)$$

and expand the integrand into partial fractions

$$\frac{t^2}{t^4-4t^2+5} = \frac{At}{t^2 + \sqrt{4+\sqrt{20}}t + \sqrt{5}} - \frac{At}{t^2 - \sqrt{4+\sqrt{20}}t + \sqrt{5}},$$

where

$$A = -B = -\frac{1}{4}\sqrt{4+\sqrt{20}}\left(-2+\sqrt{5}\right).$$

The standard integral we need is the following one

$$\int \frac{t}{t^2+bt+c}\,dt = \frac{1}{2}\ln\left|t^2+bt+c\right| - \frac{b}{\sqrt{4c-b^2}}\arctan\frac{2t+b}{\sqrt{4c-b^2}} + C, \qquad 4c-b^2>0.$$

In the case at hand $4c-b^2 = 4\sqrt{5} - \left(4+\sqrt{20}\right) = 2\sqrt{5}-4>0$. So

$$\begin{aligned}\int \frac{t}{t^2+\sqrt{4+\sqrt{20}}t+\sqrt{5}}dt &= \frac{1}{2}\ln\left|t^2+\sqrt{4+\sqrt{20}}t+\sqrt{5}\right| \\ &\quad - \frac{\sqrt{4+\sqrt{20}}}{\sqrt{2\sqrt{5}-4}}\arctan\frac{2t+\sqrt{4+\sqrt{20}}}{\sqrt{2\sqrt{5}-4}} + C,\end{aligned}$$

and

$$\begin{aligned}\int \frac{t}{t^2-\sqrt{4+\sqrt{20}}t+\sqrt{5}}dt &= \frac{1}{2}\ln\left|t^2-\sqrt{4+\sqrt{20}}t+\sqrt{5}\right| \\ &\quad + \frac{\sqrt{4+\sqrt{20}}}{\sqrt{2\sqrt{5}-4}}\arctan\frac{2t-\sqrt{4+\sqrt{20}}}{\sqrt{2\sqrt{5}-4}} + C.\end{aligned}$$

We thus get

$$\begin{aligned}\frac{I}{2} &= A\left(\frac{1}{2}\ln\left|t^2+\sqrt{4+\sqrt{20}}t+\sqrt{5}\right| - \frac{\sqrt{4+\sqrt{20}}}{\sqrt{2\sqrt{5}-4}}\arctan\frac{2t+\sqrt{4+\sqrt{20}}}{\sqrt{2\sqrt{5}-4}}\right) \\ &\quad - A\left(\frac{1}{2}\ln\left|t^2-\sqrt{4+\sqrt{20}}t+\sqrt{5}\right| + \frac{\sqrt{4+\sqrt{20}}}{\sqrt{2\sqrt{5}-4}}\arctan\frac{2t-\sqrt{4+\sqrt{20}}}{\sqrt{2\sqrt{5}-4}}\right) + C.\end{aligned}$$

Substituting back $t=\sqrt{\tan x+2}$ we get the given integral $I = \frac{2I}{2} = \int\sqrt{\tan x+2}\,dx$.

ADDED. After simplifying I've obtained

$$\begin{aligned}I &= \frac{\left(2-\sqrt{5}\right)\sqrt{4+\sqrt{20}}}{4}\ln\left|\frac{\tan x+2+\sqrt{4+\sqrt{20}}\sqrt{\tan x+2}+\sqrt{5}}{\tan x+2-\sqrt{4+\sqrt{20}}\sqrt{\tan x+2}+\sqrt{5}}\right| \\ &\quad + \frac{\sqrt{4+\sqrt{20}}}{2}\times \\ &\qquad \times\left(\arctan\frac{2\sqrt{\tan x+2}-\sqrt{4+\sqrt{20}}}{\sqrt{\sqrt{20}-4}} + \arctan\frac{2\sqrt{\tan x+2}+\sqrt{4+\sqrt{20}}}{\sqrt{\sqrt{20}-4}}\right) + C.\end{aligned}$$

3.1.3 Orthogonality of sine and cosine integrals.

Question by Frank_W (http://math.stackexchange.com/users/108145/frank-w), Orthogonality of sine and cosine integrals., URL (version: 2014-01-14): http://math.stackexchange.com/q/638308

How to prove that

$$\int_{t_0}^{t_0+T} \sin(m\omega t)\sin(n\omega t)\,\mathrm{d}t$$

will equal to 0 when $m \neq n$ and $\frac{T}{2}$ when $m = n \neq 0$? Besides

$$\int_{t_0}^{t_0+T} \cos(m\omega t)\cos(n\omega t)\,\mathrm{d}t$$

will equal to 0 when $m \neq n$ and $\frac{T}{2}$ when $m = n \neq 0$ and T when $m = n = 0$?

Answer. URL (version: 2014-01-14): http://math.stackexchange.com/q/638386

I will assume that T is the period and ω is the angular frequency[1] of the wave $\sin(\omega t)$. In such a case, which is important to obtain the final results, the following relation holds

$$\omega = \frac{2\pi}{T}. \tag{1}$$

Let $x = \omega t$, $x_0 = \omega t_0$. Then

$$I(m,n) = \int_{t_0}^{t_0+T} \sin(m\omega t)\sin(n\omega t)\,dt \tag{2}$$

$$= \frac{1}{\omega}\int_{x_0}^{x_0+2\pi} \sin(mx)\sin(nx)\,dx$$

$$= \frac{1}{2\omega}\int_{x_0}^{x_0+2\pi} \cos((m-n)\,x) - \cos((m+n)\,x)\,dx, \tag{3}$$

because in general

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta, \tag{4}$$

as can be seen by subtracting

[1] http://en.wikipedia.org/wiki/Angular_frequency

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

from

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

- For $m \neq n$, since

$$\int_{x_0}^{x_0+2\pi} \cos((m-n)\,x)\,dx = \left.\frac{\sin((m-n)\,x)}{m-n}\right|_{x_0}^{x_0+2\pi} = 0$$
$$\int_{x_0}^{x_0+2\pi} \cos((m+n)\,x)\,dx = \left.\frac{\sin((m+n)\,x)}{m+n}\right|_{x_0}^{x_0+2\pi} = 0 \qquad (5)$$

  the integral $I(m,n) = 0$.

- For $m = n \neq 0$, by (1)

$$\begin{aligned} I(m,n) = I(m,m) &= \frac{1}{2\omega}\int_{x_0}^{x_0+2\pi} 1 - \cos(2mx)\,dx \\ &= \frac{1}{2\omega}\left(x - \frac{\sin(2mx)}{2m}\right)\Bigg|_{x_0}^{x_0+2\pi} \\ &= \frac{1}{2\omega}(2\pi) = \frac{\pi}{\omega} = \frac{T}{2}. \end{aligned} \qquad (6)$$

The evaluation of the second integral is similar.

3.1.4 Help finding integral: $\int \frac{dx}{x\sqrt{1+x+x^2}}$

Question by Dave (http://math.stackexchange.com/users/28712/dave), Help finding integral: $\int \frac{dx}{x\sqrt{1+x+x^2}}$, URL (version: 2013-05-18): URL (version: 2013-05-18): http://math.stackexchange.com/q/129821

Could someone help me with finding this integral

$$\int \frac{dx}{x\sqrt{1+x+x^2}}$$

or give a hint on how to solve it.

Answer. URL (version: 2012-04-09): http://math.stackexchange.com/q/129824

Since the integrand is a quadratic irrational function of the type $R(x, \sqrt{1+x+x^2})$, you may use the Euler substitution[2] $\sqrt{1+x+x^2} = x + t$. You get

$$\begin{aligned}\int \frac{dx}{x\sqrt{1+x+x^2}} &= \int \frac{2}{t^2-1}\,dt \\ &= -2\operatorname{arctanh} t + C \\ &= -2\operatorname{arctanh}\left(\sqrt{1+x+x^2} - x\right) + C.\end{aligned}$$

3.1.5 Integration trig substitution $\int \frac{dx}{x\sqrt{x^2+16}}$

Question by Dantheman (http://math.stackexchange.com/users/84794/dantheman), Integration trig substitution $\int \frac{dx}{x\sqrt{x^2+16}}$, URL (version: 2013-07-03): http://math.stackexchange.com/q/434837

$$\int \frac{dx}{x\sqrt{x^2+16}}$$

With some magic I get down to

$$\frac{1}{4}\int \frac{1}{\sin\theta} d\theta$$

Now is where I am lost. How do I do this? I tried integration by parts but it doesn't work.

Answer. URL (version: 2013-07-03): http://math.stackexchange.com/q/434850

UPDATE 2 (...) The first integral (...)

$$\begin{aligned}I &= \int \frac{dx}{x\sqrt{x^2+16}}, \qquad x = \tan\theta, dx = \sec^2\theta d\theta \\ &= \int \frac{\sec^2\theta}{(\tan\theta)\, 4\sec\theta}\, d\theta = \int \frac{\sec\theta}{4\tan\theta}\, d\theta \\ &= \int \frac{\sec\theta}{4\tan\theta}\, d\theta = \int \frac{1}{4\sin\theta}\, d\theta.\end{aligned}$$

[2] http://www.encyclopediaofmath.org/index.php/Euler_substitutions

* * *

Use the Weierstrass substitution[3]

$$t = \tan\frac{\theta}{2}.$$

Then

$$\int \frac{1}{\sin\theta}\,d\theta = \int \frac{2}{\frac{2t}{1+t^2}\left(1+t^2\right)}\,dt = \int \frac{1}{t}\,dt = \ln|t| + C = \ln\left|\tan\frac{\theta}{2}\right| + C.$$

Comment: The Weierstrass substitution is a *universal standard substitution* to evaluate an integral of a rational fraction in $\sin\theta, \cos\theta$, i.e. a rational fraction of the form

$$R(\sin\theta,\cos\theta) = \frac{P(\sin\theta,\cos\theta)}{Q(\sin\theta,\cos\theta)},$$

where P, Q are polynomials in $\sin\theta, \cos\theta$

$$\tan\frac{\theta}{2} = t, \qquad \theta = 2\arctan t, \qquad d\theta = \frac{2}{1+t^2}dt,$$

which converts the integrand into a rational function in t. We know from trigonometry that

$$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{1-t^2}{1+t^2}, \qquad \sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{2t}{1+t^2}.$$

Proof. A possible proof is the following one, which uses the double-angle formulas and the identity $\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$:

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \frac{\frac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}}{\frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}} = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}},$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}}{\frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}} = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}.$$

[3] http://en.wikipedia.org/wiki/Weierstrass_substitution

* * *

Another possible substitution is the Euler substitution[4]

$$\sqrt{x^2+16} = t + x.$$

Then

$$\begin{aligned} I &= \int \frac{dx}{x\sqrt{x^2+16}} = \int \frac{2}{t^2-16}\,dt \\ &= \int \frac{1}{4\,(t-4)} - \frac{1}{4\,(t+4)} dt = \frac{1}{4}\ln\left|\frac{t-4}{t+4}\right| + C \\ &= \frac{1}{4}\ln\left|\frac{\sqrt{x^2+16}-x-4}{\sqrt{x^2+16}-x+4}\right| + C. \end{aligned}$$

3.1.6 Prove: $\int_0^1 \frac{\ln x}{x-1} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Question by Jozef (http://math.stackexchange.com/users/14829/jozef), Prove: $\int_0^1 \frac{\ln x}{x-1} dx = \sum_1^\infty \frac{1}{n^2}$, URL (version: 2014-01-01): http://math.stackexchange.com/q/108248

Answer. URL (version: 2012-02-11): http://math.stackexchange.com/q/108254

Hint: use the substitution $u = 1 - x$ to obtain

$$I := \int_0^1 \frac{\ln x}{x-1} dx = -\int_0^1 \frac{\ln(1-u)}{u}\,du$$

and the following Maclaurin series

$$\ln(1-u) = -u - \frac{1}{2}u^2 - \frac{1}{3}u^3 - \ldots - \frac{u^{n+1}}{n+1} - \ldots \qquad (|u| < 1).$$

[4] http://www.encyclopediaofmath.org/index.php/Euler_substitutions

3.1.7 Cat Dog problem using integration

Question by pokrate (http://math.stackexchange.com/users/50639/pokrate), Cat Dog problem using integration, URL (version: 2012-11-28): http://math.stackexchange.com/q/244333

(...)

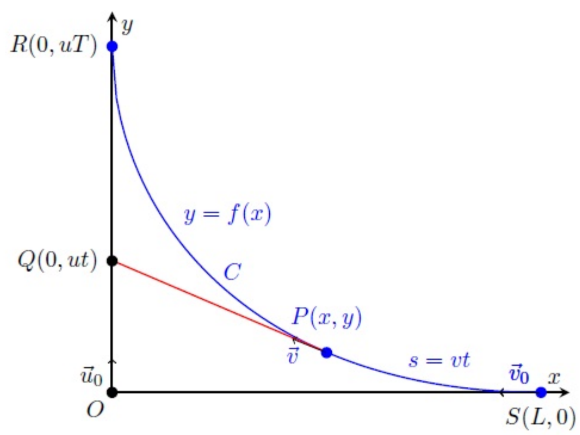
A cat sitting in a field suddenly sees a standing dog. To save its life, the cat runs away in a straight line with speed u. Without any delay, the dog starts with running with constant speed $v > u$ to catch the cat. Initially, v is perpendicular to u and L is the initial separation between the two. If the dog always changes its direction so that it is always heading directly at the cat, find the time the dog takes to catch the cat in terms of v, u and L.

(...)

Answer. URL (version: 2014-05-02): http://math.stackexchange.com/q/246113

(...)

I assume that: (a) the dog starts at the point $S = (L, 0)$ and the cat at the origin $O = (0, 0)$; (b) the cat moves in the positive direction along the y-axis, and the dog describes a *curve of pursuit* (WolframMathWorld link http://mathworld.wolfram.com/PursuitCurve.html) C in the xy-plane. I call $y = f(x)$ the equation of C.



1. At time t the tangent line to C at the point $P(x, y)$ passes through the point $Q = (0, ut)$, which means that the derivative $y' = f'(x) = dy/dx$ is

$$y' = \frac{y - ut}{x}. \tag{A}$$

Solving for t we get

$$t = \frac{y - xy'}{u}. \tag{A'}$$

2. Let s be the distance traveled by the dog from S to P, i.e. the length of the arc SP measured along C. Since the arc length formula [5] is the integral

$$s = \int_x^L \sqrt{1 + (f'(\xi))^2} d\xi = -\int_L^x \sqrt{1 + (f'(\xi))^2} d\xi, \tag{B}$$

and $s = vt$, we have

$$t = \frac{s}{v} = -\frac{1}{v} \int_L^x \sqrt{1 + (f'(\xi))^2} d\xi = \frac{y - xy'}{u}. \tag{B'}$$

Hence, equating (A') to (B'), we get

$$-\frac{u}{v} \int_L^x \sqrt{1 + (f'(\xi))^2} d\xi = y - xy' \tag{C}$$

3. Differentiate both sides and simplify

$$-\frac{u}{v}\sqrt{1 + (y')^2} = \frac{d}{dx}(y - xy')$$
$$-\frac{u}{v}\sqrt{1 + (y')^2} = y' - (y' + xy'') = -xy''.$$

to get the following differential equation

$$\sqrt{1 + (y')^2} = kxy'', \qquad k = \frac{v}{u} > 1. \tag{D}$$

[5] http://en.wikipedia.org/wiki/Arc_length

4. Set $w = y'$ and solve (D) for w applying the method of separation of variables [6]. Then

$$\sqrt{1+w^2} = kxw' = kx\frac{dw}{dx} \Leftrightarrow \frac{dw}{\sqrt{1+w^2}} = \frac{dx}{kx}. \tag{E}$$

So

$$\begin{aligned} \int \frac{dw}{\sqrt{1+w^2}} &= \int \frac{dx}{kx} + C \\ \operatorname{arcsinh} w &= \frac{1}{k}\ln x + \ln C_1. \end{aligned} \tag{F}$$

The initial condition $x = L, w = y'(L) = 0$ determines the constant C_1

$$0 = \frac{1}{k}\ln L + \ln C_1 \Rightarrow C_1 = e^{-\frac{1}{k}\ln L}.$$

Consequently,

$$\operatorname{arcsinh} w = \frac{1}{k}\ln x - \frac{1}{k}\ln L = \frac{1}{k}\ln\frac{x}{L}. \tag{G}$$

Solve (G) for w and rewrite in terms of exponentials using the definition of $\sinh z = \frac{1}{2}\left(e^z - e^{-z}\right)$

$$\frac{dy}{dx} = w = \sinh\left(\frac{1}{k}\ln\frac{x}{L}\right) = \frac{1}{2}\left(\left(\frac{x}{L}\right)^{1/k} - \left(\frac{x}{L}\right)^{-1/k}\right) \tag{H}$$

This last differential equation is easily integrable

$$\begin{aligned} y &= \frac{1}{2}\int \left(\frac{x}{L}\right)^{1/k} - \left(\frac{x}{L}\right)^{-1/k} dx \\ &= \frac{1}{2}\left(\frac{L}{1/k+1}\left(\frac{x}{L}\right)^{1/k+1} - \frac{L}{1-1/k}\left(\frac{x}{L}\right)^{1-1/k}\right) + C \end{aligned} \tag{I}$$

Find C making use of the initial condition $x = L, y = 0$

[6] http://en.wikipedia.org/wiki/Separation_of_variables

$$0 = \frac{1}{2}\left(\frac{L}{1/k+1}\left(\frac{L}{L}\right)^{1/k+1} - \frac{L}{1-1/k}\left(\frac{L}{L}\right)^{1-1/k}\right) + C$$
$$\Rightarrow C = \frac{Lk}{k^2-1}.$$

The equation of the trajectory is thus

$$y = \frac{L}{2}\left(\frac{1}{\frac{1}{k}+1}\left(\frac{x}{L}\right)^{\frac{1}{k}+1} - \frac{1}{1-\frac{1}{k}}\left(\frac{x}{L}\right)^{1-\frac{1}{k}}\right) + \frac{Lk}{k^2-1}. \qquad \text{(J)}$$

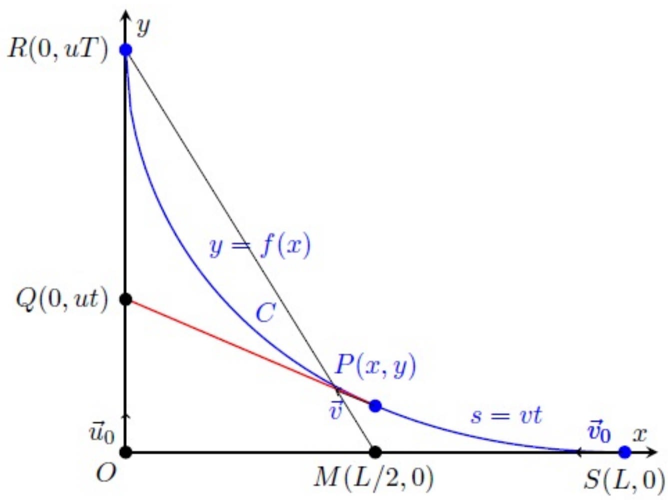
5. To obtain the time T the dog takes to catch the cat, make $x = 0$ in the last equation and observe that the cat travels the distance $y = f(0) = uT$ (point (R)):

$$y = f(0) = \frac{Lk}{k^2-1} = \frac{Lv/u}{(v/u)^2-1} = \frac{uv}{v^2-u^2}L = uT. \qquad \text{(K)}$$

Therefore

$$T = L\frac{v}{v^2-u^2}. \qquad \text{(L)}$$

ADDED. Let M be the point $(L/2, 0)$. We can easily verify that the total length of C is equal to $\overline{SM} + \overline{MR}$.



References: [1, 2, 3, 4]

3.2 Derivatives

3.2.1 Does this equality always hold?

Question by ghshtalt (http://math.stackexchange.com/users/3711/ghshtalt), Does this equality always hold?, URL (version: 2011-06-04): http://math.stackexchange.com/q/43157

Is it true in general that $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(u,x)\mathrm{d}u = \int_0^x \left(\frac{\mathrm{d}}{\mathrm{d}x}f(u,x)\right)\mathrm{d}u + f(x,x)$?

Answer. URL (version: 2013-06-20): http://math.stackexchange.com/q/43166

Yes, it is, under the conditions indicated below. Let

$$I(x) = \int_0^x f(u,x)\,\mathrm{d}u. \qquad (*)$$

If $f(u,x)$ is a continuous function and $\partial f/\partial x$ exists and is continuous, then

$$I'(x) = \int_0^x \frac{\partial f(u,x)}{\partial x}\,\mathrm{d}u + f(x,x) \qquad (**)$$

follows from the Leibniz rule and chain rule.

Note: the integrand of $(**)$ is a partial derivative. It generalizes to the integral

$$I(x) = \int_{u(x)}^{v(x)} f(t,x)\,\mathrm{d}t.$$

Under suitable conditions ($u(x), v(x)$ are differentiable functions, $f(t,x)$ is a continuous function and $\partial f/\partial x$ exists and is continuous), we have

$$I'(x) = \int_{u(x)}^{v(x)} \frac{\partial f(t,x)}{\partial x}\,\mathrm{d}t + f(v(x),x)v'(x) - f(u(x),x)u'(x).$$

3.3 Series

3.3.1 Nice proofs of $\zeta(4) = \dfrac{\pi^4}{90}$?

Question by Mike Spivey (http://math.stackexchange.com/users/2370/mike-spivey), Nice proofs of $\zeta(4) = \pi^4/90$?, URL (version: 2012-04-04): http://math.stackexchange.com/q/28329

Answer. URL (version: 2011-03-21): http://math.stackexchange.com/q/28338

In the same spirit of the 1st proof of the answer in 4.1.1 to the question "Different methods to compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$". If we substitute x for π in the Fourier trigonometric series expansion of $f(x) = x^4$, with $-\pi \leq x \leq \pi$,

$$x^4 = \frac{1}{5}\pi^4 + \sum_{n=1}^{\infty} \frac{8n^2\pi^2 - 48}{n^4} \cos n\pi \cdot \cos nx,$$

we obtain

$$\begin{aligned}\pi^4 &= \frac{1}{5}\pi^4 + \sum_{n=1}^{\infty} \frac{8n^2\pi^2 - 48}{n^4} \cos^2 n\pi \\ &= \frac{1}{5}\pi^4 + 8\pi^2 \sum_{n=1}^{\infty} \frac{1}{n^2} - 48 \sum_{n=1}^{\infty} \frac{1}{n^4}.\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{48}\left(-1 + \frac{1}{5} + \frac{8}{6}\right) = \frac{\pi^4}{48} \cdot \frac{8}{15} = \frac{1}{90}\pi^4.$$

3.4 Inequalities

3.4.1 Proving the inequality $e^{-2x} \leq 1 - x$

Question by sssuuuccc (http://math.stackexchange.com/users/22708/sssuuuccc), Proving the inequality $e^{-2x} \leq 1 - x$, URL (version: 2012-01-10): http://math.stackexchange.com/q/97909

How do I prove the inequality $e^{-2x} \leq 1 - x$ for $0 \leq x \leq 1/2$?

Answer. URL (version: 2014-05-24): http://math.stackexchange.com/q/97993

By Taylor expansion, we have

$$\ln \frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{x^n}{n},$$

whose convergence radius is $R = 1$.

The equation above can also be achieved by integrating both sides of

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

For $0 \le x \le 1/2$ we have the following upper bound

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \le \sum_{n=1}^{\infty} \frac{1}{n\,(2^n)} = \ln 2 \le 2.$$

Therefore

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \le 2x.$$

The given inequality follows.

Chap. 4

Fourier Series

4.1 Trigonometric series

4.1.1 Different methods to compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Question by AD (http://math.stackexchange.com/users/1154/ad), Different methods to compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$, URL (version: 2012-06-19): http://math.stackexchange.com/q/8337

Answer. URL (version: 2013-12-11): http://math.stackexchange.com/q/8378

We can use the function $f(x) = x^2$ with $-\pi \leq x \leq \pi$ and find its expansion into a trigonometric Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin x),$$

which is periodic and converges to $f(x)$ in $-\pi \leq x \leq \pi$. Observing that $f(x)$ is even, it is enough to determine the coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \qquad n = 0, 1, 2, 3, ...,$$

because

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0 \qquad n = 1, 2, 3, ...$$

For $n = 0$ we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2\pi^2}{3}.$$

And for $n = 1, 2, 3, ...$ we get

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \, dx \\ &= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx = \frac{2}{\pi} \times \frac{2\pi}{n^2}(-1)^n = (-1)^n \frac{4}{n^2}, \end{aligned}$$

because

$$\int x^2 \cos nx \, dx = \frac{2x}{n^2}\cos nx + \left(\frac{x^2}{n} - \frac{2}{n^3}\right)\sin nx.$$

Thus

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left((-1)^n \frac{1}{n^2}\cos nx\right).$$

Since $f(\pi) = \pi^2$, we obtain

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left((-1)^n \frac{4}{n^2}\cos(n\pi)\right)$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left((-1)^n(-1)^n \frac{1}{n^2}\right)$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Therefore

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}$$

* * *

Second method (available on-line a few years ago) by Eric Rowland. From

$$\log(1-t) = -\sum_{n=1}^{\infty}\frac{t^n}{n}$$

and making the substitution $t = e^{ix}$ one gets the series expansion

$$w = \operatorname{Log}(1 - e^{ix}) = -\sum_{n=1}^{\infty} \frac{e^{inx}}{n} = -\sum_{n=1}^{\infty} \frac{1}{n} \cos nx - i \sum_{n=1}^{\infty} \frac{1}{n} \sin nx,$$

whose radius of convergence is 1. Now if we take the imaginary part of both sides, the RHS becomes

$$\Im w = -\sum_{n=1}^{\infty} \frac{1}{n} \sin nx,$$

and the LHS

$$\Im w = \arg\left(1 - \cos x - i \sin x\right) = \arctan \frac{-\sin x}{1 - \cos x}.$$

Since

$$\begin{aligned}
\arctan \frac{-\sin x}{1 - \cos x} &= -\arctan \frac{2 \sin \frac{x}{2} \cdot \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}} \\
&= -\arctan \cot \frac{x}{2} \\
&= -\arctan \tan \left(\frac{\pi}{2} - \frac{x}{2} \right) \\
&= \frac{x}{2} - \frac{\pi}{2},
\end{aligned}$$

the following expansion holds

$$\frac{\pi}{2} - \frac{x}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx. \qquad (*)$$

Integrating the identity $(*)$, we obtain

$$\frac{\pi}{2} x - \frac{x^2}{4} + C = -\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx. \qquad (**)$$

Setting $x = 0$, we get the relation between C and $\zeta(2)$

$$C = -\sum_{n=1}^{\infty} \frac{1}{n^2} = -\zeta(2).$$

And for $x = \pi$, since

$$\zeta(2) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2},$$

we deduce

$$\frac{\pi^2}{4} + C = -\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{1}{2}\zeta(2) = -\frac{1}{2}C.$$

Solving for C

$$C = -\frac{\pi^2}{6},$$

we thus prove

$$\zeta(2) = \frac{\pi^2}{6}.$$

Note: this 2nd method can generate all the zeta values $\zeta(2n)$ by integrating repeatedly $(**)$. This is the reason why I appreciate it. Unfortunately it does not work for $\zeta(2n+1)$.

4.1.2 Fourier Series for $|\cos(x)|$

Question by Cloud15 (http://math.stackexchange.com/users/131545/cloud15, Fourier Series for $|\cos(x)|$, URL (version: 2014-03-04): http://math.stackexchange.com/q/697843

I'm having trouble figuring out the Fourier series of $|\cos(x)|$ from $-\pi$ to π. I understand its an even function, so all the b_ns are 0

$$a_0 = \frac{2}{\pi}\int_0^{\pi}|\cos(x)|\,dx = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}|\cos(x)|\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}\cos^2(x)\,dx.$$

Since for all j, k not equal the integral is zero.
So only a_1 remains. is this correct?

How would I evaluate $\sum_{n=1}^{\infty}(-1)^{n-1}/(4n^2-1)$?

Answer. URL (version: 2014-03-04): http://math.stackexchange.com/q/699257

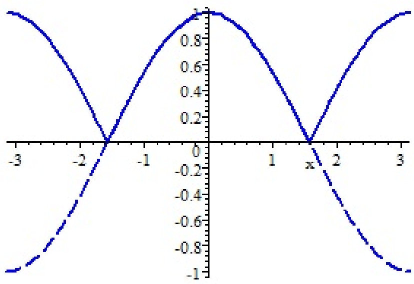
Although $\int_0^\pi \cos(x)\,dx = 0$, $a_0 \neq 0$ because

$$\int_0^{\pi/2} |\cos(x)|\,dx = \int_{\pi/2}^{\pi} |\cos(x)|\,dx.$$

We can evaluate it as follows, as can be seen in the plot below

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos(x)|\,dx = \frac{2}{\pi}\int_0^{\pi} |\cos(x)|\,dx = \frac{4}{\pi}\int_0^{\pi/2} |\cos(x)|\,dx = \frac{4}{\pi}\int_0^{\pi/2} \cos(x)\,dx = \frac{4}{\pi}. \tag{1}$$

Plot of $\cos x$ (doted line) and $|\cos x|$ (solid line) in the interval $[-\pi, \pi]$.



The coefficients $b_n = 0$ as you concluded. As for the a_n coefficients only the odd ones are equal to 0 (see below). The functions $\cos(x)$ and $\cos(nx)$ are orthogonal in the interval $[-\pi, \pi]$, but $|\cos(x)|$ and $\cos(nx)$ are not. Since

$$|\cos(x)| = \begin{cases} \cos(x) & \text{if} \quad 0 \le x \le \pi/2 \\ -\cos(x) & \text{if} \quad \pi/2 \le x \le \pi, \end{cases} \tag{2}$$

we have that

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}|\cos(x)|\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}|\cos(x)|\cos(nx)\,dx \\
&= \frac{2}{\pi}\int_0^{\pi/2}|\cos(x)|\cos(nx)\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}|\cos(x)|\cos(nx)\,dx \\
&= \frac{2}{\pi}\int_0^{\pi/2}\cos(x)\cos(nx)\,dx - \frac{2}{\pi}\int_{\pi/2}^{\pi}\cos(x)\cos(nx)\,dx. \\
a_1 &= \frac{2}{\pi}\int_0^{\pi/2}\cos^2(x)\,dx - \frac{2}{\pi}\int_{\pi/2}^{\pi}\cos^2(x)\,dx = 0.
\end{aligned}$$

Using the following trigonometric identity, with $a = x, b = nx$,

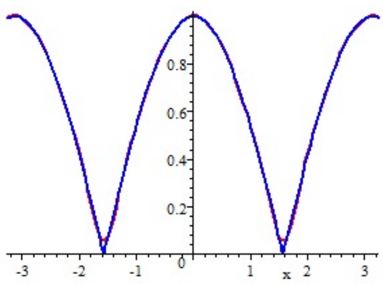
$$\cos(a)\cos(b) = \frac{\cos(a+b)+\cos(a-b)}{2}, \tag{3}$$

we find

$$\begin{aligned}
a_{2m} &= \frac{4}{\pi\,(1-4m^2)}\cos(\frac{2m\pi}{2}) = \frac{4}{\pi\,(1-4m^2)}(-1)^m \\
a_{2m+1} &= \frac{4}{\pi(1-4(2m+1)^2)}\cos(\frac{(2m+1)\pi}{2}) = 0, \qquad m = 1,2,3,\dots.
\end{aligned} \tag{4}$$

The expansion of $|\cos(x)|$ into a trigonometric Fourier series in the interval $[-\pi,\pi]$ is thus

$$|\cos x| = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx)) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{1-4m^2}\cos(2mx) \tag{5}$$



$|\sin(x)|$ (blue) and the partial sum $\frac{2}{\pi} + \frac{4}{\pi}\sum_{m=1}^{5}\frac{(-1)^m}{1-4m^2}\cos(2mx)$ (red) in $[-\pi,\pi]$

Setting $x = 0$ in (5), we obtain

$$1 = \frac{2}{\pi} + \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{1-4m^2} = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{1-4n^2}. \tag{6}$$

Hence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{1-4n^2} = \frac{1}{2} - \frac{\pi}{4}. \tag{7}$$

Chap. 5

Complex Analysis

5.1 Laurent Series

5.1.1 Finding the Laurent series of $f(z) = 1/((z-1)(z-2))$

Question by Freeman (http://math.stackexchange.com/users/11533/freeman, Finding the Laurent series of $f(z) = 1/((z-1)(z-2))$, URL (version: 2011-11-04): http://math.stackexchange.com/q/79012

Let

$$f(z) = \frac{1}{(z-1)(z-2)}$$

and let

$$R_1 = \left\{ z \;\middle|\; 1 < |z| < 2 \right\} \quad \text{and} \quad R_2 = \left\{ z \;\middle|\; |z| > 2 \right\}.$$

How do you find the Laurent series convergent on R_1? Also how do you do it for R_2?

I'm having serious trouble with this as I can't see how to expand things into series with n as any integer, not just natural number. Also how to apply Cauchy's integral formula to an annulus. If anyone can explain this to me I will be extremely grateful.

Answer. URL (version: 2011-11-04): http://math.stackexchange.com/q/79030

The function $f(z)$ can be expanded into two partial fractions

$$f(z) := \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

We now expand each partial fraction into a geometric series. On R_2 these series are

$$\begin{aligned}\frac{1}{z-2} &= \frac{1}{z(1-2/z)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n \qquad |z| > 2 \\ &= \frac{1}{z}\sum_{n=0}^{\infty} 2^n \frac{1}{z^n} = \sum_{n=0}^{\infty} 2^n \frac{1}{z^{n+1}}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{z-1} &= \frac{1}{z(1-1/z)} \\ &= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} \qquad |z| > 1.\end{aligned}$$

And so, the Laurent series is

$$\frac{1}{(z-1)(z-2)} = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}(2^n - 1) \qquad |z| > 2 > 1.$$

On R_1, the two geometric series are

$$\begin{aligned}\frac{1}{z-2} &= \frac{-1/2}{1-z/2} = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)\left(\frac{z}{2}\right)^n \qquad |z| < 2 \\ &= \sum_{n=0}^{\infty} -\frac{1}{2^{n+1}} z^n\end{aligned}$$

and

$$\begin{aligned}\frac{1}{z-1} &= \frac{1/z}{1-1/z} = \sum_{n=0}^{\infty}\frac{1}{z}\left(\frac{1}{z}\right)^n \qquad |z| > 1 \\ &= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}.\end{aligned}$$

We thus get the following Laurent series

$$\frac{1}{(z-1)(z-2)} = \sum_{n=0}^{\infty}\left(-\frac{1}{2^{n+1}}z^n - \frac{1}{z^{n+1}}\right) \qquad 1 < |z| < 2.$$

5.2 Evaluating definite integrals by the residue theorem

5.2.1 Verify integrals with residue theorem

Question by emka (http://math.stackexchange.com/users/8324/emka), Verify integrals with residue theorem, URL (version: 2011-10-13): http://math.stackexchange.com/q/71932

This is another problem that I got stuck on during my self-study of Complex Variables.

$$\int_0^\infty \frac{\cos(ax)}{(1+x^2)^2}dx \quad \text{with } a>0$$

This is equivalent to

$$\frac{1}{2}\int_{-\infty}^\infty \frac{\cos(ax)}{(1+x^2)^2}dx$$

I know the following:
(1) There is a pole at $x=i$
(2) We work with the half-circle with radius R.

Answer. URL (version: 2011-10-13): http://math.stackexchange.com/q/71994

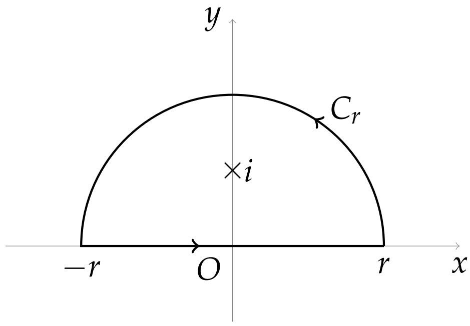
Let

$$f(z)=\frac{e^{iaz}}{(1+z^2)^2}=\frac{e^{iaz}}{(z-i)^2(z+i)^2}. \tag{1}$$

The residue of $f(z)$ at $z=i$ is[1]

$$\begin{aligned}\operatorname*{res}_{z=i} f(z) &= \frac{1}{(2-1)!}\lim_{z\to i}\frac{d}{dz}\left((z-i)^2 f(z)\right)\\ &= \lim_{z\to i}\frac{d}{dz}\left(\frac{e^{iaz}}{(z+i)^2}\right)=\lim_{z\to i}\frac{iae^{iaz}(z+i)^2-e^{iaz}2\,(z+i)}{(z+i)^4}\\ &= -\frac{1}{4}i\,(a+1)\,e^{-a}.\end{aligned} \tag{2}$$

[1] http://en.wikipedia.org/wiki/Residue_(complex_analysis)



Let C_R denote the boundary of the upper half of the disk $|z| = R$, described counterclockwise (see picture). By the residue theorem[2]

$$\begin{aligned}\int_{-R}^{R} \frac{e^{iax}}{\left(1+x^2\right)^2} dx + \int_{C_R} \frac{e^{iaz}}{\left(1+z^2\right)^2} dz &= 2\pi i \operatorname*{res}_{z=i} f(z) e^{iaz} \\ &= \frac{1}{2}\pi\,(a+1)\,e^{-a}. \end{aligned} \tag{3}$$

Then

$$\operatorname{Re}\int_{-R}^{R} \frac{e^{iax}}{\left(1+x^2\right)^2} dx + \operatorname{Re}\int_{C_R} \frac{e^{iaz}}{\left(1+z^2\right)^2} dz = \operatorname{Re}\frac{1}{2}\pi\,(a+1)\,e^{-a},$$

$$\int_{-R}^{R} \frac{\cos ax}{\left(1+x^2\right)^2} dx + \operatorname{Re}\int_{C_R} \frac{e^{iaz}}{\left(1+z^2\right)^2} dz = \frac{1}{2}\pi\,(a+1)\,e^{-a}. \tag{4}$$

When $|z| = R$, we have

$$\left|\frac{1}{(1+z^2)^2}\right| = \frac{1}{|z+i|^2\,|z-i|^2} \le \frac{1}{||z|-|i|^2\,||z|-|i||^2} = \frac{1}{(R-1)^4} =: M_R, \tag{5}$$

which means that $M_R > 0$ and

$$\lim_{R\to\infty} M_R = \lim_{R\to\infty} \frac{1}{(R-1)^4} = 0. \tag{6}$$

Then we can apply the Jordan's lemma for every *positive constant a* and conclude that

[2] http://en.wikipedia.org/wiki/Residue_theorem

$$\lim_{R\to\infty}\int_{C_R}\frac{e^{iaz}}{(1+z^2)^2}dz=0. \tag{7}$$

Consequently,

$$\int_{-\infty}^{\infty}\frac{\cos ax}{(1+x^2)^2}dx=\frac{1}{2}\pi\,(a+1)\,e^{-a} \tag{8}$$

and

$$\int_{0}^{\infty}\frac{\cos ax}{(1+x^2)^2}dx=\frac{1}{4}\pi\,(a+1)\,e^{-a}. \tag{9}$$

Note: Exercise 3 on page 265 of [7] generalizes this integral to

$$\int_{0}^{\infty}\frac{\cos ax}{(b^2+x^2)^2}dx=\frac{1}{4b^3}\pi\,(ab+1)\,e^{-ab} \qquad (a>0,b>0).$$

5.2.2 Evaluate an improper integral using complex analysis

Question by Simplyorange (https://math.stackexchange.com/users/419831/simplyorange), VEvaluate an improper integral using complex analysis, URL (version: 2020-06-21): https://math.stackexchange.com/q/3728465

I got stuck trying to find

$$\int_0^\infty \frac{\log x}{(x+a)^2+b^2}\,dx$$

using complex analysis.

My attempt is to evaluate the contour integral of

$$\int_C \frac{\log z}{(z+a)^2+b^2}\,dz$$

for some nicely chosen contour C. [...]

Answer. URL (version: 2020-07-09): https://math.stackexchange.com/q/3748855

Based on this[3] answer to the question[4] "How to evaluate $\int_0^\infty \frac{\log x}{(x^2+a^2)^2}dx$" and on this[5] answer to the question[6] "Evaluate $\int_0^\infty \frac{(\log x)^4 dx}{(1+x)(1+x^2)}$". Instead of a function with $\log z$ on the numerator, we consider a function with $\log^2 z$. This is the very same method as that pointed to in the comments.

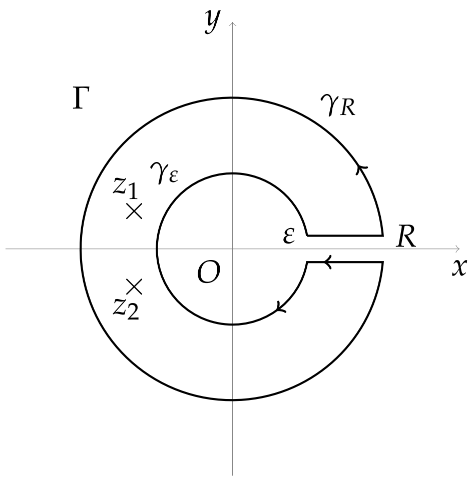
For $a, b > 0$, this method gives the closed formula

$$\int_0^\infty \frac{\log x}{(x+a)^2+b^2}\,dx = \frac{1}{2b}\arctan\left(\frac{b}{a}\right)\log\left(a^2+b^2\right), \qquad a, b > 0. \qquad (*)$$

We chose the multiple-valued function $f(z)$ with branch cut $\arg z = 0$ defined as

$$\begin{aligned} f(z) &= \frac{\log^2 z}{(z+a)^2+b^2}, \quad \text{with } 0 < \arg z < 2\pi, \quad z = re^{i\theta} \\ &= \frac{\log^2 z}{(z-z_1)(z-z_2)} \qquad z_1 = -a+ib, \quad z_2 = -a-ib, \end{aligned}$$

and integrate it counterclockwise around the closed contour Γ shown in the figure. This contour is indented around the branch point O and consists of the circles γ_R ($|z| = R$) and γ_ε ($|z| = \varepsilon$), $0 < \varepsilon < 1 < R$, and the segment $[\varepsilon, R]$ described in the positive sense above the x-axis and in the negative sense below the x-axis.



[3] https://math.stackexchange.com/a/1480810/752
[4] https://math.stackexchange.com/q/2571670/752
[5] https://math.stackexchange.com/a/2572587/752
[6] https://math.stackexchange.com/q/2571670/752

On the upper edge, $\theta = 0$ ($r \in [\varepsilon, R]$) and

$$f(z) = \frac{(\log r)^2}{(r+a)^2 + b^2}.$$

On the lower edge, $\theta = 2\pi$ ($r \in [\varepsilon, R]$) and

$$f(z) = \frac{\left(\log\left(re^{i2\pi}\right)\right)^2}{(r+a)^2 + b^2} = \frac{(\log r + i2\pi)^2}{(r+a)^2 + b^2}.$$

As such,

$$\begin{aligned}
I &= \lim_{\varepsilon \to 0, R \to \infty} \oint_\Gamma \frac{(\log z)^2}{(z+a)^2 + b^2}\, dz, \\
&= \int_0^\infty \frac{(\log r)^2}{(r+a)^2 + b^2}\, dr - \int_0^\infty \frac{\left(\log\left(re^{i2\pi}\right)\right)^2}{(re^{i2\pi} + a)^2 + b^2}\, dr \\
&\quad + \lim_{R \to \infty} \int_{\gamma_R} \frac{(\log z)^2}{(z+a)^2 + b^2}\, dz - \lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} \frac{(\log z)^2}{(z+a)^2 + b^2}\, dz \\
&= \int_0^\infty \frac{(\log r)^2 - (\log r + i2\pi)^2}{(r+a)^2 + b^2}\, dx \\
&= 4\pi^2 \int_0^\infty \frac{1}{(r+a)^2 + b^2}\, dr - i4\pi \int_0^\infty \frac{\log r}{(r+a)^2 + b^2}\, dr
\end{aligned}$$

provided that

$$\lim_{R \to \infty} \int_{\gamma_R} \frac{(\log z)^2}{(z+a)^2 + b^2}\, dz = \lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} \frac{(\log z)^2}{(z+a)^2 + b^2}\, dz = 0, \quad \text{(see below)}.$$

By the residue theorem,

$$\begin{aligned}
I &= 2\pi i\left(\mathrm{Res}_{z=z_1} f(z) + \mathrm{Res}_{z=z_2} f(z)\right)\\
&= 2\pi i\left[\mathrm{Res}_{z=z_1}\frac{(\log z)^2}{(z-z_1)(z-z_2)} + \mathrm{Res}_{z=z_2}\frac{(\log z)^2}{(z-z_1)(z-z_2)}\right]\\
&= 2\pi i\left[\frac{(\log z_1)^2}{z_1-z_2} + \frac{(\log z_2)^2}{z_2-z_1}\right]\\
&= 2\pi i\left[\frac{(\log(-a+ib))^2}{i2b} - \frac{(\log(-a-ib))^2}{i2b}\right]\\
&= \frac{\pi}{b}\left[\log(-a+ib)\right]^2 - \frac{\pi}{b}\left[\log(-a-ib)\right]^2
\end{aligned}$$

We now assume that $a, b > 0$. Then

$$\begin{aligned}
I = \frac{\pi}{b}&\left[\log(|-a+ib|) + i\left(\pi - \arctan\left(\frac{b}{a}\right)\right)\right]^2\\
&- \frac{\pi}{b}\left[\log(|-a-ib|) + i\left(\pi + \arctan\left(\frac{b}{a}\right)\right)\right]^2\\
&= \frac{\pi}{b}\left[\frac{1}{2}\log\left(a^2+b^2\right) + i\left(\pi - \arctan\left(\frac{b}{a}\right)\right)\right]^2\\
&- \frac{\pi}{b}\left[\frac{1}{2}\log\left(a^2+b^2\right) + i\left(\pi + \arctan\left(\frac{b}{a}\right)\right)\right]^2\\
&= \frac{4\pi^2}{b}\arctan\left(\frac{b}{a}\right) - i\frac{2\pi}{b}\arctan\left(\frac{b}{a}\right)\log\left(a^2+b^2\right)
\end{aligned}$$

because

$$\log(|-a+ib|) = \log(|-a-ib|) = \frac{1}{2}\log\left(a^2+b^2\right).$$

Taking the imaginary part of I we obtain $(*)$ in the form

$$\mathrm{Im}\,(I) = -4\pi\int_0^\infty \frac{\log r}{(r+a)^2+b^2}\,dr = -\frac{2\pi}{b}\arctan\left(\frac{b}{a}\right)\log\left(a^2+b^2\right)$$

Proof that $\int_{\gamma_R} f, \int_{\gamma_\varepsilon} f \to 0$. If z is any point on γ_R,

$$
\begin{aligned}
|f(z)| &= \frac{|\log z|^2}{\left|(z+a)^2+b^2\right|}, \qquad z = R\,e^{i\theta}, R > 1, 0 < \theta < 2\pi \\
&\leq \frac{(\log R + 2\pi)^2}{|z+(-z_1)|\,|z+(-z_2)|}, \\
&\leq \frac{(\log R + 2\pi)^2}{\left|R - \sqrt{a^2+b^2}\right|^2} \leq M_R
\end{aligned}
$$

where

$$
M_R = \frac{4\pi \log R + 4\pi^2 + \log^2 R}{R^2 + (a^2+b^2) - 2R\sqrt{a^2+b^2}}
$$

because

$$
|z+(-z_1)| \geq |R - |z_1||\,, |z+(-z_2)| \geq |R - |z_2||\,, |z_1| = |z_2| = \sqrt{a^2+b^2}.
$$

This means that

$$
\begin{aligned}
\left|\int_{\gamma_R} f(z)\,dz\right| &\leq M_R \times 2\pi R \\
&= \frac{4\pi \log R + 4\pi^2 + \log^2 R}{R^2 + (a^2+b^2) - 2R\sqrt{a^2+b^2}} \times 2\pi R \longrightarrow 0 \qquad (R \to \infty)\,.
\end{aligned}
$$

Similarly, if z is any point on γ_ε

$$
\begin{aligned}
|f(z)| &= \frac{|\log z|^2}{\left|(z+a)^2+b^2\right|}, \qquad z = \varepsilon\,e^{i\theta}, 0 < \varepsilon < 1, 0 < \theta < 2\pi \\
&\leq \frac{(\log \varepsilon + 2\pi)^2}{|z+(-z_1)|\,|z+(-z_2)|} \\
&\leq \frac{(\log \varepsilon + 2\pi)^2}{\left|\varepsilon - \sqrt{a^2+b^2}\right|^2} \leq M_\varepsilon,
\end{aligned}
$$

where

$$M_\varepsilon = \frac{4\pi \log \varepsilon + 4\pi^2 + \log^2 \varepsilon}{\varepsilon^2 + (a^2 + b^2) - 2\varepsilon\sqrt{a^2 + b^2}}$$

and

$$\begin{aligned}\left|\int_{\gamma_\varepsilon} f(z)\, dz\right| &\leq M_\varepsilon \times\, 2\pi\varepsilon \qquad z = \rho\, e^{i\theta}, \rho < 1 \\ &\leq \frac{4\pi \log \varepsilon + 4\pi^2 + \log^2 \varepsilon}{\varepsilon^2 + (a^2 + b^2) - 2\varepsilon\sqrt{a^2 + b^2}} \times\, 2\pi\varepsilon \longrightarrow 0 \qquad (\varepsilon \to 0)\,.\end{aligned}$$

Part II

Questions

Chap. 6

A few questions

6.1 Real Analysis and Calculus

6.1.1 Asymptotic (divergent) series

Question Asymptotic (divergent) series, URL (version: 2013-06-21): http://math.stackexchange.com/q/123632

MOTIVATION. After having read in detail an article by Alf van der Poorten I read a very short paper by Roger Apéry. I am interested in finding a proof of a series expansion in the latter, which is in not given in it. So *I assumed it should be stated or derived from a theorem on the subject*.

In Apéry, R., *Irrationalité de* $\zeta 2$ *et* $\zeta 3$, Société Mathématique de France, Astérisque 61 (1979) there is a *divergent* series expansion for a function I would like to understand. Here is my translation of the relevant part for this question

> *(...) given a real sequence* $a_1, a_2, \ldots, a_k$*, an analytic function* $f(x)$ *with respect to the variable* $\frac{1}{x}$ *tending to* 0 *with* $\frac{1}{x}$ *admits a (unique) expansion in the form*

$$f(x) \equiv \sum_{k\geq 1} \frac{c_k}{(x+a_1)(x+a_2)\ldots(x+a_k)}. \tag{A}$$

(...) and the translation by Generic Human (http://math.stackexchange.com/users/26855/generic-human) of the text after the formula:

> *(We write* $\equiv$ *instead of* $=$ *to take into account the aversions of mathematicians who, following Abel, Cauchy and d'Alembert, hold divergent series to*

be an invention of the devil; in fact, we only ever use a finite sum of terms, but the number of terms is an unbounded function of x.)

Remark. As far as I understand, based on this last text, the expansion of $f(x)$ in (A) is in general a *divergent* series and not a convergent one, but the existing answer [by WimC] seems to indicate the opposite.

The corresponding finite sum appears and is proved in section 3 of Alfred van der Poorten's article *A proof that Euler missed ... Apéry's proof of the irrationality of $\zeta(3)$* as

For all $a_1, a_2, \ldots$

$$\sum_{k=1}^{K} \frac{a_1 a_2 \cdots a_{k-1}}{(x+a_1)(x+a_2)\cdots(x+a_k)} = \frac{1}{x} - \frac{a_1 a_2 \cdots a_K}{x(x+a_1)(x+a_2)\cdots(x+a_K)}, \qquad (\mathrm{A}')$$

Questions:

1. Is series (A) indeed divergent?
2. Which is the theorem stating or from which expansion (A) can be derived?
3. Could you please indicate a reference?

Answer by robjohn (http://math.stackexchange.com/users/13854/robjohn), Asymptotic (divergent) series, URL (version: 2012-06-13): http://math.stackexchange.com/q/157890

Writing $g_1(x) = f(1/x)$ gives

$$g_1(x) \equiv \sum_{k\geq 1} \frac{c_k x^k}{(1+a_1x)(1+a_2x)\ldots(1+a_kx)} \qquad (1)$$

which vanishes at $x = 0$.

Recursively define

$$g_{n+1}(x) = \frac{(1+a_nx)g_n(x)}{x} - c_n \qquad (2)$$

where

$$c_n = \lim_{x \to 0} \frac{g_n(x)}{x} \tag{3}$$

Then

$$g_n(x) \equiv \sum_{k \geq n} \frac{c_k x^{k-n+1}}{(1 + a_n x)(1 + a_{n+1} x) \ldots (1 + a_k x)} \tag{4}$$

is another series like (1) (which vanishes at $x = 0$).

The series in (1) may or may not converge, as with the Euler-Maclaurin Sum Series. As with most asymptotic series, we are only interested in the first several terms; the remainder (not the remaining terms) can be bounded by something smaller than the preceding terms. Therefore, convergence is not an issue.

6.1.2 What is the importance of Calculus in today's Mathematics?

Question What is the importance of Calculus in today's Mathematics?, URL (version: 2013-04-29): http://math.stackexchange.com/q/58104

For engineering (e. g. electrical engineering) and physics, Calculus is important. But for a future mathematician, is the classical approach to Calculus still important? What is normally taught, as a minimum, in most Universities worldwide?

(...)

Edit: Hoping it is useful, I transcribe three comments of mine (to this question):

1. I had [have] in mind for instance Tom Apostol's books, although learning differentiation before integration. (in response to Qiaochu Yuan's "What is the classical approach to calculus?")

2. Elementary Calculus, continuous functions, functions of several variables, partial differentiation, implicit-functions, vectors and vector fields, multiple integrals, infinite series, uniform convergence, power series, Fourier series and integrals, etc. (in response to a comment by Geoff Robinson).

3. I had [have] in mind calculus for math students, although I am a retired electrical engineer. (in response to Andy's comment "Are you talking about what is usually taught to engineers and physicists, or also about a calculus curriculum for math majors? ")

(...)

Answer by Pete L. Clark (http://math.stackexchange.com/users/299/pete-l-clark), What is the importance of Calculus in today's Mathematics?, URL (version: 2012-12-20): http://math.stackexchange.com/q/58119

In a comment to his question, Américo has clarified that by "classical calculus" he means something relatively rigorous and theoretical, as for instance in Apostol's book (or Spivak's). I think the answer to the question was probably yes no matter what, but when restricted in this way it becomes a big booming **YES**.

The methods of rigorous calculus – may I say elementary real analysis? it seems more specific – are an indispensable part of the cultural knowledge of all mathematicians, pure and applied. Not all mathematicians will *directly* use this material in their work: I for one am a mathematician with relatively broad interests almost to a fault, but I have never written "by the Fundamental Theorem of Calculus" or "by the Mean Value Theorem" in any of my research papers. But nevertheless familiarity and even deep understanding of these basic ideas and themes permeates all of modern mathematics. For instance, as an arithmetic geometer the functions I differentiate are usually polynomials or rational functions, but the idea of differentiation is still there, in fact abstracted in the notion of derivations and modules of differentials. One of the most important concepts in algebraic / arithmetic geometry is *smoothness*, and although you could in principle try to swallow this as a piece of pure algebra, I say good luck with that if you have never taken multivariable calculus and understood the inverse and implicit function theorems.

Eschewing "classical" mathematics in favor of more modern, abstract or specialized topics is one of the biggest traps a bright young student of mathematics can fall into. (If you spend any time at a place like Harvard, as I did as a graduate student, you see undergraduates falling for this with distressing regularity, almost as if the floor outside your office was carpeted with banana peels.) The people who created the fancy modern machinery did so by virtue of their knowledge of classical stuff, and are responding to it in ways that are profound even if they are unfortunately not made explicit. Although I am very far from really knowing what I'm talking about here, my feeling is that the analogy to the fine arts is rather apt: abstract modern art is very much *a response* to classical, figurative, realistic (I was tempted to say "mimetic", so I had better end this digression soon!) art: if you decide to forego learning about perspective in favor of arranging black squares on a white canvas, you're severely missing the point.

The material of elementary real analysis – and even freshman calculus – is remarkably rich. I have taught more or less the same freshman calculus courses about a dozen times, and each time I find something new to think about, sometimes in resonance with my other mathematical thoughts of the moment but sometimes I just find that I have the chance to stop and think about something that never occurred to me before. Once for instance I was talking about computing volumes of solids of revolution and it occurred to me that I had never thought about *proving in general* that the method of shells will give the same answer as the method of washers. It was pretty good fun to do it, and I mentioned it to a couple of my colleagues and they had a similar reaction: "No, I never thought of that before, but it sounds like fun." There are thousands of little projects and discoveries like this in freshman calculus.

I confess though that it would be interesting to hear mathematicians talk about parts of calculus that they never liked and never had any use for. As for me, I really dread the part of the course where we do related rates problems and min / max problems. The former seems like an exercise whose only point is to exploit – sometimes to the point of cruelty – the shakiness of students' understanding of implicit differentiation, and the latter was sort of fun for me for the first ten problems but twenty years and thousands of min / max problems later I could hardly imagine something more tedious. (Moreover I am *not that good* at these problems. I had a couple of embarrassing failures as a graduate student, and ever since I look to make sure I can do the problems before I assign them, something I have stopped needing to do in most other undergraduate courses.)

Added: Let me be explicit that I am not answering the second part of the question, i.e., what is a minimum that is or should be taught. It goes hand in hand with the richness of these topics that if you tried to make a list of everything that it would be valuable for students to know, your (surely severely incomplete!) list would contain vastly more material than could be reasonably covered in the allotted courses. This is one subject where books which aim to be "comprehensive" come off as pretty daunting. For instance I own the first of Courant and John's two volumes on advanced calculus: it's more than six hundred pages! Is there anything in there which I am willing to point to as "dispensable"? Not much. (Not to mention that the second volume of their work comes in two parts, the second part of which is itself 954 pages long!) The challenge of teaching these courses lies in the fact that the potential landscape is almost infinite and virtually none of it manifestly unimportant, so you have to make hard choices about what not to do.

6.1.3 What does closed form solution usually mean?

Question What does closed form solution usually mean?, URL (version: 2011-11-25): http://math.stackexchange.com/q/9199

This is motivated by this question[1] and the fact that I have no access to Timothy Chow's paper *What Is a Closed-Form Number?*[2] indicated there by Qiaochu Yuan.

If an equation $f(x) = 0$ has no closed form solution, what does it normally mean? Added: f may depend (and normally does) on parameters.

To me this is equivalent to say that one cannot solve it for x in the sense that there is no elementary expression $g(c_1, c_2, \ldots, c_p)$ consisting only of a finite number of polynomials, rational functions, roots, exponentials, logarithmic and trigonometric functions, absolute values, integer and fractional parts, such that

$$f(g(c_1, c_2, \ldots, c_p)) = 0.$$

Answer by J. M. (http://math.stackexchange.com/users/498/498), What does closed form solution usually mean?, URL (version: 2010-11-07): http://math.stackexchange.com/q/9203

I would say it very much depends on the context, and what tools are at your disposal. For instance, telling a student who's just mastered the usual tricks of integrating elementary functions that

$$\int \frac{\exp u - 1}{u} \mathrm{d}u$$

and

$$\int \sqrt{(u+1)(u^2+1)}\mathrm{d}u$$

[1] http://math.stackexchange.com/questions/8933
[2] http://www.jstor.org/pss/2589148

have no closed form solutions is just the fancy way of saying "no, you can't do these integrals yet; you don't have the tools". To a working scientist who uses exponential and elliptic integrals, however, they do have closed forms.

In a similar vein, when we say that nonlinear equations, whether algebraic ones like $x^5 - x + 1 = 0$ or transcendental ones like $\frac{\pi}{4} = v - \frac{\sin v}{2}$ have no closed form solutions, what we're really saying is that we can't represent solutions to these in terms of functions that we know (and love?). (For the first one, though, if you know hypergeometric or theta functions, then yes, it has a closed form.)

I believe it is fair to say that for as long as we haven't seen the solution to an integral, sum, product, continued fraction, differential equation, or nonlinear equation frequently enough in applications to give it a *standard* name and notation, we just cop out and say "nope, it doesn't have a closed form".

6.2 Number Theory

6.2.1 Binomial coefficients: how to prove an inequality on the p-adic valuation?

Question Binomial coefficients: how to prove an inequality on the p-adic valuation?, URL (version: 2014-05-25): http://math.stackexchange.com/q/2791

In section 4 of the article by Afred van der Poorten's *A Proof That Euler Missed* ... the following inequality is used:

$$\nu_p\binom{n}{m} \leq \left\lfloor \frac{\ln n}{\ln p} \right\rfloor - \nu_p(m) \qquad (*)$$

(*In the original denoted* ord $_p(\cdot)$ *instead of* $\nu_p(\cdot)$),

where $\nu_p(k)$ is the p-adic valuation of $k \in \mathbb{Q}$, i. e. the exponent of the prime p in the prime factorization of k. I know some properties of the foor function and that

$$\nu_p(a/b) = \nu_p(a) - \nu_p(b),$$

$$\nu_p(a \cdot b) = \nu_p(a) \cdot \nu_p(b)$$

and

$$v_p(n!) = \sum_{i\geq 1} \left\lfloor \frac{n}{p^i} \right\rfloor$$

but I didn't convince myself on the correct argument I should use to prove $(*)$.

Question: How can this inequality be proven?

If an equation $f(x) = 0$ has no closed form solution, what does it normally mean? Added: f may depend (and normally does) on parameters.

Answer by Grigory M (http://math.stackexchange.com/users/152/grigory-m), Binomial coefficients: how to prove an inequality on the p-adic valuation?, URL (version: 2010-08-19): http://math.stackexchange.com/q/2792

Since $v_p(n!) = \sum_i \lfloor n/p^i \rfloor$, $v_p\binom{n}{m} = \sum_i (\lfloor n/p^i \rfloor - \lfloor m/p^i \rfloor - \lfloor (n-m)/p^i \rfloor)$. Observe that each summand is ≤ 1, and for $i > \left\lfloor \frac{\ln n}{\ln p} \right\rfloor$ it's clearly 0. That gives $v_p\binom{n}{m} \leq \left\lfloor \frac{\ln n}{\ln p} \right\rfloor$.

Finally observe that for $i \leq v_p(m)$ $\lfloor m/p^i \rfloor = m/p^i$, so $(\lfloor m/p^i \rfloor + \lfloor x/p^i \rfloor = \lfloor (m+x)/p^i \rfloor$ and) the corresponding summand is also 0. That gives the inequality in question.

The appendix fragment is used only once. Subsequent appendices can be created using the Chapter Section/Body Tag.

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