Problem 101

If the graph of $f : \mathbb{R} \to \mathbb{R}$ is closed and connected then f is continuous. This does not extend to maps between general connected metric spaces.

Problem 102.

Let $I = (a, b)$ be a finite or infinite open interval in $\mathbb{R}$ and d be a metric on it which is equivalent to the usual metric. Prove that there exist disjoint closed sets A and B in I such that $d(A, B) = 0$.

Problem 103

Suppose $A \subset \mathbb{R}^n$ is such that the distance between any two points of A is rational. Prove that A is at most countable.

Problem 104

Let $A \subset \mathbb{R}^n$ be countable. Show that $\mathbb{R}^n \backslash A$ is connected.

Problem 105

Let X be a separable normed linear space and f be a continuous linear functional on a subspace M of X. Show without using Zorn's Lemma (or any of its equivalents) that f can be extended to a continuous linear functional on X with the same norm.

Problem 106

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(x) > \int\limits_0^x f(t)dt$ $\forall t \in [0, 1]$. Prove that $f(x) > 0$ $\forall x \in [0, 1]$.

Problem 107

Let $p(x) = x^2 + ax + b$ and A be the 3×3 matrix with entries $p(i - j), 0 \leq i, j \leq 2$. Show that the determinant of A does not depend on the coefficients of p.

Problem 108

Let A be a bounded set in a Hilbert space. Show that there is a unique closed ball of minimal radius containing A.

Problem 109 [See also Problem 1]

Let μ be a finite positive measure on the Borel subsets of $(0,\infty)$. If $g \in L^{\infty}(\mu)$ and $\int\limits_0^\infty e^{-x}p(x)g(x)d\mu(x) = 0$ for every polynomial p show that $g = 0$ a.e. $[\mu]$. Conclude that $\{e^{-x}p(x) : p \text{ is a polynomial}\}$ is dense in $L^1(\mu)$.

Problem 110

Let k be a positive integer. Find all continuous functions $f : (0,\infty) \to (0,\infty)$ such that $x \to \int\limits_x^{kx} f(t)dt$ is constant on $(0,\infty)$.

Problem 111

Let $A \subset \mathbb{C}$ be a convex set such that $x \in A \Rightarrow -x \in A$. If $a_1, a_2, a_3 \in A$ show that at least one of the 6 numbers $a_1+a_2, a_1-a_2, a_2+a_3, a_2-a_3, a_3+a_1, a_3-a_1$ must be in A.

Problem 112

Show that every polynomial p with real coefficients and real roots satisfies the inequality $(n-1)[p'(x)]^2 \geq np(x)[p''(x)]$ where n is the degree of p.

Problem 113

Find $\sup\{\frac{(\int\limits_0^1 f(x)dx)^2(\int\limits_0^1 g(x)dx)^2}{\int\limits_0^1 [f(x)]^2dx\int\limits_0^1 [g(x)]^2dx} : f, g : [01] \to \mathbb{R}$ are continuous, $\int\limits_0^1 f(x)g(x)dx = 0\}$.

Problem 114

a) Let U be an open set in $\mathbb{R}, F$ a closed set and $U \subset F$. Show that there is a set A whose interior is U and closure is F.

b) Find all sets $A \subset \mathbb{R}$ such that $A = \partial B$ for some $B \subset \mathbb{R}$.

Problem 115

Let H be a Hilbert space and C be a closed convex subset. For any $x \in H$ let Px be the unique point of C that is closest to C. Show that $\|x-y\|^2 \geq \|x-Px\|^2 + \|y-Px\|^2 \; \forall y \in C$.

Problem 116

Let $\{x \in \mathbb{R}^n : \|x\| = 1\} \subset \bigcup_{j=1}^{n} \bar{B}(x_j, r_j)$ where $\bar{B}(x_j, r_j)$ is the closed ball with center x_j and radius r_j. Show that $0 \in \bar{B}(x_j, r_j)$ for some j. Show that the conclusion is false if the number of closed balls is allowed to exceed n.

Problem 117

Let C be a closed convex set in a Hilbert space H. Let $P(x)$ be the point of C closest to x. Show that $\|P(x) - P(y)\| \leq \|x - y\| \ \forall x, y \in H$. [See also Problem 118 below].

Problem 118

In Problem 117 show that $\|P(x_1) - P(x_2)\| < \|x_1 - x_2\|$ unless $P(x_1) - P(x_2) = x_1 - x_2$.

Problem 119

Let $f : [0, \infty) \to [1, \infty)$ be non-decreasing with $\int_1^\infty \frac{1}{f(x)} dx = \infty$. Show that $\int_1^\infty \frac{1}{x \log(f(x))} dx = \infty$. Can we also assert that $\int_1^\infty \frac{1}{x \log(f(x)) \log(\log(f(x)))} dx = \infty$?

Problem 120

a) Let (X, d) be a compact metric space and $T : X \to X$ be onto. If $d(Tx, Ty) \leq d(x, y) \ \forall x, y$ prove that $d(Tx, Ty) = d(x, y) \ \forall x, y$.

b) Let (X, d) be a compact metric space and a continuous map $T : X \to X$ satisfy $d(Tx, Ty) \geq d(x, y) \ \forall x, y$. Prove that the conclusion of part a) holds.

Remark: several improvements of these results are given in the next few problems. See also Problem 423.

Problem 121

Let (X, d) be a compact metric space and $T : X \to X$ satisfy $d(Tx, Ty) \geq d(x, y)$ for all $x, y \in X$. Then T is an isometry of X onto itself. [Thus continuity of T need not be assumed in previous problem]

[See also Problem 234 below]

Problem 122

Find an error in the following proof given in American Math. Monthly, vol. 98, no. 7, 1991 (p. 664).

Let X be a compact metric space and $T : X \to X$ be any map with $\inf\limits_{n\geq 1} d(T^n x, T^n y) > 0$ whenever $x \neq y$. Show that $T(X) = X$. Solution: let $D(x,y) = \inf\limits_{n\geq 0} d(T^n x, T^n y)$ where $T^0 = I$. D is a metric and $D \leq d$. It follows by compactness of X that the identity map $i : (X,d) \to (X,D)$ is a homeomorphism and (X,D) is a compact metric space. By definition $D(Tx,Ty) \geq D(x,y)$. By Problem 121 above T is an isometry of X onto itself.

Problem 123

Is the product of two derivatives on $\mathbb{R}$ necessarily a derivative?

Problem124

Let $p,q \in (1,\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$ and f,g be non-negative continuous functions on $\mathbb{R}$ with compact support. Show that $\int \sup\limits_{y}\{f(x-y)g(y)\}dx \geq \|f\|_p \|g\|_q$.

Problem 125

a) Find all positive numbers α such that there is a positive C^1 function f on $(0,\infty)$ with $f'(x) \geq a[f(x)]^\alpha$ for all x sufficiently large for some $a \in (0,\infty)$.

b) Does there exist a positive C^1 function f on $(0,\infty)$ with $f'(x) \geq af(f(x))$ for all x sufficiently large for some $a \in (0,\infty)$?

Problem 126

Let $a_n > 0$ and $\sum\limits_{n=1}^{\infty} a_n \log(1 + \frac{1}{a_n}) < \infty$. Show that $\sum\limits_{n=1}^{\infty} \frac{a_n}{\|x-b_n\|^k} < \infty$ almost everywhere for any sequence $\{b_n\} \subset \mathbb{R}^k$. [$\|\|$ is the norm in $\mathbb{R}^k$].

Problem 127

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f \circ g$ is Riemann integrable on $[0,1]$ whenever $g : [0,1] \to \mathbb{R}$ is continuous. Show that f is continuous on $\mathbb{R}$.

Problem 128

Is the set of all $n \times n$ invertible matrices dense in the space of all $n \times n$ matrices? Is the space of all invertible operators on a Hilbert space dense in the space of all operators on that space?

Problem 129

Let A be any $n \times n$ matrix. For any positive integer k show that there is a unique $n \times n$ matrix B such that $B(B^*B)^k = A$.

Problem 130

Let $f:\mathbb{R}\to\mathbb{R}$. Can differentiability of f at 0 be characterized by the condition $\sum f(x_n)$ converges whenever $\sum x_n$ converges? [We say f is convergence preserving (CP) if $\sum f(x_n)$ converges whenever $\sum x_n$ converges].

Problem 131

Find a necessary and sufficient condition for f to be convergence preserving. [See Problem 130 for definition of convergence preserving functions].

Problem 132

Let $f:(0,1)\to(0,1)$ be a continuous function such that for any $x\in(0,1)$ there is an integer n such that $f_{(n)}(x)=x$ where $f_{(1)}=f$ and $f_{(n)}=f\circ f_{(n-1)}$ for $n\geq 2$. Show that $f(x)=x\ \forall x\in(0,1)$. Is the result true if $(0,1)$ is replaced by $[0,1]$?

Problem 133

a) Let $f:\{0,1,2...\}\to\{0,1,2...\}$ satisfy $f(m^2+n^2)=f^2(m)+f^2(n)$ $\forall m,n\geq 0$. Show that either $f(n)=0$ for all n or $f(n)=n$ for all n

b) Let $f:[0,\infty)\to[0,\infty)$ satisfy $f(x^2+y^2)=f^2(x)+f^2(y)\ \forall x,y\geq 0$. If f is continuous show that $f\equiv 0$ or $f\equiv\frac{1}{2}$ or $f(x)=x$ for all x.

Problem 134

Let C be a bounded subset of $V\equiv\mathbb{R}^n$ or $\mathbb{C}^n$ such that for each $x\in V$ there is a unique point Px of C which is closest to it. Show that C is closed and convex.

Problem 135

In the literature there are two definitions of adjoint of an $n\times n$ matrix A. According to one definition the conjugate transpose of a matrix is called the adjoint. According to the other definition the (i,j) element of the adjoint matrix is the co-factor of $a_{i,j}$: the determinant of the matrix obtained by deleting the $i-th$ row and the $j-th$ column. Find all matrices for which the definitions lead to the same adjoint.

Problem 136

Let B be a bounded set in a Banach space X. Show that the following are equivalent:

a) B is an open ball

b) for any two points x, y in B there is an open ball V contained in B and containing x and y.

Problem 137

Give an explicit example of a Borel set in $\mathbb{R}$ which is neither an F_σ nor a G_δ.

Problem 138

Compute $\max\{\min\{|x_i - x_j| : i \neq j\} : x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, \|x\| \leq 1\}$.

Problem 139

Let H be a Hilbert space and $\{x_1, x_2, ..., x_n\}$ a finite subset of H. Find explicitly the point x in H for which $\sum_{j=1}^{n} \|x - x_j\|^2$ is minimum.

Problem 140

Let H and K be Hilbert spaces, $y_1, y_2, ..., y_n \in K$ and $A_1, A_2, ..., A_n$ be bounded operators from H to K. Show that $\sum_{j=1}^{n} \|A_j x - y_j\|^2$ is minimized at $x = x_0$ if and only if $\sum_{j=1}^{n} A_j^* A_j x_0 = \sum_{j=1}^{n} A_j^* y_j$. [$n = 1, A_1 = I$ reduces this to previous problem. If the positive operator $\sum_{j=1}^{n} A_j^* A_j$ is invertible then there is a unique x_0].

Problem 141

Consider the inequality $\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \|x - y\|$ where $\|x\| \geq 1$ and $\|y\| \geq 1$. Is the inequality true in any inner product space space? Is it true in any normed linear space?

Problem 142

Let k be a positive integer and A, B be $k \times k$ matrices with $AB^k - B^k A = B$. Show that B is nilpotent.

Problem 143

Let X be a normed linear space and define S_x as $\{y \in X : \|x + y\|^2 = \|x\|^2 + \|y\|^2\}$. Show that the following are equivalent:

i) X is an inner product space
ii) for any $x \in X$, any $y \in S_x$ and any $a \in \mathbb{R}$ we also have $ay \in S_x$.

Problem 144

Let X, Y, Z be random variables on a probability space such that $\alpha X + \beta Y + \gamma Z$ has uniform distribution on $(-1, 1)$ whenever the real numbers α, β, γ satisfy $\alpha^2 + \beta^2 + \gamma^2 = 1$. Show that $X^2 + Y^2 + Z^2 = 1$ almost surely.

Problem 145

Let $f : [0, 1] \to [0, \infty)$ be a continuously differentiable function. Let L be the length of the graph of f and A the area under the graph. Show that $A + L > \pi/4$.

Problem 146

Show that a random variable X has a symmetric distribution if and only if $\int\limits_0^\infty P\{|X - t| \leq a\} dt = a$ for all $a > 0$.

Problem 147

Show that $\mathbb{R}^n$ cannot be written as the union of a family $\{D_i : i \in I\}$ of closed balls (of positive radius) such that $(D_i)^0 \cap (D_j)^0 = \emptyset$ for $i \neq j$. [A^0 denotes the interior of A].

Problem 148

Does there exist a partition $\{A_i : i \in I\}$ of $\mathbb{R}$ such that the sigma field generated by this partition coincides with the Borel sigma field?

Problem 149

Characterize all C^∞ functions f from an open interval I in $\mathbb{R}$ into $\mathbb{R}$ such that f satisfies a differential equation of the type $f^{(n)} + g_{n-1} f^{(n-1)} + ... + g_1 f' + g_0 f = 0$ where the $g_i' s$ are all continuous.

Problem 150

Let $c_1, c_2, ..., c_N$ be distinct non-zero complex numbers. Show that $\sum_{k=1}^{N} \frac{1}{c_k} \prod_{j \neq k} \frac{1}{c_j - c_k} = \frac{(-1)^{N+1}}{c_1 c_2 ... c_N}$.

Problem 151

Let d_1 and d_2 be two metrics on a set X such that any open ball w.r.t one contains an open ball w.r.t. the other. Does it follow that the metrics are equivalent (in the sense they have the same open sets)?

Problem 152

Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms. If $b_n \uparrow 1$ in such a way that $[\log(n)][1 - b_n]$ is bounded show that $\sum_{n=1}^{\infty} a_n^{b_n} < \infty$.

Problem 153

Let P and Q be orthogonal projections on finite dimensional complex Hilbert space H. Show that PQ is an orthogonal projection if and only if all eigen values of $P + Q$ belong to $\{0\} \cup [1, \infty)$.

Problem 154

Let Ω be an open connected relatively compact subset of a metric space (X, d). Assume $\partial\Omega \neq \emptyset$. Let $f : \Omega \to \Omega$ be a continuous map such that its range $f(\Omega)$ is open. Show that $d(f(x_0), \partial\Omega) = d(x_0, \partial\Omega)$ for some $x_0 \in \Omega$.

Problem 155

Let P and Q be projections on $\mathbb{C}^n$. If $Tr(PQPQ) = Tr(PQ)$ show that PQ is a projection. [Tr stands for trace].

Problem 156

Let $0 < p < \infty, \{a_n\} \subseteq \mathbb{R}$ and $|a_1| \geq |a_2| \geq \ldots$. Then $\sup\{t^p \#\{n : |a_n| > t\} : t > 0\} = \sup\{n \, |a_n|^p : n \in \mathbb{N}\}$. (Both sides may be ∞).

Remark: given $\{a_n\} \subseteq \mathbb{R}$ we can always rearrange the sequence such that $|a_1| \geq |a_2| \geq \ldots$. Since the left side of the identity does not change under permutations we get $\sup\{t^p \#\{n : |a_n| > t\} : t > 0\} = \sup\{n^p b_n : n \in \mathbb{N}\}$ where $\{b_n : n \in \mathbb{N}\}$ is a non-increasing rearrangement of $\{|a_n|\}$.

Problem 157

In Problem 156 assume that $p > 1$. Show that $(\sup\{t^p \#\{n : |a_n| > t\} : t > 0\})^{1/p}$ is finite if $\{a_n\} \in l^p$. Also prove that finiteness of $(\sup\{t^r \#\{n : |a_n| > t\} : t > 0\})^{1/r}$ for some $r \in (0, p)$ implies $\{a_n\} \in l^p$ and that $(\sup\{t^p \#\{n : |a_n| > t\} : t > 0\})^{1/p} \leq \|\{a_n\}\|_p \leq C((\sup\{t^r \#\{n : |a_n| > t\} : t > 0\})^{1/r}$ for some constant $C = C(p, r)$.

Problem 158

If A and B are projections on $\mathbb{C}^n$ show that the following operators have the same range:

$AB - BA, ABA - BAB, (AB)^2 - (BA)^2$.

Problem 159

Prove or disprove the following:

if $f : \mathbb{R} \to \mathbb{R}$ is continuous then there exists $a \in \mathbb{R}$ such that $|f(a)| - |f(x)| < |a - x|$ for all $x \neq a$.

Problem 160

Let A and B be $n \times n$ matricies with real entries such that $A^2 + B^2 = AB - BA$. If $AB - BA$ is invertible show that 4 divides n.

Problem 161

Let τ be the topology of pointwise convergence on $C[0, 1]$. Describe all continuous linear maps $\Lambda : (C[0, 1], \tau) \to \mathbb{C}$.

Problem 162

Let P and Q be projections onto closed subspaces M and N of a Hilbert space H. Find a necessary and sufficient condition on M and N for PQ to be a projection.

Problem 163

Let $f, g : [0, 1] \to \mathbb{R}$ be continuous. If $\int_0^1 fg = 0$ show that $(\int_0^1 g^2)(\int_0^1 f^2) \geq 4[(\int_0^1 f)(\int_0^1 g)]^2$. Also show that $(\int_0^1 g^2)(\int_0^1 f)^2 + (\int_0^1 f^2)(\int_0^1 g)^2 \geq 4[(\int_0^1 f)(\int_0^1 g)]^2$.

Problem 164

Let P and Q be projections on $\mathbb{C}^n$. Show that any eigen value of $PQ+QP$ is $\geq -1/4$. Is $-1/4$ attained?

Problem 165

Let A be an $n\times n$ complex matrix such that $A^2=0$. Show that $R(A+A^*)=R(A)+R(A^*)$ where $R(T)$ denotes the range of T.

Problem 166

Let $f\in C[0,1]$ and $f(1)=0$. Show that there exists $a\in(0,1]$ with $f(a)=\int\limits_0^a f(x)dx$.

Problem 167

Prove that $\sum\limits_{k=0}^{n}\binom{n}{k}\binom{2k}{k}=\sum\limits_{k\leq n/2}\binom{n}{2k}\binom{2k}{k}3^{n-2k}$ for any positive integer n.

Problem 168

Let $f,g:[0,1]\to\mathbb{R}$ be continuous. Prove that there exists $a\in(0,1)$ such that $\int\limits_0^1 f(x)dx\int\limits_0^a xg(x)dx=\int\limits_0^1 g(x)dx\int\limits_0^a xf(x)dx$.

Problem 169

Prove or disprove that if A is set of (Lebesgue) measure 0 in $\mathbb{R}$ and $\epsilon>0$ then there exist intervals $I_1,I_2,...$ such that $A\subset\bigcup\limits_n I_n$ and the length of I_n does not exceed $\epsilon/2^n$ for any n.

Problem 170

Show that there is no continuous function $f:(0.\infty)\to\mathbb{R}$ such that $f(x)=0\Leftrightarrow f(2x)\neq 0$.

Problem 171

Prove that $x \int_x^{x+1} \sin(t^2)dt < 1$ for all $x > 1$.

Problem 172

Let A be a compact subset of $\mathbb{R}$ and $P(A)$ the collection of all non-constant polynomials with real coefficients with leading coefficient 1. [Leading coefficient of $p(x)$ is the coefficient of the highest power x in p]. Let $\|p\|$ be $\sup\{|p(x)| : x \in A\}$.

a) Show that if there exists $p \in P(A)$ with $\|p\| < 2$ then there exists $p \in P(A)$ with $\|p\| < 1$

b) Show that if $A = [-2, 2]$ then there is no $p \in P(A)$ with $\|p\| < 2$.

Problem 173

Let A be a discrete subset of $\mathbb{R}$. [i.e. $a \in A \Rightarrow \exists\ \delta > 0$ such that $A \cap (a - \delta, a + \delta) = \{a)$]. Can the closure of A be uncountable?

Problem 174

Prove Banach's Theorem that any isometric map T from one normed linear X onto another normed linear Y with $T(0) = 0$ is linear.

Problem 175

Let (X, d) be a metric space, $A \subset X$ and $f : X \to (0, \infty)$ a map such that $f(x)f(y) \leq d(x, y)$ whenever $x \in A$ and $y \in A^c$. Show that A and A^c are F_σ sets, i.e. they are countable unions of closed sets. Conversely, if A and A^c are F_σ sets show that such a function f exists.

Problem 176

If $f : (0, \infty) \to \mathbb{R}$ is a measurable function such that $f(x + y)$ lies between $f(x)$ and $f(y)$ for all x and y show that f is a constant. Give an example of a non-constant (non-measurable) function with this property.

Problem 177

Let C be a closed convex set in a normed linear space such that for some $\delta > 0$, $\|x\| \leq 1 + \delta$ implies $x = c + y$ with $c \in C$ and $\|y\| \leq 1$. Show that the interior of C is non-empty.

Problem 178

Let $f : [0,\infty) \to \mathbb{R}$ be a C^∞ function such that $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in [0,\infty)$ and for all $n \geq 0$. Show that the function g defined by $g(x) = \frac{f(0)-f(x)}{x}(x > 0), g(0) = -f'(0)$ has the same property.

Problem 179

A Lemma in Rudin's real and Complex Analysis says that if $c_1, c_2, ..., c_N$ are complex numbers then we can find $S \subset \{1, 2, ..., N\}$ such that $\left|\sum_{j \in S} c_j\right| \geq \frac{1}{\pi}\sum_{j=1}^{N} |c_j|$. Prove that for any $\epsilon > 0$ we can find an example where $\left|\sum_{j \in S} c_j\right| < (\frac{1}{\pi} + \epsilon)\sum_{j=1}^{N} |c_j|$.

Problem 180

Let $\Im$ be the collection of all $N \times N$ matrices A such that $a_{ij} \geq 0, \sum_i a_{ij} = 1$ and $\sum_j a_{ij} = 1$ for all i, j. [Matrices of this type are called Doubly Stochastic]. Find all matrices in $\Im$ that commute with all other matrices in $\Im$.

Problem 181

Show that there is a continuous map $f : \mathbb{R} \to \mathbb{R}$ which is *not* 1-1 but 1-1 on Q. Show that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous map which is 1-1 on Q^c then it is 1-1 on $\mathbb{R}$.

Problem 182

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuous. Prove that there is a non-empty proper closed C in $\mathbb{R}^2$ such that $f(C) \subset C$.

Problem 183

Let X and Y be random variables on a probability space such that $X, Y, X + Y$ and $X - Y$ all have the same distribution. Can we conclude that the common distribution is degenerate (at 0)? What if $EX^2 < \infty$? What if $E|X| < \infty$?

Problem 184

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove or disprove that there exists a continuous strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g$ is differentiable on $\mathbb{R}$.

Problem 185

Does there exist a dense set E in $\mathbb{R}^2$ such that every point in E has both coordinates rational but the distance between any two points of E is irrational?

Problem 186

Does there exist a dense subset S of the unit circle S^1 such that all points in S have rational coordinates and the distance between any two points of S is rational?

Problem 187

There is no function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = x^2 - 2 \ \forall x \in \mathbb{R}$.

Problem 188

Let $\mu_1, \mu_2, ..., \mu_n$ be non-atomic probability measures on $(\Omega, \mathcal{F})$. Show that there exist disjoint sets $A_1, A_2, ..., A_n$ in $\mathcal{F}$ such that $\mu_i(A_i) = \frac{1}{n}$ $(1 \leq i \leq n)$.

[See also Problem 295]
Problem 189

Let $\{X_i : i \in I\}$ be a family of random variables with finite mean. Which of the following condition imply which others?

a) $\{X_i : i \in I\}$ is uniformly integrable

b) There is an integrable random variable Y such that $|X_i| \leq Y$ a.s. for all $i \in I$

c) There is an integrable random variable Y such that $P\{|X_i| \geq a\} \leq P\{Y \geq a\}$ for all $i \in I$, for all $a \in [0, \infty)$.

Problem 190

Does there exist a compact set K in a normed linear space X such that every point in $X \backslash K$ has exactly two points in K closest to it?

Problem 191

If $f : (0,1) \times (0,1) \to \mathbb{R}$ is separately continuous and if f vanishes on a dense subset then it is identically 0.

Problem 192

Compute $\sup\{\inf\{\frac{f(x)}{x}\int\limits_0^x \{1-f(t)\}dt : x>0\} : f:[0,\infty)\to\mathbb{R}$ is continuous$\}$. Find all continuous functions f such that the supremum is attained at f.

Problem 193

Let P and Q be projections on a Hilbert space H. It is well known that $P \leq Q$ in the sense $< Px, x > \leq < Qx, x >$ for all $x \in H$ if and only if $P = PQ = QP$. Let $P\blacktriangledown Q$ be the glb of P and Q, i.e. the largest projection R which is $\leq P$ and $\leq Q$. If $P+Q$ is invertible show that $P\blacktriangledown Q = 2P(P+Q)^{-1}P$.

Problem 194

Let V and W be vector spaces and $T, S : V \to W$ be linear. Suppose for each $x \in V$ there is a scalar c_x such that $Tx = c_x Sx$ $(x \in V)$. Show that there is a scalar c such that $T = cS$.

Problem 195

Let $f : (a,b) \to \mathbb{R}$ satisfy the following conditions:

f is 1-1, $\liminf\limits_{y\to x+} f(y) \geq f(x)$, $\limsup\limits_{y\to x-} f(y) \leq f(x)$ for all $x \in (a,b)$ and $\limsup\limits_{y\to b-} f(y) = \inf\limits_{a<x<b} f(x)$. Show that f is strictly decreasing and continuous. Give an example to show that the last condition cannot be dropped. The same conclusion holds if $\liminf\limits_{y\to a+} f(y) = \sup\limits_{a<x<b} f(x)$.

Problem 196

Let $n \in \{2,3,4,5,6,7\}$. Does there exist an $n-$ times continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)f'(x)...f^{(n)}(x) < 0$ for all $x \in \mathbb{R}$?.

Problem 197

Let $n \in \{2,3,...\}$. If $x_1, x_2, ..., x_n \in \mathbb{R}^2, x_1 + x_2 + ... + x_n = 0$ and $\|x_i\| \leq 1$ $\forall i$ show that $\|x_i + x_j\| \leq 1$ for some i and j.

Problem 198

Let $x_1, x_2.x_3, x_{4,}x_5, x_6$ be vectors in $\Delta \equiv \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ whose sum is 0. Show there are three vectors among these whose sum belongs to Δ.

Problem 199

Show that there is no expanding continuous map from $\mathbb{R}^3$ to $\mathbb{R}^2$. [$f : \mathbb{R}^3 \to \mathbb{R}^2$ is expanding if $\|f(x) - f(y)\| \geq \|x - y\|$ for all x, y].

Problem 200

If $a_n > 0$ for all n show that $\limsup_n \frac{\log a_n}{\log n} \leq \limsup_n n(\frac{a_n}{a_{n-1}} - 1)$.