NEW KNOT INVARIANTS FROM BPS STATES

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Abstract. Counting BPS states in the 3d $\mathcal{N}=2$ theory associated to a knot complement yields a power series in two variables, $F_K(x,q)$. We review its construction and properties, as well as its relation to 3-manifold invariants.

1. Introduction

Quantum topology started with the discovery of the Jones polynomial [Jon85] and of its connection to quantum field theory [Wit89]. This led to the development of the Witten-Reshetikhin-Turaev (WRT) invariants of 3-manifolds, which were originally constructed by Witten as a path integral, and later given a rigorous mathematical definition by Reshetikhin and Turaev [RT91].

In the past twenty-five years, an important direction in quantum topology has been categorification—in particular, the replacement of numerical or polynomial invariants with vector spaces that contain more information. The prototype is Khovanov homology [Kho00], a link invariant whose Euler characteristic is the Jones polynomial. An obvious next step would be to do categorify the WRT invariants of 3-manifolds. However, these are complex numbers rather than integers, so it is not clear even what form the result should take. There is ongoing work about categorification at roots of unity [Kho16, Qi14, EQ16]. There are also invariants of 3- and 4-manifolds that generalize Khovanov homology from the perspective of skein theory [MWW22]; these are known as skein lasagna modules and have been recently proved to detect some exotic smooth structures on 4-manifolds with boundary [RW].

A different approach comes from quantum physics, namely from the 3d $\mathcal{N}=2$ theory on the fivebrane world-volume that, in three other directions, is supported on the 3-manifold. Gukov-Putrov-Vafa [GPV17] and Gukov-Pei-Putrov-Vafa [GPPV20] studied the counts of BPS states in that theory. The space of such states is a homology theory associated to a 3-manifold and a Spin^c structure a, and its graded Euler characteristic is a Laurent power series $\widehat{Z}_a(q)$. The limits of $\widehat{Z}_a(q)$ as q approaches roots of unity sometimes recover the WRT invariants, so one can think of $\widehat{Z}_a(q)$ as a substitute for WRT, with the advantage of having integral coefficients.

To be able to compute the $\widehat{Z}$ invariants for a large class of 3-manifolds, the authors of this note developed a related knot invariant $F_K(x,q)$. This is a power series in two variables, which is related to the $\widehat{Z}$ invariants of surgeries on a knot K by certain surgery formulae [GM21]. Physically, the invariant $F_K(x,q)$ counts BPS states in a theory associated to the knot complement. Mathematically, there are interesting connections to other parts of quantum topology: F_K can be obtained by resurgence from Rozansky's asymptotic expansion of the colored Jones polynomial, and satisfies a recurrence given by the quantum A-polynomial. Park [Par20b] showed that it can be computed for homogeneous knots using R-matrices. Further, Akhmechet, Johnson and Park found a common generalization of F_K and the knot lattice homology from Heegaard Floer theory [AJP].

In this paper, we summarize the construction of $F_K(x,q)$ and survey the related developments.

2. $\widehat{Z}$ invariants of 3-manifolds

We start by reviewing the $\widehat{Z}$ invariants of 3-manifolds from [GPV17], [GPPV20]. As of now there is no mathematical definition that is applicable to all 3-manifolds, but one is expected to exist, and based on physical arguments one can produce explicit formulae for various classes of 3-manifolds. The best known such formula is that for plumbed 3-manifolds from [GPPV20], which has been extensively used to test conjectures and inspire generalizations.

A plumbing is associated to a weighted graph Γ; that is, a graph with integers (weights) m_v associated to its vertices v. For each vertex v, one considers the disk bundle over S^2 with Euler number m_v, and plumbs together these bundles according to the edges of Γ. The result is a compact 4-manifold $W(\Gamma)$ whose boundary is a 3-manifold $Y = Y(\Gamma)$. Its first homology is $H_1(Y) = \mathbb{Z}^s/M\mathbb{Z}^s$, where s is the number of vertices and M is the matrix given by

$$M_{v_1,v_2} = \begin{cases} 1, & v_1, v_2 \text{ are connected}, \\ m_v, & v_1 = v_2 = v, \\ 0, & \text{otherwise.} \end{cases} \qquad v_i \in \mathrm{Vert}\,.$$

We let $\vec{\delta}$ be the vector whose entries are the degrees of the vertices of Γ.

We will focus on the case where Γ is a tree and M is negative definite, so that Y is a rational homology 3-sphere. Then, the invariants of Y are defined in [GPPV20] by the formula

$$\widehat{Z}_a(q) = q^{\frac{-3s-\sum_v m_v}{4}} \cdot \text{v.p.} \oint_{|z_v|=1} \prod_{v\in\mathrm{Vert}} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v}\right)^{2-\deg(v)} \cdot \Theta_a^{-M}(\vec{z}) \tag{1}$$

where

$$\Theta_a^{-M}(\vec{z}) = \sum_{\vec{\ell}\in 2M\mathbb{Z}^s+\vec{a}} q^{-\frac{(\vec{\ell},M^{-1}\vec{\ell})}{4}} \prod_{v\in\mathrm{Vert}} z_v^{\ell_v},$$

v.p. denotes taking the principal value of the integral, and $\vec{a} \in 2\mathbb{Z}^s + \vec{\delta}$ represents a class

$$a \in (2\mathbb{Z}^s + \vec{\delta})/(2M\mathbb{Z}^s) \cong \mathrm{Spin}^c(Y).$$

The formula (1) looks complicated, but produces concrete answers. For example, for the Poincaré sphere $P = \Sigma(2,3,5)$ (with its unique Spin^c structure), we have

$$\widehat{Z}_a(P) = q^{-3/2}(1 - q - q^3 - q^7 + q^8 + q^{14} + q^{20} + q^{29} - q^{31} - q^{42} + \cdots)$$

In [GM21], the authors checked algebraically that (1) produces an invariant of plumbed (negative-definite) manifolds. Specifically, we first generalized the formula to a larger class of weakly negative-definite plumbings. On this class we can apply a theorem of Neumann [Neu81], saying that any two plumbing presentations of the same 3-manifold are related by a sequence of the moves shown in Figure 1. We call these Neumann moves.

Theorem 2.1 (Proposition 4.6 in [GM21]). *The series $\widehat{Z}_a(q)$ is invariant under Neumann moves.*

By now, there is an extensive literature on the $\widehat{Z}$ invariants. For example:

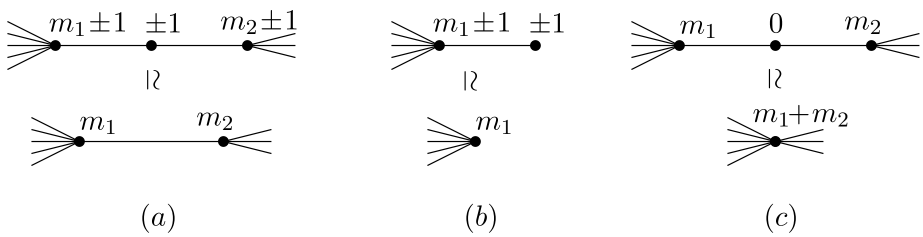

FIGURE 1. Neumann moves on weighted graphs.

- The $\widehat{Z}$ series have interesting number-theoretic (modularity) properties, which relate the invariant of a 3-manifold Y with that of $-Y$, the manifold with the opposite orientation [CCF+19, CCK+].
- There are indications that $\widehat{Z}_a(Y)$ contains interesting topological information, in particular invariants of homology cobordisms. Thus, in the limit $q \to i$ one recovers the Rokhlin invariants [GPP21], whereas in the limit $q \to 0$ one finds relations to the Ozsváth-Szabó correction terms [Har, HNS];
- The formula (1) can be generalized to some 3-manifolds with $b_1 > 0$, obtained by plumbings on graphs with cycles [CGPS20];
- The $\widehat{Z}$ invariants discussed so far are related to the Lie group $SU(2)$ (just like the Jones polynomial), but they can be generalized to any $SU(n)$. A higher rank analogue of the formula (1) is given in [Par20a].

3. A TWO-VARIABLE SERIES FOR KNOT COMPLEMENTS

A celebrated theorem of Lickorish and Wallace [Lic62, Wal60] says that every closed oriented 3-manifold can be obtained by surgery along a link in S^3. This suggests that one can define (or compute) 3-manifold invariants from link invariants. Indeed, this is how the WRT invariants were constructed in [RT91]. Furthermore, there are link surgery formulae in the context of Heegaard Floer homology [OS08, OS11, MO] and instanton Floer homology [LY25a, LY25b]. The authors' construction of the knot invariant in [GM21] was motivated by providing similar surgery formulae for the $\widehat{Z}$ series.

3.1. **Plumbed knot complements.** The starting point is adapting the formula (1) to plumbed knot complements. Consider a weighted graph Γ where we pick out a distinguished vertex v_0, for which we forget the weight. In the closed case, the plumbed manifold $\hat{Y}$ associated to Γ can be described as surgery on a link made of several framed unknots, where the framings are given by the weights, and for each edge (v, w) we link the two unknots corresponding to v and w. When we have a distinguished vertex v_0, we remove its unknot from the 3-manifold rather than fill it in by surgery. The result is a *plumbed knot complement* $Y \subset \hat{Y}$.

Given a (weakly negative definite) plumbed knot complement Y, the analogue of (1) is a power series $\widehat{Z}_a(Y; z, n, q)$ in two variables z and q, and also depending on an integer n. Up to a sign, it is given by the formula

$$(2) \qquad \left(z - \frac{1}{z}\right)^{1-\deg(v_0)} q^{\frac{3\sigma - \sum_v m_v}{4}} \oint_{|z_v|=1} \prod_{\substack{v\in \mathrm{Vert} \\ v\neq v_0}} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v}\right)^{2-\deg(v)} \Theta_a^{-M}(\vec{z})$$

where

$$\Theta_a^{-M}(\vec{z}) = \sum_{\vec{\ell}} q^{-\frac{(\vec{\ell}, M^{-1}\vec{\ell})}{4}} \prod_{v \in \mathrm{Vert}} z_v^{\ell_v}. \tag{3}$$

This time, in (2) we only integrate and sum over z_v with $v \neq v_0$, and in (3) we only sum over $\vec{\ell} = 2M\vec{n} + \vec{a}$ where the last entry of $\vec{n}$ is fixed to be n.

In the case where $\hat{Y}$ is a homology sphere, one can argue that the invariants for various a and n are determined by those for $a = n = 0$. We set

$$F_K(x, q) = \widehat{Z}_0(Y; x^{1/2}, n, q)$$

Of course, we are mostly interested in knots in S^3. The set of those whose complements are plumbed is rather restricted: it consists of algebraic knots, i.e., torus knots and some of their iterated cables. Here is the concrete formula for F_K for the torus knots $K = T(s, t)$:

$$F_K(x, q) = q^{\frac{(s-1)(t-1)}{2}} \cdot \frac{1}{2} \sum_{m \geq 1} \varepsilon_m \cdot \left(x^{\frac{m}{2}} - x^{-\frac{m}{2}}\right) q^{\frac{m^2 - (st-s-t)^2}{4st}}$$

where

$$\varepsilon_m = \begin{cases} -1 & \text{if } \ m \equiv st + s + t \ \text{ or } \ st - s - t \pmod{2st} \\ +1 & \text{if } \ m \equiv st + s - t \ \text{ or } \ st - s + t \pmod{2st} \\ 0 & \text{otherwise.} \end{cases}$$

3.2. **The recurrence method.** What can we do for other knots in S^3? Inspired by the plumbing formula and by physical considerations, we expect that, for general $K \subset S^3$, there is a two-variable series $F_K(x, q)$ with the following properties.

First, it should be related to Rozansky's asymptotic expansion of the colored Jones polynomials $J_n(q)$ [Roz98]. Rozansky shows that, in terms of the variables x and $\hbar$ (where $q = e^\hbar$ and $x = q^n$), we have

$$J_n(e^\hbar) = \frac{1}{\Delta_K(x)} + \frac{P_1(x)}{\Delta_K(x)^3}\hbar + \frac{P_2(x)}{\Delta_K(x)^5}\hbar^2 + \ldots = \sum_{m=0}^{\infty} \sum_{j=0}^{m} c_{m,j}\, n^j \hbar^m. \tag{4}$$

for some Vassiliev knot invariants $c_{m,j}$. Our conjecture is that

$$f_K(x, q) = \frac{F_K(x, q)}{x^{1/2} - x^{-1/2}}$$

is obtained from the series (4) by Borel resummation.

Second, we expect the quantum A-polynomial of K (discussed in [Gar04], [Guk05]) to annihilate the series $f_K(x, q)$. We also conjecture that

$$\lim_{q \to 1} f_K(x, q) = \mathrm{s.\,e.}\Big(\frac{1}{\Delta_K(x)}\Big), \tag{5}$$

where s.e. denotes a symmetric expansion in x and x^{-1}.

Assuming these properties, one can set up recurrence relations for the coefficients of $f_K(x, q)$. This is non-trivial, but it can be carried through for certain knots such as the trefoil, for which the answer is in agreement with the plumbed formula (2).

The recurrence method enables us to do computations for other knots, such as hyperbolic knots. For example, for the figure-eight knot we find that

$$F_K(x, q) = \frac{1}{2}(\Xi(x, q) - \Xi(x^{-1}, q)), \tag{6}$$

where

$$\Xi(x,q) = x^{1/2} + 2x^{3/2} + (q^{-1}+3+q)x^{5/2} + (2q^{-2}+2q^{-1}+5+2q+2q^2)x^{7/2} + \dots$$

Using the same method, Chae computed F_K for some cables and connected sum knots [Cha23, Cha25, Chaa].

3.3. **The surgery formula.** Suppose we have a knot complement $Y = \hat{Y} - K$, and let $Y_{p/r}$ denote the result of p/r surgery on it. In the situation where the formula (1) is applicable to $Y_{p/r}$ and (2) to Y (that is, both are given by negative definite plumbings), we can easily relate the two expressions. This justifies the following surgery formula, which we conjecture to hold for all knots K. Let $\mathcal{L}^{(a)}_{p/r}$ denote the Laplace transform

$$\mathcal{L}^{(a)}_{p/r} \;:\; x^u q^v \;\mapsto \begin{cases} q^{-u^2 r/p}\cdot q^v & \text{if } \; ru - a \in p\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the $\widehat{Z}$ invariants of $Y_{p/r}$ should be given by

$$\widehat{Z}_a(Y_{p/r}) = \varepsilon q^d \cdot \mathcal{L}^{(a)}_{p/r}\left[(x^{\frac{1}{2r}} - x^{-\frac{1}{2r}})F_K(x,q)\right], \tag{7}$$

for some $\varepsilon \in \{\pm 1\}$ and $d \in \mathbb{Q}$. (This is assuming that the right hand side gives a well-defined Laurent series in q, which happens for $-r/p$ large enough.)

In [GM21], we performed a non-trivial check that this formula is right. For the figure-eight knot $K = 4_1$, the -1 surgery is the Brieskorn sphere $-\Sigma(2,3,7)$. There are two methods for computing the $\widehat{Z}$ invariants of this manifold: one is applying the surgery formula to the expression for F_K in (6), and the other is using modularity analysis on the invariant for $+\Sigma(2,3,7)$. It turns out that these two methods give the same answer:

$$\widehat{Z}_0(-\Sigma(2,3,7)) = -q^{-1/2}(1 + q + q^3 + q^4 + q^5 + 2q^7 + q^8 + 2q^9 + q^{10} + 2q^{11} + \dots)$$

The surgery formula allowed us to also compute $\widehat{Z}$ for various hyperbolic manifolds, such as other surgeries on the figure-eight knot.

4. Further developments

4.1. **The R-matrix approach.** The Jones polynomial and its "colored" variants can be defined by taking a trace over a product of R-matrices for finite-dimensional representations of the quantum group $U_q(\mathfrak{sl}_2)$ at a root of unity. In quantum topology, the color refers to a choice of the representation and each crossing in a knot diagram contributes a copy of the R-matrix. Many quantum knot invariants can be obtain in this way, with a suitable choice of the quantum group and its representation ("color").

It is natural to ask if the quantum invariant $F_K(x,q)$ also admits such a definition. Since the $\widehat{Z}$ and F_K invariants are meant to provide a non-perturbative completion to complex Chern-Simons theory with gauge group $SL(2,\mathbb{C})$, it was clear for a long time that in the present case one needs to work with infinite-dimensional representations, rather than finite-dimensional ones [Guk05]. The highest (or lowest) weight λ of infinite-dimensional representations is a continuous complex variable, and our x is the exponential of λ, namely $x = q^\lambda$. It is this aspect of the story—the use of infinite-dimensional representations—which has been one of the main challenges, now tamed (or, regularized) by the $\widehat{Z}$ and F_K invariants.

Indeed, in taking the trace when constructing knot invariants or in summing over colors in the Reshetikhin-Turaev construction, one only encounters finite sums of representations that are finite-dimensional. (Also, working at a root of unity ensures that the dimension of representations is cut off by the order of the root of unity.) If, on the other hand, we are interested in generic complex values of q and wish to use infinite-dimensional representations of $U_q(\mathfrak{sl}_2)$ as colors, the trace will involve infinite sums and so will the surgery formula. In fact, this is the reason the surgery formula (7) is given by an integral, rather than a finite sum. In complex Chern-Simons theory, the origin of the infinite sums has to do with the non-compactness of the moduli spaces. We will return to this aspect in Section 4.9, but the reader may think of the $\widehat{Z}$ and F_K invariants as providing a regularized (tamed) version of the infinite sums via effectively compactifying the moduli spaces.

In the case of F_K, such a regularization was proposed by Park, originally for positive braid links [Par20b] and later for homogeneous braid links [Par]. The latter case is much more general[1] and is much more delicate—it involves a very non-trivial interplay of topology (diagrammatics) and representation theory (colors). As a result, one obtains the following alternative definition of $F_K(x,q)$.

Theorem 4.1. *Using Verma modules of highest and lowest weight as colors, and associating an R-matrix to each crossing in a homogeneous braid diagram of K in the usual way, one can define the trace*

$$F_K(x,q) \;=\; \mathrm{Tr}\,(R\ldots R) \tag{8}$$

such that the resulting expression is a well-defined series in q.

The fact that one can make sense of the trace over infinite-dimensional representations is the novelty of the F_K invariants.

4.2. **Higher rank.** Following [GM21], here we mostly focused on the 2-variable series $F_K(x,q)$ for knots and for the Lie algebra $\mathfrak{sl}_2$. The physical construction, however, makes it clear that there should exist generalizations $F_L^{\mathfrak{g}}(\vec{x},q)$ to links with arbitrary number of components and to more general Lie algebras $\mathfrak{g}$ of type A, D, or E. In such generalizations, the number of variables $\vec{x}$ is equal to the product of rank $(\mathfrak{g})$ and the number of components of L. The explicit formula in this generality can be found in [Par20a]. Moreover, extensions to supergroups have been studied recently [FP24, CHRY, Chab, CR23].

The R-matrix approach summarized in Section 4.1 can be extended from $U_q(\mathfrak{sl}_2)$ to more general quantum groups $U_q(\mathfrak{g})$. For $\mathfrak{g}=\mathfrak{sl}_N$, this was done in recent works by Gruen [Gru] and by Gruen and San Martín Suárez [GS].

Curiously, much like the quantum $\mathfrak{sl}_N$ invariants of knots for different N can be packed into (colored) HOMFLY-PT polynomials, the quantum knot invariants $F_K^{\mathfrak{sl}_N}$ can be unified into a HOMFLY-PT-like invariant $F_K(a,x,q)$, where the dependence on rank N is encoded in the variable

$$a = q^N.$$

Both the HOMFLY-PT polynomials and the $F_K(a,x,q)$ invariants admit the enumerative interpretation in terms of the DT invariants [EGG$^+$22]. It is remarkable that in this bridge between quantum topology and quantum geometry, all of the variables a, x, and q have a clear meaning on both sides of the correspondence. For example, one can consider the limit

[1] For example, every 3-manifold can be obtained as a surgery on a homogeneous braid link; see [Sta78] and Section 1.4 in [Par].

$q = e^{\hbar} \to 1$ as in (4), where one finds the relation to the Alexander polynomial, which in turn captures the genus-0 part of the enumerative / quantum geometry [DE25].

4.3. **Lattice homology.** One of the novel results in [GM21] was the Spin^c dependence of F_K and $\widehat{Z}$ invariants. While surprising at the time, it naturally pointed toward potential connections with other invariants labeled by Spin^c structures, notably Heegaard Floer homology and its variants.

Since for plumbed manifolds and knot complements the definition of F_K and $\widehat{Z}$ invariants essentially boils down to a certain sum over a lattice, with a quadratic form given by the adjecency matrix, the connection to lattice homology defined by the same data is especially natural. Such a connection exists and has been studied in a series of papers [AJK23, AJP, LM, Lil, HNS, MT25]. This work not only elucidates the role of Spin^c structures, but also provides valuable generalizations of the F_K and $\widehat{Z}$ invariants, and new connections to Floer-theoretic invariants. In particular, one of the main results of [AJP] is the following.

Theorem 4.2. *Let Γ_{v_0} be a negative definite marked plumbing tree representing a knot complement $\hat{Y} = Y \setminus K$. Given a relative* Spin^c *structure b on $\hat{Y}$ and a choice of sign $\varepsilon \in \{\pm 1\}$, there exists a canonically defined* weighted bigraded root

$$R^{\mathrm{bi}}_{\varepsilon}(\Gamma_{v_0}, b),$$

whose nodes carry explicit q–series weights depending only on $(\Gamma_{v_0}, b, \varepsilon)$, such that:

- *$F_K(x, q)$ can be recovered as a stabilized limit of $R^{\mathrm{bi}}_{\varepsilon}(\Gamma_{v_0}, b)$;*
- *If Γ_{v_0} and Γ'_{v_0} are related by Neumann moves, the corresponding weighted bigraded roots are canonically isomorphic;*
- *If Γ_{v_0,m_0} represents a closed 3–manifold $Y_{m_0}(K)$ by assigning an integer weight m_0 to the marked vertex v_0, then for each $b \in \mathrm{Spin}^c(Y_{m_0}(K))$ there is a canonical construction (Laplace transform and gluing along the two coordinates of the bigraded root) producing a weighted graded root $R(\Gamma_{v_0,m_0}, b)$ of a closed 3-manifold* [AJK23] *from that of a knot complement.*

4.4. **TQFT structure and affine Grassmannians.** Many quantum invariants of knots and 3-manifolds can be extended to a richer TQFT structure. The prototypical example is the Reshetikhin-Turaev construction [RT91] that relates TQFT data to that of a modular tensor category, e.g., the semisimple representation category of a quantum group at a root of unity. In our present case, q is a complex number such that $|q| < 1$ and the category in question, $\mathcal{C}$, if exists, should be much larger. In particular, based on the discussion in Section 4.1, it is expected to contain all Verma modules of the highest and lowest weight. It is not clear how one can define tensor products, which is one of the main challenges associated with this approach.

The TQFT structure underlying $\widehat{Z}$ and F_K invariants does not fit into the traditional framework of the Atiyah axioms either, since the spaces $\mathcal{H}(\Sigma)$ associated to surfaces Σ are all infinite-dimensional. For example, the surgery formula (7) is given by an integral or, equivalently, it can be expressed as an infinite sum. Luckily, the Spin^c dependence helps to tame the infinities at the cost of imposing the additional structure of a decorated TQFT [Jag23].

Spin^c-decorated TQFTs appear in the context of Rozansky-Witten theories [RW97], which are three-dimensional sigma-models with a holomorphic symplectic target X. When

X is compact, such theories admit mathematical formulations [Kap99, Kon99]. Many important holomorphic symplectic manifolds are non-compact; e.g., Coulomb branches are never compact. It is expected that the $\widehat{Z}$ and F_K invariants admit a formulation in terms of Rozansky-Witten type TQFT with the cotangent bundle to the affine Grassmannian as the target space. A specific model proposed in [GHN$^+$21, GHR25] recovers some of the desired properties and explicit examples, for which computations have been carried out. A potential benefit of this approach is that it can provide a geometric description of the sought-after category $\mathcal{C}$ and also provide a bridge to categorification of Verma modules [NV18a, NV18b, Vaz19, Rou].

4.5. **Radial limits.** The series $\widehat{Z}_a(q)$ converges in the unit disk $|q| < 1$. It was originally thought that its limits as q approaches a root of unity recover the WRT invariant (after a weighted summation over all a). This is indeed the case for negative definite plumbings, as was shown by Murakami [Mur24]. (For previous results on Seifert fibered homology spheres, see [AM22], [FIMT21].) However, for some Seifert fibrations taken with their negative orientation, the convergence only happens at *some* roots of unity [CCF$^+$19]. For example, in the simple case of homology spheres, it was originally realized through the modularity properties [CCF$^+$19], and then better understood via resurgent analysis [GJ24], that the behavior near roots of unity takes the form

$$\mathrm{WRT}(Y,k) \;=\; \Big(\widehat{Z}(q) - P(\tilde{q})\Big)\Big|_{q\to e^{2\pi i/k}}$$

where $P(\tilde{q})$ is a correction term determined by the data of complex Chern-Simons values of Y and K-theoretic Stokes coefficients. The reason is that, near roots of unity, $\widehat{Z}(q)$ may exhibit an exponential growth $\simeq \exp\Big(-\frac{4\pi^2}{\hbar}(\mathrm{CS}(\alpha_*) - m_*)\Big)$. This growth expression is familiar from the generalized volume conjecture, and is determined by the Chern-Simons value of a particular complex flat connection α_* and its integral lift, $m_* \in \mathbb{Z}$. While both α_* and m_* can be determined by the tools of resurgent analysis, it would be interesting to provide an independent definition.

This behavior by now has been well explored in a variety of concrete examples. For instance, in the case of the Seifert manifold $Y = -M(-2; \frac{1}{2}, \frac{1}{3}, \frac{1}{2})$, there are no corrections for all roots of unity of odd order, while for roots of unity of even order we have $P(q) = \eta^3(q)/\eta^2(q^2)$, where $\eta(q)$ is the Dedekind eta-function [CCF$^+$19]. More examples can be found in [GJ24, Whe25, CDGG].

4.6. **Non-semisimple quantum invariants.** Combining insights and observations from sections 4.1, 4.4, and 4.5, it should come as no surprise that the radial limits of the $\widehat{Z}$ and F_K invariants are more directly related to non-semisimple (a.k.a. logarithmic) invariants, rather than to the WRT invariants. (At the level of the representation category of the quantum group, the WRT story involves an additional step, namely the semi-simplification.)

Indeed, starting with the R-matrix definition of F_K outlined in Section 4.1, one can verify that the radial limits of the R-matrix at generic q produce the R-matrix used by Murakami [Mur08] in the definition of the ADO invariants [ADO92]. Lifting this simple observation to traces over products of R-matrices, i.e. to the definition of F_K itself (8), leads to a conjectural relation between F_K and non-semisimple ADO invariants of knots.

Conjecture 4.3. *For any knot K and $\zeta_p = e^{2\pi i/p}$,*

$$\lim_{q\to\zeta_p} F_K(x,q) = \frac{\mathrm{ADO}_p(x/\zeta_p;K)}{\Delta_K(x^p)} \tag{9}$$

This conjecture can be viewed as a generalization of (5) and is merely a tip of an iceberg that represents a large body of work on relations to non-semisimple invariants and TQFTs [BBG20, GHN$^+$21, Cha20, CGP23, MM, CHRY].

4.7. **Vertex operator algebras.** The Kazhdan-Lusztig correspondence relates the representation category of the quantum group mentioned earlier to that of a vertex algebra. Via this correspondence, the R-matrix is identified with the braiding of vertex operators. In the Reshetikhin-Turaev construction, this perspective often helps to understand the underlying category $\mathcal{C}$. In our present case, it unfortunately does not shed much light on $\mathcal{C}$ because the VOA corresponding to $U_q(\mathfrak{g})$ at generic q is not well understood. However, working at generic q offers another new and surprising connection with VOAs: the invariants $\widehat{Z}$ and F_K turn out to enjoy just the right modular properties [BMM19, CCF$^+$24, CCK$^+$] to be identified with (super)characters of logarithmic VOAs [Suga, Sugb, Sugc]. These connections are rather peculiar and crucially rely on the fact that the invariants at hand are q-series expansions of the form $q^{\Delta}(a_0 + a_1 q + a_2 q^2 + \dots)$ with integer coefficients $a_n \in \mathbb{Z}$. Upon identification with log-VOA supercharacters, the Δ-invariants, recently studied in [Har, HNS] are mapped to conformal dimensions of VOA modules. Many aspects of this correspondence remain mysterious and deserve further study.

4.8. **Categorification.** One of the main motivations for developing the invariants F_K and $\widehat{Z}$ is the hope that their categorification can produce new tools for low-dimensional topology.

There are several potential avenues for categorification of F_K and $\widehat{Z}$, all of which are equally unexplored. For example, from the representation theory point of view, a natural starting point is the categorification of Verma modules [NV18a, NV18b, Vaz19, Rou] mentioned in Section 4.1. From the viewpoint of vertex algebra, a categorification of a (super)character is the VOA module itself.

Yet another perspective comes from the observation that, for negatively plumbed manifolds, the $\widehat{Z}$ invariants coincide with particular generating series of the Stokes coefficients, which in turn can be categorified using the standard tools of the Fukaya-Seidel category [GP]. Recall that a Fukaya-Seidel category is defined by the data of a symplectic space X and a function $W : X \to \mathbb{C}$. In the present case, $X = \mathbb{C}^h \setminus H$ is the complement of a hyperplane arrangement H in the space $\mathbb{C}^h$, where h is the number of high-valency vertices in the plumbing graph Γ.

4.9. **Beyond character varieties.** As pointed out earlier, from the geometric / gauge theory perspective, the invariants F_K and $\widehat{Z}$ exist thanks to a particular compactification of the character varieties. Understanding the detailed structure of compactification divisors is absolutely crucial to a better understanding of the invariants F_K and $\widehat{Z}$ themselves, and their categorification.

Some of the work mentioned above provides indirect information about the compactification divisors. For example, the study of the decorated TQFT structure [Jag23] reveals that $\mathcal{H}(\Sigma)$ is larger than what one might expect from a naive quantization of the character variety of Σ. Similar contributions of compactification divisors can be seen for surgeries on many composite and prime knots [GP].

5. Open problems

One large overarching problem is to extend the current understanding of F_K and $\widehat{Z}$ to a larger and more encompassing structure, possibly including the full-fledged (decorated) TQFT structure. It is possible that some definitions and structures may encounter obstructions, e.g. for particular types of manifolds, as it happens in other TQFTs that do not obey Atiyah's axioms. In such cases understanding the obstructions can be extremely valuable. To pursue this general program, below we suggest a number of more concrete problems.

Problem 5.1. *Find the 0-surgery formula, i.e. a generalization of* (7) *to* $\frac{p}{r} = 0$.

Problem 5.2. *In the notation of Section 4.4, find the category* $\mathcal{C}$ *or show that it does not exist.*

Problem 5.3. *Find a family of TQFTs labeled by the admissible functions of* [AJK23].

Problem 5.4. *Study* F_K *for links with multiple components; in particular, find analogues of* (9) *and* (5).

Problem 5.5. *Following* [GP]*, construct a categorification over the integers (rather than with mod 2 coefficients).*

Problem 5.6. *Work out a construction of* $\widehat{Z}(Y)$ *based on a Heegaard splitting of* Y. *A closely related problem is to extend the developments summarized in Section 4.4 to understand* $\mathcal{H}(\Sigma)$ *for higher-genus surfaces.*

Problem 5.7. *Propose a categorification of* $F_K(x, q)$.

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