A Note on the MIR closure

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Abstract

In 1988, Nemhauser and Wolsey introduced the concept of MIR inequality for mixed integer linear programs. In 1998, Wolsey defined MIR inequalities differently. In some sense these definitions are equivalent. However, this note points out that the natural concepts of MIR closures derived from these two definitions are distinct. Dash, Günlük and Lodi made the same observation independently.

Let $S := \{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : Ax + Gy \leq b\}$ be a mixed integer set. Here $A \in \mathbb{R}^{m \times n}$ and $G \in \mathbb{R}^{m \times p}$ are matrices and $b \in \mathbb{R}^m$ is a vector. Let $P := \{(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^p : Ax + Gy \leq b\}$ be the polyhedron that arises as the natural linear relaxation of S. We assume $P \neq \emptyset$.

Nemhauser and Wolsey [6,7] define MIR^{NW} inequalities by the following procedure.

If

$$c^1 x + h y \leq c_0^1$$

and

$$c^2 x + h y \leq c_0^2$$

are valid inequalities for P, and $\pi = c^2 - c^1 \in \mathbb{Z}^n$, $\pi_0 = \lfloor c_0^2 - c_0^1 \rfloor$ and $\gamma = c_0^2 - c_0^1 - \pi_0$, then

$$\pi x + \left(c^1 x + hy - c_0^1 \right) / (1 - \gamma) \le \pi_0$$

is valid for S.

Define the MIR^{NW} closure as the intersection of all MIR^{NW} inequalities. Nemhauser and Wolsey [7] proved that the MIR^{NW} closure is identical to the split closure [1] and the Gomory mixed integer closure [4] (see [2] for another proof of the last identity).

Later, Wolsey [8] (see also Marchand and Wolsey [5]) defined the MIR^W inequality as being generated from a single constraint $ax + gy \le b$ where $(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p$. Specifically, let $f_0 := b - \lfloor b \rfloor$ and $f_j := a_j - \lfloor a_j \rfloor$. The MIR^W inequality is

$$\sum_{j=1}^{n} \left(\lfloor a_j \rfloor + \frac{(f_j - f_0)^+}{1 - f_0} \right) x_j + \frac{1}{1 - f_0} \sum_{j: g_j < 0} g_j y_j \le \lfloor b \rfloor. \tag{1}$$

For the mixed integer set S, let us define the MIR^W closure as the set of all MIR^W inequalities that can be generated from any valid inequality for the polyhedron P. (By Farkas' Lemma, every valid inequality for P is of the form $uAx + uGy - vx - wy \le ub + t$ where $u \in \mathbb{R}_+^m$, $v \in \mathbb{R}_+^n$, $w \in \mathbb{R}_+^p$ and $t \in \mathbb{R}_+$.)

In this note, we point out that $\text{MIR}^{NW} \subset \text{MIR}^W$ and that the inclusion is strict in general. This was also observed independently by Dash, Günlük and Lodi [3].

First, we give an example showing that $\text{MIR}^{NW} \ne \text{MIR}^W$. Let P be the triangle in $\mathbb{R}^2$ defined as follows

$$-2x_1 + x_2 \le 0$$
$$2x_1 + x_2 \le 2$$
$$x_2 \ge 0.$$

Let $S := P \cap \mathbb{Z}^2$. The inequality $x_2 \le 0$ is a MIR^{NW} inequality. Indeed

$$-\frac{1}{2}x_1 + \frac{1}{4}x_2 \le 0$$
$$\frac{1}{2}x_1 + \frac{1}{4}x_2 \le \frac{1}{2}$$

are valid for P. Applying the MIR^{NW} procedure we get $\pi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\pi_0 = \lfloor \frac{1}{2} \rfloor =$
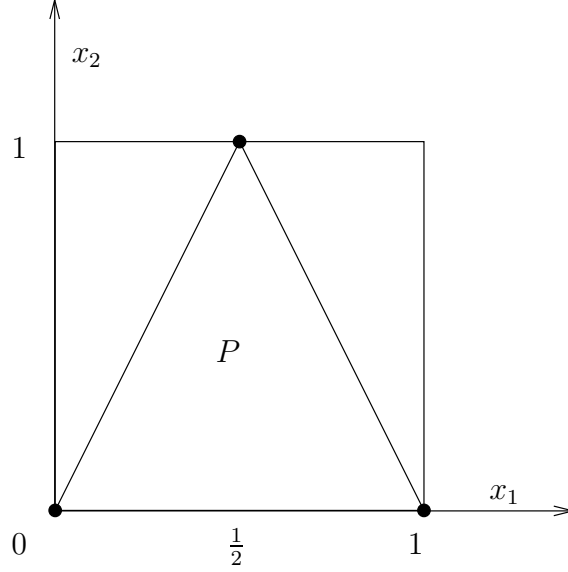
Fig. 1. Examle showing that $\mathrm{MIR}^{NW} \neq \mathrm{MIR}^{W}$.

0, $\gamma = \frac{1}{2}$. Thus the following inequality is valid for S.

$$x_1 + \frac{-\frac{1}{2}x_1 + \frac{1}{4}x_2}{\frac{1}{2}} \leq 0, \quad \text{i.e.} \quad x_2 \leq 0.$$

Therefore $x_2 \leq 0$ is valid for the MIR^{NW} closure. However $x_2 \leq 0$ is not valid for the MIR^{W} closure. We show this by contradiction. Let us assume that there exists a valid inequality $\alpha x_1 + \beta x_2 \leq \delta$ for P such that $x_2 \leq 0$ is a MIR^{W} inequality. By Farkas' Lemma, there exist multipliers $u_1, u_2, v, t \geq 0$ satisfying

$$\alpha = -2u_1 + 2u_2$$
$$\beta = u_1 + u_2 - v$$
$$\delta = 2u_2 + t.$$

Let $f(\eta) = \eta - \lfloor \eta \rfloor$. Can we generate $x_2 \leq 0$ as the MIR^{W} inequality

$$\left(\lfloor \alpha \rfloor + \frac{(f(\alpha) - f(\delta))^+}{1 - f(\delta)} \right) x_1 + \left(\lfloor \beta \rfloor + \frac{(f(\beta) - f(\delta))^+}{1 - f(\delta)} \right) x_2 \leq \lfloor \delta \rfloor?$$

For this to be the case, we must have

- $\delta < 1$ since $\lfloor \delta \rfloor = 0$,
- $\alpha \geq 0$ since $\left(\lfloor \alpha \rfloor + \frac{(f(\alpha) - f(\delta))^+}{1 - f(\delta)} \right) = 0$,
- $\delta < \beta$ since $\left(\lfloor \beta \rfloor + \frac{(f(\beta) - f(\delta))^+}{1 - f(\delta)} \right) > 0$.

$\alpha \geq 0$ is equivalent to $u_2 \geq u_1$. Furthermore $\delta < \beta$ and $v, t \geq 0$ imply $u_2 < u_1$. This is a contradiction, therefore there exists no valid inequality for P such that $x_2 \leq 0$ is a MIR^{W} inequality.

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To see that $\text{MIR}^{NW} \subset \text{MIR}^W$, we express Gomory Mixed Integer (GMI) inequalities in a form similar to (1). Recall that given an equality $ax + gy = b$ where $(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p$, the GMI inequality is

$$\sum_{j:f_j \leq f_0} \frac{f_j}{f_0} x_j + \sum_{j:f_j > f_0} \frac{1-f_j}{1-f_0} x_j + \sum_{j:g_j > 0} \frac{g_j}{f_0} y_j - \sum_{j:g_j < 0} \frac{g_j}{1-f_0} y_j \geq 1 \qquad (2)$$

where f_j and f_0 are defined as above.

Lemma 1 *Consider a mixed integer set with m constraints $S := \{(x,y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : Ax + Gy \leq b\}$. We assume that the constraints $Ax + Gy \leq b$ contain the nonnegativity constraints on x and y. Let $s := b - Ax - Gy$ be a nonnegative vector of slack variables. For any $\lambda \in \mathbb{R}^m$, let $a := \lambda A$, $g := \lambda G$, $\delta := \lambda b$, $f_j := a_j - \lfloor a_j \rfloor$ and $f_0 := \delta - \lfloor \delta \rfloor$. The Gomory mixed integer inequality generated from $\lambda Ax + \lambda Gy + \lambda s = \lambda b$ is*

$$\sum_{j=1}^n \left(\lfloor a_j \rfloor + \frac{(f_j - f_0)^+}{1-f_0} \right) x_j + \frac{1}{1-f_0} \sum_{j:g_j < 0} g_j y_j + \frac{1}{1-f_0} \sum_{i:\lambda_i < 0} \lambda_i s_i \leq \lfloor \delta \rfloor. \quad (3)$$

Proof: Applying the definition (2) to $\lambda Ax + \lambda Gy + \lambda s = \lambda b$ we get

$$\sum_{j:f_j \leq f_0} \frac{f_j}{f_0} x_j + \sum_{j:f_j > f_0} \frac{1-f_j}{1-f_0} x_j + \sum_{j:g_j > 0} \frac{g_j}{f_0} y_j - \sum_{j:g_j < 0} \frac{g_j}{1-f_0} y_j + \sum_{i:\lambda_i > 0} \frac{\lambda_i}{f_0} s_i - \sum_{i:\lambda_i < 0} \frac{\lambda_i}{1-f_0} s_i \geq 1.$$

Substituting $s = b - Ax - Gy$ in this inequality, it is straightforward to check that the result is inequality (3).

$\square$

Recall that MIR^W inequalities are obtained from valid inequalities for P. This corresponds to $\lambda \geq 0$ in Lemma 1. In this case (3) is identical to (1). Therefore MIR^W inequalities are GMI inequalities.

Other authors have defined the MIR^W closure when S is in equality form [5,3], in which case it is trivially identical to the Gomory mixed integer closure.

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