Balanced $0, \pm 1$ Matrices
Part II: Recognition Algorithm

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Abstract

In this paper we give a polynomial time recognition algorithm for balanced $0, \pm 1$ matrices. This algorithm is based on a decomposition theorem proved in a companion paper.

Keywords: balanced matrix, decomposition, recognition algorithm, 2-join, 6-join, extended star cutset

Running head: Recognition of balanced $0, \pm 1$ matrices

1 Introduction

A $0, \pm 1$ matrix is *balanced* if, in every square submatrix with two nonzero entries per row and column, the sum of the entries is a multiple of four. In [3], Conforti, Cornuéjols and Rao prove a decomposition theorem for balanced $0, 1$ matrices and they use it to obtain a polynomial time recognition algorithm for these matrices. In this paper, using a similar approach, we give a polynomial time recognition algorithm for balanced $0, \pm 1$ matrices, using a decomposition result derived in the companion paper [1]. For a survey of results on balanced matrices, see [2].

A convenient setting for working with balanced $0, \pm 1$ matrices is to consider their signed bipartite graph representations. A *signed graph G* is a graph together with an assignment of $+1$ or -1 weights to the edges. Given a $0, \pm 1$ matrix A, the *signed bipartite graph representation of A* is a signed bipartite graph G, with the two sides of the bipartition V^r and V^c

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representing respectively the rows and columns of A, and for each nonzero entry a_{ij} of A, there is an edge between nodes $i \in V^r$ and $j \in V^c$ with weight a_{ij}.

A signed bipartite graph G is *balanced* if it is the signed bipartite graph representation of a balanced $0, \pm 1$ matrix. Thus a signed bipartite graph G is balanced if and only if for every hole H of G, the sum of the weights of the edges of H is a multiple of 4. A *hole* in a bipartite graph is a chordless cycle. A hole is *balanced* if it is of weight 0 modulo 4, and it is *unbalanced* if it is of weight 2 modulo 4. A graph G *contains* a graph H, if H is an induced subgraph of G. So, a signed bipartite graph is balanced if and only if it does not contain an unbalanced hole.

In this paper we construct a recognition algorithm that takes as input a signed bipartite graph G, and outputs YES if G is balanced, and NO otherwise. The algorithm runs in time polynomial in the size of V^r and V^c. This algorithm can be used to obtain a polynomial time algorithm for finding an unbalanced hole in a graph that contains one, in the following way.

If Recognition(G)=YES, **return** "G is balanced".

Else set $H = G$.

> **While** there exists some node v in H such that Recognition($H \setminus \{v\}$) =NO, set $H = H \setminus \{v\}$.

Return "H is an unbalanced hole of G".

As mentioned above, the recognition algorithm is based on a decomposition theorem, which we state in Section 1.1. The organization of the paper is described in Section 1.2.

1.1 Decomposition Theorem

A set S of nodes (respectively edges) of a connected graph G is a *node cutset* (respectively an *edge cutset*) if the subgraph $G \setminus S$, obtained from G by removing the nodes (respectively edges) in S, is disconnected.

A *biclique* is a complete bipartite graph K_{AB} where the two sides of the bipartition A and B are both nonempty.

Extended Star Cutset

For a node x, let $N(x)$ denote the set of all neighbors of x. In a bipartite graph G, an *extended star* $(x; X; Y; R)$ consists of disjoint subsets X, Y, R of $V(G)$ and a node $x \in X$ such that

(i) $Y \cup R \subseteq N(x)$,

(ii) the node set $X \cup Y$ induces a biclique (with node set X on one side of the bipartition and node set Y on the other),

(iii) if $|X| \geq 2$, then $|Y| \geq 2$.

In a connected bipartite graph, an *extended star cutset* is an extended star $(x; X; Y; R)$ where $X \cup Y \cup R$ is a node cutset. When $R = \emptyset$ the extended star is a biclique, and the cutset is called a *biclique cutset*. When $|X| = 1$ then the extended star cutset is also called a *star cutset*.

2-Join

Let G be a connected bipartite graph with more than four nodes, containing bicliques $K_{A_1A_2}$ and $K_{B_1B_2}$, where A_1, A_2, B_1, B_2 are disjoint nonempty node sets. The edge set $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ is a *2-join* if it satisfies the following properties:

(i) The graph $G' = G \setminus (E(K_{A_1A_2}) \cup E(K_{B_1B_2}))$ is disconnected.

(ii) Every connected component of G' has a nonempty intersection with exactly two of the sets A_1, A_2, B_1, B_2 and these two sets are either A_1 and B_1 or A_2 and B_2. For $i = 1, 2$, let G'_i be the subgraph of G' containing all its connected components that have nonempty intersection with A_i and B_i.

(iii) If $|A_1| = |B_1| = 1$, then G'_1 is not a chordless path or $A_2 \cup B_2$ induces a biclique. If $|A_2| = |B_2| = 1$, then G'_2 is not a chordless path or $A_1 \cup B_1$ induces a biclique.

The purpose of Property (iii) is to exclude "improper" 2-joins.

6-Join

In a connected bipartite graph G, let A_i, $i = 1, \ldots, 6$ be disjoint, nonempty node sets such that, for each i, every node in A_i is adjacent to every node in $A_{i-1} \cup A_{i+1}$ (indices are taken modulo 6), and these are the only edges in the subgraph A induced by the node set $\cup_{i=1}^{6} A_i$. (Note that, for convenience of notation, the modulo 6 function is assumed to return values between 1 and 6, instead of the usual 0 to 5). The edge set $E(A)$ is a *6-join* if

(i) The graph $G' = G \setminus E(A)$ is disconnected.

(ii) The nodes of G can be partitioned into V_1 and V_2 so that $A_1 \cup A_3 \cup A_5 \subseteq V_1$, $A_2 \cup A_4 \cup A_6 \subseteq V_2$ and the only adjacencies between the nodes of V_1 and V_2 are the edges of $E(A)$.

(iii) $|V_i| \geq 4$ for $i = 1, 2$.

When the graph G comprises more than one connected component, we say that G has a 2-join, a 6-join or an extended star cutset if at least one of its connected components does.

Basic Classes of Graphs

A signed bipartite graph is *strongly balanced* if it is balanced and contains no cycle with exactly one chord. The recognition problem for this class of graphs is polynomial (Conforti and Rao [5]). R_{10} is the bipartite graph defined by the cycle $x_1, \ldots, x_{10}, x_1$ of length 10 with chords $x_i x_{i+5}$, $1 \leq i \leq 5$ (indices taken modulo 10). R_{10} can be signed to be balanced, say with weight $+1$ on the edges of the cycle $x_1, \ldots, x_{10}, x_1$ and -1 on the chords.

In [1] we prove the following decomposition theorem.

Theorem 1.1 *A signed bipartite graph that is balanced but not strongly balanced is either R_{10} with proper signing or it contains a 2-join, a 6-join or an extended star cutset.*

1.2 Organization of the Paper

The general idea of our recognition algorithm for balanced signed bipartite graphs is as follows. Let G be a signed bipartite graph. If G is strongly balanced or the underlying graph is R_{10}, then we are done. Else, we search for one of the three cutsets described above. If none exists, G is not balanced as a consequence of Theorem 1.1. If one exists, its removal disconnects G into several connected components. From these components, we construct *blocks* by adding some new nodes and edges with some signing. In other words, we *decompose* G into these blocks. Ideally, the blocks should be constructed so that G is balanced if and only if all the blocks are. Let $\mathcal{B}$ stand for the class of signed bipartite graphs that are balanced. We say that a decomposition is *$\mathcal{B}$-preserving* if it satisfies the following: G belongs to $\mathcal{B}$ if and only if all the blocks of the decomposition belong to $\mathcal{B}$. The three decompositions are then applied recursively to the blocks until no cutset can be found. We show that only a polynomial number of such basic blocks are generated. For each, we check whether it is R_{10} or strongly balanced. G is balanced if and only if all basic blocks are balanced (assuming all decompositions are $\mathcal{B}$-preserving).

In Section 2, we show how to construct blocks that are $\mathcal{B}$-preserving for the 2-join and the 6-join decompositions. In Section 3, we deal with the node cutset decomposition. For the extended star cutset, we are not able to construct blocks to be $\mathcal{B}$-preserving. Instead, in our recognition algorithm we first apply a certain cleaning procedure to the input graph G, which transforms it into a graph G' with the property that G' is balanced if and only if G is and, if G contains an unbalanced hole then G' contains an unbalanced hole that will either never be broken by extended star cutset decompositions or it will be detected while performing the decomposition. To construct such a procedure we need to study signed bipartite graphs that do contain unbalanced holes. In Section 3.2, we obtain certain properties of a smallest unbalanced hole which allow us to construct the cleaning procedure in Section 4.1. In Section 4, we present the recognition algorithm for signed bipartite graphs that are balanced, and prove its validity and polynomiality.

2 Edge Cutset Decompositions

Throughout the rest of the paper, we assume that G is a signed bipartite graph.

By *scaling* G at node u, we mean changing the sign of the weights on all the edges incident with u.

Remark 2.1 *Let G' be a signed bipartite graph obtained from G by scaling at node u. A hole is balanced in G' if and only if it is balanced in G.*

Let u, v be two nonadjacent nodes of G in opposite sides of the bipartition. A *3-path configuration connecting u and v*, denoted by $3PC(u, v)$, is defined by three chordless paths P_1, P_2 and P_3 with endnodes u and v, such that the node set $V(P_i) \cup V(P_j)$, $i, j \in \{1, 2, 3\}$, $i \neq j$, induces a hole. Since paths P_1, P_2 and P_3 of a 3-path configuration are of length 1 or 3 modulo 4, the sum of the weights of the edges in each path is also 1 or 3 modulo 4. It follows that two of the three paths induce a hole of weight 2 modulo 4. So a signed bipartite graph that contains a 3-path configuration is not balanced.

A *wheel*, denoted by (H, x), is defined by a hole H and a node $x \notin V(H)$ which has at least three neighbors in H, say $x_1, \ldots, x_n$. The wheel (H, x) is *even* if n is even and it is *odd* otherwise. An edge xx_i is a *spoke*. A subpath of H connecting x_i and x_j is called a *sector* if it contains no intermediate node x_l, $1 \leq l \leq n$. Consider a wheel (H, x) which is signed to be balanced. By Remark 2.1, we can assume that all spokes of the wheel are signed +1. This implies that the sum of the weights of the edges in each sector is 2 modulo 4. Hence if (H, x) is an odd wheel, the hole H has weight 2 modulo 4. So a signed bipartite graph that contains an odd wheel is not balanced.

2.1 2-Join Decomposition

A 2-join $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ is *rigid* if $A_1 \cup B_1$ or $A_2 \cup B_2$ induces a biclique. The following easy result was proved in [3].

Lemma 2.2 *Let G be a bipartite graph that has no extended star cutset. Then G has no rigid 2-join.*

Let $K_{A_1A_2}$ and $K_{B_1B_2}$ define a 2-join of G that is not rigid. The blocks G_1 and G_2 of the 2-join decomposition are defined as follows. For $i = 1, 2$, let G'_i be the subgraph of $G \setminus (E(K_{A_1A_2}) \cup E(K_{B_1B_2}))$ containing all its connected components that have nonempty intersection with A_i and B_i. To obtain G_i, we first add to G'_i a node α_i, adjacent to all the nodes in A_i and to no other node of G'_i and a node β_i, adjacent to all the nodes in B_i and to no other node of G'_i. Let Q_1 be a path in G'_2 with smallest number of edges connecting a node in A_2 to a node in B_2, and let Q_2 be a path in G'_1 with smallest number of edges connecting a node in A_1 to a node in B_1. Note that the existence of Q_1, Q_2 is guaranteed by (ii) in the definition of 2-joins. For $i = 1, 2$, add to G_i a *marker path* M_i connecting α_i and β_i with length $4 \leq |E(M_i)| \leq 5$ and edge weights +1 or −1 chosen so that the weight of M_i is congruent to the weight of Q_i modulo 4.

Theorem 2.3 *Let G_1 and G_2 be the blocks of the decomposition of the signed bipartite graph G by a 2-join $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ that is not rigid. If G does not contain an unbalanced hole of length 4, then G is balanced if and only if both G_1 and G_2 are balanced.*

The following lemma is used in the proof of Theorem 2.3.

Lemma 2.4 *Let G be a signed bipartite graph with no unbalanced hole of length four. For every biclique K_{BD} in G, we can scale G on the nodes in $B \cup D$ so that every edge in $E(K_{BD})$ has weight +1.*

Proof: If $|B| = 1$ then we can scale on nodes in D to obtain the result. Similarily, for $|D| = 1$.

We can assume $|B| \geq 2$ and $|D| \geq 2$. Let $b \in B$ and $d \in D$. Scale at nodes $d' \in D$ so that all edges bd' have weight +1. Scale at nodes $b' \in B$ so that all edges $b'd$ have weight +1. Every $d' \in D \setminus \{d\}$ and $b' \in B \setminus \{b\}$ induce a hole b, d, b', d', b of length four. By assumption this hole is balanced. Hence $b'd'$ must have weight +1. □

Remark 2.5 *Let G be a signed bipartite graph with no unbalanced hole of length 4. By Lemma 2.4 there exists a signed graph G', which is obtained from G by a sequence of scalings, such that all the edges in $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ have weight $+1$, since $K_{A_1A_2}$ and $K_{B_1B_2}$ are node disjoint.*

Proof of Theorem 2.3: By Remark 2.5 we can assume that all the edges in $E(K_{A_1A_2})$ and $E(K_{B_1B_2})$ have weight $+1$. First we show that G_1 and G_2 are balanced if G is balanced. Every hole H in G_1 corresponds to a hole H' in G, except for the case where H contains nodes α_1 and β_1 and no other nodes of M_1, and $A_2 \cup B_2$ is a biclique in G. The existence of such a biclique would contradict our assumption that $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ is a 2-join that is not rigid. The hole H' has the same weight as H, since all the edges of $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ are signed positive. Thus G_1 is balanced if G is balanced. Similarly for G_2.

Now assume that G_1 and G_2 are balanced, but G is not. Let H be an unbalanced hole of G. If it contains no edge of G'_2, there exists a hole in G_1 which is unbalanced. The same argument holds for G'_1. So H must contain both an edge of G'_1 and an edge of G'_2. Hence H must contain an edge of $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$, say an edge a_1a_2 where $a_1 \in A_1$ and $a_2 \in A_2$. Since H is a hole it cannot contain any node of $K_{A_1A_2} \setminus \{a_1, a_2\}$. So H must also contain an edge b_1b_2 where $b_1 \in B_1$ and $b_2 \in B_2$, and similarly H cannot contain any node of $K_{B_1B_2} \setminus \{b_1, b_2\}$. So $H = a_1, a_2, P_2, b_2, b_1, P_1, a_1$ where P_2 is a path in G'_2 from a_2 to b_2 having no intermediate nodes in $A_2 \cup B_2$, and P_1 is a path in G'_1 from b_1 to a_1 having no intermediate nodes in $A_1 \cup B_1$. Since the hole $a_1, \alpha_1, M_1, \beta_1, b_1, P_1, a_1$ is balanced in G_1, $w(P_2)$ and $w(M_1)$ are not congruent modulo 4. But by definition of a block, there exists a path Q_2 in G'_2 from $a'_2 \in A_2$ to $b'_2 \in B_2$, such that $w(Q_2)$ is congruent to $w(M_1)$ modulo 4. The holes $H_1 = a'_2, Q_2, b'_2, \beta_2, M_2, \alpha_2, a'_2$ and $H_2 = a_2, P_2, b_2, \beta_2, M_2, \alpha_2, a_2$ in G_2 have distinct weights modulo 4. Hence one of them must be unbalanced, contradicting our assumption. □

2.2 6-Join Decomposition

Let G be a signed bipartite graph that has a 6-join $E(A)$. *Blocks* G_1 and G_2 of a 6-join decomposition are constructed as follows. For $i = 1, \ldots, 6$ let a_i be any node of A_i. G_1 is a subgraph of G induced by the node set $V_1 \cup \{a_2, a_4, a_6\}$ and G_2 is a subgraph of G induced by the node set $V_2 \cup \{a_1, a_3, a_5\}$.

Theorem 2.6 *Let G_1 and G_2 be the blocks of the decomposition of the signed bipartite graph G by a 6-join $E(A)$. If G does not contain an unbalanced hole of length 4 or 6, then G is balanced if and only if both G_1 and G_2 are balanced.*

We first prove the following lemma.

Lemma 2.7 *If A does not contain an unbalanced hole of length 4 or 6, then there exists a signing of G which is obtained by a sequence of scalings on the nodes of A, such that for every biclique $K_{A_iA_{i+1}}, i \in \{1, \ldots, 6\}$ (where indices are taken modulo 6) the edges in the biclique are all signed +1 or they are all signed -1.*

Proof: By Lemma 2.4 we can sign all the edges in $E(K_{A_1A_2}), E(K_{A_3A_4})$ and $E(K_{A_5A_6})$ to be $+1$. W.l.o.g. let $E(K_{A_2A_3})$ contain an edge signed $+1$ and another signed -1. Now there exist in A two holes of length 6 which differ in weight by 2. Clearly one of these must be unbalanced contradicting our assumption that A contains no unbalanced hole of length 6. $\square$

Proof of Theorem 2.6: It follows from the definition of the blocks that G_1 and G_2 are induced subgraphs of G and so are balanced if G is balanced.

To prove the converse assume that G_1 and G_2 are balanced, but G contains an unbalanced hole H. By Lemma 2.7 we may assume that for every biclique $K_{A_iA_{i+1}}$, $i \in \{1, \ldots, 6\}$, the edges of the biclique are all signed $+1$ or they are all signed -1. So H must contain an edge with both ends in V_2, since otherwise there exists a hole in G_1 which is unbalanced. Similarly H must also contain an edge with both ends in V_1. Since H is a hole it must have exactly 4 nodes in common with $V(A)$. Then w.l.o.g. $H = a_1'', P_1, a_5'', a_4'', P_2, a_2'', a_1''$ where $a_1'' \in A_1$, $a_2'' \in A_2$, $a_4'' \in A_4$, $a_5'' \in A_5$, P_1 is a path with nodes in V_1 that connects a_1'' to a_5'', and P_2 is a path with nodes in V_2 that connects a_2'' to a_4''. The hole $H_1 = a_1'', P_1, a_5'', a_6, a_1''$ is a hole of G_1 and $H_2 = a_2'', P_2, a_4'', a_3, a_2''$ is a hole of G_2. Since G_1 and G_2 are balanced, both H_1 and H_2 are balanced. Also $H' = a_1'', a_2'', a_3, a_4'', a_5'', a_6, a_1''$ is a hole of G (A in particular) and by the construction of blocks the edges $a_1''a_6$ and a_6a_5'' (resp. $a_2''a_3$ and $a_4''a_3$) are signed in G the same as the corresponding edges $a_1''a_6$ and a_6a_5'' (resp. $a_2''a_3$ and $a_4''a_3$) are signed in G_1 (resp. G_2). So $w(H') \equiv (w(H) + w(H_1) + w(H_2)) \mod 4$. Since H is unbalanced and H_1 and H_2 are balanced, this implies that $w(H') \equiv 2 \mod 4$, and hence H' is an unbalanced hole of A, contradicting the assumption that G does not contain an unbalanced hole of length 6. $\square$

3 Node Cutset Decompositions

Let S be a node cutset in a signed bipartite graph G, and let $C_1, \ldots, C_k$ be the connected components of $G \setminus S$. We define the *blocks of decomposition* to be signed bipartite graphs $G_1, \ldots, G_k$, where each G_i is a subgraph of G induced by the node set $V(C_i) \cup S$.

With this definition of blocks, the decomposition by an extended star cutset is not $\mathcal{B}$-preserving. For example, consider an odd wheel (H, x) in which all the spokes have weight $+1$, and the sectors are of weight 2 modulo 4. Then the wheel is not balanced, since H is an unbalanced hole, but all the blocks of decomposition by a star cutset $N(x) \cup \{x\}$ are balanced.

In the next section we define a notion of a clean unbalanced hole and show that either some such hole is not broken by the node cutset decompositions we use in the recognition algorithm, or an unbalanced hole is detected while performing the decomposition.

To ensure that we end up with a polynomial number of blocks, instead of using extended strar cutset decompositions, we use the removal of dominated nodes together with double star cutset decompositions. A node u is said to be *dominated* if there exists a node v, distinct from u, such that $N(u) \subseteq N(v)$. A graph is said to be *undominated* if it does not contain any dominated nodes. A *double star cutset* in a graph G is a node cutset $S = N(u) \cup N(v)$, where uv is an edge of G.

Lemma 3.1 *[3] If a bipartite graph contains an extended star cutset, then it contains a dominated node or a double star cutset.*

3.1 Decompositions in Clean Graphs

Definition 3.2 *A node u is* strongly adjacent *to a hole H in the graph G, if u is not a node of H and it has at least two neighbors in H. It is* odd-strongly adjacent *if it has an odd number of neighbors in H and it is* even-strongly adjacent *if it has an even number of neighbors in H.*

Definition 3.3 *A* tent *$\tau(H, u, v)$ is a subgraph of G induced by node set $V(H) \cup \{u, v\}$, where H is a hole of G and $u \in V^r$ and $v \in V^c$ are adjacent nodes which are even-strongly adjacent to H with the following property: the nodes of H can be partitioned into two subpaths P_u and P_v containing the nodes in $N(u) \cap H$ and $N(v) \cap H$ respectively. A tent $\tau(H, u, v)$ is referred to as a* tent containing *H.*

Definition 3.4 *A hole H is said to be* clean in *G if the following three conditions hold:*

(i) No node is odd-strongly adjacent to H.

(ii) Every even-strongly adjacent node to H has exactly two neighbors in H and these two neighbors are at distance two in H.

(iii) There is no tent containing H.

Definition 3.5 *Let G be a signed bipartite graph containing a hole H. Then $\mathcal{C}_G(H) = \{H_i \mid H_i$ is obtained from H by a sequence of holes $H = H_0, H_1, \ldots, H_i$, where H_j and H_{j-1}, for $j = 1, 2, \ldots, i$, differ in one node $\}$.*

Lemma 3.6 *Let G be a signed bipartite graph which contains no unbalanced holes of length 4. Let H be an unbalanced hole in G. If H' and H differ in at most one node, then H' is unbalanced.*

Proof: Let H' be obtained from H by replacing node u by node v. Let x and y be the common neighbors of u and v in H. Since G contains no unbalanced of length four, the paths x, u, y and x, v, y have the same weight modulo 4. Thus, H' is unbalanced. $\Box$

An unbalanced hole H^* of G is *smallest* if its number of edges is smallest.

Lemma 3.7 *If H^* is a smallest unbalanced hole in G, then every even-strongly adjacent node to H^* has exactly two neighbors in H^* and these two neighbors are at distance two in H^*.*

Proof: Suppose u has an even number of neighbors, $u_1, u_2, \ldots, u_{2k}$, $k \geq 2$ in H^*. Let S_i, $i = 1, 2, \ldots, 2k$ be the sectors of (H^*, u) having nodes u_i, u_{i+1} as endnodes (where indices are taken modulo $2k$).

By scaling of the graph at every node u_i for which the edge uu_i has weight -1, we can obtain a graph in which all the spokes of (H^*, u) have weight $+1$. Now since H^* is unbalanced, there is a sector, say S_i, of weight 0 mod 4. Then the hole u, u_i, S_i, u_{i+1}, u is unbalanced and has smaller length than H^*. Hence if u is an even-strongly adjacent node in H^* it must have exactly two neighbors, say u_1 and u_2. W.l.o.g the edges uu_1 and uu_2 have weight $+1$. Clearly the two u_1u_2-subpaths of H^* say P_1 and P_2, are such that one of them is of weight 0 mod 4 and the other is of weight 2 mod 4. Suppose P_2 is of weight 2 mod 4. Then P_2 must have length two for otherwise u, u_1, P_1, u_2, u would be an unbalanced hole of smaller length than H^* . Hence u_1 and u_2 are at distance 2 in H^*. □

When referring to a tent $\tau(H^*, u, v)$ we assume that H^* is a smallest unbalanced hole. By Lemma 3.7, u has two neighbors in H^* say u_1, u_2, both adjacent to u_0 in H^*. Similarly the neighbors of v in H^* are v_1, v_2, both adjacent to v_0 in H^*. We assume that nodes $u_1, u_0, u_2, v_1, v_0, v_2$ are encountered in this order, when traversing H^*.

Definition 3.8 *A wheel with three spokes and at least two sectors of length 2 is said to be a* short 3-wheel.

Lemma 3.9 *Let G be a signed bipartite graph containing a smallest unbalanced hole H^*, but not containing a short 3-wheel and not containing an unbalanced hole of length 4. If H^* is clean in G, then every hole H_i^* in $\mathcal{C}_G(H^*)$ is clean in G.*

Proof: It suffices to show that, if H_1^* is a hole that differs from H^* in only one node, then H_1^* is clean in G.

By Lemma 3.6, H_1^* is an unbalanced hole of smallest length. By Lemma 3.7, condition (ii) of Definition 3.4 is satisfied. Hence, if the lemma is false, condition (i) or (iii) of Definition 3.4 is not satisfied. Therefore we consider the following two cases.

Case 1: Condition (i) of Definition 3.4 is not satisfied.

Now a node w must be odd-strongly adjacent to H_1^*. Since no node is odd-strongly adjacent to H^*, it follows that w has three neighbors, say w_1, w_2, w_3 in H_1^*. Two of these neighbors, say w_1 and w_2 must be in H^* and, by Lemma 3.7, they have a common neighbor, say w_0 in H^*. Since w_3 is in H_1^* but not in H^*, it follows that H_1^* is obtained from H^* by replacing some node $u \neq w_1,\ w_2$ in H^* with w_3. Let u_1 and u_2 be the neighbors of u in H^*. Note that w_3 is adjacent to u_1 and u_2 and u does not coincide with w_1 or w_2. Hence u_1 and u_2 do not coincide with w_0. Now $\tau(H^*, w_3, w)$ is a tent, contradicting the assumption that H^* is clean in G.

Case 2: Condition (iii) of Definition 3.4 is not satisfied.

There must be a tent $\tau(H_1^*, u, v)$. We first show the following claim:

Claim: *At least one of the nodes u_1, u_2, v_1, v_2 does not belong to the hole H^*.*

Proof of Claim: Assume not. Since u and v are not in H_1^*, it follows that at most one of them is in H^*. If u is in H^*, then u_0 is not in H^* and v is odd-strongly adjacent to H^*, contradicting (i) of Definition 3.4. So u is not in H^* and, by symmetry, node v is not in H^*.

Let $w \neq u_1, u_2, v_1, v_2$ be a node in H^* but not in H_1^*. Nodes w and u are not adjacent, otherwise node u is odd-strongly adjacent to H^*, contradicting the assumption that H^* is clean. By symmetry, it follows that nodes w and v are not adjacent. Now $\tau(H^*, u, v)$ is a tent, contradicting the assumption that H^* is clean and the proof of the claim is complete.

By the above claim, one of the nodes u_1, u_2, v_1, v_2 is not in H^*. Assume w.l.o.g. that u_2 is not in H^*. Clearly, node u is not in H^*. Node v is not in H^*, otherwise node v_0 is not in H^*, node u_2 coincides with v_0 and $\tau(H_1^*, u, v)$ is not a tent.

Thus the hole H_1^* is obtained from H^* by replacing a node w with u_2, where w is adjacent to u_0. Let u_3 in H^* be the other neighbor of u_2. It follows that u_3 is adjacent to w. Let Q denote the v_1u_3-subpath of H^* not containing v_2. Consider the hole $C = u, v, v_1, Q, u_3, w, u_0, u_1, u$. Now the wheel (C, u_2) is a short 3-wheel, contradicting the fact that G does not contain a short 3-wheel. $\Box$

Definition 3.10 *A signed bipartite graph G is* clean *if either G is balanced or G contains a smallest unbalanced hole H^* such that all the holes in $\mathcal{C}_G(H^*)$ are clean.*

In the next section we show how to construct, from a signed bipartite graph G, a clean graph G' that has the property that G is balanced if and only if G' is.

Lemma 3.11 *Let G be a clean graph with family $\mathcal{C}_G(H^*)$ of clean smallest unbalanced holes. Let u be a dominated node of G and let $G' = G \setminus \{u\}$. Then some hole in $\mathcal{C}_G(H^*)$ is contained in G'.*

Proof: If u is not in H^*, then H^* belongs to G'. So assume that $u \in V(H^*)$ and that it is dominated by node v. Let u_1 and u_2 be the neighbors of u in H^*. Then v is adjacent to u_1 and u_2, and since H^* is clean, these are the only neighbors of v in H^*. The hole induced by the node set $(V(H^*) \setminus \{u\}) \cup \{v\}$ is in $\mathcal{C}_G(H^*)$ and is contained in G'. $\Box$

Definition 3.12 *A $3PC(x, y)$, with the three paths P_1, P_2 and P_3, is* decomposition detectable *w.r.t. the double star cutset $S = N(u) \cup N(v)$ if $P_1 = x, u, v, y$ and the intermediate nodes of P_2 and P_3 are in different components of $G \setminus S$.*

Lemma 3.13 *Let G be a clean graph with family $\mathcal{C}_G(H^*)$ of clean smallest unbalanced holes. Furthermore assume that G does not contain an unbalanced hole of length 4. When decomposing G with a double star cutset S, then either some hole in $\mathcal{C}_G(H^*)$ is contained in one of the blocks of the decomposition or there exists a decomposition detectable $3PC(x, y)$ w.r.t. S.*

Proof: Let $S = N(u) \cup N(v)$ be a double star cutset of G. Let $C_1, \ldots, C_k$ be the connected components of $G \setminus S$ and $G_1, \ldots, G_k$ be the corresponding blocks of decomposition. We consider the following three cases.

Case 1: Both nodes u and v belong to H^*.

Let u_1 (resp. v_1) be the neighbor of u (resp. v) in H^* that is distinct from v (resp. u). The nodes of $V(H^*) \setminus \{u, v, u_1, v_1\}$ are in some connected component C_i and hence H^* is contained in G_i.

Case 2: Exactly one of the nodes u or v is in H^*.

Assume w.l.o.g. that u is in H^* and v is not. Let u_1 and u_2 be the neighbors of u in H^*. Note that, since H^* is clean, v can have at most one neighbor distinct from u in H^*. First suppose that v does not have any neighbor other than u in H^*. Then the node set $V(H^*) \setminus \{u, u_1, u_2\}$ is contained in some connected component C_i and hence G_i contains H^*. Now suppose that v has a neighbor v_1, distinct from u, in H^*. Nodes v_1 and u must have a common neighbor in H^*, say u_1. Then the node set $V(H^*) \setminus \{v_1, u, u_1, u_2\}$ is contained in some connected component C_i and hence G_i contains H^*.

Case 3: Neither u nor v is in H^*.

Assume w.l.o.g. that $|N(u) \cap V(H^*)| \leq |N(v) \cap V(H^*)|$. We consider the following three subcases.

Case 3.1: $N(u) \cap V(H^*) = \emptyset$

If $|N(v) \cap V(H^*)| = 0$ or 1, then H^* is contained in some block G_i. Suppose that $N(v) \cap V(H^*) = \{v_1, v_2\}$. Let v_0 be the common neigbor of v_1 and v_2 in H^*. The node set $V(H^*) \setminus \{v_0, v_1, v_2\}$ is contained in some connected component C_i. Let H be the hole obtained from H^* by replacing v_0 with v. Then H belongs to $\mathcal{C}_G(H^*)$ and the block G_i contains H.

Case 3.2: $N(u) \cap V(H^*) = \{u_1\}$

Then $|N(v) \cap V(H^*)| = 1$ or 2. First suppose that $N(v) \cap V(H^*) = \{v_1\}$. If u_1 and v_1 are adjacent in H^*, then H^* is contained in some block G_i. Suppose that u_1 and v_1 are not adjacent. Let P and Q be the two u_1v_1-subpaths of H^*. The nodes of $V(P) \setminus \{u_1, v_1\}$ are contained in some connected component C_i and the nodes in $V(Q) \setminus \{u_1, v_1\}$ are contained in some connected component C_j. If $i = j$ then H^* is contained in the block G_i. If $i \neq j$ then the node set $V(H^*) \cup \{u, v\}$ induces a decomposition detectable $3PC(u_1, v_1)$ w.r.t. S.

Now suppose that $N(v) \cap V(H^*) = \{v_1, v_2\}$. Let v_0 be the common neighbor of v_1 and v_2 in H^*. If $u_1 = v_0$ then H^* is contained in some block G_i. So suppose that $u_1 \neq v_0$. Scale at v_1 and v_2 to get the edges vv_1 and vv_2 to have weight $+1$. Since G does not contain an unbalanced hole of length 4, the weight of the path v_1, v_0, v_2 is congruent to $2 \bmod 4$. Scale at u and u_1 to get the edges uv and uu_1 to have weight $+1$. Let P be the u_1v_1-subpath of H^* that does not contain v_2, and let Q be the u_1v_2-subpath of H^* that does not contain v_1. Then $w(P)$ and $w(Q)$ are congruent to 1 or $3 \bmod 4$. Since the weight of the path v_1, v_0, v_2 is congruent to $2 \bmod 4$, $w(P) \not\equiv w(Q) \bmod 4$. If u_1 is not adjacent to v_1 or v_2, then either v, u, u_1, P, v_1, v or v, u, u_1, Q, v_2, v is an unbalanced hole of length smaller than H^*. So suppose w.l.o.g. that u_1 is adjacent to v_1. Then the nodes of $V(H^*) \setminus \{u_1, v_1, v_0, v_2\}$ are contained in some connected component C_i. Let H be the hole obtained from H^* by replacing v_0 with v. Then H belongs to $\mathcal{C}_G(H^*)$ and the block G_i contains H.

Case 3.3: $N(u) \cap V(H^*) = \{u_1, u_2\}$

Then $N(v) \cap V(H^*) = \{v_1, v_2\}$. Let u_0 be the common neighbor of u_1 and u_2 in H^* and let v_0 be the common neighbor of v_1 and v_2 in H^*. Since there is no tent containing H^* and $N(u) \cap V(H^*) = \{u_1, u_2\}$ and $N(v) \cap V(H^*) = \{v_1, v_2\}$, we have that u_0 is adjacent to v and v_0 is adjacent to u. Therefore H^* is contained in some block G_i. $\Box$

3.2 Properties of Smallest Unbalanced Holes

Let H be a hole of G. By $A_r(H)$ (resp. $A_c(H)$) we denote the set of all odd-strongly adjacent nodes to H which belong to V^r (resp. V^c).

Theorem 3.14 *Let G be a signed bipartite graph which does not contain an unbalanced hole of length 4. Let H^* be a smallest unbalanced hole of G. Then H^* contains two edges x_1x_2 and y_1y_2 such that*

(i) $A_r(H^*) \subseteq N(x_1) \cup N(y_1)$

(ii) $A_c(H^*) \subseteq N(x_2) \cup N(y_2)$

(iii) for every tent $\tau(H^*, u, v)$*, either* $u \in N(x_1) \cup N(y_1)$ *or* $v \in N(x_2) \cup N(y_2)$*.*

This section is devoted to the proof of the above theorem. We assume that G is a signed bipartite graph that is not balanced but does not contain an unbalanced hole of length 4. We denote by H^* a smallest unbalanced hole of G.

Lemma 3.15 *If* $u, v \in A_c(H^*)$*, then they have at least one common neighbor in* H^**. Moreover in any sector of* (H^*, v)*, node* u *has either an even number of neighbors, or exactly one neighbor adjacent to* v*.*

Proof: First we show that u cannot have an odd number, greater than one, of neighbors in any one sector of (H^*, v). Suppose not. Let u have an odd number of neighbors, greater than one in sector S_k of (H^*, v). Let $H = v, S_k, v$. Now (H, u) is an odd wheel, therefore this wheel contains an unbalanced hole which must be of smaller length than H^*. Hence u must have either an even number or exactly one neighbor in any sector of (H^*, v).

Next we show that if node u has exactly one neighbor in some sector then this node is also adjacent to v. This in turn implies that at least one node in H^* is a neighbor of both u and v since node u has an odd number of neighbors in H^*.

Suppose in sector S_k node u has a unique neighbor u_k which is not a neighbor of v. Let v_{k-1} and v_k be the end nodes of S_k, P_1 and P_2 be the $v_{k-1}u_k$ and $v_k u_k$-subpaths of S_k repectively. Since u is strongly adjacent to H^*, it has a neighbor in another sector, say S_l having one endnode v_l distinct from v_{k-1} and v_k. Let u_l be the neighbor of u closest to v_l in sector S_l. (Note that since $u, v \in V^c$, then $v_{k-1}, v_k, u_l \in V^r$ and hence u_l cannot be adjacent to v_{k-1} or v_k). Now there is a $3PC(u_k, v)$ using paths P_1, P_2 and nodes u_l and v_l. This 3-path configuration must contain an unbalanced hole which must be of smaller length than H^*, which contradicts our choice of H^*. □

Lemma 3.16 *Every three nodes in* $A_c(H^*)$ *have a common neighbor in* H^**.*

Proof: Let $U = \{u_1, u_2, u_3\} \subseteq A_c(H^*)$. Note that by Lemma 3.15 every pair of nodes in $A_c(H^*)$ has a common neighbor in H^*. Assume that there is no node of H^* that is adjacent to all three nodes of U.

Let A_{12} be the set of nodes of H^* adjacent to u_1 and u_2. A_{13} and A_{23} are analogously defined.

By our assumption $A_{12} \cap A_{23} = \emptyset$. Consider the wheel (H^*, u_1) and the strongly adjacent node u_3. For any $j, k \in \{1, 2, 3\}$ with $j \neq k$, define $A^o_{jk} = \{v \in A_{jk}|$ in the two adjacent sectors of (H^*, u_j) with the common node v, there are in total an odd number of neighbors of $u_k\}$. (Note that this definition is not symmetric, i.e. A^o_{jk} is not necessarily equal to A^o_{kj}). Now we prove two claims.

Claim 1: A^o_{jk} *contains an odd number of elements.*

Proof of Claim 1: We prove that $|A^o_{13}|$ is odd. Consider the wheel (H^*, u_1) and let $S_1, \dots, S_n$ be the sectors of this wheel, with S_i having endnodes s_i and s_{i+1} (where indices are taken modulo n). For every $i = 1, \dots, n$ let x_i denote the number of neighbors of u_3 in sector S_i. By Lemma 3.15 every sector of (H^*, u_1) either has an even number of neighbors of u_3 or exactly one neighbor, in which case the neighbor is in A_{13}. This and the definition of A^o_{13} leads to the following properties:

(a) If $s_i \in A^o_{13}$ then either $x_{i-1} = x_i = 1$, or both x_{i-1} and x_i are even.

(b) If $s_i \in A_{13} \setminus A^o_{13}$ then either $x_{i-1} = 1$ and x_i is even, or x_{i-1} is even and $x_i = 1$.

(c) If s_i and s_{i+1} are not in A_{13} then x_i is even.

Now we show that

$$\sum_{i=1}^{n} x_i \equiv |A_{13} \setminus A^o_{13}| \mod 2 \tag{1}$$

Clearly the parity of $\sum_{i=1}^{n} x_i$ is the parity of the number of sectors with an odd number of neighbors of u_3. We refer to these sectors as *odd sectors*. By Properties (a), (b) and (c), if S_i is an odd sector, then it has exactly one neighbor of u_3 (i.e. $x_i = 1$), and either s_i or s_{i+1} is an element of A_{13}. Each element in A_{13} belongs to $0, 1$ or 2 odd sectors. Clearly the parity of the number of odd sectors is equal to the parity of the number of elements in A_{13} which belong to exactly one odd sector. By Properties (a) and (b), $A_{13} \setminus A^o_{13}$ is the set of elements of A_{13} that belong to exactly one odd sector. Thus the parity of $\sum_{i=1}^{n} x_i$ is the same as the parity of $|A_{13} \setminus A^o_{13}|$.

In the summation $\sum_{i=1}^{n} x_i$, every neighbor of u_3 which is in A_{13} is counted twice, so the total number of neighbors of u_3 on H^* is

$$|N(u_3) \cap V(H^*)| = \sum_{i=1}^{n} x_i - |A_{13}| \tag{2}$$

Now by (1) and (2) we have

$$\begin{aligned} |N(u_3) \cap V(H^*)| &\equiv (|A_{13} \setminus A^o_{13}| - |A_{13}|) \mod 2 \\ &\equiv -|A^o_{13}| \mod 2 \end{aligned}$$

Since u_3 is an odd-strongly adjacent node to H^*, we have that $|A^o_{13}|$ is odd. This completes the proof of Claim 1.

Claim 2: *Let $v_1, v_2 \in V(H^*) \setminus A_{12}$ be neighbors of u_1 and u_2 respectively. If P is a v_1v_2-subpath of H^*, such that u_1 and u_2 have no neighbors in $V(P) \setminus \{v_1, v_2\}$, then u_3 has an even number of neighbors on P.*

Proof of Claim 2: Suppose that u_3 has an odd number of neighbors on P.

Assume first that u_3 has exactly one neighbor v_3 on P.

W.l.o.g $v_3 \neq v_1$. By Lemma 3.15, any two nodes of $A_c(H^*)$ have a common neighbor on H^*. Let $v_{12} \in V(H^*)$ be a common neighbor of u_1 and u_2, and let $v_{13} \in V(H^*)$ be a common neighbor of u_1 and u_3. By our assumption $A_{12} \cap A_{13} = \emptyset$, so $v_{12} \neq v_{13}$. Now there is a $3PC(v_3, u_1)$ where nodes v_1, v_{12}, v_{13} belong to distinct paths of the 3-path configuration, which must contain an unbalanced hole of length smaller than H^*. This contradicts our choice of H^*.

Assume now that u_3 has an odd number of neighbors, greater than one, on P.

Let v_{12} be defined as above. Now there is an odd wheel (C, u_3), where $C = u_1, v_1, P, v_2, u_2, v_{12}, u_1$. Since u_1 is an odd-strongly adjacent node either the v_1v_{12}-subpath of H^* that does not contain v_2 or the v_2v_{12}-subpath of H^* that does not contain v_1, is of length greater than two. Therefore the wheel contains an unbalanced hole of length smaller than H^*, which contradicts our choice of H^*. This completes the proof of Claim 2.

Now let $s_1, \ldots, s_n$ be the neighbors of u_1 on H^*, and $t_1, \ldots, t_m$ be the neighbors of u_2 on H^*. Let $P_1, \ldots, P_l$ be all the subpaths of H^*, whose endnodes belong to $\{s_1, \ldots, s_n, t_1, \ldots, t_m\}$ but have no intermediate node in this set. For every $i = 1, \ldots, l$, let x_i denote the number of neighbors of u_3 in P_i. Let the endnodes of P_i be denoted by p_i and p_{i+1} (where the indices are taken modulo l). By Lemma 3.15 and Claim 2, if x_i is odd, then $x_i = 1$. Furthermore, by property (c) in Claim 1, if $x_i = 1$ then exactly one of p_i or p_{i+1} is in $A_{13} \cup A_{23}$.

The P_i's with exactly one neighbor of u_3 are characterized as follows:

(i) If $x_i = 1$ and $p_i \in A^o_{13}$, then by Claim 2, p_{i+1} is a neighbor of u_1. Now by Property (a) in Claim 1 $x_{i-1} = 1$ and hence by Claim 2, p_{i-1} is a neighbor of u_1. Similarily if $x_i = 1$ and $p_i \in A^o_{23}$, then $x_{i-1} = 1$ and both p_{i-1} and p_{i+1} are neighbors of u_2.

(ii) If $x_i = 1$ and $p_i \in A_{13} \setminus A^o_{13}$, then by Claim 2, p_{i+1} is a neighbor of u_1. Also either by Property (b) in Claim 1 or by Claim 2, x_{i-1} is even. Similarily if $x_i = 1$ and $p_i \in A_{23} \setminus A^o_{23}$, then p_{i+1} is a neighbor of u_2 and x_{i-1} is even.

In the summation $\sum_{i=1}^{n} x_i$, every neighbor of u_3 which is in $A_{13} \cup A_{23}$ is counted twice, so the total number of neighbors of u_3 on H^* is

$$|N(u_3) \cap V(H^*)| = \sum_{i=1}^{n} x_i - |A_{13}| - |A_{23}| \tag{3}$$

Further we will show that

$$\sum_{i=1}^{n} x_i \equiv (|A_{13} \setminus A^o_{13}| + |A_{23} \setminus A^o_{23}|) \mod 2 \tag{4}$$

Now by (3) and (4) we have

$$\begin{aligned} |N(u_3) \cap V(H^*)| &\equiv (|A_{13} \setminus A^o_{13}| - |A_{13}| + |A_{23} \setminus A^o_{23}| - |A_{23}|) \mod 2 \\ &\equiv -(|A^o_{13}| + |A^o_{23}|) \mod 2 \end{aligned}$$

By Claim 1 $(|A^o_{13}| + |A^o_{23}|)$ is even, which contradicts our choice of u_3. Thus A_{13} and A_{23} cannot be disjoint.

Now we prove (4). Clearly the parity of $\sum_{i=1}^{n} x_i$ is the same as the parity of the number of sectors with an odd number of neighbors of u_3. Recall that if P_i has an odd number of neighbors of u_3, then it has exactly one neighbor (i.e. $x_i = 1$) and exactly one of p_i or p_{i+1} is an element of $A_{13} \cup A_{23}$. W.l.o.g. let $p_i \in A_{13} \cup A_{23}$. Pair off P_{i-1} and P_i if the only neighbor of u_3 in these paths is the node common to P_{i-1} and P_i, namely p_i. By Property (i) and (ii) this is possible if and only if $p_i \in A^o_{13} \cup A^o_{23}$. Notice that in this case $x_{i-1} + x_i = 2$ and the sectors together provide an even count in the sum $\sum_{i=1}^n x_i$. Hence the parity of $\sum_{i=1}^n x_i$ is the same as the parity of $|A_{13} \setminus A^o_{13}| + |A_{23} \setminus A^o_{23}|$, and so (4) holds.

This completes the proof that A_{13} and A_{23} are not disjoint. Hence we have proved the lemma. □

Lemma 3.17 *H^* contains a node adjacent to all the nodes in $A_c(H^*)$ and a node adjacent to all the nodes in $A_r(H^*)$.*

Proof: By symmetry, it suffices to prove the first statement. If H^* is of length 6 or less then the property clearly holds. Suppose now that H^* has length greater than 6. Suppose $W \subseteq A_c(H^*)$ is such that for every proper subset W' of W there exists a node of H^* which is adjacent to all nodes in W', but there exists no node of H^* adjacent to all nodes in W. By Lemma 3.15 and Lemma 3.16, $|W| > 3$. Let $W = \{w_i | i = 1, 2, \ldots, p\}$ and let $W_l = \{w_i | i = 1, \ldots, p, i \neq l\}$. Now for $l = 1, 2, \ldots, p$, all the nodes in W_l have a common neighbor say t_l, in H^*. Hence for $i = 1, \ldots, p$, node t_i is adjacent to w_j, for $j = 1, \ldots, p, j \neq i$, but t_i is not adjacent to w_i. Now there exists an odd wheel, $w_1, t_2, w_3, t_1, w_2, t_3, w_1$ with center t_4, hence it must contain an unbalanced hole smaller than H^*. This contradicts the choice of H^*. □

Lemma 3.18 *For a tent $\tau(H^*, u, v)$ the following hold:*

- *$A_r(H^*) \subseteq N(v_0) \cup N(u_1)$ or $A_r(H^*) \subseteq N(v_0) \cup N(u_2)$.*
- *$A_c(H^*) \subseteq N(u_0) \cup N(v_1)$ or $A_c(H^*) \subseteq N(u_0) \cup N(v_2)$.*

Proof: We prove the first part. Suppose $w \in A_r(H^*)$ is not adjacent to v_0. Consider the hole H^*_1 obtained from H^* by replacing v_0 with node v of $\tau(H^*, u, v)$. By Lemma 3.6, H^*_1 is unbalanced, and since it is of the same length as H^*, it also is a smallest unbalanced hole. Now w cannot be adjacent to v, for otherwise w is even-strongly adjacent to H^*_1, which violates Lemma 3.7. Node u is in $A_r(H^*_1)$ and has neighbors u_1, u_2 and v in H^*_1. Since w is not adjacent to v, by Lemma 3.17 it follows that w is adjacent to u_1 or u_2. Furthermore, by Lemma 3.17 the nodes in $A_r(H^*)$ which are not adjacent to v_0 are either all adjacent to u_1 or they are all adjacent to u_2. Therefore $A_r(H^*) \subseteq N(v_0) \cup N(u_1)$ or $A_r(H^*) \subseteq N(v_0) \cup N(u_2)$. The second part of the lemma can be proved similarly. □

Lemma 3.19 *Let $\tau(H^*, u, v)$ and $\tau(H^*, w, y)$ be two tents, where w_1, w_2 are the neighbors of w and y_1, y_2 are the neighbors of y in H^*. Let w_0 and y_0 be the common neighbors in H^* of w_1, w_2 and y_1, y_2 respectively. Then at least one of the following properties holds:*

- *Nodes u_1 and u_2 coincide with w_1 and w_2.*
- *Nodes v_1 and v_2 coincide with y_1 and y_2.*

- *Node u_0 coincides with y_1 or y_2.*
- *Node v_0 coincides with w_1 or w_2.*

Proof: Suppose the contrary. Then node u does not coincide with w, node v does not coincide with y , nodes u_0 and y are not adjacent and nodes v_0 and w are not adjacent. Let P denote the u_2v_1-subpath of H^* not containing any other neighbor of u or v. Similarly, let Q denote the v_2u_1-subpath of H^* not containing any other neighbors of u and v. Now it follows that y_1 and y_2 are contained in P or Q since they are at distance two by Lemma 3.7, and w_1 and w_2 are contained in P or Q. Assume w.l.o.g. that y_1 and y_2 are contained in P. We now prove the following two claims.

Claim 1: *Node y is not adjacent to u and node w is not adjacent to v.*

Proof of Claim 1: Suppose that y and u are adjacent. Now there is an odd wheel u_2, P, v_1, v, u, u_2 with center y. This wheel contains an unbalanced hole, which is by construction, of smaller length than H^*, which contradicts our choice of H^*. Hence y is not adjacent to u. By symmetry, it follows that w is not adjacent to v. This completes the proof of Claim 1.

Claim 2: *Nodes w_1 and w_2 belong to Q.*

Proof of Claim 2: Suppose not. Then w_1 and w_2 belong to P. By assumption, y_1 and y_2 belong to P. Let P' be the path obtained from P by substituting y for y_0. Now by Claim 1, there is an odd wheel u_2, P', v_1, v, u, u_2 with center w. This wheel contains an unbalanced hole, which is by construction, of smaller length than H^*. This contradics our choice of H^*. Hence w_1 and w_2 belong to Q. This completes the proof of Claim 2.

Now by Claim 1 and Claim 2, there is a $3PC(u, y)$ that uses at most as many edges as there are in H^*. This 3-path configuration contains an unbalanced hole, of smaller length than H^*, which contradicts our choice of H^*. □

Proof of Theorem 3.14: First assume that there is no tent in G that contains H^*. By Lemma 3.17 H^* contains a node x_2 that is adjacent to all nodes in $A_c(H^*)$. By Lemma 3.17 H^* contains a node y_1 that is adjacent to all nodes in $A_r(H^*)$. Let x_1 be a neighbor of x_2 in H^*, and let y_2 be a neighbor of y_1 in H^*. Then the edges x_1x_2 and y_1y_2 satisfy (i), (ii) and (iii).

Now assume that G contains a tent $\tau(H^*, u, v)$. By Lemma 3.18 $A_r(H^*) \subseteq N(v_0) \cup N(u_1)$ or $A_r(H^*) \subseteq N(v_0) \cup N(u_2)$, and $A_c(H^*) \subseteq N(u_0) \cup N(v_1)$ or $A_c(H^*) \subseteq N(u_0) \cup N(v_2)$. Assume that $A_r(H^*) \subseteq N(v_0) \cup N(u_1)$ and $A_c(H^*) \subseteq N(u_0) \cup N(v_1)$. By Lemma 3.19, for every tent $\tau(H^*, w, y)$ in G, either $w \in N(v_0) \cup N(u_1)$ or $y \in N(u_0) \cup N(v_1)$. Hence the edges u_0u_1 and v_0v_1 satisfy (i), (ii) and (iii). The other cases follow similarly. □

4 Recognition Algorithm and its Validity

In this section we present the algorithm that recognizes whether a signed bipartite graph is balanced.

4.1 Cleaning Procedure

CLEANING PROCEDURE

Input: A signed bipartite graph G which does not contain an unbalanced hole of length 4.

Output: A family $\mathcal{L}$ of induced subgraphs of G such that if G is not balanced, then some G' in $\mathcal{L}$ contains a smallest unbalanced hole that is clean in G'.

Step 1 Let $\mathcal{L} = \{G\}$. Let U be the set of all $(P_1; P_2)$ where P_1 and P_2 are chordless paths in G of length 3.

Step 2 For every $(P_1 = x_0, x_1, x_2, x_3; P_2 = y_0, y_1, y_2, y_3) \in U$, add to $\mathcal{L}$ the graph obtained from G by removing the node set $(N(x_1) \cup N(x_2) \cup N(y_1) \cup N(y_2)) \setminus (V(P_1) \cup V(P_2))$.

Remark 4.1 *The number of graphs in list $\mathcal{L}$ produced by the Cleaning Procedure is bounded by $|V^r|^4|V^c|^4$.*

Lemma 4.2 *The Cleaning Procedure produces the desired output.*

Proof: Assume that G is not balanced and let H^* be a smallest unbalanced hole in G. By Theorem 3.14 H^* contains edges x_1x_2 and y_1y_2 that satisfy (i), (ii) and (iii) of Theorem 3.14. Let $P_1 = x_0, x_1, x_2, x_3$ and $P_2 = y_0, y_1, y_2, y_3$ be the two subpaths of H^* with middle edges x_1x_2 and y_1y_2. Let G' be the graph obtained from G by removing the node set $(N(x_1) \cup N(x_2) \cup N(y_1) \cup N(y_2)) \setminus (V(P_1) \cup V(P_2))$. G' is one of the graphs in $\mathcal{L}$ and it contains H^*. By Lemma 3.7 and Theorem 3.14, H^* is clean in G'. $\Box$

4.2 Short 3-Wheels

SHORT 3-WHEEL PROCEDURE

Input: A signed bipartite graph G.

Output: A short 3-wheel of G or the fact that G does not contain such a node induced subgraph.

Step 1: Enumerate all distinct subsets of six nodes with three nodes in V^r and three nodes in V^c and declare them as unscanned. Go to Step 2.

Step 2: If all subsets are scanned, G does not contain a short 3-wheel, stop. Otherwise choose an unscanned subset U. If U induces a 6-cycle $C = a_1, a_2, a_3, a_4, a_5, a_6, a_1$, having unique chord a_2a_5, go to Step 3. Otherwise declare U as scanned and repeat Step 2.

Step 3: Remove the nodes in $N(a_2) \cup N(a_4) \cup N(a_5) \cup N(a_6) \setminus \{a_1, a_3\}$. If a_1 and a_3 are in the same connected component, then a short 3-wheel with spokes a_2a_1, a_2a_3, a_2a_5 is identified, stop. If not, remove the nodes in $N(a_1) \cup N(a_2) \cup N(a_3) \cup N(a_5) \setminus \{a_4, a_6\}$. If a_4 and a_6 are in the same connected component, then a short 3-wheel with spokes a_5a_2, a_5a_4, a_5a_6 is identified, stop. Otherwise declare U as scanned return to Step 2.

4.3 6-Join Decomposition

We now give an algorithm that finds a 6-join in a connected undominated graph G or shows that G does not have one.

Note that, if a connected undominated graph has a 6-join, then (using the notation given in the introduction) there exists a node in $V_1 \setminus (A_1 \cup A_3 \cup A_5)$ that is adjacent to a node of $A_1 \cup A_3 \cup A_5$ (otherwise some node in $A_1 \cup A_3 \cup A_5$ would be dominated) and there exists a node in $V_2 \setminus (A_2 \cup A_4 \cup A_6)$ that is adjacent to a node of $A_2 \cup A_4 \cup A_6$. Let $a_1, \ldots, a_6, u_1, u_2$ be 8 distinct nodes of G such that $\{a_1, \ldots, a_6\}$ induces a hole of length 6, u_1 is adjacent to at least one node in $\{a_1, a_3, a_5\}$, and u_2 is adjacent to at least one node in $\{a_2, a_4, a_6\}$ but not to u_1. The following rules yield a 6-join $E(A)$ with $\{a_1, a_3, a_5, u_1\} \subseteq V_1$ and $\{a_2, a_4, a_6, u_2\} \subseteq V_2$, or show that G does not have such a 6-join. (Note that if such a 6-join is found then, for $i = 1, \ldots, 6$, $a_i \in A_i$, $u_1 \in V_1 \setminus (A_1 \cup A_3 \cup A_5)$ and $u_2 \in V_2 \setminus (A_2 \cup A_4 \cup A_6)$).

Initially $V_1 = \{a_1, a_3, a_5, u_1\}$ and $V_2 = V(G) \setminus V_1$. Then forcing rules will be applied to move nodes from V_2 to V_1.

During the algorithm the nodes u in V_1 are partitioned into four sets:

- $u \in A_1$ if it is adjacent to a_2 and a_6 but not to a_4,
- $u \in A_3$ if it is adjacent to a_2 and a_4 but not to a_6,
- $u \in A_5$ if it is adjacent to a_4 and a_6 but not to a_2,
- $u \in V_1 \setminus (A_1 \cup A_3 \cup A_5)$ if it is not adjacent to any node a_2, a_4, a_6.

The case where some node u in V_1 is adjacent to exactly one of the nodes a_2, a_4, a_6 or to all three of them will not be permitted.

Forcing rules that move nodes from V_2 to V_1 are as follows.

- If $u \in V_2 \setminus \{a_2, a_4, a_6, u_2\}$ is adjacent to at least one node in $V_1 \setminus (A_1 \cup A_3 \cup A_5)$ then remove u from V_2 and add it to V_1.
- If $u \in V_2 \setminus \{a_2, a_4, a_6, u_2\}$ is adjacent to at least one node in $A_1 \cup A_3 \cup A_5$ and $N(u) \cap (A_1 \cup A_3 \cup A_5) \neq A_1 \cup A_3$, $A_3 \cup A_5$ or $A_1 \cup A_5$, then remove u from V_2 and add it to V_1.

Clearly, if there exists a 6-join $E(A)$ with $\{a_1, a_3, a_5, u_1\} \subseteq V_1$ and $\{a_2, a_4, a_6, u_2\} \subseteq V_2$ and u satisfies one of the above rules, then u must be in V_1.

If some node u which is moved from V_2 to V_1 does not satisfy the following: $N(u) \cap \{a_2, a_4, a_6\} = \emptyset$, $\{a_2, a_4\}$, $\{a_2, a_6\}$ or $\{a_4, a_6\}$, and $N(u) \cap \{u_2\} = \emptyset$, then the algorithm terminates since no 6-join $E(A)$ with $\{a_1, a_3, a_5, u_1\} \subseteq V_1$ and $\{a_2, a_4, a_6, u_2\} \subseteq V_2$ exists. If this situation never occurs, we continue moving nodes from V_2 to V_1 until no forcing rule applies.

At this stage the nodes of V_2 satisfy the following: no node of V_2 is adjacent to a node of $V_1 \setminus (A_1 \cup A_3 \cup A_5)$ and if a node $u \in V_2$ is adjacent to a node of $A_1 \cup A_3 \cup A_5$ then $N(u) \cap (A_1 \cup A_3 \cup A_5) = A_1 \cup A_3$, $A_3 \cup A_5$ or $A_1 \cup A_5$. Denote by A_2 the nodes of V_2 that are adjacent to all nodes in $A_1 \cup A_3$, by A_4 the nodes of V_2 that are adjacent to all nodes in $A_3 \cup A_5$ and by A_6 the nodes of V_2 that are adjacent to all nodes in $A_1 \cup A_5$. Let A be the graph induced by the node set $\cup_{i=1}^{6} A_i$. Then $E(A)$ is a 6-join of G with partition V_1 and V_2.

To determine whether a graph G has a 6-join one would apply the above algorithm to all 8-tuples $(a_1, \ldots, a_6, u_1, u_2)$ of nodes of G for which $\{a_1, \ldots, a_6\}$ induces a hole of length 6, u_1 is adjacent to at least one node in $\{a_1, a_3, a_5\}$, and u_2 is adjacent to at least one node in $\{a_2, a_4, a_6\}$ but not u_1. Clearly all of this can be implemented to run in polynomial time.

Let S be a double star cutset in a graph G, and let $G_1, \ldots, G_k$ be the blocks of decomposition. The *refined blocks of decomposition* are graphs $G_1^*, \ldots, G_k^*$, where G_i^* is obtained from G_i by removing all dominated nodes.

When we say "remove all dominated nodes from a graph F", we mean to apply the following procedure:

Step 1: If F contains a dominated node u, then go to Step 2. Otherwise, stop and output F.

Step 2: Let $F = F \setminus \{u\}$ and go to Step 1.

DOUBLE STAR CUTSET AND 6-JOIN DECOMPOSITION ALGORITHM

Input: A signed bipartite graph G that does not contain a short 3-wheel or an unbalanced hole of length 4 or 6.

Output: Either G is identified as not being balanced, or a list $\mathcal{L}$ of induced subgraphs of G with the following properties:

- The graphs in $\mathcal{L}$ do not contain a 6-join, a double star cutset or any dominated nodes.
- If the input graph G contains a family $\mathcal{C}_G(H^*)$ of clean smallest unbalanced holes, then one of the graphs G' in $\mathcal{L}$ contains a hole H' of $\mathcal{C}_G(H^*)$, and $\mathcal{C}_{G'}(H')$ is a family of clean smallest unbalanced holes in G'.

Step 1: Remove all dominated nodes from G and initialize $\mathcal{M} = \{G\}$ and $\mathcal{L} = \emptyset$.

Step 2: If $\mathcal{M}$ is empty, return $\mathcal{L}$ and stop. Otherwise, remove a graph F from $\mathcal{M}$.

Step 3: If F contains a double star cutset S go to Step 4 and otherwise go to Step 5. (Note that checking whether F contains a double star cutset involves checking for every pair of adjacent nodes u and v whether $S = N(u) \cup N(v)$ is a cutset).

Step 4: Check whether there exists a decomposition detectable $3PC(x, y)$ w.r.t. S. If it does, identify G as not balanced and stop. Otherwise, construct the refined blocks of the decomposition by S, add them to $\mathcal{M}$ and go to Step 2.

Step 5: Check whether F contains a 6-join. If it does, construct the blocks of the 6-join decomposition, remove all dominated nodes from the blocks, add these graphs to $\mathcal{M}$ and go to Step 2. Otherwise, add F to $\mathcal{L}$ and go to Step 2.

Theorem 4.3 *The Double Star Cutset and 6-Join Decomposition Algorithm produces the desired output.*

Proof: Let G be a signed bipartite graph that does not contain an unbalanced hole of length 4 or 6, or a short 3-wheel. If the algorithm terminates in Step 4, then G is correctly identified as not being balanced. So suppose that the algorithm does not terminate in Step 4. By the construction of the algorithm, the graphs in $\mathcal{L}$ do not contain a 6-join, a double star cutset or any dominated nodes. Suppose that G contains a family $\mathcal{C}_G(H^*)$ of clean smallest unbalanced holes. To prove the theorem it is enough to show the following.

(1) If G' is the graph obtained from G by removing dominated nodes, then G' contains a hole in $\mathcal{C}_G(H^*)$.

(2) If $G_1^*, \ldots, G_k^*$ are the refined blocks of decomposition of G by a double star cutset, then for some i, G_i^* contains a hole in $\mathcal{C}_G(H^*)$.

(3) If G_1 and G_2 are the blocks of decomposition of G by a 6-join $E(A)$, then for some i, G_i contains a hole in $\mathcal{C}_G(H^*)$.

(4) If G' contains a hole H' of $\mathcal{C}_G(H^*)$, then $\mathcal{C}_{G'}(H')$ is a family of clean smallest unbalanced holes in G'.

(1) and (2) follow from Lemma 3.11 and Lemma 3.13. (4) follows from the fact that if a hole H' is clean in G, then it is also clean in any induced subgraph G'. To prove (3) suppose that H^* is contained in neither G_1 nor G_2. Then H^* must contain an edge of $E(A)$. Since G does not contain an unbalanced hole of length 6, not all of the edges of H^* can be in $E(A)$. Hence w.l.o.g. we may assume that either (i) $H^* = a_1', a_2', a_3', P_1, a_1'$ where $a_i' \in A_i$ for $i = 1, 2$ and 3, and P_1 is a path with nodes in V_1 from a_1' to a_3', or (ii) $H^* = a_1', a_2', P_2, a_4', a_5', P_1, a_1'$ where $a_i' \in A_i$ for $i = 1, 2, 4$ and 5, P_1 is a path with nodes in V_1 from a_1' to a_5', and P_2 is a path with nodes in V_2 from a_2' to a_4'. If (i) holds, then the hole obtained from H^* by substituting a_2 for a_2', is a hole of $\mathcal{C}_G(H^*)$ and is contained in G_1. So assume (ii) holds. Since node a_3 has neighbors a_2' and a_4' in H^*, and H^* is clean, the path P_2 must be of length 2. Similarly path P_1 must be of length 2. Hence H^* is of length 6, contradicting our assumption. $\Box$

Lemma 4.4 *The number of graphs in the list $\mathcal{L}$ produced by the Double Star Cutset and 6-Join Decomposition Algorithm is bounded by $|V^r|^3|V^c|^3(|V^r| + |V^c|)$.*

Proof: Let G be a signed bipartite graph that does not contain a short 3-wheel, and let $\mathcal{L}$ be the list of induced subgraphs of G produced by the algorithm. Note that we are assuming that the algorithm does not terminate in Step 4, with identifying a decomposition detectable $3PC(x, y)$. We prove the lemma by showing that the number of decompositions used to decompose G by the algorithm is bounded by the number of chordless paths of length 5 in G. (So in the decomposition tree the number of parents of the leaves, i.e. graphs added to $\mathcal{L}$, is bounded by $|V^r|^3|V^c|^3$, and hence the number of graphs in $\mathcal{L}$ is bounded by $|V^r|^3|V^c|^3(|V^r| + |V^c|)$). This will be shown by proving that if F is a graph decomposed in Step 4 or Step 5 of the algorithm, F has the property that it contains a chordless path of length 5 that is not contained in any of the blocks of decomposition that are added to list $\mathcal{M}$, and that no two blocks of decomposition contain the same chordless path of length 5. So the lemma follows from the following four claims.

First suppose that F is decomposed in Step 5 by a 6-join $E(A)$. Let F_1 and F_2 be the blocks of decomposition.

Claim 1: *F_1 and F_2 do not contain the same chordless path of length 5.*

Proof of Claim 1: Any chordless path of length 5 in F_1 must contain a node of $V_1 \setminus (A_1 \cup A_3 \cup A_5)$ and hence cannot be a path of F_2. This completes the proof of Claim 1.

Claim 2: *F contains at least one chordless path of length 5 that is contained neither in F_1 nor in F_2.*

Proof of Claim 2: By (iii) of the definition of a 6-join, $|V_1| \geq 4$. Let $U_1 = V_1 \setminus (A_1 \cup A_3 \cup A_5)$. We must have $U_1 \neq \emptyset$, otherwise some node of $A_1 \cup A_3 \cup A_5$ is dominated in F, a contradiction. No node $u \in U_1$ can have neighbors in each of the sets A_1, A_3 and A_5 since, otherwise, u would be the center of a short 3-wheel. So, w.l.o.g. there exists a node u_1 with no neighbor in A_5, but at least one neighbor in A_1. In F, node u_1 is not dominated by a node of A_2. This implies that u_1 is adjacent to a node $v_1 \in U_1$ that is at distance two from $A_1 \cup A_3 \cup A_5$, i.e. v_1, u_1, a'_1 is a chordless path where $a'_1 \in A_1$. Similarly, let v_2 be a node of F_2 that is at distance two from $A_2 \cup A_4 \cup A_6$. Let v_2, u_2, a'_i be a chordless path with $u_2 \in V_2 \setminus (A_2 \cup A_4 \cup A_6)$ and $a'_i \in A_i$, $i = 2, 4$ or 6. If $i = 2$ or 6, then $v_1, u_1, a'_1, a'_i, u_2, v_2$ is the desired path. So assume that $i = 4$ and u_2 is not adjacent to any node of $A_2 \cup A_6$. Then $u_1, a'_1, a_6, a_5, a'_4, u_2$ is the desired path. This completes the proof of Claim 2.

Now assume that F is decomposed in Step 4 by a double star cutset $S = N(u) \cup N(v)$. Let $C_1, \ldots, C_k$ be the connected components of $F \setminus S$. Let $F_1, \ldots, F_k$ be the blocks of decomposition and $F_1^*, \ldots, F_k^*$ the refined blocks of decomposition. Note that F is an undominated graph, and by the defintion of refined blocks so are $F_1^*, \ldots, F_k^*$. Also, w.l.o.g. we assume that F is a connected graph.

Claim 3: *No two graphs $F_1^*, \ldots, F_k^*$ contain the same chordless path of length 5.*

Proof of Claim 3: We actualy prove a stronger statement that no two graphs $F_1^*, \ldots, F_k^*$ contain the same chordless path of length 3. Assume otherwise and let $P = a, b, c, d$ be a chordless path that is contained in both F_i^* and F_j^*, $i \neq j$. Then $\{a, b, c, d\} \subseteq S$. Since a, b, c, d must alternate between $N(u)$ and $N(v)$, and P is a chordless path, u and v cannot coincide with a or d, and similarly for v. So w.l.o.g. $a \in N(u) \setminus \{v\}$ and $d \in N(v) \setminus \{u\}$. If a does not have a neighbor in C_i then a is dominated by v in F_i, and hence a is dominated by some node in F_i^*, contradicting the assumption that F_i^* is an undominated graph. So a must have a neighbor in C_i, and similarly it must have a neighbor in C_j. By the same argument d has a neighbor in both C_i and C_j. But then there is a decomposition detectable $3PC(a, d)$ w.r.t. S, contradicing our assumption. This completes the proof of Claim 3.

Claim 4: *F contains at least one chordless path of length 5 that is not contained in any of the graphs $F_1^*, \ldots, F_k^*$.*

Proof of Claim 4: Each of the connected components $C_1, \ldots, C_k$ must contain at least two nodes, since F is an undominated graph. Since F is connected, a node of C_i, $i = 1, \ldots, k$, must have a neighbor in S.

First assume that there exist nodes $p_1 \in V(C_1)$ and $p_2 \in V(C_2)$ such that they have a common neighbor $a_1 \in N(u)$. Since $|V(C_1)| \geq 2$, C_1 contains a node q_1 adjacent to p_1. Similarly C_2 contains a node q_2 adjacent to p_2. Since q_2 is not dominated by a_1, q_2 must have a neighbor t_2 that a_1 is not adjacent to. If t_2 is adjacent to q_1, then $t_2 \in N(v)$ and hence there is a decomposition detectable $3PC(a_1, t_2)$ w.r.t. S, contradicting our assumption. So t_2 is not adjacent to q_1, and hence $P = q_1, p_1, a_1, p_2, q_2, t_2$ is the desired path.

Now assume that no two nodes, one from C_1 and one from C_2, have a common neighbor in S. Let p_1 (resp. p_2) be a node of C_1 (resp. C_2) that is adjacent to $a_1 \in S$ (resp. $a_2 \in S$). Let q_1 (resp. q_2) be a neighbor of p_1 (resp. p_2) in C_1 (resp. C_2). If $a_1, a_2 \in N(u)$, then $P = q_1, p_1, a_1, u, a_2, p_2$ is the desired path. So we may assume that $a_1 \in N(u)$ and $a_2 \in N(v)$. If a_1a_2 is not an edge then $P = p_1, a_1, u, v, a_2, p_2$ is the desired path. Otherwise, $P = q_1, p_1, a_1, a_2, p_2, q_2$ is the desired path. This completes the proof of Claim 4. $\Box$

4.4 2-Join Decomposition

In [6] an algorithm that either finds a 2-join in a graph G or concludes that G does not have one is given. We outline here this algorithm for the sake of completeness, in the case where G contains no extended star cutset.

Lemma 4.5 *Let G be a bipartite graph that has no extended star cutset. Then, in every 2-join, $|V(G'_i)| \geq 4$, for $i = 1, 2$.*

Proof: By Lemma 2.2, the 2-join is not rigid. Suppose $|V(G'_i)| \leq 3$, for $i = 1$ or 2.

If there exists a node $u \in V(G'_i) \setminus (A_i \cup B_i)$, then $|A_i| = |B_i| = 1$ and, since the 2-join is not rigid, (ii) of the definition of 2-join implies that u is adjacent to both these nodes. This contradicts (iii) of the definition of 2-join.

So $V(G'_i) \setminus (A_i \cup B_i) = \emptyset$. By (ii) of the definition of 2-join, every node of A_i has a neighbor in B_i and vice versa, every node in B_i has a neighbor in A_i. Since the 2-join is not rigid, this implies that $|A_i| \geq 2$ and $|B_i| \geq 2$. $\Box$

Let a_1, b_1, a_2, b_2 be 4 distinct nodes of a bipartite graph G, such that a_1a_2 and b_1b_2 are edges, but a_1b_2, a_2b_1 are not. The following procedure yields a 2-join $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ with $a_1 \in A_1$, $b_1 \in B_1$, $a_2 \in A_2$ and $b_2 \in B_2$, or shows that no such 2-join exists.

For every 2 distinct nodes $u_1, v_1 \in V(G) \setminus \{a_1, a_2, b_1, b_2\}$, each adjacent to at most one node in $\{a_2, b_2\}$, the following rules identify a partition of $V(G)$ into V_1 and V_2, where $a_1, b_1, u_1, v_1 \in V_1$ and $a_2, b_2 \in V_2$, such that the edges with one endnode in V_1 and the other in V_2 induce two disjoint bicliques $K_{A_1A_2}$ and $K_{B_1B_2}$ satisfying Properties (i) and (ii) in the definition of 2-join, or show that no such partition exists.

Initially we let $V_1 = \{a_1, b_1, u_1, v_1\}$ and $V_2 = V(G) \setminus \{a_1, b_1, u_1, v_1\}$. Then forcing rules will be applied that will move nodes from V_2 to V_1.

During the algorithm, the nodes u in V_1 are partitioned into three sets:

- $u \in A_1$ if ua_2 is an edge but ub_2 is not,
- $u \in B_1$ if ub_2 is an edge but ua_2 is not,
- $u \in V_1 \setminus (A_1 \cup B_1)$ if neither ua_2 nor ub_2 is an edge.

The case where some node u in V_1 is adjacent to both a_2 and b_2 will not be permitted.

The forcing rules that move nodes from V_2 to V_1 are as follows.

- If $u \in V_2 \setminus \{a_2, b_2\}$ is adjacent to at least one node in $V_1 \setminus (A_1 \cup B_1)$, add u to V_1 and remove it from V_2.
- If $u \in V_2 \setminus \{a_2, b_2\}$ is adjacent to some node in $A_1 \cup B_1$ and $N(u) \cap (A_1 \cup B_1) \neq A_1$ or B_1, then add u to V_1 and delete it from V_2.

Note that if there is a 2-join $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ with $a_1, b_1, u_1, v_1 \in V_1$ and $a_2, b_2 \in V_2$, and u satisfies one of the above rules, then u would have to be in V_1.

If some node u which is moved from V_2 to V_1 is adjacent to both a_2 and b_2, then the algorithm terminates since no 2-join $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ with $a_1, b_1, u_1, v_1 \in V_1$ and $a_2, b_2 \in V_2$ exists. If this situation never occurs, we continue moving nodes from V_2 to V_1 until no forcing rule applies.

At this stage the nodes of V_2 satisfy the following: no node of V_2 is adjacent to a node of $V_1 \setminus (A_1 \cup B_1)$, and if a node u of V_2 is adjacent to a node of $A_1 \cup B_1$ then $N(u) \cap (A_1 \cup B_1) = A_1$ or B_1. Denote by A_2 the nodes of V_2 that are adjacent to all nodes in A_1, and by B_2 the nodes of V_2 that are adjacent to all nodes in B_1. The edge set $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ satisfies (i) of the definition of 2-join. By our assumption that G has no extended star cutset, (ii) of the definition of 2-join holds as well.

Now, if (iii) also holds, we have a 2-join with $a_1, b_1, u_1, v_1 \in V_1$ and $a_2, b_2 \in V_2$. On the other hand, if no choice of u_1, v_1 yields an edge set $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ satisfying (iii), then no 2-join with $a_1, b_1 \in V_1$ and $a_2, b_2 \in V_2$ exists. Indeed, the only way in which a choice u_1, v_1 can fail to yield a 2-join with $a_1, b_1, u_1, v_1 \in V_1$ and $a_2, b_2 \in V_2$ when such a 2-join exists is if, at termination, $|A_1| = |B_1| = 1$ and V_1 induces a chordless path P. Furthermore, any 2-join with $a_1, b_1, u_1, v_1 \in V_1'$ and $a_2, b_2 \in V_2'$ satisfies $V_1 \subset V_1'$. Therefore, the choice u_1, v_1', where $v_1' \in V_1' \setminus V(P)$ yields the desired 2-join.

To determine whether a bipartite graph G without extended star cutsets has a 2-join, one would apply the above algorithm to all 4-tuples (a_1, b_1, a_2, b_2) of nodes of G for which a_1a_2 and b_1b_2 are edges, but a_1b_2, a_2b_1 are not. Clearly all of this can be implemented to run in polynomial time.

2-JOIN DECOMPOSITION ALGORITHM

Input: A signed bipartite graph G that does not contain a short 3-wheel, an unbalanced hole of length 4, an extended star cutset or a 6-join.

Output: A list of signed bipartite graphs $\mathcal{L}$ with the following properties:

- The graphs in $\mathcal{L}$ do not contain an extended star cutset, a 6-join or a 2-join.
- G is balanced if and only if all the graphs in $\mathcal{L}$ are balanced.

Step 1: Let $\mathcal{M} = \{G\}$ and $\mathcal{L} = \emptyset$.

Step 2: If $\mathcal{M}$ is empty, return $\mathcal{L}$ and stop. Otherwise remove a graph M from $\mathcal{M}$.

Step 3: Check whether M has a 2-join. If it does not, then add M to $\mathcal{L}$ and go to Step 2. Otherwise, the 2-join is not rigid (we justify this in Theorem 4.6). Construct the blocks of the 2-join decomposition, add them to $\mathcal{M}$ and go to Step 2.

Theorem 4.6 *The 2-Join Decomposition Algorithm produces the desired output.*

Proof: Let G be a signed bipartite graph that does not contain a short 3-wheel, an unbalanced hole of length 4, an extended star cutset or a 6-join. Let $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ be a 2-join of G. Let G_1 and G_2 be the blocks of the decomposition. To prove the validity of the algorithm it is enough to show that (i) G_1 and G_2 do not contain a short 3-wheel, an unbalanced hole of length 4, an extended star cutset or a 6-join, and (ii) G is balanced if and only if G_1 and G_2 are balanced.

By Lemma 2.2, the 2-join is not rigid. So by Theorem 2.3, (ii) holds. By the construction of the blocks, there is no hole of length less than 7 in the blocks that uses the marker paths. Hence G_1 and G_2 do not contain an unbalanced hole of length 4, a short 3-wheel or a 6-join.

We now show that G_1 and G_2 do not contain an extended star cutset. Suppose w.l.o.g. that G_1 contains an extended star cutset $S = (x; X; Y; R)$. Recall that the marker path M_1 of G_1 is of length 4 or 5. Let $G_i' = G_i \setminus V(M_i)$.

Case 1: Node x coincides with α_1 or β_1.

Assume w.l.o.g. that x coincides with α_1. Since $|E(M_1)| \geq 4$, β_1 is not in S. So, S is a cutset that separates β_1 from a node in $G_1' \setminus S$. We can assume w.l.o.g. that the neighbor of α_1 in M_1 is not in S, since the set obtained by removing that neighbor from S would also be an extended star cutset of G_1. So $Y \cup R \subseteq A_1$. If S is a star cutset, i.e. $X = \{x\}$ and $R = \emptyset$, then $S^* = Y \cup A_2$ is a biclique cutset of G, separating B_2 from a node in $G_1' \setminus S$. So assume that $|X| \geq 2$. Then at least two nodes of A_1 are contained in Y. Let x^* be any node of A_2. Then $S^* = (x^*; (X \cup A_2) \setminus \{x\}; Y; R)$ is an extended star cutset of G separating B_2 from a node in $G_1' \setminus S$.

Case 2: Node x is an intermediate node of M_1.

Since M_1 has length at least 4, we must have $|X| = 1$, i.e. S is a star cutset. W.l.o.g. assume $\beta_1 \notin S$. Then S separates β_1 from a node in $G_1' \setminus S$. But then $S' = \{\alpha_1\}$ is also a star cutset of G_1. So, by Case 1, we are done.

Case 3: Node x is in A_1 or B_1.

W.l.o.g. assume that x is in A_1. If $\beta_1 \notin S$, then S separates β_1 from a node in $G_1' \setminus S$. If $\alpha_1 \notin Y \cup R$, let $S^* = S$. If $\alpha_1 \in R$, let $S^* = (x; X; Y; (R \setminus \{\alpha_1\}) \cup A_2)$ and if $\alpha_1 \in Y$, let $S^* = (x; X; (Y \setminus \{\alpha_1\}) \cup A_2; R)$. Then S^* is an extended star cutset of G separating B_2 from a node in $G_1' \setminus S$. So $\beta_1 \in S$ and hence $\beta_1 \in X$. Thus $Y \subseteq B_1$. Now $S^* = (x; (X \setminus \{\beta_1\}) \cup B_2; Y; (R \setminus \{\alpha_1\}) \cup A_2)$ is an extended star cutset of G separating a node of $G_1' \setminus S$ from a node of $G_2' \setminus (A_2 \cup B_2)$. Indeed, this graph is nonempty by the following claim.

Claim: $V(G_2') \setminus (A_2 \cup B_2) \neq \emptyset$.

Proof: Assume otherwise, namely $V(G_2') \setminus (A_2 \cup B_2) = \emptyset$. By (ii) in the definition of a 2-join, every node of A_2 has a neighbor in B_2 and, vice versa, every node in B_2 has a neighbor in A_2. Since the 2-join is not rigid, this implies that $|A_2| \geq 2$ and $|B_2| \geq 2$. Furthermore, every node in A_2 has a node in B_2 that it is not adjacent to (otherwise, there is a star cutset) and every node in B_2 has a node in A_2 that it is not adjacent to. Let u be a node of largest degree in the graph induced by $A_2 \cup B_2$. W.l.o.g. assume $u \in A_2$. Let Q be the set of neighbors of u in B_2 and let $v \in B_2 \setminus Q$. Let $w \in A_2$ be a neighbor of v. Then w is not adjacent to some node $q \in Q$, by our choice of u. Since the 2-join is not rigid, $A_1 \cup B_1$ is not a biclique, i.e. there exist $a_1 \in A_1$ and $b_1 \in B_1$ which are not adjacent. So ua_1wvb_1qu is a 6-hole. Now, if x is adjacent to b_1, it induces a short 3-wheel with this 6-hole, a contradiction. Therefore x is not adjacent to b_1 and $uxwvb_1qu$ is a 6-hole. But then, any $y \in Y$ induces a short 3-wheel with this 6-hole, a contradiction. This completes the proof of the claim.

Case 4: Node x is in $G_1' \setminus (A_1 \cup B_1)$.

Not both α_1 and β_1 can be in S. Assume w.l.o.g. that $\beta_1 \notin S$. Then S is a cutset separating β_1 from a node in $G_1' \setminus S$. If $\alpha_1 \notin S$, then S is a cutset of G separating B_2 from a node in $G_1' \setminus S$. So $\alpha_1 \in S$. Then $\alpha_1 \in X$, $Y \subseteq A_1$ and hence $S^* = (x; (X \setminus \{\alpha_1\}) \cup A_2; Y; R)$ is an extended star cutset of G separating B_2 from a node in $G_1' \setminus S$. $\Box$

Lemma 4.7 *The number of graphs in the list $\mathcal{L}$ produced by the 2-Join Decomposition Algorithm is linear in the size of the input graph G.*

Proof: For a graph G, let $\Phi(G) = |E(G)| - |V(G)| - 1$.

First, we show that, if a connected graph G has a 2-join with blocks G_1, G_2, then $\Phi(G_1) + \Phi(G_2) < \Phi(G)$. Consider a 2-join of G, say $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$, and let G_1', G_2' be the graphs described in the definition of a 2-join. Then

$$\Phi(G) = |E(G_1')| + |E(G_2')| + |A_1| \times |A_2| + |B_1| \times |B_2| - |V(G_1')| - |V(G_2')| - 1$$

and

$$\Phi(G_i) = |E(G_i')| + |A_i| + |B_i| - |V(G_i')| - 2.$$

Now $\Phi(G_1) + \Phi(G_2) < \Phi(G)$ follows by observing that any positive integers p, q satisfy $p + q \leq p \times q + 1$.

Now we show that, if G has a 2-join but no extended star cutset, then $\Phi(G) > 0$, $\Phi(G_1) \geq 0$ and $\Phi(G_2) \geq 0$. Since G has a 2-join, it has more than four nodes and therefore it is 2-connected. Thus, for $i = 1, 2$, G_i is 2-connected as well and its number of edges is at least $|V(G_i)|$, i.e. $\Phi(G_i) \geq -1$. If $\Phi(G_i) = -1$, then G_i is a hole, but this is impossible by Property (iii) in the definition of a 2-join. Therefore $\Phi(G_i) \geq 0$. Since $\Phi(G_1) + \Phi(G_2) < \Phi(G)$, it follows that $\Phi(G) > 0$.

This implies that the total number of blocks created in the 2-join decomposition algorithm is at most $2\Phi(G)$, i.e. it is linear in the size of the input graph. $\Box$

4.5 Recognition Algorithm

We now give the recognition algorithm, prove its validity and polynomial time bound.

RECOGNITION ALGORITHM

Input: A signed bipartite graph G.
Output: YES if G is balanced and NO otherwise.

Step 1: Check whether G contains an unbalanced hole of length 4 or 6. If it does output NO.

Step 2: Apply the Short 3-Wheel Procedure to check whether G contains a short 3-wheel. If it does, output NO.

Step 3: Apply the Cleaning Procedure to G and let $\mathcal{L}_1$ be the output family of graphs.

Step 4: For each $L \in \mathcal{L}_1$, apply the Double Star Cutset and 6-Join Decomposition Algorithm. If L is identified as not being balanced output NO, and otherwise union the output with $\mathcal{L}_2$.

Step 5: For each $L \in \mathcal{L}_2$, apply the 2-Join Decomposition Algorithm and union the output with $\mathcal{L}_3$.

Step 6: For each $L \in \mathcal{L}_3$, check whether L is strongly balanced. If some $L \in \mathcal{L}_3$ is not strongly balanced, then output NO. If every $L \in \mathcal{L}_3$ is strongly balanced, output YES.

Remark 4.8 *An algorithm that tests whether a signed bipartite graph is strongly balanced is given in [5]. Hence the details of Step 6 are omitted in this paper.*

Theorem 4.9 *The Recognition Algorithm produces the desired output and it can be implemented to run in time polynomial in the size of the input graph G.*

Proof: If G contains an unbalanced hole of length 4 or 6, a short 3-wheel or a 3-path configuration, then the algorithm correctly identifies G as not being balanced. So suppose that the algorithm does not terminate in Step 1, 2 or 4.

Claim 1: *No $L \in \mathcal{L}_3$ contains an extended star cutset, a 6-join or a 2-join.*

Proof of Claim 1: The graphs in $\mathcal{L}_2$ do not contain a 6-join, a double star cutset or any dominated nodes. By Lemma 3.1, they do not contain an extended star cutset. So by the 2-Join Decomposition Algorithm, graphs in $\mathcal{L}_3$ do not contain an extended star cutset, a 6-join or a 2-join. This completes the proof of Claim 1.

Claim 2: *G is balanced if and only if all the graphs in $\mathcal{L}_3$ are balanced.*

Proof of Claim 2: If G is balanced, then all the induced subgraphs of G are balanced, and hence all the graphs in $\mathcal{L}_3$ are balanced. Suppose that G is not balanced. Then G contains a smallest unbalanced hole H^*. By the Cleaning Procedure, some graph $G' \in \mathcal{L}_1$ contains H^* and H^* is clean in G'. By Lemma 3.9 all the holes in $\mathcal{C}_{G'}(H^*)$ are clean in G'. By the Double Star Cutset and 6-Join Decomposition Algorithm, some graph $G'' \in \mathcal{L}_2$ contains an unbalanced hole in $\mathcal{C}_{G'}(H^*)$. So G is balanced if and only if all the graphs in $\mathcal{L}_2$ are balanced. Then, by the 2-Join Decomposition Algorithm, G is balanced if and only if all the graphs in $\mathcal{L}_3$ are balanced. This completes the proof of Claim 2.

So by Claim 1, Claim 2 and Theorem 1.1, G is balanced if and only if every $L \in \mathcal{L}_3$ is strongly balanced. Hence the algorithm correctly identifies G as balanced or not balanced.

Now we show that the Recognition Algorithm can be implemented to run in time polynomial in the size of the input graph G. Steps 1 and 2 can clearly be implemented to run in polynomial time. By Remark 4.1, Lemma 4.4 and Lemma 4.7, the Cleaning Procedure, the Double Star Cutset and 6-Join Decomposition Algorithm and the 2-Join Decomposition Algorithm can be implemented to run in polynomial time. Furthermore, the number of graphs in $\mathcal{L}_3$ is polynomial in the size of G. So by Remark 4.8, Step 6 can also be implemented to run in polynomial time. $\Box$

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