Decomposition of Balanced Matrices

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Abstract

A $0,1$ matrix is *balanced* if it does not contain a square submatrix of odd order with two ones per row and per column. We show that a balanced 0,1 matrix is either totally unimodular or its bipartite representation has a cutset consisting of two adjacent nodes and some of their neighbors. This result yields a polytime recognition algorithm for balancedness. To prove the result, we first prove a decomposition theorem for balanced 0,1 matrices that are not strongly balanced.

to the memory of Ray Fulkerson, who planted the seeds of this research twenty five years ago, in a graduate course at Cornell University.

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1 Introduction

1.1 Decomposition theorem

In an undirected graph G, a cycle is *balanced* if its length is a multiple of 4. The graph G is *balanced* if all its chordless cycles are balanced. Clearly, a balanced graph is simple and bipartite.

We prove a decomposition theorem for balanced graphs and we give a polytime recognition algorithm based on this decomposition theorem. The theorem states that every balanced graph is either "strongly balanced" or contains a cutset, namely an "extended star cutset" or a "2-join". These three concepts are defined next.

A *biclique* is a complete bipartite graph K_{AB} where the two sides of the bipartition A and B are both nonempty.

Extended star cutsets

In a connected bipartite graph G, an *extended star* $(x; T; A; R)$ is defined by disjoint node sets T, A, R and a node $x \in T$ such that all nodes in $A \cup R$ are neighbors of x and the node set $T \cup A$ induces a biclique. Furthermore, if $|T| \geq 2$, then $|A| \geq 2$. The set R may be empty. See Figure 1. The node set $S = T \cup A \cup R$ is an *extended star cutset* if $G \setminus S$ is a disconnected graph.

Since the nodes in $T \cup A$ induce a biclique, an extended star cutset with $R = \emptyset$ is called a *biclique cutset*. An extended star cutset having $T = \{x\}$ is called a *star cutset*, since it is composed by a node x and a subset of its neighbors. Note that a star cutset is a special case of a biclique cutset.

2-join

Let G be a connected bipartite graph with more than four nodes, containing bicliques $K_{A_1A_2}$ and $K_{B_1B_2}$, where A_1, A_2, B_1, B_2 are disjoint nonempty node sets. The edge set $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ is a *2-join* if it satisfies the following properties (see Figure 1):

(i) The graph $G' = G \setminus (E(K_{A_1A_2}) \cup E(K_{B_1B_2}))$ is disconnected.

(ii) Every connected component of G' has a nonempty intersection with exactly two of the sets A_1, A_2, B_1, B_2 and these two sets are either A_1 and B_1 or A_2 and B_2. For $i = 1, 2$, let G'_i be the subgraph of G' containing all its connected components that have nonempty intersection with A_i and B_i.

(iii) If $|A_1| = |B_1| = 1$, then G'_1 is not a chordless path or $A_2 \cup B_2$ induces a biclique. If $|A_2| = |B_2| = 1$, then G'_2 is not a chordless path or $A_1 \cup B_1$ induces a biclique.

The purpose of Property (iii) is to exclude "improper" 2-joins, as we discuss later, in Section 2.

When the graph G comprises more than one connected component, we say that G has an extended star cutset or a 2-join if at least one of its connected components does.

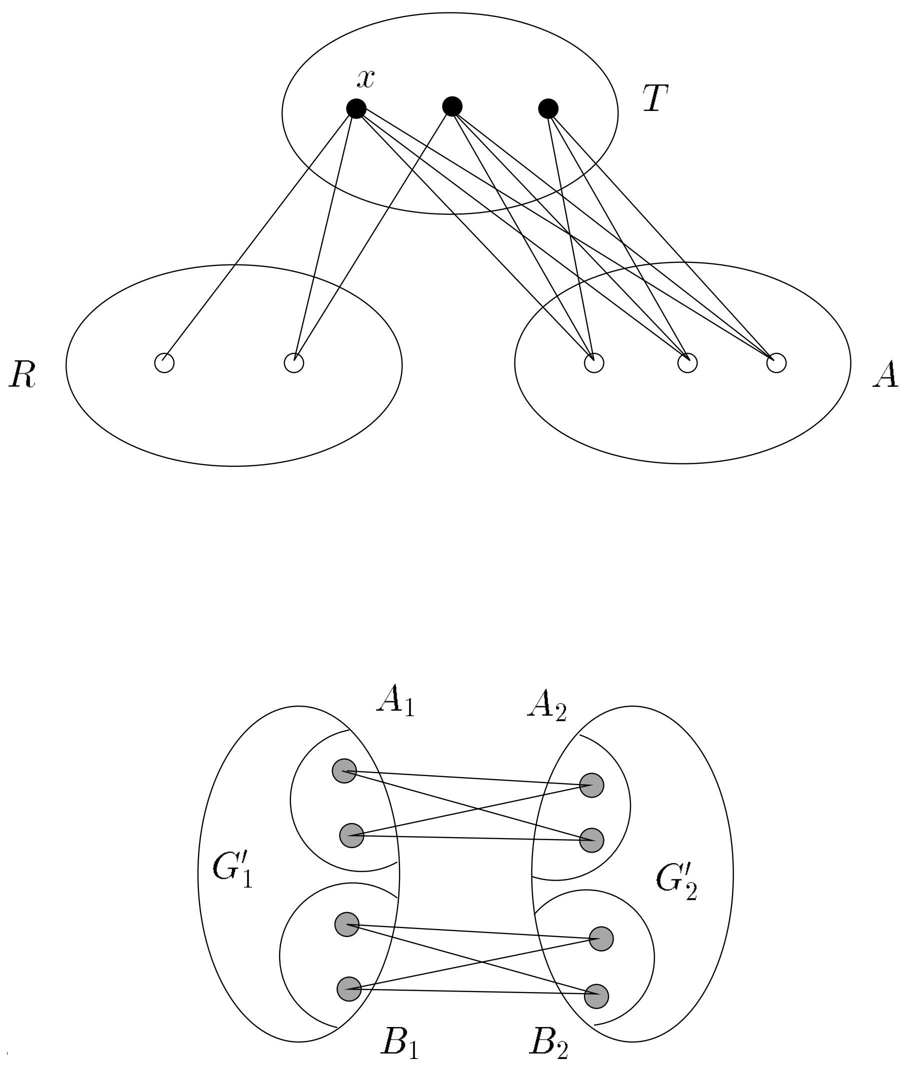


Figure 1: An extended star and a 2-join

Strongly balanced graphs

A graph is *strongly balanced* if it is balanced and contains no cycle with exactly one chord.

Theorem 1.1 *If G is a balanced graph that is not strongly balanced, then G has an extended star cutset or a 2-join.*

Balanced matrices

Given a $0,1$ matrix A, the *bipartite representation of* A is the bipartite graph $G = (V^r \cup V^c, E)$ having a node in V^r for every row of A, a node in V^c for every column of A and an edge ij joining nodes $i \in V^r$ and $j \in V^c$ if and only if the entry a_{ij} of A equals 1. Conversely, let $G = (V^r \cup V^c, E)$ be a bipartite graph with no parallel edges. Up to permutations of rows and columns and taking transpose, there is a unique 0,1 matrix A having G as bipartite representation.

A $0,1$ matrix A is *balanced* if its bipartite representation is a balanced graph. Equivalently, A is balanced if and only if A does not contain a square submatrix of odd order with exactly two ones per row and per column. Balanced matrices were first introduced by Berge [2] and we summarize here their relevance in combinatorial optimization.

Berge [3] showed that if A is balanced, the polyhedra $P(A) = \{x \geq 0 \mid Ax \leq 1\}$ and $Q(A) = \{y \geq 0 \mid yA \geq 1\}$ have only vertices with 0,1 components. Berge and Las Vergnas [8] showed that if A is balanced, then the maximum number of 1's in a $0,1$ vector $x \in P(A)$ is equal to the minimum number of 1's in a $0,1$ vector $y \in Q(A)$. More generally, Fulkerson, Hoffman and Oppenheim [23] showed that the inequalities defining $P(A)$ and $Q(A)$ are totally dual integral systems. A hypergraph is *balanced* if its node-edge incidence matrix is a balanced matrix. Balanced hypergraphs can be viewed as a natural generalization of bipartite graphs. This is the motivation that led Berge to introduce the notion of balancedness. Indeed, a hypergraph with exactly two nodes in each edge is balanced if and only if it is a bipartite graph. Balanced hypergraphs can be characterized by a bicoloring theorem [2]: The nodes of a balanced hypergraph can be colored either red or blue in such a way that every edge with at least two nodes contains both a red node and a blue node. Hall's theorem [26] about the existence of a perfect matching in a bipartite graph extends to balanced hypergraphs [14]. Further results on balanced matrices are surveyed in [12].

Totally unimodular matrices

A matrix is *totally unimodular* if every square submatrix has a determinant equal to 0, +1 or -1. A consequence of Theorem 1.1 is the following result, proved in Section 2 of this paper.

Theorem 1.2 *If a 0,1 matrix is balanced but not totally unimodular, then its bipartite representation has an extended star cutset.*

Let (P) be a property of 0,1 matrices that is invariant upon permutation of rows, permutation of columns and taking transpose (such as total unimodularity, balancedness, etc). It will be convenient to refer to a "bipartite graph with Property (P)" to mean that it is the bipartite representation of a 0,1 matrix with Property (P).

In Section 3, we use Theorem 1.2 to recognize in polytime whether a 0,1 matrix is balanced. The algorithm is presented in terms of bipartite representations. Given a connected bipartite graph G, let S be a node set such that $G \setminus S$ is a disconnected graph. Let $G'_1, \ldots, G'_k$ denote the connected components of $G \setminus S$. The *blocks* of the decomposition of G by the node cutset S are the graphs G_i induced by $V(G'_i) \cup S$. The idea of the algorithm is to use extended star cutsets to decompose G, and then its blocks, recursively, until no block contains an extended star cutset. This yields a polytime algorithm to recognize whether G is balanced provided that:

- total unimodularity can be recognized in polytime,
- extended star cutsets can be found in polytime,
- the total number of blocks generated is polynomial,
- G is balanced if and only if the blocks are balanced.

Total unimodularity can be recognized in polytime, see [28]. In Section 3, we show how to find extended star cutsets in polytime, and we prove that the number of blocks is polynomial. In general, when G has an extended star cutset, it is false that G is balanced if and only if its blocks are balanced. The major task of Section 3 is to show how to overcome this difficulty.

The rest of the paper (Sections 4 to 8) is devoted to the proof of Theorem 1.1. In fact, we prove that this theorem holds for a more general class of graphs.

Balanceable graphs

A *signed graph* is a graph whose edges are labelled with $+1$ or -1. In a signed graph, a cycle is *balanced* if the sum of its labels is a multiple of 4. A signed graph G is *balanced* if all its chordless cycles are balanced. Note that every balanced signed graph is bipartite and that, given a balanced undirected graph, we obtain a balanced signed graph by labelling $+1$ all the edges.

A graph is *balanceable* if there exists an assignment of ± 1 labels to its edges so that the resulting signed graph is balanced. Balanced graphs are obviously balanceable. Two examples of bipartite graphs that are not balanceable are odd wheels and 3-path configurations, which we define next.

Odd wheels, 3-path configurations and a theorem of Truemper

A *hole* of a bipartite graph is a chordless cycle. A *wheel* (H, v) is a bipartite graph consisting of a hole H and a node v having at least three neighbors in H. The wheel (H, v) is *odd* if v has an odd number of neighbors in H.

A *3-path configuration* is a bipartite graph consisting of three internally node-disjoint chordless paths, connecting two nonadjacent nodes u and v in opposite sides of the bipartition, and containing no edge other than those of the paths (see Figure 2). In all figures of this paper, solid lines represent edges and dotted lines represent paths with at least one edge.

Both a 3-path configuration and an odd wheel have the following properties: They contain an odd number of edges and each edge belongs to exactly two holes. Therefore in any signing,

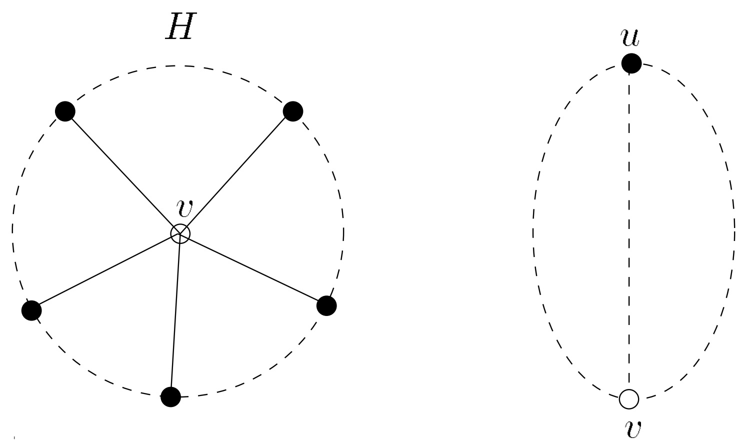


Figure 2: An odd wheel and a 3-path configuration

the sum of the labels of all holes is equal to 2 mod 4. This implies that at least one of the holes is not balanced, showing that neither 3-path configurations nor odd wheels are balanceable. These are in fact the only minimal bipartite graphs that are not balanceable, as a consequence of a theorem of Truemper [30].

Theorem 1.3 *A bipartite graph is balanceable if and only if it does not contain an odd wheel or a 3-path configuration as an induced subgraph.*

An easy proof of Truemper's theorem can be found in [15], as well as a discussion of its consequences.

Weakly balanced graphs

Two examples of graphs that are balanceable but not balanced are connected 6-holes and R_{10}, to be defined next.

A *triad* is a bipartite graph consisting of three internally node-disjoint paths $t, \dots, u$; $t, \dots, v$ and $t, \dots, w$, where nodes t, u, v, w are distinct and belong to the same side of the bipartition. Furthermore, the graph induced by the nodes of the triad contains no other edges than those of the three paths. The nodes u, v, w are the *attachments* of the triad.

A *fan* consists of a chordless path $P = x, \dots, y$ together with a node z not in P, adjacent to a positive even number of nodes in P, where nodes x, y, z belong to the same side of the bipartition and are the *attachments* of the fan.

A *connected 6-hole* is a bipartite graph induced by two disjoint node sets A and B such that each induces either a triad or a fan, their attachments induce a 6-hole and there are no other adjacencies between the nodes of A and B.

R_{10} is the graph defined by a cycle $C = x_1, x_2, \dots, x_{10}, x_1$ with chords $x_i x_{i+5}$, $1 \leq i \leq 5$.

A bipartite graph G is *weakly balanced* if G contains no odd wheel, no 3-path configuration, no connected 6-hole and no R_{10} as induced subgraphs. Equivalently, G is weakly balanced if G is balanceable and contains no connected 6-hole and no R_{10} as induced subgraphs. Clearly, every balanced graph is weakly balanced and every weakly balanced graph is balanceable.

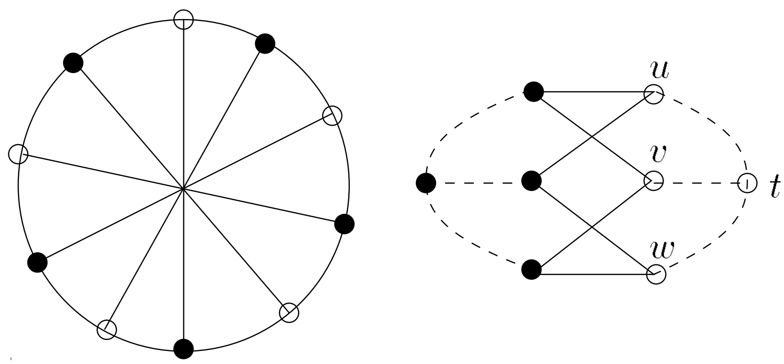


Figure 3: R_{10} and a connected 6-hole

A graph is *strongly balanceable* if it is balanceable and contains no cycle with exactly one chord. Since R_{10} and connected 6-holes contain cycles with a unique chord, every strongly balanceable graph is weakly balanced.

In this paper, we prove the following stronger version of Theorem 1.1.

Theorem 1.4 *If G is a weakly balanced graph that is not strongly balanceable, then G has an extended star cutset or a 2-join.*

In [13], a decomposition theorem for the class of balanceable graphs is presented. This result builds on Theorem 1.4: A third decomposition is introduced to deal with connected 6-holes, and the class of strongly balanceable graphs, which is used as "basic class" in Theorem 1.4, is enlarged to contain R_{10}.

1.2 Additional Definitions and Notation

Adjacency

Let G be a graph. Given a node set $S \subset V(G)$, node $u \notin S$ is *strongly adjacent* to S if u has at least two neighbors in S. (Throughout this paper $\subset$ denotes strict inclusion while $\subseteq$ denotes inclusion.) We denote by $N(u)$ the set of neighbors of u in G, and by $N_H(u)$ the set of neighbors of u in H, for any subgraph H of G. For a node $v \in V(H)$, we say that node $u \notin V(H)$ is a *twin* of v relative to H, if $N_H(u) = N_H(v)$.

Paths and Cycles

A path P may be denoted by the sequence of its distinct nodes $x_1, x_2, \ldots, x_n$, $n \geq 1$. A path having x_1 and x_n as endnodes is an *x_1x_n-path*. Let x_i and x_l be two nodes of P, where $l \geq i$. The path $x_i, x_{i+1}, \ldots, x_l$ is called the *x_ix_l-subpath* of P and is denoted by $P_{x_ix_l}$. We write $P = x_1, \ldots, x_i, P_{x_ix_l}, x_l, \ldots, x_n$. When $n \geq 3$, we denote by $\tilde{P}$ the x_2x_{n-1}-subpath of P.

For a path P (a cycle C), the edges connecting consecutive nodes of P (of C) are called the *edges of P* (*edges of C*) and this edge set is denoted by $E(P)$ ($E(C)$ respectively). The length of P or C is equal to the cardinality of $E(P)$ or $E(C)$.

Direct Connections

Let A, B, C be three disjoint node sets such that no node of A is adjacent to a node of B. A path $P = x_1, x_2, \ldots, x_n$ *connects* A *and* B if x_1 is adjacent to at least one node in A and x_n is adjacent to at least one node in B. The path P, connecting A and B, is a *direct connection between* A *and* B if, in the subgraph induced by the nodes $V(P) \cup A \cup B$, no path connecting A and B is shorter than P. A direct connection between A and B *avoids* C if $V(P) \cap C = \emptyset$.

1.3 Classes of balanced graphs and decomposition theorems

The decomposition approach adopted in this paper was already used in the literature for several classes of balanced graphs and for related classes of matrices. The result of Yannakakis [31] for restricted unimodular matrices, the results of Anstee and Farber [1], Hoffman, Kolen and Sakarovitch [27], Golumbic and Goss [24] for totally balanced matrices, the results of Conforti and Rao for strongly balanced matrices [16] and linear balanced matrices [17] and Seymour's [29] characterization of totally unimodular matrices are all in this spirit.

Restricted balanced graphs

A graph is *restricted balanced* if every cycle is balanced. Restricted balanced graphs were introduced by Commoner [10]. A graph is *basic* if it is bipartite and all the nodes in one side of the bipartition have degree at most two. Testing whether a graph is basic is trivial and testing whether a basic graph is restricted balanced amounts to testing whether a graph is bipartite. Yannakakis [31] proved the following decomposition theorem:

Theorem 1.5 *Let G be a biconnected restricted balanced graph that is not basic. Then G has a 2-join consisting of two edges.*

An algorithm for checking whether a graph is restricted balanced follows from this theorem, see [31]. See [16] for a different algorithm.

1-Joins and strongly balanced graphs

Let K_{AB} be a biclique of a connected graph G with the property that $G \setminus E(K_{AB})$ is disconnected. Let V_A be the node set of the connected components of $G \setminus E(K_{AB})$ with at least one node in A. Similarly, let V_B be the node set of the connected components with at least one node in B. The set $E(K_{AB})$ forms a *1-join* if $|V_A| \geq 2$ and $|V_B| \geq 2$. This concept was introduced by Cunningham and Edmonds [22]. Conforti and Rao [16] prove the following decomposition theorem for strongly balanced graphs:

Theorem 1.6 *Let G be a strongly balanced graph that is not restricted balanced. Then G has a 1-join.*

An algorithm for checking whether a graph is strongly balanced follows from this theorem, see [16].

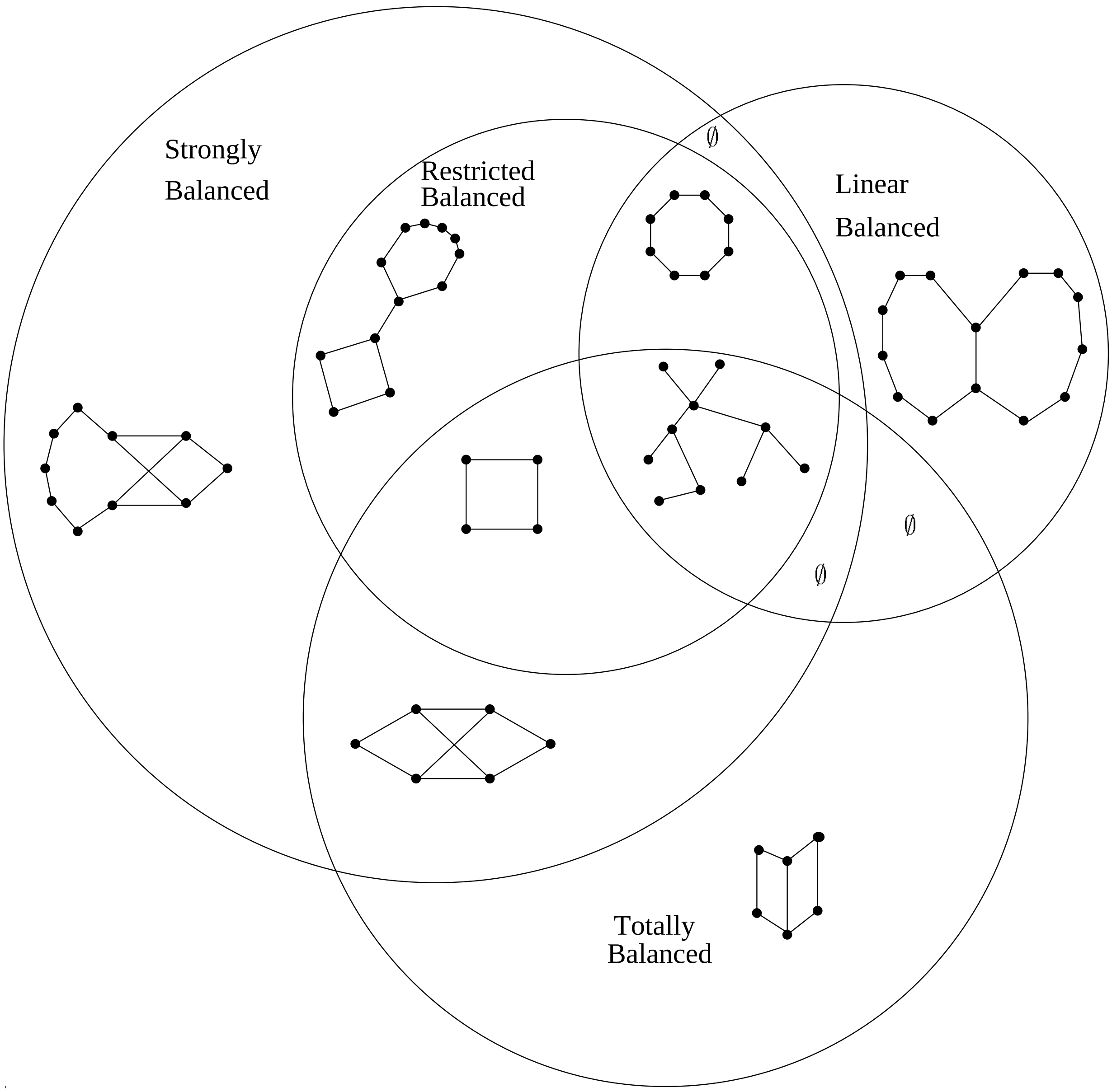


Figure 4: Classes of balanced graphs

Totally balanced graphs

A graph is *totally balanced* if it is bipartite and every hole has length 4. Totally balanced graphs arise in location theory and were the first balanced graphs to be the object of an extensive study. Several authors (Golumbic and Goss [24], Golumbic [25], Anstee and Farber [1] and Hoffman, Kolen and Sakarovitch [27]) gave properties of these graphs. An edge uv of a bipartite graph is *bisimplicial* if the node set $N(u) \cup N(v)$ induces a biclique. Note that if uv is a bisimplicial edge and nodes u and v have degree at least 2, then G has a 2-join formed by the edges connecting u and v to nodes in $G \setminus \{u, v\}$. The following theorem of Golumbic and Goss characterizes totally balanced graphs and can be used in a recognition algorithm, see [24].

Theorem 1.7 *A totally balanced graph has a bisimplicial edge.*

Linear balanced graphs

A bipartite graph is *linear* if it does not contain a cycle of length 4. Note that an extended star cutset in a linear bipartite graph is always a star cutset. Conforti and Rao [17] prove the following decomposition theorem for linear balanced graphs. This can be used to check whether a linear bipartite graph is balanced, see [19], [20].

Theorem 1.8 *Let G be a linear balanced graph that is not strongly balanced. Then G has a star cutset.*

Figure 4 shows the Venn diagram for the classes of balanced graphs defined above.

1.4 Even wheels, parachutes, connected squares, goggles

Here we define four weakly balanced graphs that play an important role in the proof of Theorem 1.4.

Even wheels

A wheel (H, v) is *even* if v has an even number (≥ 4) of neighbors in H.

Parachutes

A *parachute* is defined by four paths of positive length $P_1 = v_1, \ldots, z$, $P_2 = v_2, \ldots, z$, $M = v, \ldots, z$ and $T = v_1, \ldots, v_2$, where nodes v and z are in the same side of the bipartition, nodes v_1 and v_2 are adjacent to v and $|E(P_1)| + |E(P_2)| \geq 3$. No other adjacency exists in a parachute. See Figure 5.

Connected Squares

Connected squares are defined by four chordless paths of positive lengths $P_1^c = s_1^c, \ldots, t_1^c$; $P_2^c = s_2^c, \ldots, t_2^c$; $P_1^r = s_1^r, \ldots, t_1^r$; $P_2^r = s_2^r, \ldots, t_2^r$, where nodes s_1^c and s_2^c are adjacent to both s_1^r and s_2^r and nodes t_1^c and t_2^c are adjacent to both t_1^r and t_2^r, as in Figure 6(a). No other adjacency exists in connected squares. The nodes $s_1^c, s_2^c, t_1^c, t_2^c$ are in one side of the bipartition and $s_1^r, s_2^r, t_1^r, t_2^r$ are in the other.

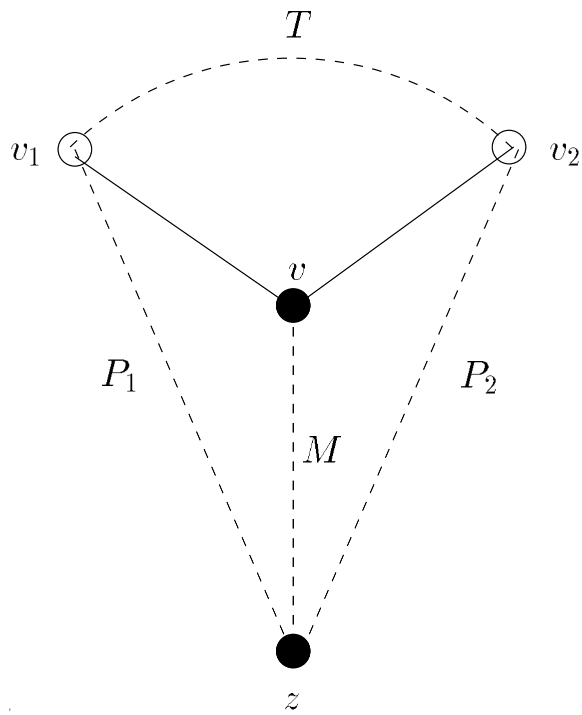


Figure 5: Parachute

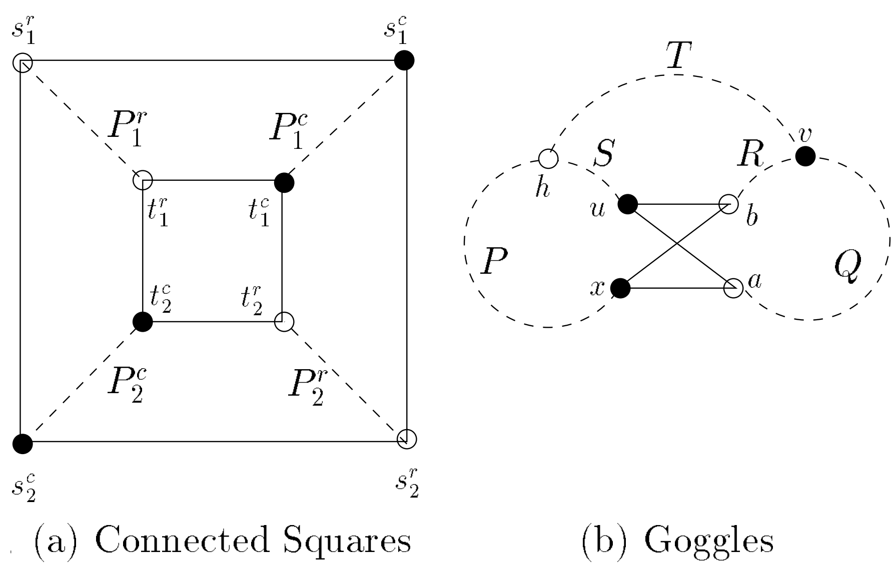


Figure 6: Connected squares and goggles

Goggles

Goggles are defined by a chordless path $T = h, \ldots, v$ and a cycle $C = h, P, x, a, Q, v, R, b, u, S, h$, with chords ua and xb (and hv when T has length 1), where P, Q, R, S denote chordless paths of length greater that one, such that no intermediate node of T belongs to C. No other edge with both endnodes in $V(C) \cup V(T)$ exists in goggles. The nodes a, b, h are on one side of the bipartition and x, u, v are in the other, see Figure 6(b).

1.5 Outline of the proof of the main theorem

In this subsection, we state the results used in the proof of Theorem 1.4. We first introduce the following classes of graphs:

- A bipartite graph is *wheel-free* if it contains no wheel.
- A bipartite graph is *wheel-and-parachute-free* if it contains no wheel and no parachute.

Let G be a weakly balanced graph. In Section 4, we study the case where G is wheel-and-parachute-free and we prove the following theorem:

Theorem 1.9 *If G is a wheel-and-parachute-free weakly balanced graph that is not strongly balanceable, then G has a 2-join.*

Section 5 proves a decomposition result when G contains a wheel:

Theorem 1.10 *If G is a weakly balanced graph that contains a wheel, then G has an extended star cutset.*

Sections 6, 7 and 8 deal with the remaining case, namely the case when G contains a parachute but no wheel. Section 6 proves the following result:

Theorem 1.11 *Let G be a wheel-free weakly balanced graph containing a parachute. Then G has an extended star cutset, or G contains connected squares or goggles.*

Section 7 studies connected squares:

Theorem 1.12 *Let G be a wheel-free weakly balanced graph containing connected squares. Then G has a biclique cutset or a 2-join.*

Section 8 studies goggles:

Theorem 1.13 *Let G be a wheel-free weakly balanced graph containing goggles but no connected squares. Then G has an extended star cutset or a 2-join.*

Clearly, Theorem 1.4 follows from these five results.

1.6 Some conjectures and open questions

The following conjecture has been formulated in [17]:

Conjecture 1.14 *Every balanced graph G has an edge e with the property that $G \setminus e$ remains balanced.*

In other words, if a 0,1 matrix is balanced, the 1's can be turned into 0's sequentially so that all intermediate matrices are balanced. Conjecture 1.14 is obviously equivalent to the following:

Conjecture 1.15 *Every balanced graph contains an edge that is not the unique chord of a cycle.*

This property holds for totally balanced graphs, as a consequence of Theorem 1.7. Note that every edge of the graph R_{10} is the unique chord of a cycle of length 8, hence the above conjectures cannot be extended to the class of balanceable graphs.

A biclique cutset is a special case of an extended star cutset, hence the question arises whether Theorem 1.1 can be strengthened, by showing that every balanced graph that is not strongly balanced has a biclique cutset or a 2-join.

Conjecture 1.16 *If G is a wheel-free balanced graph that is not strongly balanced, then G has a biclique cutset or a 2-join.*

Note that Theorem 1.12 proves this conjecture when G contains connected squares. Such a result would be interesting since biclique cutsets preserve balancedness in wheel-free graphs.

Theorem 1.17 *If G is a wheel-free graph that contains a biclique cutset, then G is balanced if and only if all the blocks are.*

Proof: The "only if" part is obvious, since the blocks are induced subgraphs of G.

To prove the "if" part, assume G has biclique cutset $A \cup B$, G is not balanced but all the blocks are. Let H be an unbalanced hole of G. At least two nonconsecutive nodes of H, say v_i and v_j, belong to $A \cup B$, else H is contained in some block. Furthermore, nodes v_i and v_j belong to the same set, else H has a chord. Assume w.l.o.g. that $v_i, v_j \in A$. Let w be a node of B. If $w \in V(H)$, then wv_1 and wv_2 are edges of H and H contains no other node of $A \cup B$, else H has a chord. Now, it follows that H is contained in some block, a contradiction. So $w \notin V(H)$. Assume that w has no neighbor in $V(H)$ other than nodes v_i, v_j, and let P_1, P_2 be the two subpaths of H connecting v_i and v_j. Then the holes $H_1 = v_i, w, v_j, P_1, v_i$ and $H_2 = v_i, w, v_j, P_2, v_i$ have distinct length mod 4, and each one belongs to a block, a contradiction to the assumption that all blocks are balanced. Hence w has at least three neighbors in H, and (H, w) is a wheel. $\square$

The graph in Figure 7 shows that Conjecture 1.16 cannot be extended to all balanced graphs. More generally, we define an infinite family of graphs as follows. Let H be a hole where nodes $u_1, \ldots, u_p, v_1, \ldots, v_q, w_1, \ldots, w_p, x_1, \ldots, x_q$ appear in this order when traversing H, but are not necessarily adjacent. Let $Y = \{y_1, \ldots, y_p\}$ and $Z = \{z_1, \ldots, z_q\}$ be two node

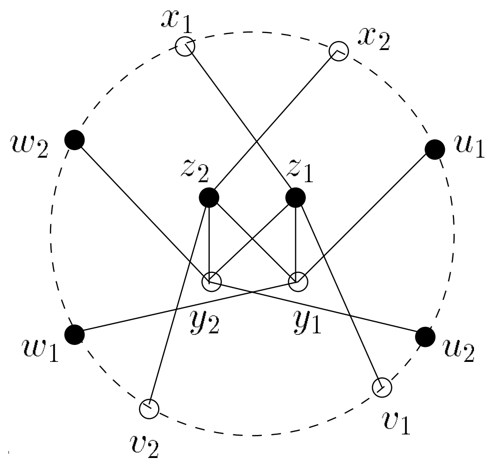


Figure 7: W_{22}

sets having empty intersection with $V(H)$ and inducing a biclique K_{YZ}. Node y_i is adjacent to u_i and w_i for $1 \le i \le p$. Node z_i is adjacent to v_i and x_i for $1 \le i \le q$. Any balanced graph of this form for $p, q \ge 2$ is called a W_{pq}. For all values of $p, q \ge 2$, the graph W_{pq} has no 2-join and no biclique cutset.

Since the graphs W_{pq} contain a wheel, a stronger form of the above conjecture is the following.

Conjecture 1.18 *If G is a balanced graph that is not strongly balanced, then G is either a W_{pq} or has a biclique cutset or a 2-join.*

2 2-Joins, Balancedness and Total Unimodularity

2.1 Introduction

In this section, we define the blocks G_1 and G_2 of a 2-join decomposition and we show that G is balanced if and only if G_1 and G_2 are balanced.

A 2-join $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ is *rigid* if $A_1 \cup B_1$ or $A_2 \cup B_2$ induces a biclique. For a 2-join that is not rigid, we show that G is totally unimodular if and only if the blocks G_1 and G_2 are totally unimodular.

These results are then used to deduce Theorem 1.2 from Theorem 1.1.

2.2 2-Join decomposition

Let $K_{A_1A_2}$ and $K_{B_1B_2}$ define a 2-join of G. The blocks G_1 and G_2 of the 2-join decomposition are defined as follows. For $i = 1, 2$, let G'_i be the subgraph of $G \setminus (E(K_{A_1A_2}) \cup E(K_{B_1B_2}))$ containing all its connected components that have nonempty intersection with A_i and B_i. To obtain G_i, we first add to G'_i a node α_i, adjacent to all the nodes in A_i and to no other node of G'_i and a node β_i, adjacent to all the nodes in B_i and to no other node of G'_i.

If neither $A_1 \cup B_1$ nor $A_2 \cup B_2$ induces a biclique, let Q_1 be a shortest path in G'_2 connecting a node in A_2 to a node in B_2, and let Q_2 be a shortest path in G'_1 connecting a node in A_1 to a node in B_1. Note that the existence of Q_1, Q_2 is guaranteed by (ii) in the definition of 2-joins. For $i = 1, 2$, add to G_i a *marker path* M_i connecting α_i and β_i with length $3 \leq |E(M_i)| \leq 6$ and $|E(M_i)|$ congruent to $|E(Q_i)|$ modulo 4.

If exactly one set $A_1 \cup B_1$ or $A_2 \cup B_2$ induces a biclique, say $A_1 \cup B_1$, then no marker path is added in G_1 and a marker path M_2 consisting of a single edge, connecting α_2 and β_2, is added to G_2.

If both $A_1 \cup B_1$ and $A_2 \cup B_2$ induce bicliques, then no marker path is added in G_1 and G_2.

The graphs G_1 and G_2 are the *blocks* of the 2-join decomposition of G. It follows from (iii) in the definition of a 2-join that the blocks G_1 and G_2 are both distinct from G. Furthermore, some graph invariant decreases. In that sense, the 2-join decomposition is "proper". For example, let $\Phi(G) = |E(G)| - |V(G)| - 1$.

Lemma 2.1 *If a connected graph G has a 2-join with blocks G_1, G_2, then $\Phi(G_1) + \Phi(G_2) < \Phi(G)$. Furthermore, if G has no extended star cutset, then $\Phi(G_1) \geq 0$ and $\Phi(G_2) \geq 0$.*

Proof: Consider a 2-join of G, say $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$, and let G'_1, G'_2 be defined as above. Then

$$\Phi(G) = |E(G'_1)| + |E(G'_2)| + |A_1| \times |A_2| + |B_1| \times |B_2| - |V(G'_1)| - |V(G'_2)| - 1$$

and

$$\Phi(G_i) \leq |E(G'_i)| + |A_i| + |B_i| - |V(G'_i)| - 2.$$

Now $\Phi(G_1) + \Phi(G_2) < \Phi(G)$ follows by observing that any positive integers p, q satisfy $p + q \leq p \times q + 1$.

Now assume that G has no extended star cutset. Since G has a 2-join, it has more than four nodes and therefore it is 2-connected. Thus, for $i = 1, 2$, G_i is 2-connected as well and its number of edges is at least $|V(G_i)|$, i.e. $\Phi(G_i) \geq -1$. If $\Phi(G_i) = -1$, then G_i is a hole, but this is impossible by Property (iii) in the definition of a 2-join. Therefore $\Phi(G_i) \geq 0$. □

Theorem 2.2 *Let G_1, G_2 be the blocks of a 2-join decomposition of a connected bipartite graph G. Then G is a balanced graph if and only if both G_1 and G_2 are balanced graphs.*

Proof: We first prove that, if G is balanced, then G_1 and G_2 are balanced. Assume not and let H be an unbalanced hole of G_1 or G_2.

Case 1 Neither $A_1 \cup B_1$ nor $A_2 \cup B_2$ induces a biclique.

Assume w.l.o.g. that H is in G_1. Then H must contain nodes α_1 and β_1, but not the marker path connecting them, otherwise G would contain a hole with the same length mod 4 as H. Since G contains nonadjacent nodes $a_2 \in A_2$ and $b_2 \in B_2$, the hole induced by $(V(H) \setminus \{\alpha_1, \beta_1\}) \cup \{a_2, b_2\}$ is an unbalanced hole of G.

Case 2 Exactly one of $A_1 \cup B_1$ or $A_2 \cup B_2$ induces a biclique.

Then G_1 and G_2 are induced subgraph of G and therefore they are balanced.

Case 3 Both $A_1 \cup B_1$ and $A_2 \cup B_2$ induce bicliques.

W.l.o.g. H is in G_1. H must contain nodes α_1 and β_1, otherwise G would contain a hole with the same length as H. But then H has chords, a contradiction.

We now prove that, if G_1 and G_2 are balanced, then G is balanced. Assume not and let H be an unbalanced hole of G. Then H must contain at least one node in each of the set A_1, A_2, B_1, B_2, otherwise H is also a hole contained in G_1 or G_2. If H contains no edge of G'_2, then $H = a'_1, a'_2, a''_1, \ldots, b'_1, b'_2, b''_1, \ldots, a'_1$ where $a'_1, a''_1 \in A_1$ and $b'_1, b''_1 \in B_1$, $a'_2 \in A_2$ and $b'_2 \in B_2$. Then neither $A_1 \cup B_1$ nor $A_2 \cup B_2$ induces a biclique and $H' = a'_1, \alpha_1, a''_1, \ldots, b'_1, \beta_1, b''_1, \ldots, a'_1$ is an unbalanced hole of G_1, a contradiction. So $H = a'_1, a'_2, P_2, b'_2, b'_1, P_1, a'_1$, where $a'_1 \in A_1$ and $b'_1 \in B_1$, $a'_2 \in A_2$ and $b'_2 \in B_2$. If both sets $A_1 \cup B_1$ and $A_2 \cup B_2$ induce bicliques, then H has length 4, contradicting the choice of H. If exactly one set $A_1 \cup B_1$ or $A_2 \cup B_2$ induces a biclique, say $A_1 \cup B_1$, then G_2 contains a hole of the same length as H. Now consider the case where neither $A_1 \cup B_1$ nor $A_2 \cup B_2$ induces a biclique. Since G_1 contains no unbalanced hole, the length of its marker path M_1 is not congruent to the length of P_2 mod 4. It follows that G'_2 contains a chordless path Q_2 connecting a node $a''_2 \in A_2$ to a node $b''_2 \in B_2$ whose length is not congruent to the length of P_2 mod 4. The holes $a'_2, P_2, b'_2, \beta_2, M_2, \alpha_2, a'_2$ and $a''_2, Q_2, b''_2, \beta_2, M_2, \alpha_2, a''_2$ have distinct lengths mod 4. Hence one of them is unbalanced, contradicting the assumption that G_2 is balanced. □

Next, we prove a lemma which is a graphical analog of the fact that 3-sums preserve regularity in matroid theory.

Lemma 2.3 *Let G be a connected bipartite graph with a 2-join that is not rigid, and let G_1 and G_2 be the blocks of the corresponding 2-join decomposition of G. Then G is totally unimodular if and only if both G_1 and G_2 are totally unimodular.*

Proof: A graph is *Eulerian* if all its nodes have even degree. By Camion's theorem [9], a bipartite graph is totally unimodular if and only if it contains no Eulerian induced subgraph with 2 (mod 4) edges. If G_1 or G_2 contains an Eulerian induced subgraph with 2 (mod 4) edges, then so does G since the length of a marker path is the same (mod 4) as that of a path in G.

So, to prove the lemma, it suffices to show that, if G contains an Eulerian induced subgraph with 2 (mod 4) edges, then so does G_1 or G_2. Let H be an Eulerian induced subgraph of G with 2 (mod 4) edges. Let $E(K_{A_1^* A_2^*}) \cup E(K_{B_1^* B_2^*})$ be the 2-join of G and let $A_i = V(H) \cap A_i^*$, $B_i = V(H) \cap B_i^*$, for $i = 1, 2$. We remove from H subsets of edges that form 4-cycles in $E(K_{A_1 A_2})$ or in $E(K_{B_1 B_2})$, as follows (the resulting graph H' is Eulerian with 2 (mod 4) edges, but may not be induced in G):

(i) If $|A_1|$ or $|A_2|$ are both even, then all edges of $K_{A_1 A_2}$ are removed.

(ii) If $|A_1|$ is even and $|A_2|$ is odd, then choose $a_2 \in A_2$ and remove all edges of $K_{A_1 A_2 \setminus \{a_2\}}$.

(iii) If both $|A_1|$ and $|A_2|$ are odd, then choose $a_1 \in A_1$ and $a_2 \in A_2$, and remove all edges of $K_{A_1 \setminus \{a_1\}\, A_2 \setminus \{a_2\}}$.

Edges with endnodes in $B_1 \cup B_2$ are removed in the same way, based on the parity of $|B_1|$ and $|B_2|$. Note that (iii) occurs for A_1, A_2 if and only if (iii) occurs for B_1, B_2, since H is Eulerian and therefore the number of edges in $E(K_{A_1 A_2}) \cup E(K_{B_1 B_2})$ is even.

Case 1 (i) or (ii) occurs.

Then the edges of H' partition into two Eulerian graphs H_1 and H_2 that are induced subgraphs of G_1 and G_2. One of these subgraphs has 2 (mod 4) edges.

Case 2 (iii) occurs.

For $i = 1, 2$, define H_i' to be the subgraph of $H' \setminus (E(K_{A_1 A_2}) \cup E(K_{B_1 B_2}))$ containing the connected components that have nonempty intersection with A_i and B_i. Define H_i to be the graph induced by $V(H_i')$ and the nodes of the marker path M_i of G_i. Then H_1 and H_2 are Eulerian induced subgraphs of G_1 and G_2 respectively. Furthermore, $|E(H_1)| + |E(H_2)|$ is equal to $|E(H')| + |E(M_1)| + |E(M_2)| + 2$, since the edges $a_1 a_2$ and $b_1 b_2$ of H' correspond to edges in both H_1 and H_2 and all the other edges appear exactly once.

If $|E(M_1)| + |E(M_2)|$ equals 2 (mod 4), it follows that $|E(H_1)|$ or $|E(H_2)|$ equals 2 (mod 4) and we are done.

Otherwise, $|E(M_1)| + |E(M_2)|$ equals 0 (mod 4), since this sum must be even. Now M_2 has the same length (mod 4) as a shortest path Q_2 in G_1' connecting a node in A_1 to a node in B_1. Therefore the nodes of Q_2 and M_1 induce a hole of length 2 (mod 4) in G_1. This is the required Eulerian induced subgraph.

□

2.3 Bipartite graphs without extended star cutsets

Lemma 2.4 *Let G be a bipartite graph that has no extended star cutset. Then G has no rigid 2-join.*

Proof: Assume G has a rigid 2-join $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ and let $A_1 \cup B_1$ induce a biclique. If $|A_1| > 1$ or $|B_1| > 1$ or $V(G'_1) \setminus (A_1 \cup B_1) \neq \emptyset$, then G has a star cutset with center in A_1 or B_1. So $|A_1| = |B_1| = 1$ and $V(G'_1) \setminus (A_1 \cup B_1) = \emptyset$. Now, $A_2 \cup B_2$ must be a biclique by (iii) of the definition of a 2-join. So, as above, it follows that $|A_2| = |B_2| = 1$ and $V(G'_2) \setminus (A_2 \cup B_2) = \emptyset$. But then G has only four nodes, contradicting the definition of 2-join. □

Lemma 2.5 *Let G be a balanced graph that has no extended star cutset. Then, in every 2-join, $V(G'_i) \setminus (A_i \cup B_i) \neq \emptyset$, for $i = 1, 2$.*

Proof: Assume otherwise, say $V(G'_1) \setminus (A_1 \cup B_1) = \emptyset$. By (ii) in the definition of a 2-join, every node of A_1 has a neighbor in B_1 and, vice versa, every node in B_1 has a neighbor in A_1. By Lemma 2.4, the 2-join is not rigid. These two facts imply that $|A_1| \geq 2$ and $|B_1| \geq 2$. Furthermore, every node in A_1 has a node in B_1 that it is not adjacent to (otherwise, there is a star cutset) and every node in B_1 has a node in A_1 that it is not adjacent to. Let u be a node of largest degree in the graph induced by $A_1 \cup B_1$. W.l.o.g. assume $u \in A_1$. Let Y be the set of neighbors of u in B_1 and let $v \in B_1 \setminus Y$. Let $w \in A_1$ be a neighbor of v. Then w is not adjacent to some node $y \in Y$, by our choice of u. Since the 2-join is not rigid, $A_2 \cup B_2$ is not a biclique, i.e. there exist $a_2 \in A_2$ and $b_2 \in B_2$ that are not adjacent. Now ua_2wvb_2yu is a 6-hole, a contradiction. □

Lemma 2.6 *Let G be a balanced graph that has no extended star cutset. If G has a 2-join, then the blocks G_1, G_2 of the 2-join decomposition do not have an extended star cutset.*

Proof: Assume otherwise, i.e. one of the blocks, say G_1, has an extended star cutset $S = (x; T; Q; R)$. By Lemma 2.4, the 2-join is not rigid. So, for $i = 1, 2$, G_i contains a marker path $M_i = \alpha_i, \dots, \beta_i$ of length $|E(M_i)| \geq 3$. Let $G'_i = G_i \setminus V(M_i)$.

Case 1 Node x coincides with α_1 or β_1.

Assume w.l.o.g. that x coincides with α_1. Since $|E(M_1)| \geq 3$, β_1 is not in S. So, S is a cutset that separates β_1 from a node in $G'_1 \setminus S$. We can assume w.l.o.g. that the neighbor of α_1 in M_1 is not in S, since the set obtained by removing that neighbor from S would also be an extended star cutset of G_1. So $Q \cup R \subseteq A_1$. If S is a star cutset, i.e. $T = \{x\}$ and $Q = \emptyset$, then $S^* = R \cup A_2$ is a biclique cutset of G, separating B_2 from a node in $G'_1 \setminus S$. So assume that $|T| \geq 2$. Then at least two nodes of A_1 are contained in Q. Let x^* be any node of A_2. Then $S^* = (x^*; (T \cup A_2) \setminus \{x\}; Q; R)$ is an extended star cutset of G separating B_2 from a node in $G'_1 \setminus S$.

Case 2 Node x is an intermediate node of M_1.

Since M_1 has length at least 3, we must have $|T| = 1$, i.e. S is a star cutset. W.l.o.g. assume $\beta_1 \notin S$. Then S separates β_1 from a node in $G'_1 \setminus S$. But then $S' = \{\alpha_1\}$ is also a star cutset of G_1. So, by Case 1, we are done.

Case 3 Node x is in A_1 or B_1.

W.l.o.g. assume that x is in A_1. If $\beta_1 \notin S$, then S separates β_1 from a node in $G'_1 \setminus S$. If $\alpha_1 \notin Q \cup R$, let $S^* = S$. If $\alpha_1 \in R$, let $S^* = (x; T; Q; (R \setminus \{\alpha_1\}) \cup A_2)$ and if $\alpha_1 \in Q$, let $S^* = (x; T; (Q \setminus \{\alpha_1\}) \cup A_2; R)$. Then S^* is an extended star cutset of G separating B_2 from a node in $G'_1 \setminus S$. So $\beta_1 \in S$ and hence $\beta_1 \in T$. Thus $Q \subseteq B_1$. Now $S^* = (x; (T \setminus \{\beta_1\}) \cup B_2; Q; (R \setminus \{\alpha_1\}) \cup A_2)$ is an extended star cutset of G separating a node of $G'_1 \setminus S$ from a node of $G'_2 \setminus (A_2 \cup B_2)$. (Note that this graph is nonempty by Lemma 2.5.)

Case 4 Node x is in $G'_1 \setminus (A_1 \cup B_1)$.

Not both α_1 and β_1 can be in S. Assume w.l.o.g. that $\beta_1 \notin S$. Then S is a cutset separating β_1 from a node in $G'_1 \setminus S$. If $\alpha_1 \notin S$, then S is a cutset of G separating B_2 from a node in $G'_1 \setminus S$. So $\alpha_1 \in S$. Then $\alpha_1 \in T$, $Q \subseteq A_1$ and hence $S^* = (x; (T \setminus \{\alpha_1\}) \cup A_2; Q; R)$ is an extended star cutset of G separating B_2 from a node in $G'_1 \setminus S$.

□

Now we prove that Theorem 1.2 follows from Theorem 1.1 and the above results.

Proof of Theorem 1.2: Assume that G is a balanced graph that has no extended star cutset. Decompose G into blocks using 2-join decompositions, recursively, until no 2-joins exist. This process is finite, by Lemma 2.1. All the blocks are balanced by Lemma 2.2. By Lemma 2.6, the blocks have no extended star cutset. So, by Theorem 1.1, all the blocks are strongly balanced. Strongly balanced graphs are totally unimodular [16]. By Lemma 2.4, the 2-joins used in the decomposition are not rigid, and, by Lemma 2.3, 2-joins that are not rigid preserve total unimodularity. It follows that G is totally unimodular. □

3 Polytime Recognition Algorithm

In this section, we use Theorem 1.2 to recognize in polytime whether a bipartite graph G is balanced. Since it is not true that the blocks of an extended star cutset decomposition are balanced if and only if G is balanced, we decompose a family of subgraphs of G, say $G_1, \ldots, G_p$, instead of just G. Then, by applying extended star cutset decompositions to all the G_i's, we can show that the desired property holds for this larger family of blocks, namely, all blocks in this family are balanced if and only if G is balanced. To describe the appropriate family of graphs G_i, we need to study smallest unbalanced holes in bipartite graphs G that are not balanced.

3.1 Undominated Graphs

A node u is said to be *dominated* if there exists a node v, distinct from u, such that $N(u) \subseteq N(v)$. The graph G is said to be *undominated* if it contains no dominated nodes. A *double star* is a node set $S = N(u) \cup N(v)$ where u, v are adjacent nodes. S is a *double star cutset* if $G \setminus S$ is nonempty and contains more connected components than G. The lemma below shows a relation between double star cutsets and extended star cutsets for undominated bipartite graphs.

Lemma 3.1 *If an undominated bipartite graph G has an extended star cutset, then it has a double star cutset.*

Proof: Let $S = (x; T; A; R)$ be an extended star cutset of G and let $G'_1, G'_2, \ldots, G'_k$ be the connected components of $G \setminus S$. By definition, $A \neq \emptyset$. Define $S^* = N(x) \cup N(v)$, where v is a node in A. Clearly, $S \subseteq S^*$. Suppose S^* is not a double star cutset of G. Then all the nodes in one of the connected components of $G \setminus S$, say G'_i, belong to $S^* \setminus S$. Hence $V(G'_i) \subset N(x) \cup N(v)$, i.e. each node in G'_i is dominated by x or by v. $\Box$

Lemma 3.1 and Theorem 1.2 imply:

Theorem 3.2 *If G is an undominated balanced graph that is not totally unimodular, then G has a double star cutset.*

3.2 Smallest unbalanced holes

Let G be a bipartite graph that is not balanced and let H^* be a smallest unbalanced hole in G. In this subsection, we study properties of strongly adjacent nodes to H^*.

A strongly adjacent node u to a hole H in G is *odd-strongly adjacent* if u has an odd number of neighbors in H, and it is *even-strongly adjacent* if it has an even number of neighbors in H. The sets $A^r(H)$ and $A^c(H)$ contain the odd-strongly adjacent nodes to H that belong to V^r and V^c respectively.

The following properties of the sets $A^r(H^*)$ and $A^c(H^*)$, associated with a smallest unbalanced hole H^* were proven by Conforti and Rao in [19].

Property 3.3 *There exists a node $x^r \in V^r \cap V(H^*)$ that is adjacent to all the nodes in $A^c(H^*)$. There exists a node $x^c \in V^c \cap V(H^*)$ that is adjacent to all the nodes in $A^r(H^*)$.*

Property 3.4 *Every even-strongly adjacent node to H^* is a twin of a node in H^**

The above properties were used in [20] to design a polytime algorithm to test whether a linear bipartite graph is balanced. To test balancedness of a bipartite graph, we need the following additional properties of strongly adjacent nodes.

Definition 3.5 *A* tent $\tau(H, u, v)$ *is a subgraph of G induced by a hole H and two adjacent nodes u and v that are even-strongly adjacent to H with the following property:*

The nodes of H can be partitioned into two subpaths containing the nodes in $N(u) \cap V(H)$ and $N(v) \cap V(H)$ respectively.

A tent $\tau(H, u, v)$ is referred to as a *tent containing H*. We now study properties of a tent $\tau(H^*, u, v)$ containing a smallest unbalanced hole H^*. By Property 3.4, both u and v are twins of nodes of H. We assume throughout that the first node, say u in the definition of a tent $\tau(H, u, v)$ belongs to V^r and that the second node, say v, belongs to V^c. We use the notation of Figure 8, where nodes $u_1, u_0, u_2, v_1, v_0, v_2$ are encountered in this order, when traversing H^*.

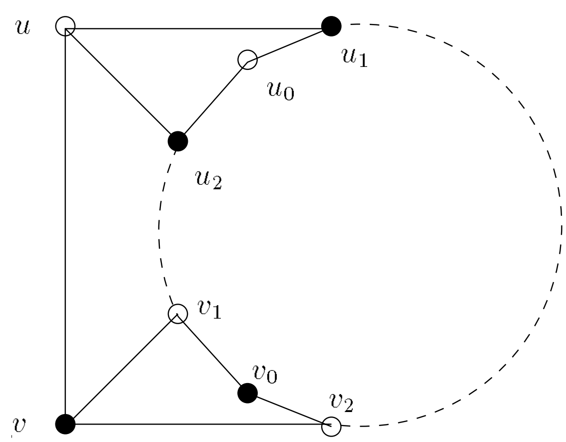


Figure 8: Tent

Lemma 3.6 *Let H^* be a smallest unbalanced hole and $\tau(H^*, u, v)$ be a tent containing it. At least one of the two sets $N(v_0) \cup N(u_1)$, $N(v_0) \cup N(u_2)$ contains $A^r(H^*)$. At least one of the two sets $N(u_0) \cup N(v_1)$, $N(u_0) \cup N(v_2)$ contains $A^c(H^*)$.*

Proof: By symmetry, we only need to prove the first statement. Suppose v_0 is not adjacent to a node $w \in A^r(H^*)$. Consider the hole H_1^* obtained from H^* by replacing v_0 with node v. Now w in not adjacent to v, for otherwise w is even-strongly adjacent to H_1^*, violating Property 3.4. Therefore, w is in $A^r(H_1^*)$. Node u is in $A^r(H_1^*)$ and has neighbors u_1, u_2 and v in H_1^*. By Property 3.3, all nodes in $A^r(H_1^*)$ have a common neighbor in H_1^*. So it follows that this common neighbor must be u_1 or u_2. □

Lemma 3.7 *Let H^* be a smallest unbalanced hole and $\tau(H^*, u, v)$, $\tau(H^*, w, y)$ be two tents containing H^*, where w_1, w_2 are the neighbors of w and y_1, y_2 are the neighbors of y in H^*. Let w_0 and y_0 be the common neighbors in H^* of w_1, w_2 and y_1, y_2 respectively. Then at least one of the following properties holds:*

- *Nodes u_1 and u_2 coincide with w_1 and w_2.*
- *Nodes v_1 and v_2 coincide with y_1 and y_2.*
- *Nodes u_0 and y are adjacent.*
- *Nodes v_0 and w are adjacent.*

Proof: Suppose the contrary. Then u, v, w, y are all distinct nodes and one of the following two cases occurs. The edges of H^* can be partitioned in two paths P_1, P_2 with common endnodes so that either (Case 1:) P_1 contains u_1, u_2, v_1, v_2 and P_2 contains w_1, w_2, y_1, y_2 or (Case 2:) P_1 contains u_1, u_2, y_1, y_2 and P_2 contains v_1, v_2, w_1, w_2.

Assume u and y are adjacent and consider the hole H^*_{wy} contained in $V(H^*) \cup \{w, y\}$, containing w, y, u_1, u_2. Then (H^*_{wy}, u) is an odd wheel and all the holes of (H^*_{wy}, u) are smaller than H^*. Since one of them is unbalanced, we have a contradiction to the minimality of H^*. By symmetry, w and v are nonadjacent as well.

In Case 1, consider the hole H^*_{vwy} contained in $V(H^*) \cup \{v, w, y\}$, containing v, w, y, u_1, u_2. Then (H^*_{vwy}, u) is an odd wheel and all the holes of (H^*_{vwy}, u) are smaller than H^*, a contradiction. In Case 2, nodes u and y are connected by a $3PC(u, y)$. The three holes of this 3-path configuration are smaller than H^* and at least one of them is unbalanced. □

3.3 A Recognition Algorithm

In this subsection, we give an algorithm to test whether a bipartite graph is balanced.

Definition 3.8 *A hole H is said to be* clean *in G if the following three conditions hold:*

(i) *No node is odd-strongly adjacent to H.*

(ii) *Every even-strongly adjacent node is a twin of a node in H.*

(iii) *There is no tent containing H.*

In a wheel (W, v), a subpath of W having two nodes of $N_W(v)$ as endnodes and only nodes of $V(W) \backslash N_W(v)$ as intermediate nodes is called a *sector* of (W, v). A *short 3-wheel* is a wheel with three sectors, at least two of which have length 2.

RECOGNITION ALGORITHM

Input: A bipartite graph G.
Output: G is identified as balanced or not balanced.

Step 1 Apply Procedure 1 to check whether G contains a short 3-wheel. If so, G is not balanced, otherwise go to Step 2.

Step 2 Apply Procedure 2 to create at most $|V^r|^4|V^c|^4$ induced subgraphs $G_1, \ldots, G_p$ such that, if G is not balanced, one of the induced subgraphs created, say G_i, contains an unbalanced hole H^* that is clean in G_i.

Step 3 Apply Procedure 3 to each of the graphs $G_1, \ldots, G_p$ to decompose them into undominated induced subgraphs $B_1, \ldots, B_s$ that do not contain a double star cutset. While decomposing a graph with a double star cutset $N(u) \cup N(v)$, Procedure 3 also checks for the existence of a 3-path configuration containing nodes u and v and nodes in two distinct connected components resulting from the decomposition. If such a configuration is found, then G is not balanced, otherwise go to Step 4.

Step 4 Test whether all the blocks $B_1, \ldots, B_s$ are totally unimodular. If so, G is balanced, otherwise G is not balanced.

An algorithm to test whether a bipartite graph is totally unimodular can be found in [28]. Hence the details of Step 4 are omitted in this paper.

3.4 Short 3-Wheels

PROCEDURE 1, for identifying whether G contains a short 3-wheel, can be described as follows: Let $C = a_1, a_2, a_3, a_4, a_5, a_6, a_1$ be a 6-cycle of G having unique chord a_2a_5. If a_1 and a_3 are in the same connected component of $G \setminus (N(a_2) \cup N(a_4) \cup N(a_5) \cup N(a_6) \setminus \{a_1, a_3\})$, or if a_4 and a_6 are in the same connected component of $G \setminus (N(a_1) \cup N(a_2) \cup N(a_3) \cup N(a_5) \setminus \{a_4, a_6\})$, then a short 3-wheel containing C is identified. Otherwise G has no short 3-wheel containing C. Perform such a test for all 6-cycles of G with a unique chord.

The complexity of this procedure is of order $O(|V^r|^4|V^c|^4)$.

3.5 Clean Unbalanced Holes

In this subsection, we show how to create at most $|V^r|^4|V^c|^4$ induced subgraphs of G such that, if G is not balanced, one of these subgraphs, say G_t, contains an unbalanced hole that is smallest in G and is clean in G_t.

Given a graph F and nodes i, j, k, l of F that induce the chordless path i, j, k, l, we define F_{ijkl} to be the induced subgraph obtained from F by removing the nodes in $N_F(j) \cup N_F(k) \setminus \{i, j, k, l\}$.

PROCEDURE 2

Input: A bipartite graph G.

Output: A family $\mathcal{L} = G_1, \ldots, G_p$, where $p \leq |V^r|^4|V^c|^4$, of induced subgraphs of G such that if G is not balanced, one of the subgraphs in $\mathcal{L}$, say G_t, contains an unbalanced hole that is smallest in G and is clean in G_t.

Step 1 Let $\mathcal{L}^* = \{G_{ijkl}$: nodes i, j, k, l of G induce the chordless path $i, j, k, l\}$

Step 2 Let $\mathcal{L} = \{Q_{ijkl}$: the graph Q is in $\mathcal{L}^*$ and nodes i, j, k, l of Q induce the chordless path $i, j, k, l\}$.

We now prove the validity of Procedure 2.

Lemma 3.9 *If G is not balanced, one of the graphs in $\mathcal{L}$, say G_t, contains an unbalanced hole H^* that is smallest in G and is clean in G_t.*

Proof: Let H^* be a smallest unbalanced hole in G. Choose two induced paths u_1, u_0, u_2, u_3 and v_1, v_0, v_2, v_3 on H^* as follows:

- If no tent contains H^*: $A^r(H^*) \subseteq N(u_0)$ and $A^c(H^*) \subseteq N(v_0)$. (This choice is possible by Property 3.3.)
- If some tent $\tau(H^*, u, v)$ contains H^*: $\{u_1, u_2\} = N(u) \cap V(H^*)$ and $\{v_1, v_2\} = N(v) \cap V(H^*)$. By Lemma 3.6, we can index u_i, $i = 1, 2$, so that $A^r(H^*) \subseteq N(v_0) \cup N(u_2)$ and we can index v_i, $i = 1, 2$, so that $A^c(H^*) \subseteq N(u_0) \cup N(v_2)$. By Lemma 3.7, for every tent $\tau(H^*, w, y)$ that contains H^*, w or y is adjacent to one of the nodes in $\{u_0, u_2, v_0, v_2\}$.

So $(G_{u_1 u_0 u_2 u_3})_{v_1 v_0 v_2 v_3}$ belongs to $\mathcal{L}$ and contains H^* but in $(G_{u_1 u_0 u_2 u_3})_{v_1 v_0 v_2 v_3}$, no tent contains H^* and H^* has no strongly adjacent nodes other than twins. □

3.6 Double Star Cuset Decompositions

We describe a procedure to decompose a bipartite graph G into blocks that are induced subgraphs and do not have a double star cutset.

Definition 3.10 *Let H be a hole in a graph. Then $\mathcal{C}(H) = \{H_i \mid H_i$ is a hole that can be obtained from H by a sequence of holes $H = H_0, H_1, \ldots, H_i$, where $|V(H_j) \setminus V(H_{j-1})| = 1$, for $j = 1, 2, \ldots, i\}$.*

Lemma 3.11 *Let G be a bipartite graph that is not balanced and contains no short 3-wheel. If H is a smallest unbalanced hole and is clean in G, then every hole H_i in $\mathcal{C}(H)$ is clean in G, $|H_i| = |H|$ and $\mathcal{C}(H_i) = \mathcal{C}(H)$.*

Proof: Let $H = H_0, H_1, \ldots, H_i$ be a sequence of holes as in Definition 3.10. It suffices to show the lemma for H_1. Since $A^r(H) \cup A^c(H) = \emptyset$, by Property 3.4, H_1 has been obtained from $H = x_1, x_2, x_3, \ldots, x_n, x_1$ by substituting one node, say x_3, with its twin y_3. We assume w.l.o.g. that $x_3, y_3 \in V^r$. So $|H_1| = |H|$ and $\mathcal{C}(H_1) = \mathcal{C}(H)$.

Assume $A^r(H_1) \cup A^c(H_1) \neq \emptyset$. Since H is clean, then H must contain a twin y_i of x_i in V^c, where y_i is adjacent to y_3 but not to x_3. Now $\tau(H, y_3, y_i)$ is a tent, a contradiction to the assumption that H is clean, and this proves that H_1 satisfies (i) of Definition 3.8.

The fact that H is clean shows that H_1 satisfies (ii) of Definition 3.8.

Finally, assume H_1 contains a tent $\tau(H_1, x, y)$, where $x \in V^r$. Then $x_3 \neq x$, else $y \in A^r(H)$. So x is a twin of a node in H and y is adjacent to y_3. We can assume that x_1 is the unique neighbor of y in H. Now let x_i be the neighbor of x with lowest index and let $C = x_1, x_2, x_3 \ldots, x_i, x, y, x_1$. The neighbors of y_3 in C are y, x_2, x_4 and (C, y_3) is a short 3-wheel. This proves that H_1 satisfies (iii) of Definition 3.8. □

PROCEDURE 3

Input: A bipartite graph F not containing a short 3-wheel.

Output: Either a 3-path configuration of F, or a list of undominated induced subgraphs $F_1^*, \ldots, F_q^*$ of F each containing an induced path of length 3, where $q \leq |V^c(F)|^2 |V^r(F)|^2$ with the following properties:

- The graphs $F_1^*, \ldots, F_q^*$ do not contain a double star cutset.
- If the input graph F is not balanced and contains a clean unbalanced hole that is smallest in F, then one of the graphs in the list, say F_i^*, contains an unbalanced hole H^* in $\mathcal{C}(H)$, that is smallest and clean in both F and F_i^*.

Step 1 Delete dominated nodes in F until F becomes undominated. Let $\mathcal{M} = \{F\}$, $\mathcal{T} = \emptyset$.

Step 2 If $\mathcal{M}$ is empty, stop. Otherwise remove a graph R from $\mathcal{M}$. If R has no double star cutset, add R to $\mathcal{T}$ and repeat Step 2. Otherwise, let $S = N_R(u) \cup N_R(v)$ be a double star cutset of R. Let $R_1', \ldots, R_l'$ be the connected components of $R \setminus S$ and let $R_1, \ldots, R_l$ be the corresponding blocks, that is R_i is the graph induced by $V(R_i') \cup S$. Go to Step 3.

Step 3 Consider every pair of nonadjacent nodes u_p and v_q adjacent to u and v respectively and distinct from v and u. If there exist two distinct connected components of $R \setminus S$ that each contain neighbors of u_p and neighbors of v_q, there is a $3PC(u_p, v_q)$ and F is not balanced. Otherwise go to Step 4.

Step 4 From each block R_i, remove dominated nodes until the resulting graph R_i^* becomes undominated. Add to $\mathcal{M}$ all the graphs R_i^* that contain at least one chordless path of length 3. Go to Step 2.

Lemma 3.12 *Let F be a bipartite graph that does not contain a short 3-wheel and let H^* be a smallest unbalanced hole that is clean in F.*

If Procedure 3, when applied to F, does not detect a 3-path configuration in Step 3, then one of the graphs F_i^*, obtained as ouput of Procedure 3, contains an unbalanced hole in $\mathcal{C}(H^*)$.

Proof: Let $N(u) \cup N(v)$ be a double star cutset of F, used in Procedure 3. Let $F_1', \ldots, F_t'$ be the connected components of $F \setminus (N(u) \cup N(v))$ and $F_1, \ldots, F_t$ be the corresponding blocks. We first show that if no 3-path configuration is detected in Step 3, an unbalanced hole $H' \in \mathcal{C}(H^*)$ is contained in some block F_i obtained at the end of Step 3.

Choose $H' \in \mathcal{C}(H^*)$ such that $V(H') \cap \{u, v\}$ is maximal. By Lemma 3.11, H' is clean in F, so u is either in H' or has at most one neighbor in H' and the same holds for v.

Let W be the subgraph induced by $V(H') \setminus (N(u) \cup N(v))$. We have three possibilities for W:

(i) If H' contains no neighbor of u and v, then $W = H'$.

(ii) If both u and v have a single neighbor u_1 and v_1 in H' and u_1, v_1 are nonadjacent, then W consists of two paths.

(iii) In all the remaining cases, it is easy to check that W consists of a single path.

If H' does not belong to any of the blocks $F_1, \ldots, F_t$, the graph W must be disconnected and have a component in, say, F_i' and another in, say, F_j'. So (ii) holds. Let u_1 and v_1 be the neighbors of u and v in H'. Then $V(W) \cup \{u, v\}$ induces a $3PC(u_1, v_1)$ which is detected in Step 3 of the algorithm.

So, at the end of Step 3, one block F_i contains H' and, by Lemma 3.11, $H' \in \mathcal{C}(H^*)$ is clean in F_i. Since H' is clean, the graph F_i^*, obtained from F_i by removing dominated nodes, contains a hole $H'' \in \mathcal{C}(H') = \mathcal{C}(H^*)$, where possibly $H' = H''$. □

Lemma 3.13 *The number of graphs $F_1^*, \ldots, F_q^*$ produced by Procedure 3 applied to F is bounded by $|V^r(F)|^2|V^c(F)|^2$. So is the number of double star cutsets used by Procedure 3.*

Proof: Let $N(u) \cup N(v)$ be a double star cutset of F. Let $F_1', \ldots, F_t'$ be the connected components of $F \setminus (N(u) \cup N(v))$ and let $F_1^*, \ldots, F_t^*$ be the corresponding undominated blocks.

Claim 1 *No two distinct undominated blocks contain the same chordless path of length 3.*
Proof of Claim 1: Suppose by contradiction that a chordless path $P = a, b, c, d$ belongs to two distinct undominated blocks F_i^* and F_j^*. Then $\{a, b, c, d\} \subseteq N_F(u) \cup N_F(v)$.

Node u is distinct from a and d for otherwise a and d are adjacent and P is not a chordless path. By symmetry, v is also distinct from a and d. Since both F_i^* and F_j^* are undominated, both nodes a and d have at least one neighbor in both the connected components F_i' and F_j'. Now Step 3 of Procedure 3 detects a 3-path configuration. This completes the proof of Claim 1.

Claim 2 *The graph F contains at least one chordless path of length 3 that is not contained in any of the undominated blocks F_i^*.*
Proof of Claim 2: Each of the connected components $F_1', \ldots, F_t'$ must contain at least two nodes, since F is an undominated graph. At least one node in F_i' must be adjacent to a node in $N_F(u) \cup N_F(v)$. Assume w.l.o.g. that node p_i in F_i' is adjacent to a neighbor of v, say v_i.

Suppose now no node in F_i' is adjacent to a node in $N(u)$. Then the nodes in $N(u) \setminus \{v\}$ are dominated by v in $F_i' \cup N(u) \cup N(v)$. Thus, the undominated block F_i^* does not contain any neighbor of u except v. This in turn implies that node u is dominated by v_i. Thus u would have been deleted from F_i^*. Now $P = p_i, v_i, v, u$ is a chordless path of length 3 in F but P is not in any of the undominated blocks $F_1^*, \ldots, F_t^*$.

So a node in F_i' must be adjacent to a node, say u_i, that is a neighbor of u. Repeating the same argument for $j = 1, \ldots, t$, it follows that each connected component F_j' contains a node, say w_j, that is adjacent to a node, say $u_j \in N_F(u)$. Suppose now u_j has a neighbor, say g in a connected component F_k', distinct from F_j'. Let q be a neighbor of g in F_k'. Then $P = q, g, u_j, w_j$ is a chordless path of length 3 contained in F but not in any of the undominated blocks $F_1^*, \ldots, F_t^*$. Suppose now that u_j does not have any neighbor in F_k', $k \neq j$. Then, in Step 4 of Procedure 3, node u_j is deleted from the undominated block F_k^*. Now the path w_k, u_k, u, u_j is a chordless path of length 3 contained in F but not in any of the undominated blocks $F_1^*, \ldots, F_t^*$. This completes the proof of Claim 2.

Every undominated block that is added to the list $\mathcal{M}$ in Step 4 of Procedure 3 contains a chordless path of length 3. Hence every undominated block that is added to the list $\mathcal{T}$ in Step 2 contains a chordless path of length 3. By Claim 1, the same chordless path of length 3 is not in any other undominated block that is added to the list $\mathcal{T}$. So the number of graphs in the list $F_1^*, \ldots, F_q^*$ is at most $|V^r(F)|^2|V^c(F)|^2$. By Claim 2, it follows that the number of double star cutsets used to decompose the graph F with Procedure 3 is at most $|V^r(F)|^2|V^c(F)|^2$. $\square$

3.7 Validity of the Algorithm

We now prove the validity of the recognition algorithm given in Subsection 3.3.

Theorem 3.14 *The running time of the recognition algorithm is polynomial in the size of the input graph G, and the algorithm correctly identifies G as balanced or not.*

Proof: The recognition algorithm described in Subsection 3.3 applies first Procedures 1, 2 and 3. The running time of each of these procedures has been shown to be polytime in its respective subsection. Finally, in Step 4, the algorithm checks whether each of the (polynomially many) blocks is totally unimodular. Total unimodularity can be checked in polytime [28]. Hence the running time of the recognition algorithm described in Subsection 3.3 is polynomial.

Suppose G is balanced. Then G does not contain a short 3-wheel or a 3-path configuration. All the induced subgraphs of G are balanced, so the graphs produced by Procedures 2 and 3 are balanced. Now, by Theorem 1.2, every graph in the list $B_1, \ldots, B_s$ is totally unimodular. Then Step 4 of the algorithm identifies G as balanced.

Suppose G is not balanced. If G contains a short 3-wheel, Step 1 of the algorithm identifies G as not balanced. Suppose G does not contain a short 3-wheel. Clearly G contains an unbalanced hole of smallest cardinality. Now, by Lemma 3.9, one of the induced subgraphs, say G_i, of G, in the list produced by Procedure 2 contains an unbalanced hole H^*, of smallest cardinality, that is clean in G_i. Now G_i is one of the graphs considered for double star cutset decompositions by Procedure 3. By Lemma 3.12, Procedure 3 either detects a 3-path configuration or one of the undominated blocks, say B_j, in the final list produced by Procedure 3 contains an unbalanced hole in the family $\mathcal{C}(H^*)$. In the former case G is correctly identified as not balanced. In the latter case, B_j is not totally unimodular and Step 4 of the algorithm identifies G as not balanced. $\square$

Remark 3.15 *A polytime recognition algorithm for balancedness that does not use total unimodularity testing can be obtained as follows. Instead of stopping the recognition algorithm in Step 4, where the blocks are checked for total unimodularity, continue the decomposition process, using 2-join decompositions for each B_j. A polytime algorithm for identifying 2-joins is given in [21] for a slightly different definition. This algorithm can be adapted to the 2-join decomposition used here. By pursuing the decomposition process until the blocks contain no 2-join, one can show, using Theorem 1.1, that G is balanced if and only if all the blocks are strongly balanced. In fact, by Theorems 1.6 and 1.5, one can show that G is balanced if and only if all the blocks are basic. This property can be checked in polytime and the number of blocks is polynomial by Lemma 2.1.*

4 Wheel-and-Parachute-Free Graphs

4.1 Introduction

In this section, we consider weakly balanced graphs that are wheel-and-parachute-free.

Remark 4.1 *The class of wheel-and-parachute-free weakly balanced graphs properly contains totally balanced graphs and strongly balanceable graphs.*

Proof: The cycle H of a wheel (H, v) and the cycle induced by the paths T, P_1, P_2 in a parachute $Par(T, P_1, P_2, M)$ are holes of length strictly greater than 4. Hence totally balanced graphs are wheel-and-parachute-free.

In a wheel (H, v), a cycle with a unique chord is induced by v and an appropriate subpath of H. In a parachute, assume w.l.o.g. that P_1 has length greater than 1. Then the graph obtained from the parachute by removing the intermediate nodes of P_1 is a cycle with a unique chord. Hence strongly balanceable graphs are wheel-and-parachute-free.

To see that the inclusion is proper, note that a cycle C with a unique chord having length 10 or more, is neither strongly balanceable nor totally balanced. Yet, the cycle C is a wheel-and-parachute-free weakly balanced graph. $\square$

A 2-join $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ is said to be *rigid* if $A_1 \cup B_1$ or $A_2 \cup B_2$ induces a biclique. The main result of this section is the following.

Theorem 4.2 *If G is a wheel-and-parachute-free weakly balanced graph that is not strongly balanceable, then G has a rigid 2-join.*

Assume uv is a bisimplicial edge of a bipartite graph G and both $A = N(u) \setminus \{v\}$ and $B = N(v) \setminus \{u\}$ are nonempty. Then G contains a rigid 2-join $E(K_{A,\{u\}}) \cup E(K_{B,\{v\}})$ (note that Property (iii) of the definition of a 2-join is satisfied since $A \cup B$ induces a biclique). So, Theorem 4.2 can be viewed as an extension of Theorem 1.7 about totally balanced graphs.

The next result describes the different types of possible strongly adjacent nodes to a cycle with a unique chord.

Theorem 4.3 *Let C be a cycle with a unique chord uv in a weakly balanced graph and let C_1 and C_2 be the two holes of the graph induced by $V(C)$. If x is a strongly adjacent node to C, then x is of one of the following types:*

Type 1 *The set $N_C(x)$ is contained in $V(C_1)$ or in $V(C_2)$.*

Then $|N_C(x)|$ is even.

Type 2 *The set $N_C(x)$ is not contained in $V(C_1)$ or in $V(C_2)$ and $N(x) \cap \{u, v\} \neq \emptyset$.*

Then $|N_{C_1}(x)|$ and $|N_{C_2}(x)|$ are both even.

Type 3 *The set $N_C(x)$ is not contained in $V(C_1)$ or in $V(C_2)$ and $N(x) \cap \{u, v\} = \emptyset$.*

Then either $|N_{C_1}(x)|$ is even and $|N_{C_2}(x)| = 1$ or $|N_{C_2}(x)|$ is even and $|N_{C_1}(x)| = 1$. Furthermore the unique neighbor of x in C_1 or C_2 is adjacent to u or v.

Proof: If, for $i = 1$ or 2, $N_C(x)$ is contained in $V(C_i)$, then $|N_C(x)|$ is even, else (C_i, x) is an odd wheel. Thus x of Type 1.

If $N_C(x)$ is not contained in $V(C_1)$ or $V(C_2)$ and x is adjacent to u or v, then $|N_{C_i}(x)|$ is even for $i = 1, 2$, else (C_i, x) is an odd wheel. Thus x of Type 2.

If $N_C(x)$ is not contained in $V(C_1)$ or $V(C_2)$ and x is not adjacent to u or v, then assume w.l.o.g. that u and x are in opposite sides of the bipartition.

Assume that $|N_{C_1}(x)|$ or $|N_{C_2}(x)|$ is even. Then there is a $3PC(u, x)$ unless x has a unique neighbor, adjacent to v, in $V(C_1)$ or in $V(C_2)$. Thus x of Type 3.

Assume that $|N_{C_i}(x)| = 1$ for $i = 1, 2$. If both neighbors of x in $V(C)$ are adjacent to v, then there is an odd wheel with center v. If one neighbor of x in $V(C)$, say y, is not adjacent to v, then there is a $3PC(y, v)$. □

4.2 Decomposition

Let G be a wheel-and-parachute-free weakly balanced graph. In this subsection we show that, for every edge uv that is the unique chord of at least one cycle, the graph G has a biclique cutset K_{AB} with $u \in A$ and $v \in B$. This result will then be used to prove Theorem 4.2.

For a cycle C with unique chord uv, we use the notation of Figure 9. It will be convenient to write $C = (C_1, C_2)$, where C_1 and C_2 are the two holes of the graph induced by $V(C)$. We assume w.l.o.g. that u is in V^r and that v is in V^c.

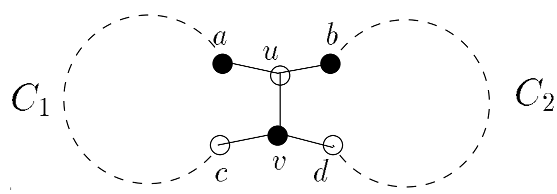


Figure 9: Cycle with unique chord $C = (C_1, C_2)$

Lemma 4.4 *Every node x that is strongly adjacent to C is either of Type 1 [4.3] and has two neighbors in C_1 or in C_2, or is a twin of u or v relative to C.*

Proof: Every strongly adjacent node x is of Type 1, 2 or 3[4.3] and has at most two neighbors in C_1 and in C_2, since G contains no wheel.

If x is of Type 2 [4.3], assume w.l.o.g. that x is adjacent to u. Then x has exactly two other neighbors in C, one in C_1 and one in C_2, say x_1 and x_2 respectively. If x_1 is distinct from c (see Figure 9), then there is a parachute with paths $T = u, a, \ldots, x_1$, $P_1 = u, v$, $P_2 = x_1, \ldots, c, v$ and $M = x, x_2, \ldots, d, v$. So $x_1 = c$. By symmetry, it follows that $x_2 = d$.

If x is of Type 3 [4.3], assume w.l.o.g. that x is adjacent to b. Then x has exactly two neighbors in $V(C_1) \setminus \{u, v\}$, say x_1 and x_2. The nodes of $V(C_1) \cup \{b, x\}$ induce a parachute, a contradiction. □

A node x *fits* C if it is adjacent to u or v and has no neighbor in $V(C) \setminus \{u, v, a, b, c, d\}$.

Lemma 4.5 *Each node in a direct connection from $V(C_1) \setminus \{u, v\}$ to $V(C_2) \setminus \{u, v\}$ fits C.*

Proof: We prove the lemma by induction on the length of the direct connection. In case it has only one node, then, by Lemma 4.4, if consists of u, v or one of their twins; clearly, such nodes fit C. Let $P = x_1, \ldots, x_n$ be a direct connection from $V(C_1) \setminus \{u, v\}$ to $V(C_2) \setminus \{u, v\}$.

Claim *If a node of P fits C or is strongly adjacent to C, all other nodes of P fit C.*

Proof of Claim: Let x_i fit C or be strongly adjacent to C. By symmetry between C_1 and C_2, it suffices to show that all x_j with $j > i$ fit C. We may as well assume that we have chosen i as small as possible. By symmetry between u and v we may assume that x_i is not adjacent to v. So if $i \neq 1$, x_i is adjacent to u, and if $i = 1$, $x_1 = x_i$ has two neighbors in C_1.

Let y_1 be the neighbor of x_1 on C_1 closest to v. If $i \neq 1$, let $y_2 = u$, and if $i = 1$, let y_2 be the neighbor of x_1 in C_1 distinct from y_1. If y_1 and y_2 are adjacent, then $y_2 = u$ and $y_1 = a$. But, in that case, the subgraph induced by $V(P) \cup V(C) \setminus \{u\}$ contains a hole with at least three neighbors of u, namely x_i, a and v, which contradicts the fact that G is wheel-free. So y_1 and y_2 are nonadjacent. Consequently, the cycle C^*, obtained from C by replacing the y_1y_2-subpath of C_1 not using uv by $y_1, x_1, P_{x_1x_i}, x_i, y_2$, has a unique chord. Let C_1^* and C_2^* $(= C_2)$ be the two holes in C^*. Then $P_{x_{i+1}x_n}$ is a direct connection from $V(C_1^*) \setminus \{u, v\}$ to $V(C_2^*) \setminus \{u, v\}$. As it is shorter than P, all its nodes fit C^*, by the induction hypothesis. Hence, as $C_2^* = C_2$, the nodes of $P_{x_{i+1}x_n}$ fit C. This completes the proof of the claim.

Assume that some node of P does not fit C. Then by the claim, no node of P fits C nor is strongly adjacent to C. So, P avoids $N(u) \cup N(v)$. and x_1 (x_n) has a unique neighbor in C, say y_1 (y_n). Then $y_1 \neq a$ as otherwise C would have a wheel with center u, or a parachute with center u or a $3PC(u, y_n)$. Hence, by symmetry, $y_1 \neq c$ and $y_n \notin \{b, d\}$. But then $V(C) \cup V(P)$ contains a $3PC(y_1, u)$ or a $3PC(y_1, v)$, a contradiction. □

Let $S(C_1, C_2)$ denote the set of all nodes x in G such that there exists a direct connection from $V(C_1) \setminus \{u, v\}$ to $V(C_2) \setminus \{u, v\}$ starting with x (that is, with x adjacent to $V(C_1) \setminus \{u, v\}$). Note that $S(C_1, C_2)$ contains u, v and their twins relative to C and that, by Lemma 4.5, all the other nodes in $S(C_1, C_2)$ are adjacent to u and c or to v and a.

Theorem 4.6 *Let G be a wheel-and-parachute-free weakly balanced graph. Let C be a cycle with a unique chord uv and let C_1 and C_2 be the two holes induced by C. Then $S(C_1, C_2)$ is a biclique cutset of G separating $V(C_1) \setminus \{u, v\}$ from $V(C_2) \setminus \{u, v\}$.*

Proof: Clearly, $S(C_1, C_2)$ separates $V(C_1) \setminus \{u, v\}$ from $V(C_2) \setminus \{u, v\}$ as it intersects all direct connections. It remains to prove that $S(C_1, C_2)$ induces a biclique. Assume not; then there exist two nonadjacent nodes $x \in V^r \cap S(C_1, C_2)$ and $y \in V^c \cap S(C_1, C_2)$. By Lemma 4.5, x is adjacent to both a and v and y is adjacent to both c and u. Let P and Q be direct connections from $V(C_1) \setminus \{u, v\}$ to $V(C_2) \setminus \{u, v\}$ starting with x and y respectively. Let L denote the graph induced by the nodes in P, Q and $V(C_2) \setminus \{u, v\}$. Let R be the shortest path connecting x and y in L. Let w be the node of R adjacent to x. As x and y are not adjacent, $w \neq y$. If $w \in V(C_2) \setminus \{u, v\}$, it is b (as x fits C); otherwise it lies on $(V(P) \cup V(Q)) \setminus \{x, y\}$. So, by Lemma 4.5, w is adjacent to u. Now, let T be the ac-subpath of C_1 not using uv. Then $H = x, R, y, c, T, a, x$ is a hole containing at least three neighbors of u (namely a, y and w), contradicting the fact that G is wheel-free. □

Let $G(C_1, C_2)$ be the block (as defined in Subsection 1.1) containing C_1 in the decomposition of G by $S(C_1, C_2)$. Let $W(C_1, C_2) = V(G(C_1, C_2)) \setminus S(C_1, C_2)$.

Proof of Theorem 4.2: Let $C = (C_1, C_2)$ be a cycle with a unique chord uv such that $G(C_1, C_2)$ has the smallest possible number of nodes. Let R be the set of nodes in $W(C_1, C_2)$ that are adjacent to all the nodes in $S(C_1, C_2) \cap V^r$ or to all the nodes in $S(C_1, C_2) \cap V^c$. Moreover, let Z be the set of nodes in $W(C_1, C_2) \setminus R$ that have at least one neighbor in $S(C_1, C_2)$. If $R \cup Z$ and $S(C_1, C_2)$ do not induce a rigid 2-join, Z is not empty. So we may assume that $Z \cap V^c$ is nonempty. Let Q be a shortest path in $W(C_1, C_2)$ connecting a node $z \in Z \cap V^c$ with a node $r \in R \cap V^c$. (Note that, by Theorem 4.6, $R \cap V^c \neq \emptyset$ as it contains a.) Let $x \in S(C_1, C_2)$ be adjacent to z and let $y \in S(C_1, C_2) \cap V^r$ be nonadjacent to z. Clearly, as Q was chosen as short as possible, no intermediate node of Q is adjacent to x or to y. Hence the cycle $H_1 = x, z, Q, r, x$ is a hole. Let Γ be the component of $G \setminus S(C_1, C_2)$ containing C_2. By the definition of $S(C_1, C_2)$, each member of $S(C_1, C_2)$ is adjacent to a node in Γ. Let T be the shortest xy-path in $\Gamma \cup \{x, y\}$ and $H_2 = r, y, T, x, r$. Then (H_1, H_2) is a cycle with a unique chord, namely xr. As $H_1 \setminus \{x\}$ is in $W(C_1, C_2)$, we have that $S(H_1, H_2)$ is contained in $G(C_1, C_2)$. On the other hand, $H_2 \setminus \{x, y, r\}$ is in Γ, and so, as each node of $S(C_1, C_2)$ has a neighbor in Γ, we see that $S(C_1, C_2) \cap W(H_1, H_2) = \emptyset$. Hence, $G(H_1, H_2)$ in contained in $G(C_1, C_2)$. As y lies in $G(C_1, C_2)$ and not in $G(H_1, H_2)$, that containment is proper, contradicting the minimality of $G(C_1, C_2)$. $\square$

5 Even Wheels

5.1 Introduction

In this section, we prove Theorem 1.10, which states that if a graph is weakly balanced and contains an even wheel, then it has an extended star cutset. The proof is in two parts, treated in Subsections 5.2 and 5.3 respectively. In Subsection 5.2, we give properties of the strongly adjacent nodes to even wheels. In Subection 5.3 we prove Theorem 1.10. In fact, we prove a stronger result. In order to state it, we first introduce some notation.

Given an even wheel (H, v), a subpath of H having two nodes of $N_H(v)$ as endnodes and only nodes of $V(H) \setminus N_H(v)$ as intermediate nodes is called a *sector* of (H, v). Two sectors are *adjacent* if they have a common endnode and two nodes of $N_H(v)$ are *consecutive* if they are the endnodes of some sector. We paint the nodes of $V(H) \setminus N_H(v)$ with two colors, say blue and green, in such a way that nodes of $V(H) \setminus N_H(v)$ have the same color if they are in the same sector, and have distinct colors if they are in adjacent sectors. The nodes of $N_H(v)$ are left unpainted.

In a bipartite graph G, an even wheel is *small* if no even wheel of G contains strictly fewer nodes.

Theorem 5.1 *Let G be a weakly balanced graph that contains a wheel. Then G contains an extended star cutset $(x; T; A; R)$ and a small even wheel (H, x) such that $|N_H(x) \cap A| \geq 2$ and the extended star cutset separates the blue nodes of H from the green nodes.*

5.2 Strongly Adjacent Nodes to an Even Wheel

The goal of this subsection is to prove the following two theorems.

Theorem 5.2 *Let (H, v) be an even wheel of a weakly balanced graph G. Assume that $v \in V^r$. Let $u \in V^c \backslash N(v)$ be a node with neighbors in at least two distinct sectors of H. Then u has exactly two neighbors in H, say u_j and u_k, belonging to two distinct sectors of the same color.*

For an even wheel (H, v), define the set of nodes:

$T(H, v) = \{u \in V(G)|$ No sector of (H, v) entirely contains $N_H(u)$ and $|N_H(v) \cap N_H(u)| \geq 2\}$.

Theorem 5.3 *If a weakly balanced graph contains an even wheel, then it contains a small even wheel (H, v) such that*

$$\Big|\bigcap_{u \in T(H,v)} N_H(u)\Big| \geq 2.$$

Proof of Theorem 5.2: Assume that node u has neighbors in at least three different sectors, say S_i, S_j, S_k. If none of these sectors is adjacent to the other two, then there exist three unpainted nodes v_i, v_j, v_k, such that $v_i \in V(S_i) \backslash (V(S_j) \cup V(S_k))$, $v_j \in V(S_j) \backslash (V(S_i) \cup V(S_k))$, $v_k \in V(S_k) \setminus (V(S_i) \cup V(S_j))$. This implies the existence of a $3PC(u, v)$, where each of the nodes v_i, v_j, v_k belongs to a distinct path of the 3-path configuration. If H has four sectors each containing neighbors of u, then each sector has exactly one neighbor of u (otherwise there is a $3PC(u,v)$). But now Theorem 4.3 is contradicted in the cycle formed by two adjacent sectors. So u has neighbors in exactly three sectors and one of them is adjacent to the other two, say S_j is adjacent to both S_i and S_k. Let v_i be the unpainted node in $V(S_i) \cap V(S_j)$ and v_k the unpainted node in $V(S_j) \cap V(S_k)$. Then, there is a $3PC(u,v)$ unless node u has a unique neighbor u_i in S_i which is adjacent to v_i and a unique neighbor u_k in S_k which is adjacent to v_k. When this is the case, node u has an even number of neighbors in S_j (else (H,u) is an odd wheel) and therefore the nodes in (H,v) together with u induce a connected 6-hole with fan sides and 6-hole $u, u_i, v_i, v, v_k, u_k, u$. See Figure 10. This contradicts the assumption that G is weakly balanced.

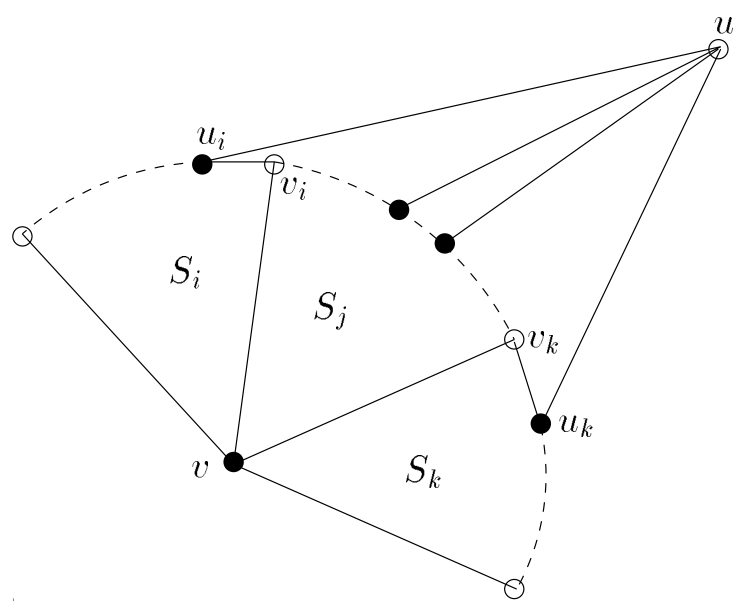


Figure 10:

So u has neighbors in at most two sectors of the wheel, say S_j and S_k. If these two sectors are adjacent, let v_i be their common endnode and v_j, v_k the other endnodes of S_j and S_k respectively. Let H' be the hole obtained from H by replacing $V(S_j) \cup V(S_k)$ by the shortest path in $V(S_j) \cup V(S_k) \cup \{u\} \setminus \{v_i\}$. The wheel (H', v) is an odd wheel. So the sectors S_j and S_k are not adjacent.

If u has three neighbors or more on H, say two or more in S_j and at least one in S_k, then denote by v_j and v_{j-1} the endnodes of S_j and by v_k one of the endnodes of S_k. There exists a $3PC(u,v)$ where each of the nodes v_j, v_{j-1}, and v_k belongs to a different path. Therefore u has only two neighbors in H, say $u_j \in V(S_j)$ and $u_k \in V(S_k)$. Let C_1 and C_2 be the holes formed by the node u and the two $u_j u_k$-subpaths of H, respectively. In order for both (C_1, v) and (C_2, v) to be even wheels, the sectors S_j and S_k must be of the same color. $\square$

Let (H,v) be an even wheel of a weakly balanced graph G. In the remainder of this subsection, we assume that $v \in V^r$. Before proving Theorem 5.3, we need to establish results about nodes $u \in V^r$ that are strongly adjacent to H. Node u can be of four types:

Type 1 There exists a sector of (H, v) containing all the nodes of $N_H(u)$.

Type 2 Node u is not of Type 1 and all its neighbors in H are unpainted. Note that, in particular, the center v of the wheel is of Type 2.

Type 3 Node u is not of Types 1 or 2 and all its painted neighbors in H have the same color.

Type 4 Node u has painted neighbors of both colors.

Lemma 5.4 *If $u \in V^r$ is strongly adjacent to H and has a unique neighbor w in some sector of (H, v), then node w is unpainted.*

Proof: Assume that node u has a unique neighbor w in sector S_i and that w is painted. Let v_i and v_{i-1} be the endnodes of S_i. Since node u is strongly adjacent to H, it has at least one neighbor in the path induced by $V(H) \setminus V(S_i)$. Choose u^* among the nodes of $N_H(u) \setminus V(S_i)$ and choose v^* among the nodes of $N_H(v) \setminus V(S_i)$ in such a way that the u^*v^*-subpath of H not containing S_i is shortest. Note that $u^* \in V^c$, hence u^* cannot be adjacent to v_i or v_{i-1}. This implies a $3PC(w, v)$, where each of the nodes v_i, v_{i-1} and v^* belongs to a different path. □

Lemma 5.5 *Let $u \in V^r$ be a Type 4 strongly adjacent node to an even wheel (H, v). Let s and t be a green and a blue neighbor of u, respectively. Each of the st-subpaths of H contains at least one unpainted neighbor of u. Hence $u \in T(H, v)$.*

Proof: Assume that one of the two st-subpaths of H contains no unpainted neighbor of u. Let Q be this subpath. Let P be a $s't'$-subpath of Q such that s' is a green neighbor of u, t' is a blue neighbor of u, and P contains no other painted neighbor of u. P contains an odd number of unpainted nodes, none of which are adjacent to u. If this number is three or more, then v is the center of an odd wheel with hole induced by the nodes of P and u.

So P contains exactly one neighbor of v, say x. Consider the cycle C with unique chord vx induced by v and the two sectors of H having x as an endnode. Node u is a strongly adjacent node relative to C and therefore must be of one of the three types of Theorem 4.3. It is not of Type 1[4.3] since s' and t' are in different sectors. It is not of Type 2[4.3] either since u is not adjacent to v or x. Since s' and t' are painted, they are not adjacent to v and since $s', t', x \in V^c$, the nodes s' and t' are not adjacent to x either. So the node u is not of Type 3[4.3] relative to C, a contradiction to Theorem 4.3. Therefore Q must contain an unpainted node adjacent to u. □

Now assume the even wheel (H, v) is small.

Remark 5.6 *Every strongly adjacent node to H has at most two neighbors in each sector of (H, v).*

Lemma 5.7 *If (H, v) is a small even wheel, x is an unpainted node of (H, v) and $u \in V^r$ is a Type 3 or 4 node not adjacent to x, then u has at least one painted neighbor in one of the two sectors of (H, v) adjacent to x and no painted neighbor in the other.*

Proof: If u has no painted neighbor in the two sectors of (H, v) adjacent to x, then v has at least three neighbors in a sector of (H, u), a contradiction to (H, v) being small. Since u is not adjacent to x, by Lemma 5.5, u cannot have painted neighbors in both sectors of (H, v) adjacent to x. $\square$

Lemma 5.8 *Let (H, v) be a small even wheel and let u be a Type 2, 3 or 4 node. Then $|N_H(u)| = |N_H(v)|$ and therefore (H, u) is a small even wheel.*

Proof: When u is a Type 2 node, the result is immediate. When u is a Type 3 node, it follows from Remark 5.6, Lemma 5.4 and Lemma 5.7.

Now consider the case when u is a Type 4 node. By Lemma 5.5, we have that $|N_H(u) \cap N_H(v)| \geq 2$. Let P be a subpath of H with endnodes in $N_H(u) \cap N_H(v)$, say x and y but no intermediate node in $N_H(u) \cap N_H(v)$. The nodes x and y are said to be *consecutive nodes* of $N_H(u) \cap N_H(v)$ in H. Now Lemma 5.5 implies that $V(P) \cap N(u)$ does not contain nodes of distinct colors. Assume w.l.o.g. that $V(P) \cap N(u)$ contains no green node. Then Lemmas 5.4, 5.7 and the fact that (H, v) is a small wheel imply that u has exactly two neighbors in every blue sector of P. This shows that $|N_H(u) \cap V(P)| = |N_H(v) \cap V(P)|$. As this holds for each pair of consecutive nodes of $N_H(u) \cap N_H(v)$ in H, we get the equality claimed in the lemma. $\square$

Lemma 5.9 *Let (H, v) be a small even wheel and let u be a Type 4 node having painted neighbors u_i and u_{i+1} in two adjacent sectors, say S_i, S_{i+1}. Then every Type 2, 3 or 4 node is adjacent to the common endnode v_i of S_i, S_{i+1}.*

Proof: Node v_i belongs to $N(u)$, as a consequence of Lemma 5.5. Assume by contradiction that there exists a node w, of Type 2, 3 or 4, that is not adjacent to v_i. By Lemma 5.7, node w has a painted neighbor in S_i or S_{i+1}. If w has a painted neighbor in both S_i and S_{i+1}, then Lemma 5.5 implies that w is adjacent to v_i. Therefore we assume w.l.o.g. that w has a painted neighbor in S_i but no painted neighbor in S_{i+1}. Remark 5.6 applied to (H, w) and v implies that w has a neighbor in S_{i+2}. Let w_i be a painted neighbor of w in S_i that is closest to u_i and let w_{i+2} be the neighbor of w in S_{i+2} that is closest to the common endnode v_{i+1} of S_{i+1}, S_{i+2}. (Possibly $w_i = u_i$ or $w_{i+2} = v_{i+1}$). Since u is adjacent to v_i and has a painted neighbor in S_{i+1}, it follows from Remark 5.6 that u is not adjacent to v_{i+1}. Thus, u has no painted neighbor in S_{i+2} by Lemma 5.5. Now there is a $3PC(v_{i+1}, u)$: $P_1 = v_{i+1}, v, v_i, u$; $P_2 = v_{i+1}, (S_{i+2})_{v_{i+1}w_{i+2}}, w_{i+2}, w, w_i, (S_i)_{w_i u_i}, u_i, u$; $P_3 = v_{i+1}, (S_{i+1})_{v_{i+1}u_{i+1}}, u_{i+1}, u$. $\square$

Lemma 5.10 *Let (H, v) be a small even wheel and assume that a Type 4 node exists. Then $T(H, v)$ contains all Type 2, 3 and 4 nodes and*

$$\left| \bigcap_{u \in T(H,v)} N_H(u) \right| \geq 2.$$

Proof: Let z be a Type 4 node, having painted neighbors u_i and u_j of distinct colors in (H, v). We show that each of the two $u_i u_j$-subpaths of H contains a node in $\bigcap_{u \in T(H,v)} N_H(u)$.

Let P be one of the two $u_i u_j$-subpaths of H and let X be the set of Type 2, 3, 4, nodes in (H, v). Pick a pair of nodes $x,\ y \in X$ such that y is of Type 4 relative to (H, x) and y has two painted neighbors y_l, y_m in P of distinct colors in (H,x). Furthermore nodes x and y are chosen such that the $y_l y_m$-subpath $P_{y_l y_m}$ of P contains the smallest number of unpainted nodes. (Note that (H,x) and (H,v) have the same set of nodes of Type 2, 3, 4.) If $P_{y_l y_m}$ contains exactly one unpainted node of (H,x), then the proof follows from Lemma 5.9. Now consider the case where $P_{y_l y_m}$ contains more than one unpainted neighbor. This number is odd, say $2k+1$, and let x^* be the $k+1^{st}$ unpainted node in $P_{y_l y_m}$, starting from either end. We show that every Type 2, 3, 4 node with respect to (H,x) is adjacent to x^*.

Assume not. Then there exists a node w of Type 3 or 4 that is not adjacent to x^*. Since y has no painted neighbors in the two sectors adjacent to x^*, Lemma 5.7 shows that y is adjacent to x^*. So $w \neq y$. Let S_l and S_m be the sectors of (H,x) containing y_l and y_m respectively and having x_l, x_{l+1} and x_m, x_{m+1} as endnodes, see Figure 11.

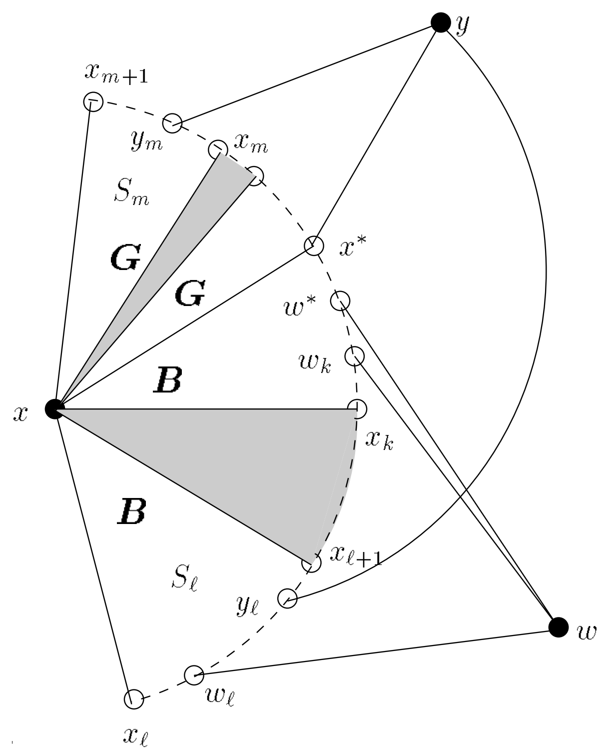


Figure 11:

Let $P_{x_l x_{m+1}}$ be the $x_l x_{m+1}$-subpath of H containing x^*. Then in $P_{x_l x_{m+1}}$, node w has either two neighbors in every green sector of (H,x) or two neighbors in every blue sector. For, if not, let w_1 and w_2 be neighbors of w that are painted with distinct colors and are closest in $P_{x_l x_{m+1}}$. Since w is not adjacent to x^*, w_1 and w_2 cannot belong to S_l and S_m by Lemma 5.7. But this contradicts our assumption on $P_{y_l y_m}$.

Assume w.l.o.g. that node w has two neighbors in every blue sector of (H,x) and that S_l is painted blue. Let x_k be the other endnode of the blue sector having x^* as endnode. In this sector, let w^* be the neighbor of w closest to x^* and w_k the one closest to x_k.

If w has at least one painted neighbor in S_l, then let w_l be such a neighbor and assume w.l.o.g. that the $y_l w_l$-subpath $P_{y_l w_l}$ of S_l does not contain another neighbor of w or y. Then the following three paths induce a $3PC(x^*, w)$: $P_1 = x^*, y, y_l, P_{y_l w_l}, w_l, w$; $P_2 = x^*, x, x_k, \ldots, w_k, w$; $P_3 = x^*, \ldots, w^*, w$.

Hence w is adjacent to x_l and x_{l+1}. In the wheel (H, w), node y is of Type 4, having neighbors x^* and y_l in sectors of (H, w) of distinct colors. But now the number of neighbors of w in the $x^* y_l$-subpath of P is smaller than the number of neighbors of x in $P_{y_l y_m}$, a contradiction to the choice of the pair x, y. $\square$

Let u, w be Type 2 or 3 nodes relative to a small even wheel (H, v). We say that $u \succ w$ relative to the wheel (H, v) if, in every sector S of (H, v) where each of the nodes u and w has two neighbors, both neighbors of w in S belong to the path connecting the two neighbors of u in S.

Lemma 5.11 *Assume that no small even wheel has a Type 4 node. Let (H, v) be a small even wheel and let u, w be Type 2 or 3 nodes relative to (H, v). Then $w \succ u$ or $u \succ w$ relative to (H, v).*

Proof: By Lemma 5.8, (H, u) and (H, w) are small even wheels. Assume that neither $w \succ u$ nor $u \succ w$ relative to (H, v). There are three cases.

Case 1 In some sector, the neighbors of u, w, say u_1, u_2 and w_1, w_2 respectively, appear in the order u_1, w_1, u_2, w_2 where $u_1 \neq w_1, u_2 \neq w_1$ and $u_2 \neq w_2$.

Then, in the wheel (H, u), node w has a unique neighbor, namely w_1, in one sector and w_1 is painted in (H, u). This contradicts Lemma 5.4.

Case 2 In some sector S_i, the neighbors of u, w, say u_1, u_2 and w_1, w_2 respectively, appear in the order u_1, u_2, w_1, w_2 (possibly $u_2 = w_1$).

Let S_i have endnodes v_i, v_{i+1} where w.l.o.g. $v_i, u_1, u_2, w_1, w_2, v_{i+1}$ appear in this order in H. Let S_{i+1} be the sector adjacent to S_i with endnodes v_{i+1}, v_{i+2} and S_{i+2} the sector adjacent to S_{i+1} distinct from S_i. Let u_3 be the neighbor of u in S_{i+2} that is closest to v_{i+2}, in S_{i+2}. Similarly, let w_3 be the neighbor of w in S_{i+2} closest to v_{i+2}. The nodes u_3 and w_3 exist by Lemma 5.8. By Remark 5.6, node w can have at most two neighbors in a sector of (H, u). This implies that u_3 is closer to v_{i+2} than w_3, in S_{i+2}. Define the hole H' as follows: $w, w_1, (S_i)_{w_1 v_i}, v_i, v, v_{i+2}, (S_{i+2})_{v_{i+2} w_3}, w$. H' is shorter than H and node u has at least three neighbors in H, contradicting the assumption that (H, v) is small.

Case 3 There exist two sectors S_i and S_j in which the nodes u and w have their neighbors as in Figure 12 where, possibly, either $u_{i_1} = w_{i_1}$ or $u_{i_2} = w_{i_2}$, but not both, and possibly, either $u_{j_1} = w_{j_1}$ or $u_{j_2} = w_{j_2}$, but not both.

In other words, at least one of u_{i_1} and u_{i_2} is painted in (H, w) and lies in a sector W adjacent to the $w_{i_1} w_{i_2}$-sector W_i of (H, w). Similarly, at least one of u_{j_1} and u_{j_2} is painted in (H, w) and lies in the $w_{j_1} w_{j_2}$-sector W_j of (H, w). However, as w is a Type 3 node with respect to (H, v), sector W_i of (H, w) has the same color in (H, w) as sector W_j. Hence, W and W_j are painted differently in (H, w). So u is of Type 4 with respect to (H, w), a contradiction.

□

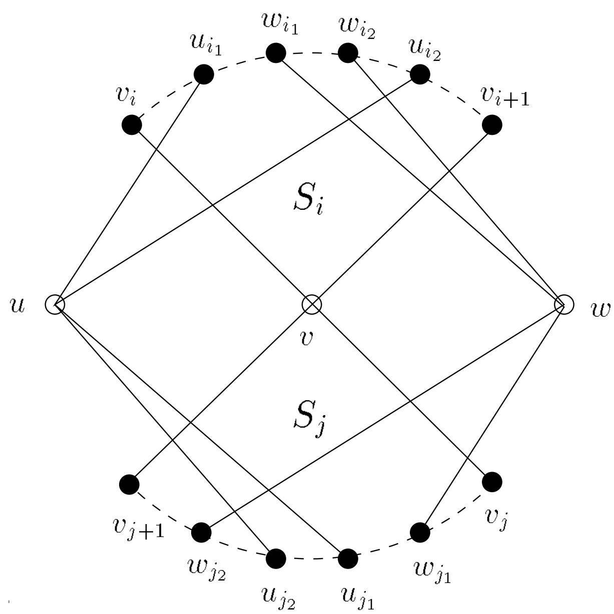


Figure 12:

We are now ready to prove the main result of this subsection.

Proof of Theorem 5.3: If there exists a small even wheel having at least one Type 4 node, the result holds as a consequence of Lemma 5.10. Moreover, by Lemma 5.8, the theorem follows also if there is a small even wheel (H, v) with no Type 3 node in $T(H, v)$. So we assume that no small wheel has Type 4 nodes and that each small wheel has a Type 3 node in $T(H, v)$.

The key observation of this proof is the fact that if u and w are Type 3 nodes of a small even wheel (H, v) such that the painted neighbors of u and w have the same color, then

$$w \succ u \Longrightarrow N_H(w) \cap N_H(v) \supseteq N_H(u) \cap N_H(v).$$

Claim 1 *There exists a small even wheel (H, v^*) and a blue-green painting of its sectors such that all Type 3 nodes of (H, v^*) have blue neighbors.*

Proof of Claim 1: Let (H, v) be a small even wheel, then the relation $\succ$ is transitive in the family $\mathcal{B}(H, v)$ of all the Type 3 nodes with blue neighbors on (H, v). Hence, by Lemma 5.11, there is a Type 3 node v^* in $\mathcal{B}(H, v)$ such that each Type 3 node w in $\mathcal{B}(H, v)$ satisfies $w \succ v^*$. Assume the wheel (H, v^*) is painted such that $v \in \mathcal{B}(H, v^*)$. Then, from the choice of v^*, it is easy to see that all Type 3 nodes in (H, v^*) lie in $\mathcal{B}(H, v^*)$. So Claim 1 follows.

Claim 2 *There exists a Type 3 node u^* in $T(H, v^*)$ such that each Type 3 node w in $T(H, v^*)$ satisfies $w \succ u^*$ relative to (H, v^*).*

Proof of Claim 2: Observe that (H, v^*) has no Type 4 nodes and at least one node in $\mathcal{B}(H, v^*) \cap T(H, v^*)$. Hence by Lemma 5.11, $\succ$ is a linear order on the family $\mathcal{B}(H, v^*) \cap T(H, v^*)$. So Claim 2 follows.

Claims 1 and 2 and the earlier observation show that (H, v^*) has the property that all nodes in $T(H, v^*)$ are adjacent to all the nodes in $N_H(u^*) \cap N_H(v^*)$. $\square$

5.3 An Extended Star Cutset Theorem for Small Even Wheels

In this subsection, we prove the following key result concerning the decomposition of weakly balanced graphs that contain an even wheel.

Theorem 5.12 *Let (H, v) be a small even wheel in a weakly balanced graph. Then every path connecting a blue node to a green node of (H, v) contains a node in $N(v) \cup T(H, v)$.*

Before proving this result, we observe that Theorem 5.1 follows as a corollary.

Proof of Theorem 5.1: There exists a small even wheel (H, v) such that $| \cap_{u \in T(H,v)} N_H(u)| \geq 2$, as a consequence of Theorem 5.3. Now, for any $a_1, a_2 \in \cap_{u \in T(H,v)} N_H(u)$, Theorem 5.12 implies that $N(v) \cup (N(a_1) \cap N(a_2))$ is an extended star cutset of G separating the blue sectors of (H, v) from the green sectors. $\square$

The remainder of this subsection is devoted to the proof of Theorem 5.12. We make use of the following lemma.

Lemma 5.13 *Let (H, v) be an even wheel in a weakly balanced graph G and let P be a chordless path with nodes in $V(G) \setminus (V(H) \cup N(v))$ such that any $x \in V(P)$ is adjacent to at most one node in $N_H(v)$ and to no painted node of H. Then at most two nodes of $N_H(v)$ have a neighbor in P.*

Proof: Assume the lemma is not true and let $P' = y_1, y_2, \ldots, y_n$ be a shortest subpath of P with the property that three distinct nodes of $N_H(v)$ have a neighbor in P'. Denote by v_1, v_2, v_3 the three nodes of $N_H(v)$ with a neighbor in P'. We can assume w.l.o.g. that v_1 is adjacent to y_1 and no other node of P', v_3 is adjacent to y_n and no other node of P' and that v_2 is adjacent to some intermediate nodes of P'. Let y_s and y_t be such nodes, such that the y_1y_s-subpath $P'_{y_1y_s}$ of P' and the y_ty_n-subpath $P'_{y_ty_n}$ of P' are as short as possible.

Let P_{ij} be the v_iv_j-subpath of H not containing v_k, for $i, j, k \in \{1, 2, 3\}$ and $i \neq j \neq k$. Let H_{12} be the hole induced by the nodes in P_{12} and in $P'_{y_1y_s}$, H_{13} the hole induced by the nodes in P_{13} and in P' and let H_{23} be the hole induced by the nodes in P_{23} and in $P'_{y_ty_n}$. Since (H, v) is an even wheel, at least one of the paths P_{ij} contains an odd number of intermediate nodes in $N(v)$. Hence there exists an odd wheel. $\square$

Proof of Theorem 5.12: If the theorem does not hold for (H, v) let $P = s^*, s, \ldots, t, t^*$ be a shortest path connecting nodes of H with distinct colors, and containing no node of $N(v) \cup T(H, v)$. W.l.o.g. assume that $v \in V^r$, s^* is green and t^* is blue. The following possibilities can occur for nodes s and t.

(a) Node s (or t) has only one neighbor in H, namely s^* (t^* respectively),

(b) Node s (or t) belongs to $V^c \setminus N(v)$, is strongly adjacent to H, but all its neighbors are in the same sector of (H, v),

(c) Node s (or t) belongs to $V^c \setminus N(v)$ and has exactly two neighbors in (H, v), one in sector S_i and one in sector S_j, $i \neq j$, where S_i and S_j have the same color,

(d) Node s (or t) belongs to $V^r \setminus T(H, v)$ and is a Type 1 node,

(e) Node s (or t) belongs to $V^r \setminus T(H, v)$ and is a Type 3 node with at most one neighbor in $N_H(v)$.

It follows from Theorem 5.2, Lemma 5.5 and the definition of $T(H, v)$ that no other possibility can occur for the node s (or t).

Next, we show that we can dispose of the possibilities (b) and (d) by modifying the wheel (H, v) and the path P.

Claim 1 *There exists a small even wheel (H', v) and a path $P' = s^{*\prime}, s', \ldots, t', t^{*\prime}$ connecting nodes of distinct colors in (H', v), containing no node of $N(v) \cup T(H', v)$, such that the nodes s' and t' satisfy one of the properties (a), (c) or (e) above and, furthermore, the nodes of $V(P') \setminus \{s^{*\prime}, s', t', t^{*\prime}\}$ have at most one neighbor in H'.*

Proof of Claim 1: First, assume that some node u of $V(P) \setminus \{s^*, s, t, t^*\}$ has at least two neighbors in H. These neighbors are unpainted, otherwise a shorter path P would exist. All Type 2 nodes are in $T(H, v)$, so u must be of Type 1. Denote by v_i and v_{i-1} the nodes of H adjacent to u and by S_i the $v_i v_{i-1}$-sector of (H, v). Assume w.l.o.g. that S_i is a blue sector. Construct H' from H by replacing the sector S_i by the sector v_{i-1}, u, v_i and let P' be the s^*u-subpath of P. Note that (H', v) is small and $T(H', v) = T(H, v)$. Therefore, P' connects sectors of distinct colors in (H', v) and contains no node of $N(v) \cup T(H', v)$. In P', the node t' adjacent to u is different from s (if $s = t'$, then this node violates Theorem 5.2 with respect to (H', v)). Note also that P' is shorter than P. So by repeating the above procedure, we can dispose of all the nodes of $V(P) \setminus \{s^*, s, t, t^*\}$ with at least two neighbors in H. In the remainder, we assume w.l.o.g. that the nodes of $V(P) \setminus \{s^*, s, t, t^*\}$ have at most one neighbor in H and, if this neighbor exists, it is unpainted.

Assume that s satisfies property (b) or (d) and let S_i be the sector containing s^*. Denote by v_i and v_{i-1} the endnodes of S_i and by s_i and s_{i-1} the neighbors of s in S_i that are closest to v_i and v_{i-1} respectively. Let (H', v) be the wheel obtained from (H, v) by substituting the $s_{i-1}s_i$-subpath of S_i with s_{i-1}, s, s_i and let P' be the subpath obtained from P by removing the node s^*, namely $P' = s, s', \ldots, t, t^*$. The wheel (H', v) is small and, since $T(H', v) = T(H, v)$, the path P' connects two sectors of (H', v) with distinct colors and contains no node of $N(v) \cup T(H', v)$. Note that $s' = t$ cannot occur, since this node would either violate Theorem 5.2 with respect to (H', v) or would be Type 4 relative to (H', v), a contradiction to the fact that P' contains no node of $T(H', v)$. Therefore, s' has at most two neighbors in H'. If s' does have two neighbors, it must be of Type 1 relative to (H', v), i.e. Property (d) holds. In this case the above procedure can be repeated and P' can be shortened again. The proof of Claim 1 is now complete.

As a consequence of this claim, we can assume w.l.o.g. that (H, v) and $P = s^*, s, \ldots, t, t^*$ have the following properties, in addition to those already stated at the beginning of the proof: s and t satisfy Properties (a), (c) or (e) and the nodes of $V(P) \setminus \{s^*, s, t, t^*\}$ have at most one neighbor in H.

Claim 2 *Assume s is a Type 3 node.*

(i) If $N_H(s) \cap N_H(v) = \emptyset$, then no node of $V(P) \setminus \{s^*, s, t, t^*\}$ is adjacent to a node of H.

(ii) If $N_H(s) \cap N_H(v) = \{a\}$, then no node of $V(P) \setminus \{s^*, s, t, t^*\}$ is adjacent to a node of $H \setminus \{a\}$.

Proof of Claim 2: Assume not and let u be the node of $V(P) \setminus \{s^*, s, t, t^*\}$ that is closest to s in P and adjacent to a node of H (Case (i)) or of $H \setminus \{a\}$ (Case (ii)). By Claim 1, node u can only be adjacent to one node of H and this node is unpainted. Let $x \in N_H(v)$ be this node. By Lemma 5.8, node s has exactly two neighbors in each green sector of (H, v). Let S_i be the green sector having x as endnode and let s_i be the neighbor of s closest to x in S_i. Let S_j be a green sector distinct from S_i, say with endnodes v_j and v_{j-1} and let s_j and s_{j-1} be the neighbors of s in S_j, closest to v_j and v_{j-1} respectively. Assume w.l.o.g. that s_j is painted. Then $x \in V^c$ and $s \in V^r$ are connected by a $3PC(x, s)$: $P_1 = x, u, P_{us}, s$; $P_2 = x$, $(S_i)_{xs_i}$, s_i, s; $P_3 = x, v, v_j$, $(S_j)_{v_j s_j}$, s_j, s. This completes the proof of Claim 2.

A similar statement to Claim 2 holds when t is of Type 3.

Claim 3 *Neither node s nor node t is of Type 3.*

Proof of Claim 3: Assume s is a Type 3 node. Let S_i be the blue sector containing t^* where we assume that t^* is chosen such that t has no neighbor in $N_H(v) \setminus V(S_i)$. (This choice is possible since t satisfies Properties (a) or (c) or (e).)

Let S_{i-1} and S_{i+1} be the green sectors adjacent to S_i. Let v_i be the common endnode of S_i and S_{i+1} and let v_{i-2} be the endnode of S_{i-1} not on S_i. As we have a symmetry between S_{i-1} and S_{i+1}, Property (e) implies that we may choose the numbering so that s is not adjacent to v_i nor to v_{i-2} and, by Claim 2, no intermediate node of P_{ts} is adjacent to v_i or v_{i-2}. Let s_{i-1} be the neighbor of s on S_{i-1} closest to v_{i-2} and let s_{i+1} be the neighbor of s on S_{i+1} closest to v_i. Moreover, let $\hat{t}$ be the neighbor of t on S_i closest to v_i. (Note that possibly $\hat{t} = v_i$.) Now the following three paths form a $3PC(v_i, s)$: $P_1 = v_i, (S_i)_{v_i\hat{t}}, \hat{t}, t, P_{ts}, s$; $P_2 = v_i, (S_{i+1})_{v_i s_{i+1}}, s_{i+1}, s$; $P_3 = v_i, v, v_{i-2}, (S_{i-2})_{v_{i-2} s_{i-1}}, s_{i-1}, s$. This completes the proof of Claim 3.

Let U be the set of unpainted nodes of H adjacent to at least one node of $V(P) \setminus \{s^*, s, t, t^*\}$.

Claim 4 *If s satisfies Property (a), let v_{i-1} and v_i be the endnodes of the sector S_i containing s^*. Then $U \subseteq \{v_{i-1}, v_i\}$.*

Proof of Claim 4: Assume not and let x be the node of $V(P) \setminus \{s^*, s, t, t^*\}$ adjacent to v_k, $k \neq i-1$, i such that P_{sx} is shortest.

Case 1 No intermediate node of P_{sx} is adjacent to an unpainted node of H.

Let P_1 and P_2 be the $v_k s^*$-subpaths of H and let $C_1 = v_k, P_1, s^*, s, P_{sx}, x, v_k$, $C_2 = v_k, P_2, s^*, s, P_{sx}, x, v_k$. Then either (C_1, v) or (C_2, v) is an odd wheel.

Case 2 Some intermediate node of P_{sx} is adjacent to an unpainted node of H.

By Lemma 5.13, only one unpainted node of H is adjacent to some intermediate node of P_{sx}, and by the choice of node x, we can assume w.l.o.g. this node to be v_i. Let y be the neighbor of v_i in P_{sx} such that P_{yx} is shortest, let P_1 be the $v_k v_i$-subpath of H not containing v_{i-1} and P_2 be the $v_k s^*$-subpath of H not containing v_i. Consider the cycles $C_1 = v_k, P_1, v_i, y, P_{yx}, x, v_k$ and $C_2 = s^*, s, P_{sx}, x, v_k, P_2, s^*$. An easy counting argument shows that either (C_1, v) or (C_2, v) is an odd wheel.

Claim 5 *If s satisfies Property (c), let v_{i-1} v_i and v_{j-1} v_j be the endnodes of sectors S_i and S_j containing the two neighbors $s^* = s_i$ and s_j of s in H. Then U is contained either in $\{v_{i-1},\ v_i\}$ or in $\{v_{j-1},\ v_j\}$. Furthermore if $U \neq \emptyset$, at least one node of U is adjacent to s_i or s_j.*

Proof of Claim 5: We first show $U \subset \{v_{i-1}, v_i, v_{j-1}, v_j\}$.

Assume not and let v_k, $k \neq i-1, i, j-1, j$ be the node of U adjacent to the node $x \in V(P) \setminus \{s^*, s, t, t^*\}$ such that the xs-subpath P_{xs} of P is shortest. Lemma 5.13 shows $|U| \leq 2$. Hence by symmetry, we assume w.l.o.g. that, among v_{i-1}, v_i, v_{j-1} and v_j, only node v_i can be adjacent to an intermediate node of P_{xs}. Now the following three paths induce a $3PC(v, s)$:

$P_1 = v, v_{i-1}, (S_i)_{v_{i-1}s_i}, s_i, s$; $P_2 = v, v_{j-1}, (S_j)_{v_{j-1}s_j}, s_j, s$; $P_3 = v, v_k, x, P_{xs}, s$.

This shows $U \subset \{v_{i-1}, v_i, v_{j-1}, v_j\}$.

Assume that nodes v_{i-1}, v_i, v_{j-1}, v_j, appear in this order when traversing H clockwise starting in v_{i-1}. We now show $U \neq \{v_i, v_j\}$.

Assume not and let x and y be nodes of $V(P) \setminus \{s^*, s, t, t^*\}$ adjacent to v_i, v_j respectively such that the xy-subpath P_{xy} of P is shortest and let P_1 be a $v_j v_i$-subpath of H. Then the wheel (C, v), induced by the cycle $C = v_j, P_1, v_i, x, P_{xy}, y, v_j$ is odd since S_i and S_j are sectors with the same color.

We now show $U \neq \{v_{i-1}, v_j\}$.

Assume w.l.o.g. that when traversing P starting from s we first encounter nodes that are adjacent to v_{i-1}. Let y be the first encountered neighbor of v_j. Then s_j is adjacent to v_j, else the following three paths induce a $3PC(v, s)$:

$P_1 = v, v_i, (S_i)_{v_i s_i}, s_i, s$; $P_2 = v, v_{j-1}, (S_j)_{v_{j-1}s_j}, s_j, s$; $P_3 = v, v_j, y, P_{ys}, s$.

A similar argument shows that s_i is adjacent to v_{i-1}. Now the nodes of P_{ys} together with the nodes in (H, v) induce a connected 6-hole with fan sides and 6-hole $s_i, s, s_j, v_j, v, v_{i-1}, s_i$.

By symmetry, the above two arguments show that $U \neq \{v_{i-1},\ v_{j-1}\}$ and $U \neq \{v_i,\ v_{j-1}\}$. Hence U is contained either in $\{v_{i-1},\ v_i\}$ or in $\{v_{j-1},\ v_j\}$.

Finally, if U is nonempty, let x be the node of $V(P) \setminus \{s^*, s, t, t^*\}$ adjacent to a node of U, such that the sx-subpath of P is shortest. Assume w.l.o.g. that v_i is adjacent to x. Then s_i and v_i are adjacent, else there exists a $3PC(v, s)$. This completes the proof of Claim 5.

Claim 6 *$U = \emptyset$.*

Proof of Claim 6: Lemma 5.13 shows $|U| \leq 2$. Since s and t satisfy Property (a) or (c) the above two claims, applied to both s and t rule out the possibility that $|U| = 2$. So $|U| = 1$. Assume $U = \{v_i\}$, let S_i and S_{i+1} be the sectors of (H, v) having v_{i-1}, v_i and v_i, v_{i+1} as endnodes, where S_i is green and consider the following cases:

Case 1 Both s and t satisfy Property (a).

Claim 4, applied to both s and t shows that s^* belongs to S_i and t^* belongs to S_{i+1}. Let P_1 be the t^*s^*-subpath of H not containing v_i and $C = t^*, P_1, s^*, P, t^*$. Then (C, v) is an odd wheel.

Case 2 Node s satisfies Property (a) and node t satisfies Property (c).

Claim 4 applied to s and Claim 5 applied to t shows that s^* belongs to S_i and t has neighbor $t^* = t_{i+1}$ in S_{i+1} which is adjacent to v_i. Let t_j be the second neighbor of t in H and let v_j be an endnode of the sector S_j containing t_j, distinct from v_{i-1}. If t_j is not adjacent to v_{i-1}, the following three paths induce a $3PC(v,t)$: $P_1 = v, v_{i+1}, (S_{i+1})_{v_{i+1}t_{i+1}}, t_{i+1}, t$; $P_2 = v, v_j, (S_j)_{v_j t_j}, t_j, t$; $P_3 = v, v_{i-1}, (S_i)_{v_{i-1}s^*}, s^*, s, P_{st}, t$.

If t_j is adjacent to v_{i-1} (i.e. $S_j = S_{i-1}$), then the nodes in P together with the ones in (H, v), induce a connected 6-hole with fan sides and 6-hole $v, v_i, t_{i+1}, t, t_j, v_{i-1}, v$.

Case 3 Both s and t satisfy Property (c).

Claim 5 applied to both s and t shows that one neighbor of s in H, say s^*, belongs to S_i and one neighbor of t in H, say t^*, belongs to S_{i+1} and both s^* and t^* are adjacent to v_i. Let S_j and S_k be the sectors containing the other neighbors s_j and t_k of s and t respectively. Let P_1 be the $t_k s_j$-subpath of H not containing v_i. Consider the cycle $C_1 = s_j, s, P_{st}, t, t_k, P_1, s_j$. Since s_j and t_k belong to sectors with distinct colors, P_1 contains an odd number of neighbors of v. In fact, P_1 contains a unique neighbor of v, say z_1, else (C_1, v) is an odd wheel. Consider now the cycle $C_2 = s, P_{st}, t, t^*, \ldots, v_{i+1}, v, v_{i-1}, \ldots, s^*, s$. Then P_{st} has an odd number of neighbors of v_i, else the wheel (C_2, v_i) is odd. In fact, P_{st} contains a unique neighbor of v_i, say z_2, else (C_1, v_i) is an odd wheel. Now the two z_1z_2-subpaths of C_1, together with z_1, v, v_i, z_2, induce a $3PC(z_1, z_2)$.

By symmetry, this completes the proof of Claim 6.

Claim 7 *The path P cannot exist.*

Proof of Claim 7: Assume P exists. Then Claim 6 shows that $U = \emptyset$.

Case 1 Both s and t satisfy Property (a).

Let P_1 be a t^*s^*-subpath of H containing more that one unpainted node and let $C = s^*, P, t^*, P_1, s^*$. Then since s^* and t^* belong to sectors with distinct colors, (C, v) is an odd wheel.

Case 2 Node s satisfies Property (a) and node t satisfies Property (c).

Let v_{i-1} and v_i be the endnodes of the sector S_i containing $s^*$and let t^* and t_k be the neighbors of t in H. Then t^* is adjacent to v_{i-1} and t_k is adjacent to v_i (or viceversa), else there is a $3PC(v,t)$. Note that $s^* \in V^c$, as otherwise there is a $3PC(s^*, t)$. Now the nodes in P together with the ones in (H, v), induce a connected 6-hole with a fan side and a triad side and 6-hole $v, v_{i-1}, t^*, t, t_k, v_i, v$.

Case 3 Both s and t satisfy Property (c).

Let S_i, S_j be the sectors containing the neighbors $s^* = s_i$ and s_j of s in H and S_k, S_l be the sectors containing the neighbors t_k and t_l of t in H. Then S_k is adjacent to both S_i and S_j, else each of these three sectors has one endnode that is not the endnode of any of the other two and this implies the existence of a $3PC(v, s)$. By symmetry, S_l is also adjacent to both S_i and S_j. So (H, v) is a wheel with four spokes. Furthermore t_k is adjacent to both endnodes of S_k, else there is a $3PC(v, t)$. By symmetry, s_i is adjacent to both endnodes of S_i. Now the common neighbor of t_k and s_i is the center of a wheel with three spokes.

By symmetry, this completes the proof of Claim 7 and of the theorem. $\square$

6 Parachutes

6.1 Introduction

In this section, we consider a wheel-free weakly balanced graph G that contains a parachute. We use the notation introduced in Section 1. Let $\Pi = Par(P_1, P_2, M, T)$ be a parachute of G. The node z is called *bottom node*, v_1 and v_2 are called *side nodes* and v is called *center node*. We assume w.l.o.g. that $v, z \in V^c$ and $v_1, v_2 \in V^r$. Let m be the neighbor of v in the path M. The nodes of $V(\Pi) \setminus \{v, v_1, v_2, m\}$ induce two connected components called the *top* of Π, induced by $V(T) \setminus \{v_1, v_2\}$, and the *bottom* of Π, induced by $V(P_1) \cup V(P_2) \cup V(M) \setminus \{v, v_1, v_2, m\}$.

Recall that, for a path $P = x_1, x_2, \ldots, x_{n-1}, x_n$ where $n \geq 3$, we denote by $\tilde{P}$ the $x_2 x_{n-1}$-subpath of P. With this notation, the top of Π is $\tilde{T}$ and the bottom of Π is induced by $V(\tilde{P}_1) \cup V(\tilde{P}_2) \cup (V(\tilde{M}) \setminus \{m\}) \cup \{z\}$.

When $|E(T)| = 2$, the parachute Π is said to have a *short top*; the top is *long* when $|E(T)| \geq 4$. Similarly, the parachute Π is said to have a *short middle* when $|E(M)| = 2$, and *long middle* otherwise. Finally, when $|E(P_1)| = 1$ or $|E(P_2)| = 1$, the parachute Π is said to have one *short side*; otherwise, we say that Π has *long sides*.

This section is organized as follows. In Subsection 6.2, we list all possible strongly adjacent nodes to Π. In Subsections 6.3 and 6.4, we list all possible direct connections from the bottom of Π to the top of Π, avoiding $N(v) \cup ((N(v_1) \cap N(v_2)) \setminus V(\tilde{T}))$. When no such path exists, the graph G has an extended star cutset disconnecting the top of Π from the bottom. When there is such a path, at least one of the following possibilities arises.

- The graph G contains no parachute with long sides. This case is treated in Subsection 6.5 where we prove the existence of an extended star cutset in G (Theorem 6.6).
- The graph G contains a stabilized parachute. This concept is defined in Subsection 6.6 and G is shown to have an extended star cutset (Theorem 6.10).
- The graph G contains a parachute with short middle path and long sides, but G contains no stabilized parachute and no connected squares. This case is treated in Subsection 6.8 where we prove the existence of an extended star cutset (Theorem 6.16).
- The graph G contains connected squares. This case is treated in Section 7.
- The graph G contains goggles. This case is treated in Section 8.

6.2 Strongly Adjacent Nodes

Theorem 6.1 *Let $\Pi = Par(P_1, P_2, M, T)$ be a parachute in a wheel-free weakly balanced graph G. Let $w \in V(G) \setminus V(\Pi)$ be a strongly adjacent node to Π. Then w satisfies one of the following properties:*
Node w has exactly two neighbors in Π and both belong to one of the paths P_1, P_2, M or T.
Node w is of one of the following types, see Figure 13.

- **Type a** *Node $w \in V^c$ is adjacent to the neighbors of z in $\tilde{P}_1$ and $\tilde{P}_2$ respectively and to no other node of Π.*

- **Type b** *Node* $w \in V^r$ *is adjacent to one node in* $\tilde{P}_1$*, to one node in* $\tilde{P}_2$ *and to no other node of* Π.
- **Type c** *Node* $w \in V^c$ *is adjacent to exactly two nodes of* Π*, one of which is the neighbor of* z *in* $\tilde{M}$ *and the other is the neighbor of* z *in* P_1 *or in* P_2.
- **Type d** *Node* $w \in V^r$ *is adjacent to one node in* $V(M) \setminus \{z\}$*, to one node in either* $\tilde{P}_1$ *or* $\tilde{P}_2$ *(but not both) and to no other node of* Π.
- **Type e** *Node* $w \in V^r$ *is adjacent to* v*, to one node in* $\tilde{T}$ *and to no other node of* Π.
- **Type f** *Node* $w \in V^c$ *is adjacent to one node in* P_1*, to one node in* P_2*, to one node of* $\tilde{M}$ *and to no other node of* Π.
- **Type g** *Node* $w \in V^c$ *is adjacent to* m*, to two nodes in* T *at least one of which is in* $\tilde{T}$*, and to no other node of* Π.
- **Type h** *Node* $w \in V^r$ *is adjacent to* v*, to one node in* $\tilde{T}$*, one node in* $V(M) \setminus \{v\}$ *and to no other node of* Π.
- **Type i** *Node* $w \in V^r$ *is adjacent to* v*, to one node in* $\tilde{T}$*, to one node in either* $\tilde{P}_1$ *or* $\tilde{P}_2$ *(but not both) and to no other node of* Π.

When Π *has a short side, say* P_2*, the following additional types of strongly adjacent nodes can occur.*

- **Type j** *Node* $w \in V^c$ *is adjacent to* v_2*, to one node in* $\tilde{M}$ *distinct from the neighbor of* z*, and to no other node of* Π.
- **Type k** *Node* $w \in V^c$ *is adjacent to one node in* $V(T) \setminus \{v_1\}$*, to one node in* $\tilde{P}_1$ *and to no other node of* Π.
- **Type l** *Node* $w \in V^r$ *is adjacent to the neighbors of* v_1 *in* $\tilde{T}$ *and* $\tilde{P}_1$ *respectively and to no other node of* Π.
- **Type m** *Node* $w \in V^r$ *is adjacent to two nodes of* $V(M) \setminus \{v\}$*, to the neighbor of* v_2 *on* $\tilde{T}$ *and to no other node of* Π.
- **Type n** *Node* $w \in V^c$ *is adjacent to* v_2*, to one node in* $\tilde{T}$*, to one node in* $V(\tilde{M}) \setminus \{m\}$ *and to no other node of* Π.

When Π *has a short top and long sides, one additional type of strongly adjacent node can occur.*

- **Type o** *Node* $w \in V^r$ *is adjacent to two nodes of* $V(M) \setminus \{v\}$*, to the unique node of* $V(\tilde{T})$ *and to no other node of* Π.

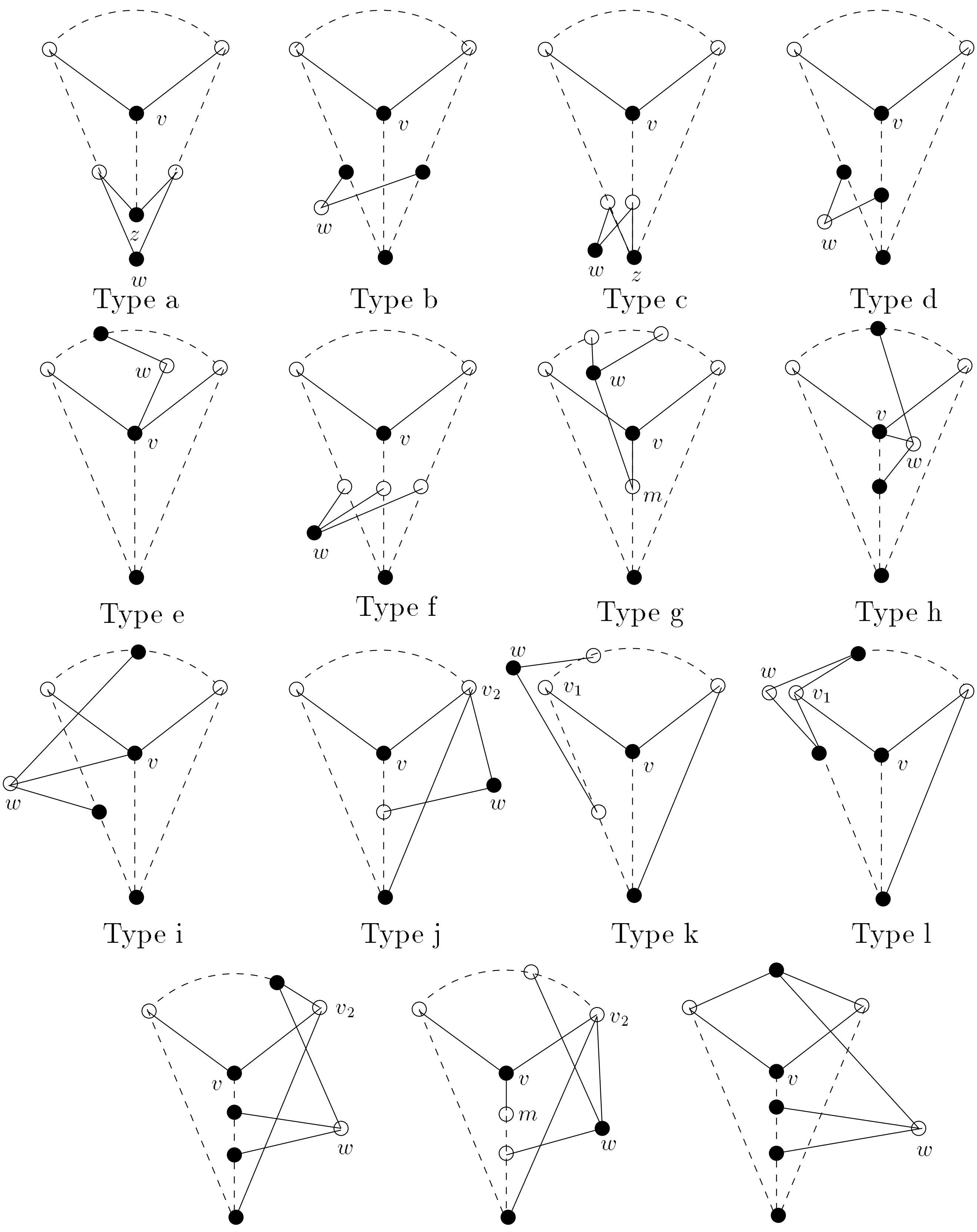


Figure 13: Strongly adjacent nodes

Proof: Assume that w is a strongly adjacent node to Π contradicting the theorem. Then no path P_1, P_2, M or T contains all the neighbors of w else there is a wheel.

Claim 1 *w has at most one neighbor in P_1, at most one neighbor in P_2, at most two in each of $P_1 \cup P_2 \cup T$, $P_1 \cup M$ and $P_2 \cup M$, and at most three neighbors in Π overall.*
Proof of Claim 1: If one of the first five assertions does not hold, w is the center of a wheel. In particular, w has at most four neighbors in Π overall. Suppose w has four neighbors in Π. Then two are in $\tilde{T}$ and two are in $\tilde{M}$. Now w violates Theorem 4.3 in the cycle with unique chord induced by $V(\Pi) \setminus V(\tilde{P}_1)$ or $V(\Pi) \setminus V(\tilde{P}_2)$.

Claim 2 $N(w) \not\subseteq V(\tilde{P}_1) \cup V(\tilde{P}_2) \cup \{z\}$.
Proof of Claim 2: If the claim is wrong then, by Claim 1, node w has exactly two neighbors in Π, one in $\tilde{P}_1$ and one in $\tilde{P}_2$. As w is not of Type a or b, it has a neighbor in $V^r \setminus N(z)$, say w_1. But then, there is a $3PC(w_1, z)$.

Claim 3 $N(w) \cap (V(M) \setminus \{z\}) \neq \emptyset$.
Proof of Claim 3: If the claim is wrong, w has exactly two neighbors in $V(P_1) \cup V(P_2) \cup V(T)$ by Claim 1. By Claims 1 and 2 and by symmetry, we may assume that w has exactly one neighbor in $V(P_1) \setminus \{v_1\}$, say w_1, and exactly one in $V(T) \setminus \{v_1\}$, say w_2. P_2 is short, as otherwise there is a $3PC(v_2, z)$. As w is not of Type k, $w \in V^r$. Moreover, $w_1 \in N(v_1)$, as otherwise there is a $3PC(w_1, v_1)$. As P_1 is long, $w_1 \neq z$. As w is not of Type l, $w_2 \notin N(v_1)$. But this implies the existence of a $3PC(w_2, v_1)$.

Let w_1 be the neighbor of w on M closest to v.

Claim 4 $N(w) \cap V(\tilde{T}) \neq \emptyset$.
Proof of Claim 4: Suppose the claim is wrong. By symmetry, we may assume that w has a neighbor in $V(P_2) \setminus \{z\}$, say w_2. By Claim 1, the only other possible neighbor of w lies on $V(P_1) \setminus \{z\}$. First, suppose that such a third neighbor exists. As w is not of Type f, $w \in V^r$. Moreover, $w_1 \neq v$ as otherwise there is a wheel with center v. But then there is a $3PC(w, v)$. So w has only two neighbors on Π. As w is not of Type d, $w \in V^c$. Moreover, $w_2 \in N(z)$, as otherwise there is a $3PC(w_2, z)$. As w is not of Type c or j, $w_1 \notin N(z)$ and $w_2 \neq v_2$. But then there is a $3PC(w_1, z)$.

Let w_2 be a neighbor of w on $\tilde{T}$.

Claim 5 *w has exactly three neighbors in Π.*
Proof of Claim 5: If not, by Claim 1, w_1 and w_2 are the only neighbors of w in Π. Then $w_1 \neq v$ since w is not of Type e. But now there is a $3PC(w_2, v)$ if $w \in V^c$, or $3PC(w_1, v_1)$ if $w \in V^r$.

Let w_3 be the third neighbor of w in Π.

Claim 6 *$w_1 \neq m$ and, if P_i is long, then $w_2 \in N(v_{3-i})$ and $w_3 \in V(M) \cup V(\tilde{P}_{3-i})$.*
Proof of Claim 6: W.l.o.g. assume P_1 is long. Then $\Pi \setminus \tilde{P}_1$ is a cycle with a unique chord to which w is strongly adjacent. First, observe that w is not of Type 1[4.3] with respect to

$\Pi \setminus \tilde{P}_1$, because if it were, then $w_1 = v$ and $w_3 \in V(\tilde{P}_1)$, so w would be of Type i with respect to Π. Suppose w is of Type 2[4.3]. Since w is not of Type h or i, $w_1 \neq v$. Therefore $w_3 = v_2$ (as $w_3 \neq v$, by the choice of w_1). $w_1 \neq m$ since w is not of Type g, and P_2 is long since w is not of Type h or f. Now there is a $3PC(z, v_2)$. So w is of Type 3[4.3]. As w is not of Type g and $w_1 \neq z$, it follows that $w_3 \notin V(T)$. Since $w_2 \neq v_1$, we have that $w_2 \in N(v_2)$ and $w_3 \in M \cup \tilde{P}_2$. As w is not of Type h or i, $w_1 \neq m$.

We may assume that P_1 is long. Hence, by Claim 6, so is P_2, as otherwise w would be of Type m. Now Claim 6 again implies that T is short and $w_3 \in M$. But then w is of Type o, which yields a contradiction. □

6.3 Parachute Modifications

Let $\Pi = Par(P_1, P_2, M, T)$ be a parachute with center node $v \in V^c$ and side nodes v_1, v_2. Let $S(\Pi) = N(v) \cup ((N(v_1) \cap N(v_2)) \setminus V(\tilde{T}))$. In this subsection and the next, we enumerate all possible direct connections from the bottom of Π to the top of Π avoiding $S(\Pi)$, in two cases:

Π has long sides,

Π has a short side and G contains no parachute with long sides.

Let $Q = x_1, \dots, x_n$ denote a direct connection from bottom to top avoiding $S(\Pi)$, where x_1 is adjacent to a node in the bottom of Π and x_n is adjacent to a node in the top of Π. It follows from the definition of a direct connection that no node of $\tilde{Q}$ is adjacent to a node of $V(\Pi) \setminus \{v_1, v_2, m\}$. Furthermore, since Q avoids $S(\Pi)$, a node of $\tilde{Q}$ is adjacent to at most one of v_1, v_2. To reduce the number of possible path types that need to be enumerated in the main theorem of this subsection (Theorem 6.4), we introduce the concept of *parachute modification.*

Definition 6.2 *Assume $y \in V(G) \setminus V(\Pi)$ has exactly two neighbors in Π, both are in T and at least one is in $\tilde{T}$. A* parachute modification at the top *consists of replacing Π by the unique parachute Π' that is induced by a subset of $V(\Pi) \cup \{y\}$ and is distinct from Π.*

Assume $y \in V(G) \setminus V(\Pi)$ has exactly two neighbors in Π that are both in P_1, or both in P_2 or both in $V(M) \setminus \{v\}$. A parachute modification at the bottom *consists of replacing Π by the unique parachute Π' that is induced by a subset of $V(\Pi) \cup \{y\}$ and is distinct from Π.*

Remark 6.3 *(i) If Π' is obtained from Π by a parachute modification at the top, then both Π and Π' have long top. If Π' is obtained from Π by parachute modification at the bottom, then Π' has long sides if and only if Π has long sides.*

(ii) Let Π' be obtained from Π by a parachute modification using a node of a direct connection $Q = x_1, \dots, x_n$ from bottom to top avoiding $S(\Pi)$. Then $n > 1$ and if the modification is at the top, it involves node x_n. If the modification is at the bottom, it involves node x_1. Furthermore $S(\Pi) = S(\Pi')$.

Theorem 6.4 *Let G be a wheel-free weakly balanced graph. Let $\Pi = Par(P_1, P_2, M, T)$ be a parachute in G and $Q = x_1, \dots, x_n$ a direct connection from bottom to top avoiding $S(\Pi)$ such that no parachute modification exists relative to a node of Q.*

(i) If Π has long top and long sides, then $n \geq 2$ and, up to symmetry between P_1 and P_2, Q is of one of the following types, see Figure 14.

- **Type a** *Node x_1 is a strongly adjacent node of Type f[6.1], adjacent to v_1, m and one node of $\tilde{P}_2$. Node x_n is not strongly adjacent and its unique neighbor is adjacent to v_1 in $\tilde{T}$. Exactly one node of $\tilde{Q}$ is adjacent to m and none is adjacent to v_1, v_2.*

- **Type b** *Node x_1 is not strongly adjacent and its unique neighbor is the node of $\tilde{P}_2$ adjacent to v_2. Node x_n is of Type g[6.1] adjacent to v_2, m and one node of $\tilde{T}$. No node of $\tilde{Q}$ has a neighbor in Π. Furthermore, Π has a short middle path.*

- **Type c** *Node $x_1 \in V^c$ is not strongly adjacent and its unique neighbor is in $\tilde{P}_2$. Node x_n is a strongly adjacent node of Type g[6.1], adjacent to v_2, m and one node of $\tilde{T}$. Exactly one node of $\tilde{Q}$ is adjacent to v_2 and none is adjacent to v_1, m. Furthermore, Π has a short middle path.*

- **Type d** *Node x_1 is a strongly adjacent node of Type c[6.1], with neighbors in $\tilde{P}_1$ and $\tilde{M}$. Node x_n is a strongly adjacent node of Type g[6.1], adjacent to v_2, m and one node of $\tilde{T}$. Exactly one node of $\tilde{Q}$ is adjacent to v_2 and none is adjacent to v_1, m. Furthermore, Π has a short middle path.*

- **Type e** *Node x_1 is a strongly adjacent node of Type f[6.1], adjacent to v_2, m and one node of $\tilde{P}_1$. Node x_n is a strongly adjacent node of Type g[6.1], adjacent to v_2, m and one node of $\tilde{T}$. No node of $\tilde{Q}$ has a neighbor in Π.*

(ii) If Π has short top and long sides, then either $n = 1$ and x_1 is a strongly adjacent node of Type o[6.1], or $n \geq 2$ and, up to symmetry between P_1 and P_2, Q is a direct connection of Type a above or is of the types described below, see Figures 14 and 15.

- **Type f** *Node $x_1 \in V^r$ is not strongly adjacent and its unique neighbor is adjacent to m in $\tilde{M}$. Node x_n is not strongly adjacent. Exactly one node of $\tilde{Q}$ is adjacent to m and none is adjacent to v_1 or v_2.*

- **Type g** *Node $x_1 \in V^c$ is not strongly adjacent and its unique neighbor is in $V(\tilde{M}) \setminus \{m\}$. Node x_n is not strongly adjacent. Exactly two nodes of $\tilde{Q}$ are adjacent to m and none is adjacent to v_1 or v_2.*

- **Type h** *Node x_1 is a strongly adjacent node of Type f[6.1] adjacent to m, one node in $\tilde{P}_1$ and one node in $\tilde{P}_2$. Node x_n is not strongly adjacent. Exactly one node of $\tilde{Q}$ is adjacent to m and none is adjacent to v_1 or v_2.*

- **Type i** *Node x_1 is a strongly adjacent node of Type a[6.1]. Node x_n is not strongly adjacent. Exactly two nodes of $\tilde{Q}$ are adjacent to m and none is adjacent to v_1 or v_2.*

- **Type j** *Node $x_1 \in V^c$ is not strongly adjacent and its unique neighbor belongs to $V(\tilde{M}) \setminus \{m\}$. Node x_n is not strongly adjacent. No node of $\tilde{Q}$ has a neighbor in Π.*

- **Type k** *Node x_1 is a strongly adjacent node of Type a[6.1]. Node x_n is not strongly adjacent. No node of $\tilde{Q}$ has a neighbor in Π.*

(iii) If Π has a short side, say P_2, and G contains no parachute with long sides, then Π has short top and either $n = 1$ and x_1 is a strongly adjacent node of Type l[6.1], or $n \geq 2$ and Q is of Type l described below, see Figure 15.

- **Type l** *Nodes $x_1, x_n \in V^r$ are not strongly adjacent and their respective neighbors $b \in V(\tilde{P}_1)$ and $t \in V(\tilde{T})$ are adjacent to v_1. No node of $\tilde{Q}$ has a neighbor in Π.*

Proof: Throughout this proof, we assume that either Π has long sides, or Π has a short side and G contains no parachute with long sides.

First, we consider the case $n = 1$, i.e. Q consists of a single node x_1 that is strongly adjacent to Π. Then it follows from Theorem 6.1 that Π has a short side and x_1 is of Type k, l, m, n[6.1] or that Π has a short top and long sides and x_1 is of Type o[6.1]. When x_1 is of Type k, m or n[6.1], Π has a short side and the graph induced by $V(\Pi) \cup \{x_1\}$ contains a parachute with long sides, a contradiction. (For example, if x_1 is of Type n[6.1], assume w.l.o.g. that x_1 is adjacent to v_2. Then P_2 is short, x_1 is not adjacent to v_1 and the parachute with long sides has center node v_2, bottom node v_1 and side nodes x_1 and z). When x_1 is of Type l[6.1], Π has a short side and the graph induced by $V(\Pi) \cup \{x_1\}$ contains a parachute with long sides unless Π has short top. This proves the theorem when $n = 1$.

Now consider the case $n \geq 2$. By Theorem 6.1, either x_n is not strongly adjacent to Π or it is a strongly adjacent node of Type g[6.1]. Similarly, either x_1 is not strongly adjacent to Π or it is a strongly adjacent node of Type a, b, c, d, f, j or k[6.1]. When x_1 is of Type j[6.1], Π has a short side, say P_2, and $\Pi \cup \{x_1\}$ contains a parachute with long sides with center node v_2, side nodes v, z and top path induced by $V(P_1) \cup \{v\}$, since the neighbor of x_1 in $\tilde{M}$ is distinct from m (as Q connects bottom to top) and from the neighbor of z (by the definition of Type j[6.1]). So the case where x_1 is of Type j[6.1] does not occur. When x_1 is of Type k[6.1], Π has a short side, say P_2, and x_1 is adjacent to v_2 and the neighbor of z in $\tilde{P}_1$ (since, otherwise, there is a parachute with long sides).

We will divide the proof into two parts, depending on whether x_n is of Type g[6.1] or is not strongly adjacent to Π. Then, in each of the two parts, the proof will be broken down further based on the adjacencies between the intermediate nodes of Q and $\{v_1, v_2, m\}$. Finally, subcases will occur depending on the node type of x_1. The two following claims reduce the number of cases that have to be considered.

We say that node $x_i \in V(\tilde{Q})$ adjacent to m and node $x_j \in V(\tilde{Q})$ adjacent to v_1 or v_2, say v_2, are *consecutive* in Q if $N(v_1) \cap V(Q_{x_i x_j}) = \emptyset$, $N(v_2) \cap V(Q_{x_i x_j}) = \{x_j\}$ and $N(m) \cap V(Q_{x_i x_j}) = \{x_i\}$. We allow $x_i = x_j$, and $Q_{x_i x_j} = x_i$ in this case.

Claim 1 *If $x_i, x_j \in V(\tilde{Q})$ are consecutive in Q, where x_i is adjacent to m and x_j is adjacent to v_2, then P_2 is short.*

Proof of Claim 1: $V(Q_{x_i x_j}) \cup V(T) \cup V(P_1) \cup V(M)$ induces an odd wheel with center v unless P_2 is short. This proves Claim 1.

Claim 2 *At most two of the nodes v_1, v_2, m are adjacent to a node of $\tilde{Q}$.*

Proof of Claim 2: For any pair of nodes $x_i, x_j \in V(\tilde{Q})$ such that node x_i is adjacent to v_1, node x_j is adjacent to v_2 and the subpath $Q_{x_i x_j}$ of Q connecting them contains no other node adjacent to v_1 or v_2, the following property holds: $x_i \neq x_j$, no intermediate node of $Q_{x_i x_j}$ is adjacent to m, and at most one of x_i, x_j is adjacent to m. (The first statement follows from

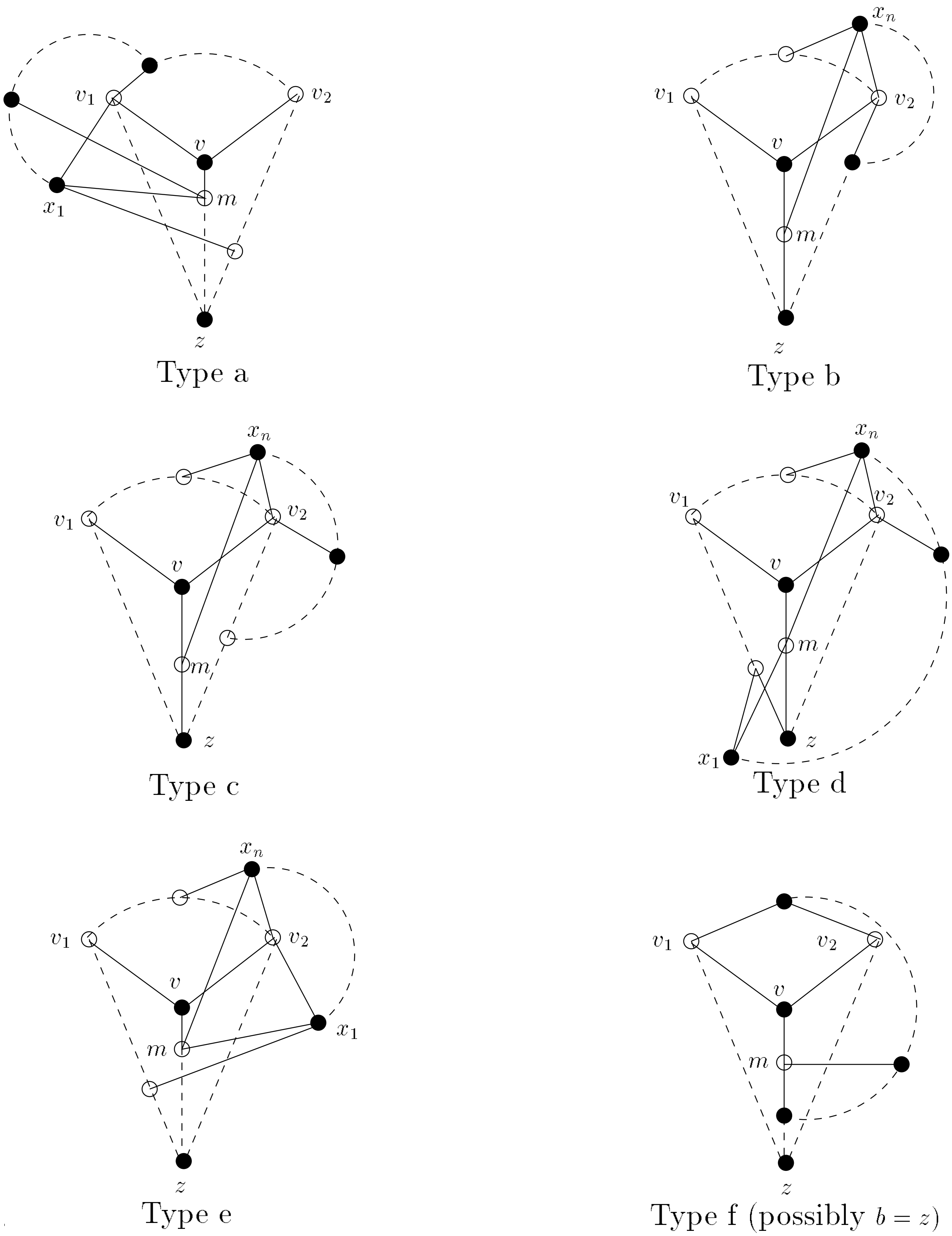
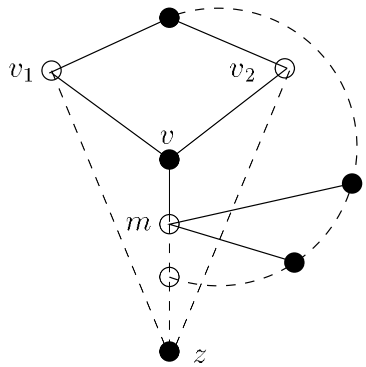
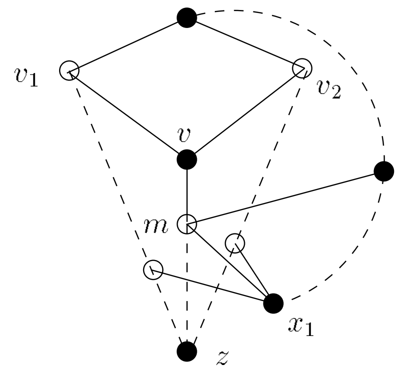


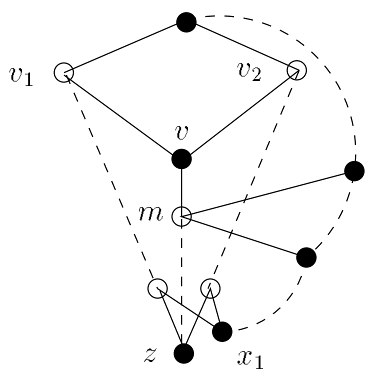
Figure 14: Direct connections from bottom to top



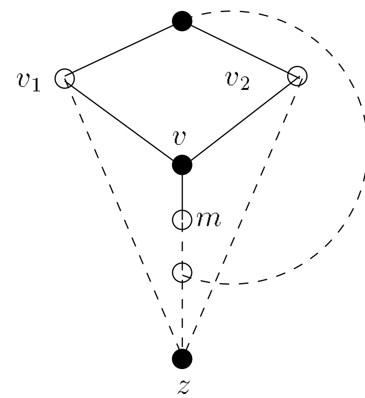
Type g



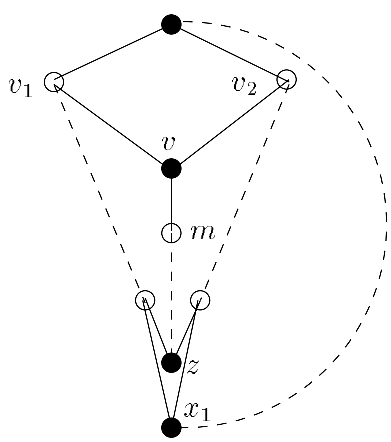
Type h



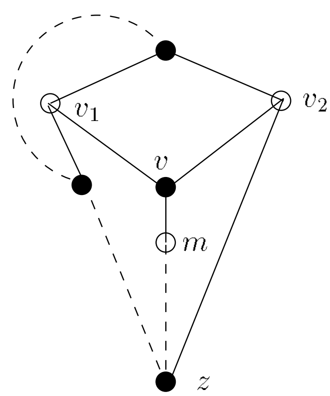
Type i



Type j



Type k



Type l

Figure 15: Direct connections from bottom to top (continued)

the fact that Q contains no node of $N(v_1) \cap N(v_2)$, the second and third follow from Claim 1 and the fact that P_1 and P_2 cannot both be short).

Therefore, if each of the nodes v_1, v_2, m has a neighbor in $\tilde{Q}$, then by symmetry, we can assume that $\tilde{Q}$ contains a node x_k adjacent to v_1 but not m and a node x_ℓ adjacent to m but not v_1 with the property that $Q_{x_k x_\ell}$ contains at least one node (possibly x_ℓ itself) adjacent to v_2 but no intermediate node of $Q_{x_k x_\ell}$ is adjacent to m or v_1. Now P_2 is short, by Claim 1. In fact, $Q_{x_k x_\ell}$ contains at exactly one node, say x^*, adjacent to v_2, else there is a wheel with center v_2. But then $V(Q_{x_k x_\ell}) \cup V(\Pi) \setminus \{v\}$ induces a parachute with long sides: the center node is v_2, the side nodes are x^* and z and the bottom node is v_1. This proves Claim 2.

Part 1 Node x_n is of Type g[6.1]

By symmetry, we may assume x_n is not adjacent to v_1.

Claim 3 *If P is any chordless $x_n v_1$-path in $(V(Q) \cup V(P_1) \cup V(P_2) \cup V(\tilde{M})) \setminus \{m, v_2\}$, then v_2 and m both have a neighbor in $\tilde{P}$. Moreover, no node of $\tilde{Q}$ is adjacent to v_1.*

Proof of Claim 3: Let T_1 and T_2 be the chordless paths from x_n to v_1 and from x_n to v_2 that only use nodes of $V(T) \cup \{x_n\}$. Since the paths P, T_1 and x_n, T_2, v_2, v, v_1 do not form a $3PC(x_n, v_1)$, node v_2 is adjacent to at least one node of $\tilde{P}$. Since the paths P, T_1 and x_n, m, v, v_1 do not form a $3PC(x_n, v_1)$, node m is adjacent to at least one node of $\tilde{P}$. Finally, no node of $\tilde{Q}$ is adjacent to v_1. For otherwise, by construction of P, we have that $\tilde{P}$ is a subpath of $\tilde{Q}$ and by the above argument, v_1, v_2 and m all have neighbors in $\tilde{Q}$, a contradiction to Claim 2. This proves Claim 3.

Claim 4 *Node x_n is adjacent to v_2.*

Proof of Claim 4: Suppose that neither v_1 nor v_2 are neighbors of x_n. Then, by Claim 3 and symmetry, $\tilde{Q}$ contains no neighbor of v_1 and no neighbor of v_2. Also by Claim 3 and symmetry, we see that x_1 is not adjacent to v_1 or v_2. So Q contains no neighbor of v_1 or v_2. Again by Claim 3 and symmetry, one now deduces that both P_1 and P_2 are short, a contradiction. This proves Claim 4.

Case 1 No intermediate node of Q is adjacent to v_1, v_2 or m.

Assume first that x_1 is not strongly adjacent to Π and let b be the node of Π adjacent to x_1. By Claim 3, it follows that either b is adjacent to v_2 and m is adjacent to z, or b is adjacent to m and v_2 is adjacent to z. When the first possibility occurs, the path Q is of Type b. When the second possibility occurs, P_2 is short and there is a parachute with long sides and center node m, bottom node v_1 and side nodes x_n, b.

Assume now that x_1 is strongly adjacent to Π. By Claim 3, it follows that x_1 is not of Type a, b, c, d or k[6.1]. Assume x_1 is of Type f[6.1]. Then x_1 is adjacent to v_2 and to m, by Claim 3. Hence the path Q is of Type e.

Case 2 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \{v_2\}$.

Since there is no wheel, $\tilde{Q}$ contains exactly one neighbor of v_2.

Assume first that x_1 is not strongly adjacent to Π and let b be the node of Π adjacent to x_1. By Claim 3, it follows that either b belongs to P_2 and m is adjacent to z, or b is the neighbor of m in $\tilde{M}$. If b belongs to P_2 and is adjacent to v_2, then there is a wheel with center v_2. If b belongs to P_2 and is not adjacent to v_2, then there is a $3PC(b, v_2)$ when $b \in V^c$ or Q is of Type c when $b \in V^r$. If b is the neighbor of m in $\tilde{M}$, then either there is a $3PC(z, v_2)$ when P_2 is long, or there is a wheel with center v_2 when P_2 is short.

Assume now that x_1 is strongly adjacent to Π. By Claim 3, it follows that x_1 is not of Type a, b, d or k[6.1]. If x_1 is of Type c[6.1], the middle path M must be short, i.e. x_1 is adjacent to m. If x_1 is adjacent to m and to a node b in $\tilde{P}_2$, there is a $3PC(x_1, v_2)$. If x_1 is adjacent to m and to a node b in $\tilde{P}_1$, the path Q is of Type d. If x_1 is of Type f[6.1] and is adjacent to v_2, there is a wheel with center v_2. If x_1 is of Type f[6.1] and is not adjacent to v_2, there is a $3PC(x_1, v_2)$.

Case 3 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \{m\}$.

Since there is no wheel, $\tilde{Q}$ contains exactly one neighbor of m.

Assume first that x_1 is not strongly adjacent to Π and let b be the neighbor of x_1 in Π. By Claim 3, we have that b is adjacent to v_2. If b is not adjacent to m, there is a $3PC(m, b)$; else there is a wheel with center m.

Assume now that x_1 is strongly adjacent to Π. By Claim 3, it follows that x_1 is not of Type a, b or d[6.1]. If x_1 is of Type c[6.1], then it is adjacent to v_2. So, if x_1 is of Type c or f[6.1], there is an odd wheel with center m (if x_1 and m are adjacent), or there is a $3PC(m, x_1)$ (if x_1 and m are not adjacent). If x_1 is of Type k[6.1], there exists a parachute with long sides with center m and bottom node v_1.

Case 4 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \{v_2, m\}$.

$\tilde{Q}$ has exactly one neighbor of m (or else there is a wheel with center m). By Claim 1, P_2 is short.

Assume first that x_1 is not strongly adjacent to Π and let b be the neighbor of x_1 in Π. If $b \in V(M) \setminus \{v, m\}$, there is a wheel with center v_2. So b belongs to $\tilde{P}_1$ and there exists a parachute with long sides and center node m.

Assume now that x_1 is strongly adjacent to Π. If x_1 is of Type c[6.1], with neighbors in $\tilde{P}_1$ and $\tilde{M}$, then there is a wheel with center v. If x_1 is of Type c[6.1] with neighbors v_2 and in $\tilde{M}$, or if x_1 is of Type f or k[6.1], then there is a wheel with center v_2. If x_1 is of Type d[6.1], with neighbors in $\tilde{P}_1$ and $\tilde{M}$, then there is a $3PC(x_1, z)$.

Part 2 Node x_n has a unique neighbor, say t in Π.

Case 1 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \emptyset$.

Case 1.1 Node x_1 has a unique neighbor, say b in Π.

Assume first that b is in $\tilde{P}_1$. Then z is adjacent to v_2, else there is a $3PC(z, v_2)$. The nodes b and t belong to the same side of the bipartition, else there is a $3PC(b, t)$. If $b, t \in V^c$, then they are both adjacent to v_1, else there is a $3PC(b, v_1)$ or a $3PC(t, v_1)$. Furthermore T is short, else there exists a parachute with long sides. Hence Q is of Type l. If $b, t \in V^r$, then there is a parachute with long sides.

Assume now that $b \in V(M) \setminus \{v, m\}$. Then $b \in V^r$, else there is a $3PC(b, v_1)$ or $3PC(b, v_2)$.

If the top path of Π is long, assume w.l.o.g. that t is not adjacent to v_1. Then P_2 is short, else there is a $3PC(z, v_2)$. Node t is adjacent to v_2, else there is a $3PC(z, v_1)$. This yields a parachute with long sides induced by $V(Q) \cup V(\Pi) \setminus V(\tilde{P}_1)$.

If the top path of Π is short, then the path Q is of Type j when the side paths are long and, when P_2 is short, there is a parachute with long sides induced by $V(Q) \cup V(\Pi) \setminus V(\tilde{P}_1)$.

Case 1.2 Node x_1 is strongly adjacent to Π.

Assume first that x_1 is of Type a[6.1]. If the top path T is long, there exists a $3PC(x_1, v_1)$ or a $3PC(x_1, v_2)$. So T is short and Q is a path of Type k.

Assume that x_1 is of Type b[6.1]. Let b_1, b_2 be the neighbors of x_1 in $\tilde{P}_1$ and $\tilde{P}_2$ respectively. If b_1 is not adjacent to v_1 or b_2 is not adjacent to v_2, there exists a $3PC(z, x_1)$. If b_1 is adjacent to v_1 and b_2 is adjacent to v_2 then $t \in V^r$, else there is a $3PC(t, x_1)$. This yields a connected 6-hole, contradicting the assumption that G is weakly balanced.

Assume x_1 is of Type c[6.1], say with neighbors in $\tilde{M}$ and P_2. If P_2 is long, then x_1 is not adjacent to v_2 and there is a $3PC(x_1, v_2)$. If P_2 is short, then x_1 is adjacent to v_2. If t is adjacent to v_2, then v_2 is the center of a wheel and if t and v_2 are nonadjacent then $t \in V^r$, else there is a $3PC(t, v_2)$. Now there is a parachute with long sides, center node v_2, bottom node t.

If x_1 is of Type d[6.1], there is a $3PC(x_1, z)$.

If x_1 is of Type f[6.1]. Then x_1 is not adjacent to both v_1, v_2. If x_1 is not adjacent to v_1, there is a $3PC(x_1, v_1)$, and if x_1 is not adjacent to v_2, there is a $3PC(x_1, v_2)$.

Finally, if x_1 is of Type k[6.1], there is a wheel with center v_2.

Case 2 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \{m\}$.

Case 2.1 Node x_1 has a unique neighbor, say b in Π.

If b is in $\tilde{P}_1$, then $b \in V^r$, else there is a $3PC(m, b)$ and $t \in V^c$, else there is a $3PC(v, t)$. This implies the existence of a $3PC(b, t)$.

If $b \in V(M) \setminus \{v, m\}$, then $V(Q) \cup \{b\}$ contains at most two nodes adjacent to m, otherwise there is a wheel. If it contains only one neighbor of m, say x_i, there is a $3PC(x_i, v_1)$. So, $V(Q) \cup \{b\}$ contains exactly two neighbors of m. If one side of Π is short, there exists a parachute with long sides with center m and bottom v_1. So Π has long sides. Now Π has short top, else there is a wheel with center m. If $b \in V^r$, Q is of Type g. If $b \in V^c$ and b is not adjacent to m, there is a $3PC(b, m)$. If b is adjacent to m, then Q is of Type f.

Case 2.2 Node x_1 is strongly adjacent to Π.

Assume first that x_1 is of Type a [6.1]. Then Q has at most two nodes adjacent to m, otherwise there is a wheel. If Q has only one neighbor of m, say x_i, there is a

$3PC(x_i, v_1)$. So, Q has exactly two neighbors of m. Now, if Π has long top, there is a wheel with center m. If Π has short top, Q is of Type i.

If x_1 is of Type b or d[6.1], there is a $3PC(x_1, z)$.

If x_1 is of Type c [6.1], then x_1 is adjacent to m, else there is a $3PC(x_1, m)$. But now there is a wheel with center m, since x_1 is not adjacent to v_1 or v_2.

If x_1 is of Type f [6.1], then x_1 is adjacent to m, else there is a $3PC(x_1, m)$. Hence $\tilde{Q}$ contains exactly one neighbor of m, as otherwise there is a wheel with center m. Let n_1 and n_2 be the neighbors of x_1 in P_1 and P_2 respectively. Assume w.l.o.g. that $n_2 \neq v_2$. If $n_1 \neq v_1$, then either T is long and there is a wheel with center m, or T is short and Q is of Type h. If $n_1 = v_1$, then the neighbor t of x_n is adjacent to v_1, otherwise there is a wheel with center m. Thus the path Q is of Type a.

Finally, assume that x_1 is of Type k[6.1]. Then Q has at most two nodes adjacent to m, otherwise there is a wheel. If Q has only one neighbor of m, say x_i, there is a $3PC(x_i, v_1)$. So, Q has exactly two neighbors of m. But now, there is a parachute with long sides, middle node m and bottom node v_1.

Case 3 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \{v_1\}$ and v_1 is adjacent to t.

Case 3.1 Node x_1 has a unique neighbor, say b, in Π.

If $b \in V(P_1) \setminus \{v_1\}$, then v_1 has exactly one neighbor in $\tilde{Q}$ and v_1 is not adjacent to b, else there is a wheel with center v_1. Now $b \in V^r$, else there is a $3PC(b, v_1)$. P_2 is short, else there is a $3PC(v_2, z)$. Now the graph induced by $V(\Pi) \cup V(Q)$ contains a parachute with long sides, center v_1 and bottom b.

If $b \in V(\tilde{P}_2) \cup V(\tilde{M}) \setminus \{m\}$, then there is a wheel with center v_1 when b is not adjacent to v_2, and there is a $3PC(b, v_1)$ otherwise.

Case 3.2 Node x_1 is strongly adjacent to Π.

If x_1 is of Type a[6.1], there is a wheel with center v_1.

If x_1 is of Type b or d[6.1], then there is a wheel with center v_1.

Assume x_1 is of Type c[6.1]. If x_1 is not adjacent to v_2, then there is a wheel with center v_1. If x_1 is adjacent to v_2, then x_1 is not adjacent to m and P_2 is short. Hence there is a parachute with long sides, center node v_2, bottom node v_1.

Assume that x_1 is of Type f [6.1]. Then, if x_1 is adjacent to v_1, there is a wheel with center v_1, and if x_1 is not adjacent to v_1, there is a $3PC(x_1, v_1)$.

Finally, if x_1 is of Type k[6.1], there is a wheel with center v_2.

Case 4 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \{v_1\}$ and v_1 is not adjacent to t.

Then $t \in V^r$, else there is a $3PC(t, v_1)$. Consider the parachute Π' obtained from Π by replacing the tv_1-subpath of T by the path $x_i, \ldots, x_n, t$, where x_i is the node of $\tilde{Q}$ of highest index adjacent to v_1. As $Q_{x_1 x_{i-1}}$ is shorter than Q, we may assume by induction that $Q_{x_1 x_{i-1}}$ and Π' satisfy Theorem 6.4. Since Π' has long top and $Q_{x_1 x_{i-1}}$ contains no neighbor of m, we get a contradiction. So Case 4 cannot occur.

Case 5 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \{v_2, m\}$.

As a consequence of Claim 1, P_2 is short. Let $x_j \in V(\tilde{Q})$ be the neighbor of v_2 closest to t in Q and $x_i \in V(\tilde{Q})$ the neighbor of m closest to t. Note that $i = j$ is possible. Suppose that $j > i$. If $t \in V^c$, there is a $3PC(v_2, t)$ or a wheel with center v_2. So $t \in V^r$. If v_2 has more than one neighbor in $Q_{x_i x_n}$, there is a wheel with center v_2. If x_j is the unique neighbor of v_2 in $Q_{x_i x_n}$, there is parachute with long sides, center node v_2, bottom node t and side nodes v, x_j.

So $j \leq i$. If there is no other neighbor of m on the tx_j-subpath of Q, then there is a $3PC(v_2, x_i)$ and if there are more than two neighbors, then there is a wheel with center m. So there are exactly two neighbors of m on the tx_j-subpath of Q, say x_i and x_k. Now there is a parachute with long sides, center node m and bottom node v_2.

Case 6 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \{v_1, v_2\}$.

Let x_k be the node of smallest index in $\tilde{Q}$ adjacent to v_h for $h = 1$ or 2 such that at least one of the nodes $x_2, \ldots, x_{k-1}$ is adjacent to v_{3-h}, and let x_l be such a node with highest index, $l < k - 1$. W.l.o.g. assume x_k is adjacent to v_2. Let Π' be parachute obtained from Π by replacing T by $v_1, x_l, Q_{x_l x_k}, x_k, v_2$. Let $Q' = Q_{x_1 x_{l-1}}$. Then Q' is a direct connection from bottom to top of Π' avoiding $S(\Pi')$. By induction, as Q' is shorter than Q, we can assume that Π' and Q' satisfy Theorem 6.4. However, since Π' has long top and m has no neighbor on Q', we get a contradiction. So Case 6 cannot occur.

$\square$

6.4 Connections from Bottom to Top

In this subsection, we continue the study of direct connections Q from bottom to top of a parachute. These connections were considered in Theorem 6.4 under the assumption that parachute modifications relative to Q had been performed. Here we describe the possible direct connections before parachute modifications are performed.

Theorem 6.5 *Let G be a wheel-free weakly balanced graph. Let $\Pi = Par(P_1, P_2, M, T)$ be a parachute and let $Q = x_1, \ldots, x_n$ be a direct connection from bottom to top avoiding $S(\Pi)$.*

(i) If Π has long top and long sides, then $n \geq 2$ and, up to symmetry between P_1 and P_2, Q is of Type a, b, c, d or e[6.4] or of one of the following types, see Figure 16.

- **Type a1** *Node x_1 is a strongly adjacent node of Type f[6.1], adjacent to v_1, m and one node in $\tilde{P}_2$. Node x_n is strongly adjacent to Π, adjacent to v_1 and to one node in $\tilde{T}$. Exactly one node of $\tilde{Q}$ is adjacent to m and none is adjacent to v_1, v_2.*

- **Type b1** *Node x_1 is strongly adjacent to Π, adjacent to v_2 and to one node in $\tilde{P}_2$. Node x_n is a strongly adjacent node of Type g[6.1], adjacent to v_2, m and one node in $\tilde{T}$. No node of $\tilde{Q}$ has a neighbor in Π. Furthermore, Π has a short middle path.*

- **Type b2** *Node $x_1 \in V^r$ is strongly adjacent to Π, with two neighbors in $V(P_2) \setminus \{v_2\}$. Node x_2 is adjacent to v_2 and to no other node of Π. Node x_n is a strongly adjacent* *node of Type g[6.1], adjacent to v_2, m and one node in $\tilde{T}$. No node of $Q_{x_3x_{n-1}}$ has a neighbor in Π. Furthermore, Π has a short middle path.*

- **Type c1** *Node $x_1 \in V^r$ is strongly adjacent to Π, adjacent to two nodes in P_2. Node x_n is a strongly adjacent node of Type g[6.1], adjacent to v_2, m and one node in $\tilde{T}$. Exactly one node of $\tilde{Q}$ is adjacent to v_2 and none is adjacent to v_1, m. Furthermore, Π has a short middle path.*

(ii) If Π has short top and long sides, then either $n = 1$ and the only node of Q is of Type o[6.1], or $n \geq 2$ and, up to symmetry between P_1 and P_2, Q is of Types a, f, g, h, i, j or k[6.4] or of one of the following types. See Figure 16.

- **Type f1** *Node $x_1 \in V^c$ is strongly adjacent to Π, adjacent to m and a node in $\tilde{M}$. Node x_n is not strongly adjacent to Π. Exactly one node of $\tilde{Q}$ is adjacent to m and none is adjacent to v_1 or v_2.*

- **Type g1** *Node $x_1 \in V^r$ is strongly adjacent to Π and its two neighbors both belong to $V(M) \setminus \{v, m\}$. Node x_n is not strongly adjacent to Π. Exactly two nodes of $\tilde{Q}$ are adjacent to m and none is adjacent to v_1 or v_2.*

- **Type j1** *Node $x_1 \in V^r$ is strongly adjacent to Π and its two neighbors both belong to $V(M) \setminus \{v, m\}$. Node x_n is not strongly adjacent to Π. No node of $\tilde{Q}$ has a neighbor in Π.*

(iii) If Π has a short side, say P_2, and G contains no parachute with long sides, then Π has short top and either $n = 1$ and the only node of Q is of Type l[6.1], or $n \geq 2$ and Q is of Type l[6.4] or as described below. See Figure 16.

- **Type l1** *Node x_1 is strongly adjacent to Π and is adjacent to v_1 and one node in $\tilde{P}_1$. Node x_n is not strongly adjacent to Π. No node of $\tilde{Q}$ has a neighbor in Π.*

- **Type l2** *Node $x_1 \in V^r$ is strongly adjacent to Π and has exactly two neighbors in P_1. Furthermore, one of these neighbors is adjacent to v_1. Node x_2 is adjacent to v_1 and to no other node of Π. Node x_n is not strongly adjacent to Π. No node of $Q_{x_3x_{n-1}}$ has a neighbor in Π.*

Proof: A parachute Π' with direct connection $Q' = x'_1, \ldots, x'_n$ is called *top-initial for Q'* if Π' and Q' cannot be obtained from a parachute Π with direct connection Q by one parachute modification at the top using one node of Q. It follows from Remark 6.3 that if Π' is not top-initial for Q', then $Q = x'_1, \ldots, x'_n, x'_{n+1}$, node x'_{n+1} is in the top of Π', and $V(\Pi') \setminus V(\Pi) = x'_{n+1}$.

Parachute Π' with direct connection $Q' = x'_1, \ldots, x'_n$ is *bottom-initial for Q'* if Π' and Q' cannot be obtained from a parachute Π with direct connection Q by one parachute modification at the bottom using one node of Q. Again, it follows from Remark 6.3 that if Π' is not bottom-initial for Q', then $Q = x'_0, x'_1, \ldots, x'_n$, node x'_0 is in the bottom of Π', and $V(\Pi') \setminus V(\Pi) = x'_0$.

The proof of the theorem uses the following sufficient conditions for Π' being top-initial or bottom-initial for $Q' = x'_1, \ldots, x'_n$.

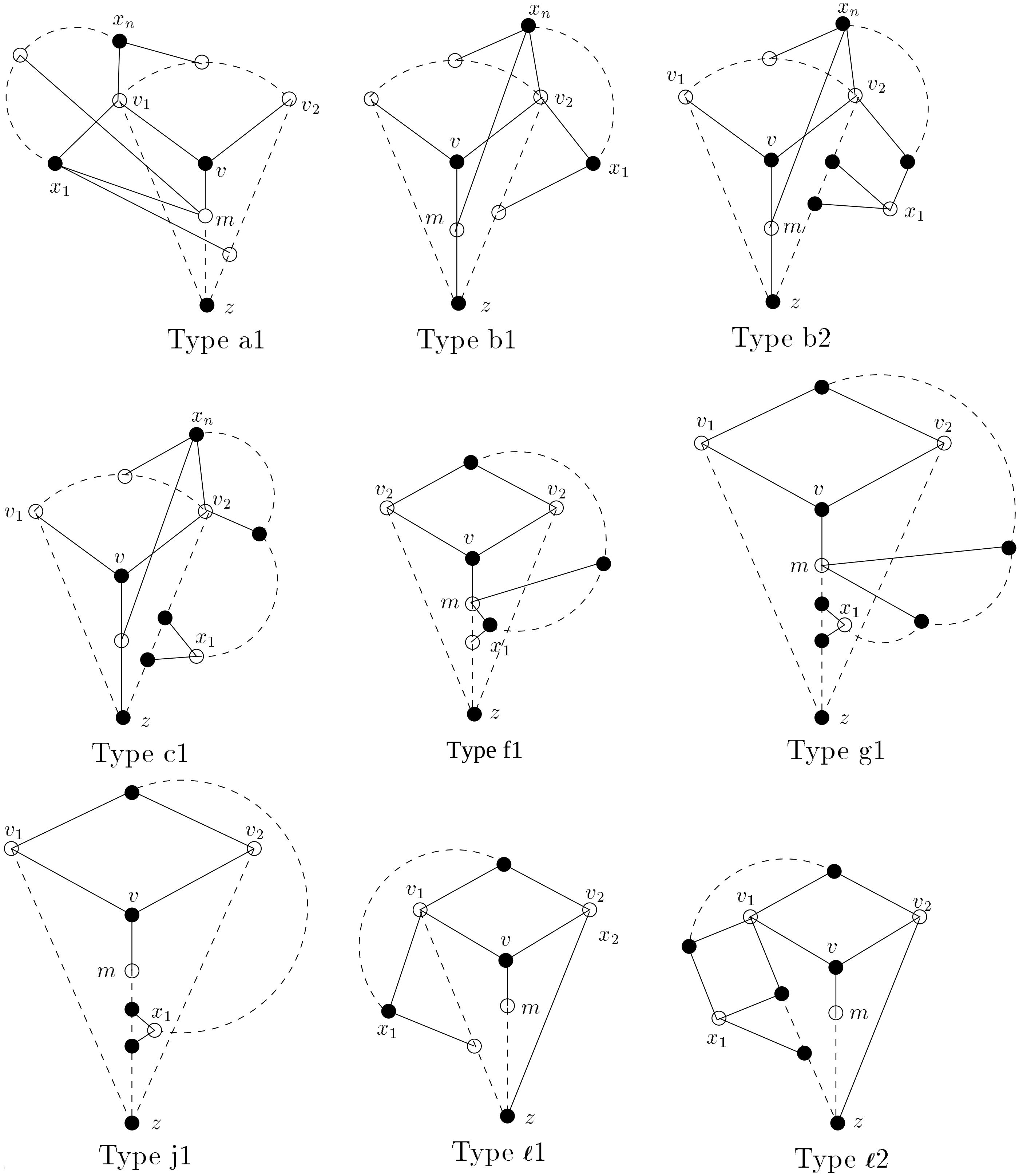


Figure 16: Direct connections from bottom to top

(1) If the top of Π' is short, then Π' is top-initial for Q'.

(2) If x'_1 has more than one neighbor in the bottom of Π', then Π' is bottom-initial for Q'.

(3) If Π' has long sides and x'_n is adjacent to one of its side nodes v_1 or v_2 and to node m, then Π' is top-initial for Q'.

(4) If Π' has long sides and x'_1 is adjacent to one of its side nodes v_1 or v_2 and to node m, then Π' is bottom-initial for Q'.

(1) follows from Remark 6.3. (2) follows from the definitions of direct connection and parachute modification. To prove (3): Assume that Π' has long sides and that x'_n is adjacent to m and to one of v_1 and v_2. If Π', Q' come from Π, Q by a parachute modification at the top, then x'_n is a strongly adjacent node to Π of Type j[6.1], but this contradicts Theorem 6.4 as Π has long sides. Similarly, one proves (4).

Now, we prove the theorem as follows: Given a parachute Π', we list all direct connections Q' from bottom to top given by Theorem 6.4. For the ones that are not bottom and top-initial, we add to the list the ones that give them after one parachute modification at the top or at the bottom. We repeat this until all the direct connections added are both bottom and top-initial. Two rounds of this procedure will suffice. In the proof of (iii), the procedure also stops when a parachute with long sides is detected.

(i) and (ii): Π has long sides.
Assume Π' with direct connection Q' is derived from Π, Q with one parachute modification at the top.

By (1) and (3) above, Q' is of Type a[6.4]. It follows that Q is of Type a1.

Assume now Π' with direct connection Q' is derived from Π, Q with one parachute modification at the bottom. So by (2) and (4) above, Q' is of Type b, c, d, f, g, j[6.4].

If Q' is of Type b[6.4], then Q is of Type b1.

If Q' is of Type c[6.4], then Q is of Type c1.

If Q' is of Type d[6.4], then there is a wheel with center v.

If Q' is of Type f[6.4], then Q is of Type f1.

If Q' is of Type g[6.4], then Q is of Type g1.

If Q' is of Type j[6.4], then Q is of Type j1.

We now examine all the newly added direct connections.

If Q' is of Type a1, then Π' is bottom-initial for Q' by (4) and Π' is top-initial for Q', else there is a wheel with center v.

If Q' is of Type b1, then Π' is top-initial for Q' by (2) and one parachute modification at the bottom gives Type b2.

If Q' is of Type c1, then Π' is top and bottom-initial for Q' by (3) and (2).

If Q' is of Type f1, then Π' is top-initial for Q' by (1) and one parachute modification at the bottom gives Type g1.

If Q' is of Type g1, then Π' is top and bottom-initial for Q' by (1) and (2).

If Q' is of Type j1, then Π' is top and bottom-initial for Q' by (1) and (2).

We now examine all the newly added direct connections.

If Q' is of Type b2, then Π' is top and bottom-initial for Q' by (3) and (2).

(iii): Π has a short side and G contains no parachute with long sides.

Assume Π' with direct connection Q' is derived from Π, Q with one parachute modification at the top.

Then by Theorem 6.4, Q' is of Type l[6.1] or l[6.4]. In both cases, Π' is top-initial for Q' by (1) and one parachute modification at the bottom gives Type l1.

Next, consider the case where Q' is of Type l1. Then Π' is top-initial for Q' by (1). Moreover, if there has been one parachute modification at the bottom, then the first node x'_0 of Q has two neighbors in P_1. One of these neighbors is adjacent to v_1, as otherwise there is a parachute with long sides. Hence Q is of Type l2.

Finally, if Q' is of Type l2, then Π' is top and bottom-initial for Q' by (1) and (2).

□

6.5 Parachutes with a Short Side

As in the earlier subsections, G is a wheel-free weakly balanced graph. We show that, if G contains a parachute with one short side but no parachute with long sides, then G has an extended star cutset.

Theorem 6.6 *Let G be a wheel-free weakly balanced graph containing no parachute with long sides. Let $\Pi = Par(P_1, P_2, M, T)$ be a parachute with a short side, say $P_2 = v_2, z$ and let its middle path be $M = v, m, \ldots, z$. Then $S(\Pi)$ or $N(v_2) \cup (N(z) \cap N(v)) \setminus \{m\}$ is an extended star cutset of G.*

Proof: If $S(\Pi)$ is not an extended star cutset then, by Theorem 6.5(iii) and Remark 6.3, Π has short top $T = v_1, t, v_2$, and we can assume that, after possibly parachute modifications at the bottom, there is a direct connection $Q = x_1, \ldots, x_n$ from bottom to top of Type l[6.4] or of Type l[6.1], where x_1 is adjacent to the neighbor, say a, of v_1 in P_1. Note that Π induces another parachute with short side, namely the parachute with center node v_2, side nodes v, z, top path M, middle path T and side paths P_1 and v, v_1. Denote by Π^* this parachute. Now, if $S(\Pi^*) = N(v_2) \cup (N(z) \cap N(v)) \setminus \{m\}$ is not an extended star cutset then, by Theorem 6.5(iii), Π^* has short top v, m, z and, there is a direct connection $R = y_1, \ldots, y_r$ from the bottom of Π^* to the top m of Type l[6.1], Type l[6.4], Type l1 or l2[6.5].

No node of $V(R) \setminus \{y_1\}$ is adjacent to or coincident with a node of Q, since otherwise Π^* would contain a direct connection from bottom to top violating Theorem 6.5. Now, if R is of Type l1 or l2[6.5], there is a wheel with center z. So R is of Type l[6.1] or Type l[6.4]. Node y_1 has a neighbor in Q since, otherwise, there is a wheel with center z. Finally, y_1 is adjacent to x_1 but to no other node of Q since, otherwise, Π would contain a direct connection from bottom to top violating Theorem 6.5. Let b be the neighbor of z in P_1, see Figure 17.

Node x_1 is adjacent to t, else there is a $3PC(x_1, t)$. Similarly, y_1 is adjacent to m, else there is a $3PC(y_1, m)$. Finally, a is adjacent to b, else there is a $3PC(a, b)$. Now the graph induced by $V(\Pi) \cup V(Q) \cup V(R)$ is an R_{10} configuration. It follows that G is not weakly balanced, a contradiction. □

6.6 Stabilized Parachutes

In the remainder of this subsection, we consider wheel-free weakly balanced graphs G that contain a parachute with long sides and short middle. In this subsection, we make the further assumption that G contains a stabilized parachute, as defined below.

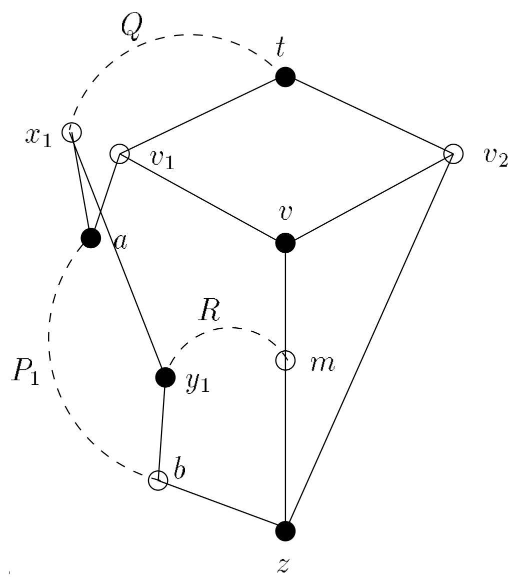


Figure 17: Connections Q and R

Definition 6.7 *A* stabilized parachute (Π, R), *see Figure 18, consists of a parachute* $\Pi = Par(P_1, P_2, M, T)$ *with long sides* $P_1 = v_1, a, \ldots, z$ *and* $P_2 = v_2, \ldots, z$, *a short middle* $M = v, m, z$ *and of a chordless path* $R = r_1, \ldots, r_k$ *(possibly* $k = 1$*), where* $r_i \in V \setminus V(\Pi)$ *for* $i = 1, \ldots, k$, *such that node* r_1 *is adjacent to node* a *and node* r_k *is adjacent to* v. *Nodes* r_1 *and* r_k *do not have any other adjacencies in* Π *than those just mentioned and nodes* r_i *for* $i = 2, \ldots, k-1$, *are not adjacent to any node of* Π. *Furthermore,*

(i) any strongly adjacent node of Type f[6.1] relative to Π *that is adjacent to* v_2 *must also be adjacent to* v_1, *and*

(ii) any node in $V \setminus (V(\Pi) \cup V(R))$ *that has two neighbors in* T *and is adjacent to* r_k *must also be adjacent to* m.

We now prove that if G contains a stabilized parachute, then G has an extended star cutset.

Lemma 6.8 *Let* G *be a wheel-free weakly balanced graph. If* Π *is a stabilized parachute then the only possible direct connections from bottom to top avoiding* $S(\Pi)$ *are of Type b, c, d, e[6.4] or Type b1, b2, c1[6.5].*

Proof: A direct connection of Type o[6.1] or Type g or j[6.4] cannot occur since a stabilized parachute has middle path of length 2. Similarly for Types f1, g1 and j1[6.5]. Now we show that a direct connection $Q = x_1, \ldots, x_n$ of Type a, f, h, i or k[6.4] and Type a1[6.5] cannot occur.

Case 1 Path Q is of Type a[6.4].

It follows from Condition (i) of Definition 6.7 that x_1 is adjacent to v_1, m and a node in $\tilde{P}_2$. Nodes $x_2, \ldots, x_n$ are not adjacent or equal to any of the nodes $r_1, \ldots, r_{k-1}$, else

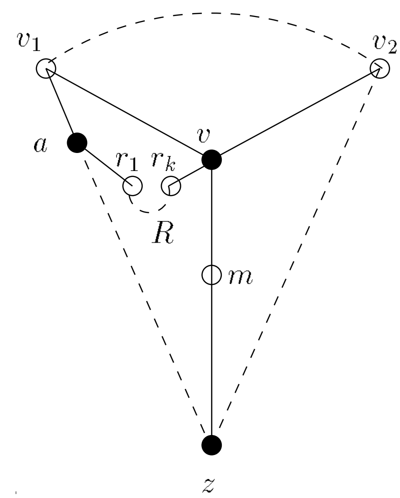


Figure 18: A stabilized parachute

there is a direct connection from bottom to top that contradicts Theorem 6.5. Clearly, x_1 is not in R and r_k is not in Q.

If r_k is not adjacent to any node in $\tilde{Q}$, there is a wheel with center v_1 whether or not x_1 is adjacent at least one node in the set $\{r_1, \ldots, r_k\}$.

If r_k is adjacent to at least one node in $\tilde{Q}$, let x_j be the node of $\tilde{Q}$ that is adjacent to m and let q be a neighbor of r_k in $Q_{x_2 x_n}$ such that $Q_{x_j q}$ contains no other neighbor of r_k. Note that $x_j, q, x_1 \in V^c$, so $H = x_j, Q_{x_j q}, q, r_k, R, r_1, a, v_1, x_1, m, x_j$ is a hole. Now (H, v) is a wheel.

Case 2 Path Q is of Type a1[6.5].

By Theorem 6.5, x_n is not adjacent or equal to $r_1, \ldots, r_{k-1}$ and obviously, $x_n \neq r_k$. By Condition (ii) of Definition 6.7, node x_n is not adjacent to r_k. Therefore, after parachute modification, we are back in Case 1.

Case 3 Path Q is of Type f, h, i or k[6.4].

If some node of $V(Q) \setminus \{x_1\}$ is adjacent or equal to at least one node of $V(R) \setminus \{r_k\}$, then there is a direct connection from the bottom to the top of Π violating Theorem 6.5(ii). So no such adjacency exists. Clearly, x_1 is not in R and r_k is not in Q. If Q is of Type h, i or k[6.4] and x_1 is adjacent to a node in R, there is a wheel with center x_1. If Q is of Type f[6.4], then x_1 is adjacent to at most one node of R, else there is a wheel with center x_1. If x_1 is adjacent to exactly one node of R, there is a $3PC(x_1, a)$, where a is the neighbor of v_1 in P_1. So, in all cases, x_1 is not adjacent to a node in R. If r_k is not adjacent to a node in Q, then there is a wheel with center v_1. So r_k is adjacent to a node in Q. Now, if Q is of Type f[6.4], there is a wheel with center v, and if Q is of Type h, i or k[6.4], there is a $3PC(x_1, r_k)$.

$\square$

Lemma 6.9 *Let G be a wheel-free weakly balanced graph. If G contains a parachute Π with long sides having a direct connection of Type b, c, d[6.4] or b1, b2, c1[6.5], then G has a stabilized parachute with top shorter than the top of Π.*

Proof: If the direct connection is of Types c, d[6.4] or b1, b2, c1[6.5] there exists a parachute with long sides with a direct connection of Type b[6.4]. So we assume that the direct connection $Q = x_1, \ldots, x_n$ is of Type b[6.4]. Assume w.l.o.g. that x_1 is adjacent to the neighbor b of v_2 in P_2. Construct the parachute Π' as follows. The middle path of Π' is $M' = x_n, m, z$. The top path T' of Π' is the subpath of T connecting the two neighbors of x_n in T, say $t \in \tilde{T}$ and v_2. The side path P_2' is identical to P_2 and the side path P_1' connects t to z, using nodes of $V(T) \cup V(P_1)$. We will show that Π', with extra path induced by $Q_{x_1 x_{n-1}}$, defines a stabilized parachute with shorter top than Π. In order to prove that Π' defines a stabilized parachute, we need to check Conditions (i) and (ii) of Definition 6.7. Condition (i) holds since $t \in V(\tilde{T})$ and a node w of Type f[6.1] relative to Π' that is adjacent to t must also be adjacent to v_2, else w violates Theorem 6.1 relative to Π. To see that Condition (ii) holds, consider a node y adjacent to x_{n-1} and to two nodes of T'. There is a direct connection that violates Theorem 6.5(i) with respect to Π, unless node y is adjacent to m. This completes the proof that Π' is a stabilized parachute. $\Box$

Theorem 6.10 *Let G be a wheel-free weakly balanced graph. If G contains a stabilized parachute, then G has an extended star cutset.*

Proof: Among all parachutes that give rise to a stabilized parachute, let Π be one with shortest top. If Π has no extended star cutset, every direct connection from bottom to top avoiding $S(\Pi)$ is of Type e[6.4] by Lemmas 6.8 and 6.9.

Consider $Q = x_1, \ldots, x_n$ of Type e[6.4] and assume w.l.o.g. that x_1 and x_n are adjacent to v_1. Then the first node r_1 of the extra path $R = r_1, \ldots, r_k$ is adjacent to the neighbor of v_1 in P_1, by Condition (i) of Definition 6.7. Note that the nodes $x_2, \ldots, x_n$ are not adjacent or equal to $r_1, \ldots, r_{k-1}$, because otherwise Π would have a direct connection of Type b[6.5], which contradicts, by Lemma 6.9, the fact that Π is a stabilized parachute with shortest top.

If r_k is not adjacent to any node in $Q_{x_2 x_n}$, there is a wheel with center v_1 whether or not x_1 has neighbors in R.

If r_k is adjacent to at least one node in $Q_{x_2 x_n}$, then there is a parachute with shorter top path obtained by replacing the center node v by the node x_n and replacing the extra path R by a chordless path from x_n to r_1 only involving nodes of $(V(Q) \setminus \{x_1\}) \cup V(R)$. The new parachute satisfies Condition 6.7(i) as any node violating it would violate Theorem 6.1 with respect to Π. To see that Condition (ii) holds, consider a node w adjacent to x_{n-1} and to two nodes of the new top. There is a direct connection that violates Theorem 6.5(i) with respect to Π, unless node w is adjacent to m. $\Box$

6.7 Parachutes with Long Top and Long Sides

In this subsection, we show that, if G is a wheel-free weakly balanced graph that contains a parachute with long top and long sides, then G has an extended star cutset.

Lemma 6.11 *Let G be a wheel-free weakly balanced graph that contains no stabilized parachute. Let Π be a parachute with long sides having a direct connection of Type a[6.4]. Then Π has short top and a direct connection of Type f[6.4].*

Proof: Let $Q = x_1, \ldots, x_n$ be a direct connection of Type a[6.4] where x_1 is adjacent to v_1 and let x_j be the node of $\tilde{Q}$ adjacent to m. Consider the parachute Π' in $\Pi \cup Q$ with center m, side nodes x_1 and x_j and bottom node v_2. Then Π' has long sides and short middle. Let R' be the path in $M \cup P_2$ from m to the neighbor of x_1 in P_2. By assumption, $\Pi' \cup R'$ is not a stabilized parachute. If w would be a node violating Condition 6.7(i) with respect to Π', then it would be adjacent to x_j, v and some node in P_2. So, there would be a wheel (H, m) with $H = w, x_j, Q_{x_j x_1}, v_1, v, w$. Hence Π' satifies Condition 6.7(i). So Condition 6.7(ii) is not satisfied for Π' and R'. Hence, there exists a node w adjacent to the neighbor of m distinct from v in M, to two neighbors x_f, x_i on Q for $1 \leq f < i \leq j$, and not adjacent to v. So $w, x_i, Q_{x_i x_n}, x_n$ is a direct connection from bottom to top of Π. By Theorem 6.5, it has to be of Type f[6.4], since it contains x_j. □

Lemma 6.12 *Let G be a wheel-free weakly balanced graph that contains no stabilized parachute. Then a parachute with long sides cannot have a direct connection of Type e[6.4] or Type a1[6.5].*

Proof: If $Q = x_1, \ldots, x_n$ is a direct connection of Type e[6.4], where x_1 is adjacent to v_2, m and $b \in \tilde{P}_1$ and x_n is adjacent to v_2 and $t \in \tilde{T}$. We show that a stabilized parachute occurs by taking v_2 as the center node, x_1, x_n as the side nodes and v_1 as the bottom node. There exists no pair d_1, d_2 of strongly adjacent nodes of Type f[6.1] relative to this parachute such that d_1 is adjacent to x_1 but not to x_n and d_2 is adjacent to x_n but not to x_1 (else there is a wheel with center v). Since there are two possibilities for the path R, namely the subpath of T from v_2 to t and the path from v_2 to b, Condition (i) of Definition 6.7 is satisfied by one of the choices for R. Next, we consider Condition (ii). First, consider the case when R is the subpath of T connecting v_2 to t. If there is a node w adjacent to two nodes in Q and to the neighbor of v_2 in T and w is not adjacent to v, then there is a direct connection from bottom to top of Π that contradicts Theorem 6.5, for $w \in V^r$ cannot be adjacent to m. Therefore Condition (ii) holds.

Now, consider the case when R connects v_2 to b. If there is a node w adjacent to two nodes in Q and to the neighbor q of v_2 in P_2, then one of three possibilities occurs.

If w is also adjacent to v, Condition (ii) holds.

If w is not adjacent to v but is adjacent to at least one node of $V(\Pi) \setminus \{v, q\}$, then there is a direct connection from bottom to top of Π that contradicts Theorem 6.5.

If w is not adjacent to any node of $V(\Pi) \setminus \{q\}$, then there is a direct connection of Type b[6.4] from bottom to top of Π, and the result follows from Lemma 6.9.

If the direct connection is of Type a1[6.5], we get, after a parachute modification at the top, a parachute with long sides, long top and a direct connection of Type a[6.4], contradicting Lemma 6.11. □

Theorem 6.13 *Let G be a wheel-free weakly balanced graph. If G contains a parachute with long top and long sides, then G has an extended star cutset.*

Proof: By Theorem 6.5(i), Π must have a direct connection Q of Type a, b, c, d, e[6.4] or Type a1, b1, b2, c1[6.5]. Now, we have a stabilized parachute. Indeed, if Q is of Type b, c, d[6.4] or Type b1, b2, c1[6.5], this is guaranteed by Lemma 6.9. If Q is of Type a[6.4], this is guaranteed by Lemma 6.11 since Π has long top. Finally, if Q is of Type e[6.4] or Type a1[6.5], this follows from Lemma 6.12. Now, by Theorem 6.10, G contains an extended star cutset. □

6.8 Parachutes with Short Middle Path

In this subsection, we assume again that G is a wheel-free weakly balanced graph. We show that, if G contains a parachute with long sides and short middle but G contains no connected squares, then G has an extended star cutset.

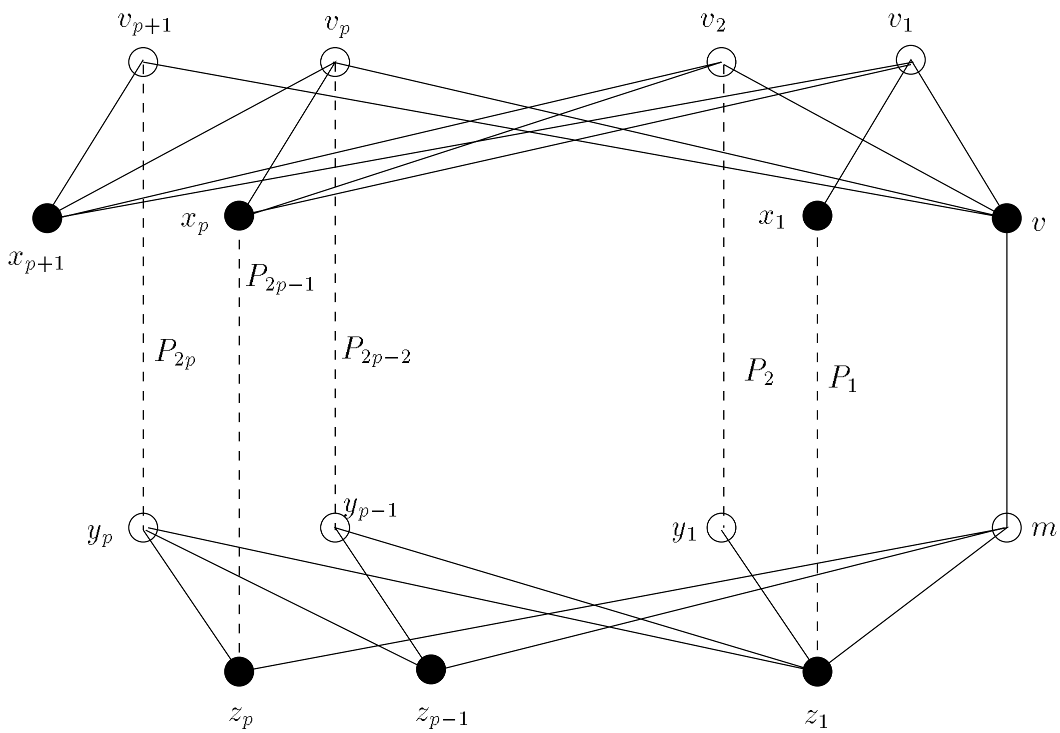
A direct connection $Q = x_1, \ldots, x_n$ from bottom to top of a parachute Π is of Type f_{short} if Q is of Type f[6.4] and x_2 is adjacent to m.

Lemma 6.14 *Let G be a wheel-free weakly balanced graph that contains no connected squares. Suppose G contains no extended star cutset and let Π be a parachute with long sides and short middle. Then Π has short top and each direct connection from bottom to top is of Type a[6.4], f_{short} or h, i[6.4]. Moreover, Π has at least one direct connection of Type f_{short} or h, i[6.4].*

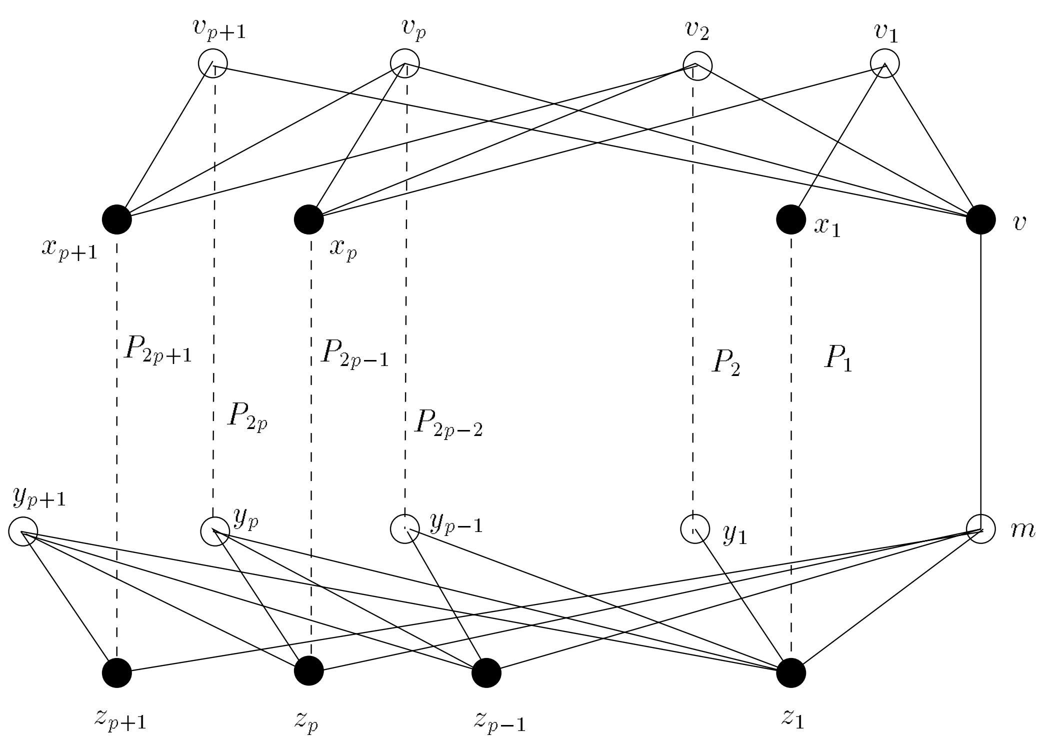
Proof: By Theorem 6.13, Π has short top. Since $S(\Pi)$ is not an extended star cutset of G and Π has a short middle path, by Theorem 6.5(ii), its direct connections are of Type a, f, h, i, k[6.4]. As there are no connected squares, Π has no direct connection of Type k[6.4]. By Theorem 6.10, G contains no stabilized parachute. By Lemma 6.11, Π has a direct connection of Type f[6.4] if it has one of Type a[6.4]. Now let Q be a direct connection of Type f[6.4]. Let Π' be the unique parachute in $\Pi \cup Q$ with center node m, middle path m, v, v_2 and side nodes z and the neighbor x_j of m on Q. The top of Π' is the subpath of Q connecting x_j with z and this top must be short by Corollary 6.13. So Q is of Type f_{short} relative to Π. □

Definition 6.15 *For $k \geq 2$, a k-*parachute *Π^k is defined as follows, see Figure 19. For k even, say $k = 2p$, Π^k consists of nodes $v, m, v_1, \ldots, v_{p+1}, x_1, \ldots, x_{p+1}, y_1, \ldots, y_p, z_1, \ldots, z_p$ and chordless paths P_j, for $j = 1, \ldots, 2p$ where:*

- *node v is adjacent to nodes m and $v_1, \ldots, v_{p+1}$,*
- *node m is adjacent to nodes v and $z_1, \ldots, z_p$,*
- *for $t = 1, \ldots, p+1$, node x_t is adjacent to nodes $v_1, \ldots, v_t$,*
- *for $t = 1, \ldots, p$, node y_t is adjacent to nodes $z_1, \ldots, z_t$,*
- *for $t = 1, \ldots, p$, path P_{2t-1} connects x_t to z_t and path P_{2t} connects y_t to v_{t+1},*
- *for $i \neq j$, $V(P_i) \cap V(P_j) = \emptyset$,*
- *there are no adjacencies between the nodes of Π^k other than those indicated above.*



2p-parachute



$(2p+1)$-parachute

Figure 19:

For k odd, say $k = 2p+1$, the k-parachute Π^k is obtained from Π^{k-1} by adding a node z_{p+1} adjacent to m, a node y_{p+1} adjacent to nodes $z_1, \ldots, z_{p+1}$ and a chordless path P_{2p+1} connecting x_{p+1} to z_{p+1} whose inner nodes are distinct from $V(\Pi^{k-1}) \cup \{z_{p+1}, y_{p+1}\}$ and are not adjacent to $\{y_{p+1}\} \cup V(\Pi^{k-1}) \setminus \{x_{p+1}\}$.

This definition implies that a 2-parachute is a parachute with long side paths P_1, P_2, short middle path v, m, z_1 and short top path v_1, x_2, v_2.

Theorem 6.16 *Assume that G is a wheel-free weakly balanced graph that contains no connected squares. If G contains a parachute with long sides and short middle, then G has an extended star cutset.*

Proof: Assume that G has no extended star cutset. By Theorem 6.10, G contains no stabilized parachute. Let Π^k is a k-parachute with k maximal.

Claim 1 Π^k *exists and $k \geq 3$.*

Proof of Claim 1: By Lemma 6.14, each parachute with long sides and short middle has short top, so is a 2-parachute. Hence, Π^k exists.

Suppose $k = 2$. A parachute with long sides and short middle and with a direct connection of Type f_{short} induces a 3-parachute. So, by Lemmas 6.11 and 6.14, we get

(*) *Direct connections of parachutes with long sides and short middle are of Type h, i[6.4].*

Let Π be a parachute with long sides and short middle. Then the top of Π is short. Let t denote its single node. Let $Q = x_1, \ldots, x_n$ be a direct connection from the bottom to the top of Π. By (*), Q is of Type h, i[6.4]. Hence for both $\ell = 1$ and $\ell = 2$, x_1 has a neighbor y_ℓ on $\tilde{P}_\ell$. Moreover, m has two neighbors x_i and x_j on Q (with $j < i$, say). Note that $j = 1$ if Q is of Type h[6.4]. W.l.o.g. assume that Π and Q are chosen such that, if possible, Q is of Type h[6.4].

For $\ell = 1$ and $\ell = 2$, we define the parachute Π_ℓ with long sides and short middle as follows: the middle path is m, v, v_l, the top path is $Q_{x_j x_i}$, and the side paths are $Q_{x_i x_n} \cup \{t, v_\ell\}$ and $Q_{x_j x_1} \cup (P_\ell)_{y_\ell v_\ell}$. Let $R = r_1, \ldots, r_{n'}$ be the shortest path that is a direct connection from the bottom to the top of Π_1 or Π_2. Then, by (*), R is of Type h or i[6.4]. By construction, R is node disjoint from $\Pi \cup Q$. So R is a direct connection from the bottom to the top of both Π_1 and Π_2 and it has the same type relative to both Π_1 and Π_2. If R is of Type i[6.4] with respect to both Π_1 and Π_2, then r_1 is adjacent to t, a node of $\tilde{P}_1$ and a node of $\tilde{P}_2$. But this contradicts Theorem 6.1. So R is of Type h[6.4] with respect to both Π_1 and Π_2 and, therefore, r_1 is adjacent to v. By Theorem 6.1, r_1 cannot have neighbors in both $\tilde{P}_1$ and $\tilde{P}_2$. So r_1 has a neighbor in $Q_{x_1 x_j} \setminus \{x_j\}$. This implies that Q is of Type i[6.4] with respect to Π. But this contradicts the choice of Π, Q since R is of Type h[6.4] with respect to Π_1. This proves Claim 1.

From now on, we will assume that k is odd ($k = 2p+1$ say), as the proof of the even case is essentially the same. Let Π^* be the parachute with top path z_{p+1}, y_{p+1}, z_p, middle path m, v, v_{p+1} and side paths $P_{2p+1} \cup \{v_{p+1}\}$ and $P_{2p} \cup \{z_p\}$. Let P_{2p+2} be a direct connection of Π^* of Type f_{short}, h or i[6.5]. Denote by v_{p+2} the node of P_{2p+2} adjacent to v and closest to y_{p+1}. Moreover, let x_{p+2} be the first node of P_{2p+2} (so adjacent to the bottom of Π^*).

For $i = 1, \ldots, p$, we denote by Π_i the parachute with middle path m, v, v_i, side paths $P_{2i-2} \cup \{z_{i-1}\}$ and $P_{2p+1} \cup \{v_i\}$ and top path $z_{p+1}, y_{p+1}, z_{i-1}$, and we denote by Π'_i the parachute with middle path m, v, v_i, side paths $P_{2i-1} \cup \{v_i\}$ and $P_{2p+1} \cup \{v_i\}$ and top path z_{p+1}, y_{p+1}, z_i.

Claim 2: *Let $i = 1, \ldots, p$ and $\Pi \in \{\Pi_i, \Pi'_i\}$. Then P_{2p+2} is a direct connection for Π. Moreover, P_{2p+2} is of Type f_{short} for Π if and only if it is of Type f_{short} for Π^*.*

Proof of Claim 2: If P_{2p+2} would contain no direct connection for Π, then P_{2p+2} would be of Type f_{short} for Π^* and $P_{2p+2} \cup \{v_{p+1}\}$ would be a direct connection for Π that is not of Type a[6.4], f_{short} or h, i [6.4]. So P_{2p+2} contains a direct connection Q for Π.

Suppose that Q is a proper subpath of P_{2p+2}. Then Q is not a direct connection for Π^*, hence it has to be of Type f_{short} for Π. However this implies that $Q \cup \{v_i\}$ is a direct connection for Π^* that is not of Type a[6.4], f_{short} or h, i[6.4], a contradiction. So $Q = P_{2p+2}$, and P_{2p+2} is a direct connection for Π. The remainder of the claim now follows because, for both Π^* and Π, we have that P_{2p+2} is of Type f_{short} if and only if P_{2p+2} has no neighbor on P_{2p+1}. This proves Claim 2.

Claim 3: *P_{2p+2} is of Type f_{short} for all the parachutes Π_i and Π'_i with $i = 1, \ldots, p$.*

Proof of Claim 3: Suppose not. Then, by Claim 2, P_{2p+2} is not of Type f_{short} for Π^* and Π'_p. Hence x_{p+2}, the first node on P_{2p+2}, has a neighbor on P_{2p} and one on P_{2p-1} and either it is adjacent to v (if P_{2p+2} is of Type h[6.4] for Π^*) or to x_{p+1} (if P_{2p+2} is of Type i[6.4] for Π^*). In any case $x_{p+2} \in V^r$. But then, x_{p+2} is a strongly adjacent node violating Theorem 6.1 with respect to the parachute with top path v_{p+1}, x_{p+1}, v_p, middle path v, m, z_p and side paths $P_{2p} \cup \{z_p\}$ and $P_{2p-1} \cup \{v_p\}$. This proves Claim 3.

From the last claim, it follows immediately that $\Pi^k \cup P_{2p+2}$ is a $k+1$-parachute, contradicting our choice of k. □

6.9 The Parachute Theorem

Theorem 6.17 *Let G be a weakly balanced graph that is not strongly balanceable. Assume G contains no extended star cutset. Then G is wheel-free and contains a parachute with long sides. Furthermore, if G contains no connected squares, any parachute Π with long sides has a short top, a long middle and a direct connection of Type j[6.4] or j1[6.5]. In addition, any direct connection from bottom to top avoiding $S(\Pi)$ is of one of these two types.*

Proof: Assume G contains no extended star cutset. By Theorems 4.6, 5.1 and 6.6, G is wheel-free and contains a parachute with long sides. Let Π be such a parachute and let Q be a direct connection from bottom to top avoiding $S(\Pi)$. There exists a parachute with long sides and short middle in $\Pi \cup Q$ when Q is of Type o[6.1], Types a, b, c, d, e, f, g, h, i[6.4] or a1, b1, b2, c1, f1, g1[6.5]. So by Theorem 6.16, none of these direct connections can occur. Now, by Theorem 6.5, Q must be of Type j, k[6.4] or j1[6.5]. This proves the theorem, since Type k[6.4] yield connected squares. □

7 Connected Squares

7.1 A Classification of Nodes and Paths

In this section, we prove the following result:

Theorem 7.1 *Let G be a wheel-free weakly balanced graph that contains connected squares. Then G has a biclique cutset or a 2-join.*

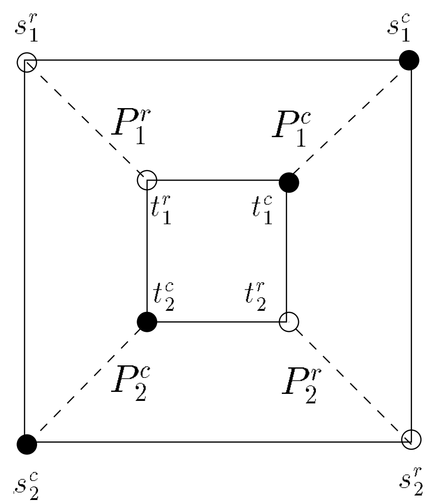


Figure 20: Connected squares

Throughout this section, G denotes a wheel-free weakly balanced graph that contains connected squares, and $CS(P_1^c, P_2^c; P_1^r, P_2^r)$ denotes connected squares in G, see Figure 20. We assume $s_1^c, s_2^c, t_1^c, t_2^c \in V^c$ and $s_1^r, s_2^r, t_1^r, t_2^r \in V^r$. Recall that $\tilde{P}_1^c$ denotes the subpath obtained from P_1^c by removing its endnodes s_1^c, t_1^c. The subpaths $\tilde{P}_2^c$, $\tilde{P}_1^r$, $\tilde{P}_2^r$ are analogously defined.

The following lemma characterizes the strongly adjacent nodes to connected squares.

Lemma 7.2 *Let $\Sigma = CS(P_1^c, P_2^c; P_1^r, P_2^r)$ be connected squares and $v \in V(G) \setminus V(\Sigma)$ be a strongly adjacent node to Σ. Then one of the following holds:*

- *Node v has exactly two neighbors in Σ, both contained in P_1^c or in P_2^c or in P_1^r or in P_2^r.*
- *Node v is of one of the following types, see Figure 21:*

  Type a *Node v has three neighbors in Σ, two of them being s_1^c, s_2^c or t_1^c, t_2^c or s_1^r, s_2^r or t_1^r, t_2^r. If $v \in V^c$, the third neighbor is in $\tilde{P}_1^c$ or in $\tilde{P}_2^c$ and if $v \in V^r$, the third neighbor is in $\tilde{P}_1^r$ or in $\tilde{P}_2^r$.*

  By extension, we define s_1^c, s_2^c, s_1^r, s_2^r, t_1^c, t_2^c, t_1^r, t_2^r to be Type a nodes.

Type b *Node v has exactly two neighbors in Σ that are s_1^c, s_2^c or t_1^c, t_2^c or s_1^r, s_2^r or t_1^r, t_2^r.*

Type c *Node v has exactly two neighbors in Σ and if $v \in V^r$, then v has one neighbor in $\tilde{P}_1^r$ and one in $\tilde{P}_2^r$. If $v \in V^c$, then v has one neighbor in $\tilde{P}_1^c$ and one in $\tilde{P}_2^c$.*

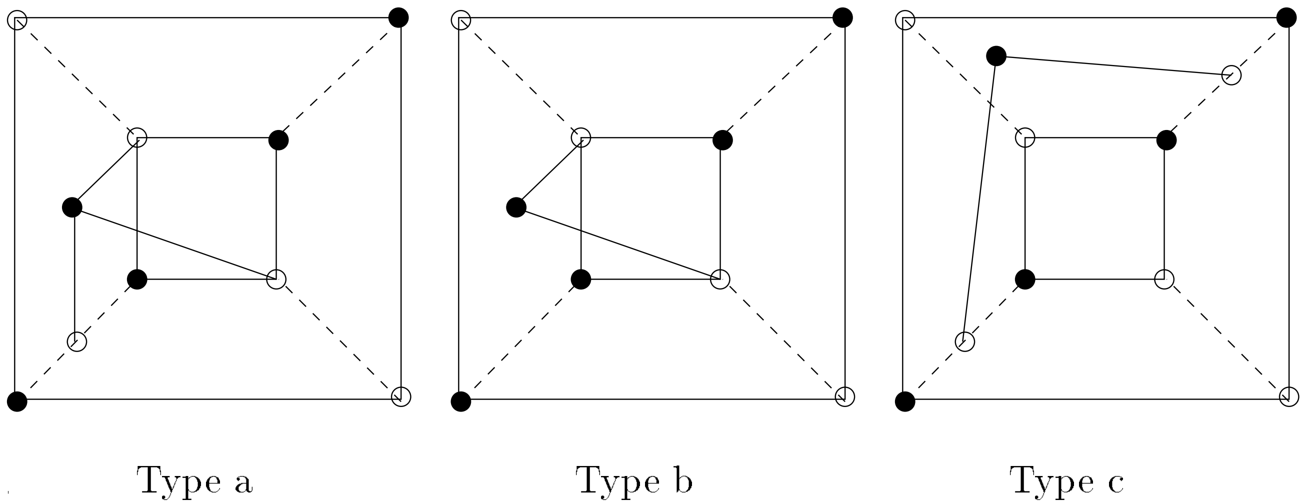


Figure 21: Strongly adjacent nodes

Proof: If all the neighbors of v in Σ belong to one of the paths P_1^c, P_2^c, P_1^r, P_2^r, then v has two neighbors in this path, else v is the center of a wheel. If v has neighbors in more than one path, then v has a unique neighbor in each of these paths, else v is again the center of a wheel.

In the rest of the proof, assume w.l.o.g. that $v \in V^c$. If v has (unique) neighbors in both P_1^c and P_2^c, say v_1 and v_2, then v has no neighbors in P_1^r and P_2^r. For, if v is adjacent to, say v_3 in P_1^r, then assume w.l.o.g. that v_3 is distinct from t_1^r. Then v_1, v_2 and v_3 are intermediate nodes in the three paths of a $3PC(v, t_1^r)$. So v is of Type c in this case.

Consider now the case in which v has no neighbors in P_1^c or P_2^c, say P_2^c. If v has (unique) neighbors in both P_1^r and P_2^r, say v_3 and v_4, then either $v_3 = s_1^r$ and $v_4 = s_2^r$, or $v_3 = t_1^r$ and $v_4 = t_2^r$. For, if not, assume $v_3 \neq s_1^r$. Then $v_4 = t_2^r$, else nodes t_2^c, s_1^r and s_2^r are intermediate nodes in a $3PC(v_3, s_2^c)$. The same argument, applied to $v_4 \neq s_2^r$, shows $v_3 = t_1^r$. Thus node v is of Type a if it has a neighbor in $\tilde{P}_1^c$, and v is of Type b if it has no neighbor in $\tilde{P}_1^c$. Finally, if v has no neighbors in P_1^r or P_2^r, assume w.l.o.g. that v has no neighbor in P_2^r, is adjacent to v_1 in P_1^c, to v_3 in P_1^r and $v_3 \neq t_1^r$. Then v_1 is adjacent to s_1^c, else we have a $3PC(v_1, s_1^c)$. Now s_1^c is the center of an odd wheel. $\square$

Let $\mathcal{S}^c$ comprise all Type a and Type b[7.2] nodes adjacent to s_1^r and s_2^r. Node sets $\mathcal{S}^r$, $\mathcal{T}^c$ and $\mathcal{T}^r$ are defined analogously. Let $\mathcal{S} = \mathcal{S}^r \cup \mathcal{S}^c$ and $\mathcal{T} = \mathcal{T}^r \cup \mathcal{T}^c$.

Lemma 7.3 *The sets $\mathcal{S}$ and $\mathcal{T}$ are disjoint and no node of $\mathcal{S}$ is adjacent to a node of $\mathcal{T}$.*

Proof: The first property follows immediately from Lemma 7.2. If the second property does not hold, there is a 3-path configuration connecting a node in $\{s_1^c, s_2^c, s_1^r, s_2^r\}$ and a node in $\{t_1^c, t_2^c, t_1^r, t_2^r\}$. $\Box$

Let v be a Type b[7.2] node in $\mathcal{S}$ and let $\mathcal{P}_v(\Sigma)$ be the family of direct connections between v and $\mathcal{T}$, avoiding the set $\mathcal{S} \setminus \{v\}$. When no confusion arises, we write $\mathcal{P}_v$ instead of $\mathcal{P}_v(\Sigma)$. Consider the following classification of the paths in $\mathcal{P}_v$.

Classification 7.4 *Let $P = x_1, x_2, \ldots, x_n$ be a direct connection in $\mathcal{P}_v$ where x_1 is adjacent to a Type b[7.2] node v in S and x_n is adjacent to a node in $\mathcal{T}$.*

- *P is* attached *if x_n is adjacent to a Type a[7.2] node in $\mathcal{T}$.*
- *P is* detached *if x_n is not adjacent to any Type a[7.2] node in $\mathcal{T}$. Hence x_n is adjacent only to Type b[7.2] nodes in $\mathcal{T}$.*

The above classification induces a classification of the strongly adjacent nodes of Type b[7.2]:

Classification 7.5 *Let v be a Type b[7.2] node in $\mathcal{S}$.*

- *Node v is* attached *if v has at least one attached direct connection in $\mathcal{P}_v$.*
- *Node v is* detached *if $\mathcal{P}_v$ is nonempty and all the direct connections in $\mathcal{P}_v$ are detached.*
- *Node v is* separable *if $\mathcal{P}_v$ is empty.*

Similarly, each Type b[7.2] node $w \in \mathcal{T}$ is classified as attached, detached or separable, based on the direct connections in $\mathcal{P}_w$ between w and $\mathcal{S}$, avoiding $\mathcal{T} \setminus \{w\}$.

An attached direct connection $P \in \mathcal{P}_v$ is *minimal* if there exists no direct connection $P' \in \mathcal{P}_v$ with nodes in $V(P) \cup V(\Sigma)$ such that $V(P') \setminus V(\Sigma) \subset V(P) \setminus V(\Sigma)$.

Lemma 7.6 *Let $v \in \mathcal{S}$ be an attached Type b[7.2] node, and let $P = x_1, x_2, \ldots, x_n$ be a minimal attached direct connection in $\mathcal{P}_v$, where x_n is adjacent to a Type a[7.2] node t in $\mathcal{T}^c$. Say t has a neighbor in $\tilde{P}_1^c$. Let x_h be the node of highest index in $V(P) \setminus V(\Sigma)$. Then the following holds:*

(i) $N(x_h) \cap V(\Sigma) \subset V(P_1^c)$.

(ii) $v \in \mathcal{S}^c$.

(iii) *At most one node of $\{x_1, \ldots, x_h\}$ is adjacent to s_1^c and none is adjacent to s_2^c, s_1^r or s_2^r.*

(iv) *Node x_n is not adjacent to any Type a[7.2] node t' with a neighbor in $\tilde{P}_2^c$.*

Proof: Since $P \in \mathcal{P}_v$ is minimal, no node in $\{x_1, \ldots, x_{h-1}\}$ is adjacent to a node in $V(\Sigma) \setminus \{s_1^c, s_2^c, s_1^r, s_2^r\}$.

Claim 1 *If x_h is a strongly adjacent node of Type c[7.2], then no node in $\{x_1, \ldots, x_{h-1}\}$ is adjacent to a node in $\{s_1^c, s_2^c, s_1^r, s_2^r\}$.*

Proof of Claim 1: By Lemma 7.2, $h < n$. As $P \in \mathcal{P}_v$ and $t \in \mathcal{T}^c$, node x_h has one neighbor in $\tilde{P}_1^c$, say z_1, and one in $\tilde{P}_2^c$, say z_2. Let x_i, $i < h$, be the node of highest index adjacent to a node $x^* \in \{s_1^c, s_2^c, s_1^r, s_2^r\}$. By symmetry, we may assume $x^* \in \{s_1^c, s_1^r\}$. If $x^* = s_1^r$, then nodes z_1, z_2 and x_i are intermediate nodes in the three paths of a $3PC(x_h, s_1^r)$. So $x^* = s_1^c$. Now z_1 is adjacent to s_1^c, else there is a $3PC(z_1, s_1^c)$. Let Q be the shortest path from x_i to s_1^r, contained in $V(P_{x_{i-1}x_1}) \cup \{v, s_2^c\}$. Then the hole $H = x_h, P_{x_h x_i}, x_i, Q, s_1^r, P_1^r, t_1^r, t_1^c, (P_1^c)_{t_1^c z_1}, z_1, x_h$ induces a wheel with center s_1^c. This proves Claim 1.

Claim 2 $N(x_h) \cap V(\Sigma) \subset V(P_1^c)$.

Proof of Claim 2: Assume not. Then, by Lemma 7.2, x_h is a strongly adjacent node of Type c[7.2] with neighbors in $\tilde{P}_1^c$ and $\tilde{P}_2^c$ and, by Claim 1, $P_{x_1 x_{h-1}}$ has no neighbors in Σ. If v would be in $\mathcal{S}^c$, there would be a $3PC(x_h, s_1^r)$. So $v \in \mathcal{S}^r$. If v is adjacent to x_h, there is a wheel with center x_h. If v is not adjacent to x_h, there is a $3PC(x_h, v)$. This proves Claim 2.

Claim 3 *No node in* $\{x_1, \dots, x_{h-1}\}$ *is adjacent to* s_2^c, s_1^r *or* s_2^r.

Proof of Claim 3: Let x_i, $i < h$, be the node of highest index adjacent to a node $x^* \in \{s_2^c, s_1^r, s_2^r\}$. By symmetry we may assume $x^* \in \{s_2^c, s_1^r\}$. However, if $x^* = s_1^r$, there is a $3PC(t, s_1^r)$ and if $x^* = s_2^c$, there is a $3PC(t_1^r, s_2^c)$. This proves Claim 3.

By Claim 2, (i) holds. If v would be in $\mathcal{S}^r$ then, by Claim 3, there would be a $3PC(t_1^r, s_2^c)$ in $(V(\Sigma) \cup V(P)) \setminus \{s_1^c\}$. So (ii) follows as well. Claims 2 and 3 establish the second part of (iii). The first part of (iii) holds since, when s_1^c has more than one neighbor in $\{x_1, \dots, x_h\}$, there is a wheel with center s_1^c. Finally, suppose (iv) is false. Then by (iii) and symmetry between t and t', no node among $x_1, \dots, x_{n-1}$ has a neighbor in Σ. So there is a $3PC(t, s_1^r)$. $\square$

Lemma 7.6 shows that, up to symmetry, Figure 3 depicts the possible attached direct connections in $\mathcal{P}_v$ where, in Figure 22(a), node s_1^c may not be adjacent to a node x_i of P and node x_h may have two neighbors in P_1^c.

We now characterize the direct connections in $\mathcal{P}_v$, when v is a detached Type b[7.2] node.

Lemma 7.7 *Let* $P = x_1, x_2, \dots, x_n$ *be a direct connection in* $\mathcal{P}_v$, *where* x_1 *is adjacent to a detached Type b[7.2] node* $v \in \mathcal{S}$ *and* x_n *is adjacent to a Type b[7.2] node* $t \in \mathcal{T}^c$. *Then* P *satisfies the following properties:*

- *No node* x_i, $1 \le i \le n$, *is adjacent to a node in* Σ.
- *Node* v *belongs to* $\mathcal{S}^c$.

Proof: Since v is a detached node, no node x_i, $1 \le i \le n$, is adjacent to a node in $V(\Sigma) \setminus \{s_1^c, s_2^c, s_1^r, s_2^r\}$. Let x_l be the node with highest index adjacent to a (unique) node $x^* \in \{s_1^c, s_2^c, s_1^r, s_2^r\}$. By symmetry, we may assume $x^* \in \{s_1^c, s_1^r\}$. If $x^* = s_1^c$, there is a $3PC(s_1^c, t_1^r)$ and if $x^* = s_1^r$, there is a $3PC(s_1^r, t)$. Hence the first part of the lemma follows. The second part now follows immediately for, if $v \in \mathcal{S}^r$, there is a $3PC(t_1^r, s_1^c)$. $\square$

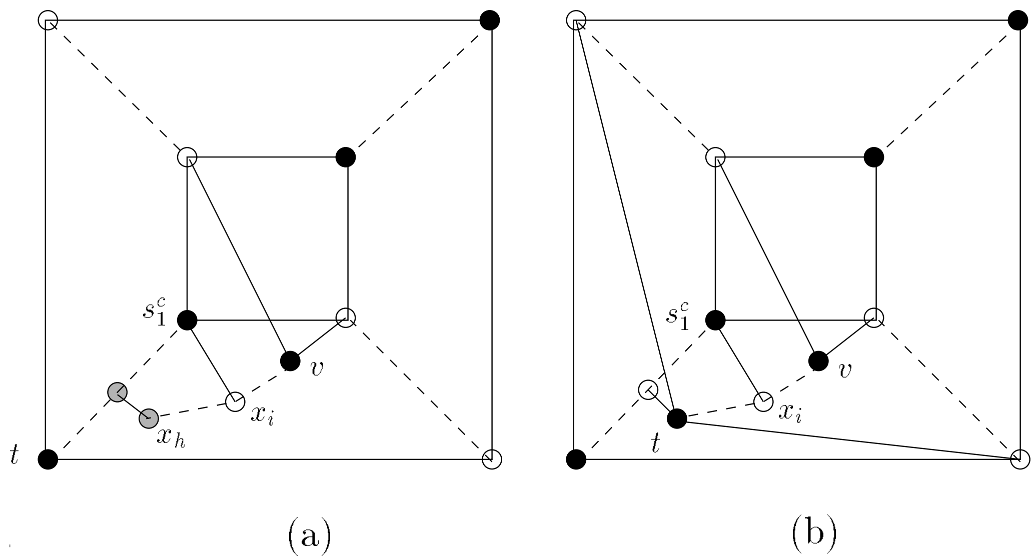


Figure 22: Attached direct connections

7.2 Extreme Connected Squares

Definition 7.8 *A subgraph Σ of G is called* extreme connected squares *if it has the following properties and no other subgraph Σ' of G with $V(\Sigma) \subset V(\Sigma')$ does.*

Two disjoint node sets $S, T \subset V(\Sigma)$ induce bicliques K_S and K_T respectively, and $E(K_S) \cup E(K_T)$ is a 2-join of Σ. Each connected component G_j of $\Sigma \backslash (E(K_S) \cup E(K_T))$ has a nonempty intersection S_j with S and a nonempty intersection T_j with T, and either $S_j \cup T_j \subset V^r$ or $S_j \cup T_j \subset V^c$. Furthermore, at least two connected components G_j satisfy $S_j \cup T_j \subset V^r$ and at least two satisfy $S_j \cup T_j \subset V^c$. Finally, every node of Σ belongs to a chordless path of $\Sigma \setminus (E(K_S) \cup E(K_T))$ with endnodes in S and T.

Using the notation introduced in Definition 7.8, we let $S^r = S \cap V^r$ and we define S^c, T^r and T^c analogously. Σ^r denotes the subgraph of Σ comprising all connected components G_j that have nonempty intersection with $S^r \cup T^r$. We define Σ^c analogously. A chordless path of $\Sigma \setminus (E(K_S) \cup E(K_T))$ with endnodes in S and T is called an *ST-path*. It follows from the definition of extreme connected squares that, if P_i^c, P_j^c are ST-paths in distinct components of Σ^c and P_k^r, P_l^r are ST-paths in distinct components of Σ^r, then $V(P_i^c) \cup V(P_j^c) \cup V(P_k^r) \cup V(P_l^r)$ induces connected squares. Connected squares $CS(P_i^c, P_j^c; P_k^r, P_l^r)$ constructed in this fashion are said to be *contained in* Σ. Let $\mathcal{S}^c$ be the set of nodes in G adjacent to all the nodes in S^r. The sets $\mathcal{S}^r$, $\mathcal{T}^c$ and $\mathcal{T}^r$ are analogously defined. Let $\mathcal{S} = \mathcal{S}^c \cup \mathcal{S}^r$ and $\mathcal{T} = \mathcal{T}^c \cup \mathcal{T}^r$. Note that $S \subseteq \mathcal{S}$ and $T \subseteq \mathcal{T}$.

Lemma 7.9 *Let Σ be extreme connected squares and $\Sigma_{ijkl} = CS(P_i^c, P_j^c; P_k^r, P_l^r)$ be connected squares contained in Σ. Let $Q_i = s, x_1, \ldots, x_n, t$ be a path in G where*

- *node s is adjacent to all nodes of S^r, node t is adjacent to all the nodes of T^r and these are the only adjacencies between nodes of Q_i and Σ^r,*

- *at least one node of Q_i is coincident with or adjacent to a node in the component G_i that contains P_i^c, and*

- *no node of Q_i is adjacent to a node of $V(\Sigma^c) \setminus V(G_i)$.*

Then $V(Q_i) \subseteq V(G_i)$.

Proof: Let Σ' be the graph obtained from Σ by adding $V(Q_i)$ to $V(G_i)$. Define $S' = S \cup \{s\}$, $T' = T \cup \{t\}$. The first condition in Lemma 7.9 shows that $E(K_{S'}) \cup E(K_{T'})$ is a 2-join of Σ'. The second and third conditions in Lemma 7.9 guarantee that the number of connected components of $\Sigma' \setminus (E(K_{S'}) \cup E(K_{T'}))$ is the same as for Σ. Finally, every node of Σ' belongs to a chordless path of $\Sigma' \setminus (E(K_{S'}) \cup E(K_{T'}))$ with endnodes in S' and T'. So Σ' satisfies all the properties of Definition 7.8. Thus $\Sigma' = \Sigma$, since Σ is extreme. $\Box$

Lemma 7.10 *Let Σ be extreme connected squares and let v be a Type a[7.2] node with respect to some connected squares contained in Σ. Then $v \in S \cup T$.*

Proof: Let v be a Type a[7.2] node with respect to $\Sigma_{ijkl} = CS(P_i^c, P_j^c; P_k^r, P_l^r)$ contained in Σ. Assume w.l.o.g. that v is adjacent to a node in $\tilde{P}_i^c$ and to the endnodes s_k^r, s_l^r of P_k^r and P_l^r in S^r. There exists a unique chordless path Q_i contained in $(V(P_i^c) \setminus \{s_i^c\}) \cup \{v\}$, where s_i^c is the endnode of P_i^c in S^c. By Lemma 7.9 applied to Σ_{ijkl} and Q_i, it suffices to show that

(i) v is adjacent to all the nodes in S^r and no other node of Σ^r, and

(ii) v has no neighbor in $V(\Sigma^c) \setminus V(G_i)$.

If (i) does not hold, then either v is not adjacent to some node $s_m^r \in S^r$ (by Definition 7.8, s_m^r belongs to an ST-path P_m^r), or v has a neighbor in the interior of an ST-path P_m^r in Σ^r. Assume w.l.o.g. that the component G_m that contains P_m^r is distinct from the component G_k that contains P_k^r. Now, v is a strongly adjacent node violating Lemma 7.2 in connected squares $CS(P_i^c, P_j^c; P_k^r, P_m^r)$.

If (ii) does not hold, v has a neighbor in an ST-path P_m^c in $\Sigma^c \setminus G_i$. Then v is a strongly adjacent node violating Lemma 7.2 in connected squares $CS(P_i^c, P_m^c; P_k^r, P_l^r)$. $\Box$

Lemma 7.11 *Let Σ be extreme connected squares and let v be an attached Type b[7.2] node with respect to some connected squares contained in Σ. Then $v \in S \cup T$.*

Proof: By Lemma 7.10, we can assume that v is not a Type a[7.2] node with respect to any connected squares contained in Σ. Choose $\Sigma_{ijkl} = CS(P_i^c, P_j^c; P_k^r, P_l^r)$ such that v is attached in Σ_{ijkl} with an attached direct connection $P = x_1, \ldots, x_n$ and $V(P) \setminus V(\Sigma_{ijkl})$ has smallest cardinality among all possible choices of Σ_{ijkl} and P. Let s_i^c and t_i^c denote the endnodes of P_i^c in S^c and T^c respectively. s_j^c, t_j^c, s_k^r, t_k^r, s_l^r and t_l^r are defined analogously. Let x_h be the node of highest index in $V(P) \setminus V(\Sigma_{ijkl})$ (possibly $h = n$). By Lemma 7.10, we can also assume that x_n is adjacent to t_i^c, t_j^c, t_k^r or t_l^r, say t_i^c.

By Lemma 7.9 applied to Σ_{ijkl} and the chordless path contained in $(V(P_i^c) \setminus \{s_i^c\}) \cup V(P) \cup \{v\}$, it suffices to show the following claims.

Claim 1 *No node of P is adjacent to a node in Σ^r.*

Proof of Claim 1: Assume that a node of P is adjacent to a node $z \in \Sigma^r$ and let P^r_m be an ST-path containing z. Let s^r_m and t^r_m be the endnodes of P^r_m. Assume w.l.o.g. that $k \neq m$. Let $\Sigma_{ijkm} = CS(P^c_i, P^c_j; P^r_k, P^r_m)$ and choose the shortest subpath P' of P with endnodes x, y satisfying the following property. $\emptyset \neq N(x) \cap V(\Sigma_{ijkm}) \subseteq V(P^c_i)$, $\emptyset \neq N(y) \cap V(\Sigma_{ijkm}) \subseteq V(P^r_m)$ and $(N(x) \cup N(y)) \cap V(\Sigma_{ijkm}) \neq \{s^c_i, s^r_m\}, \{t^c_i, t^r_m\}$. Note that the existence of such a path is obvious when $h < n$, or when $h = n$ and x_n is strongly adjacent to Σ_{ijkl}. If $h = n$ and x_n has t^c_i as unique neighbor in Σ_{ijkl}, then s^c_i must have a neighbor in P, otherwise there is a $3PC(s^r_k, t^c_i)$. So again a path P' with the desired property exists. Since x has neighbors only in P^c_i and y has neighbors only in P^r_m, by possibly shortening P' and modifying P^c_i and P^r_m accordingly, we can assume that x has a unique neighbor x^* in P^c_i and y has a unique neighbor y^* in P^r_m. Lemma 7.2 shows that even after shortening, P' has length at least 1.

To complete the proof of Claim 1, we show that, for any connected squares $CS(P^c_i, P^c_j; P^r_k, P^r_m)$, the existence of a path P' with the above properties leads to a contradiction.

Assume first that no intermediate node of P' is adjacent to s^c_i, s^r_m, t^c_i or t^r_m. Then x^*, y^* belong to the same side of the bipartition, else $V(\Sigma_{ijkm}) \cup V(P')$ contains a $3PC(x^*, y^*)$. So we can assume w.l.o.g. that $x^*, y^* \in V^r$ and $y^* \neq s^r_m$. Let $H = x^*, x, P', y, y^*, (P^r_m)_{y^* t^r_m}, t^r_m, t^c_j, t^r_k, P^r_k, s^r_k, s^c_i, (P^c_i)_{s^c_i x^*}, x^*$. Now, t^c_i has two neighbors in H, namely t^r_k, t^r_m. So, if t^c_i is adjacent to x^*, we have an odd wheel, and if t^c_i is not adjacent to x^*, we have a $3PC(t^c_i, x^*)$.

So s^c_i, s^r_m, t^c_i or t^r_m is adjacent to some intermediate node of P'. We assume w.l.o.g. that the last case occurs and $x^* = t^c_i$ by the minimality of P'. Then $y^* \in V(P^r_m) \setminus \{t^r_m\}$. Now (H', t^r_m) is a wheel where $H' = x^*, x, P', y, y^*, (P^r_m)_{y^* s^r_m}, s^r_m, s^c_j, P^c_j, t^c_j, t^r_k, x^*$. This proves Claim 1.

Claim 2 *Node v is adjacent to all the nodes in S^r and to no other node in Σ^r.*

Proof of Claim 2: Let P^r_m be any ST-path in Σ^r. By Lemma 7.2, if v has a neighbor in P^r_m, then this neighbor is unique, namely s^r_m. So assume v has no neighbor in P^r_m. It follows from Claim 1 that v, s^r_m and t^r_k are intermediate nodes in the three paths of a $3PC(s^r_k, t^c_i)$. This proves Claim 2.

Claim 3 *No node of $V(P) \cup \{v\}$ is adjacent to a node in $V(\Sigma^c) \setminus V(G_i)$.*

Proof of Claim 3: Node v is not adjacent to any node in $V(\Sigma^c)$ since we have assumed that it is not of Type a[7.2] with respect to any connected squares contained in Σ. Assume next that a node x_q of P is adjacent to a node $v_m \in V(\Sigma^c) \setminus (V(G_i) \cup S^c)$. Let P^c_m be an ST-path containing v_m and let $\Sigma_{imkl} = CS(P^c_i, P^c_m; P^r_k, P^r_l)$. By the choice of P and Σ_{ijkl}, $x_q = x_h$ and no node $x_1, \ldots, x_{h-1}$ is adjacent to a node of $V(P^c_i) \cup V(P^c_m) \setminus S^c$. Now v is an attached Type b[7.2] node for Σ_{imkl} and P is a minimal attached direct connection in Σ_{imkl} violating Lemma 7.6(i). Finally, assume that a node x_q of P is adjacent to a node $v_m \in S^c \setminus V(G_i)$. Let P^c_m be an ST-path containing v_m and let $\Sigma_{imkl} = CS(P^c_i, P^c_m; P^r_k, P^r_l)$. Then again v is an attached Type b[7.2] node for Σ_{imkl} and P is a minimal attached direct connection in Σ_{imkl} violating Lemma 7.6(iii). $\square$

7.3 Biclique Cutsets and 2-Joins

In this subsection, we prove the following theorem:

Theorem 7.12 *Let Σ be extreme connected squares in a wheel-free weakly balanced graph G. Then either $E(K_S) \cup E(K_T)$ is a 2-join of G separating Σ^c from Σ^r, or K_S or K_T is a biclique cutset of G.*

Proof: Note first that by Definition 7.8, the only edges connecting a node of Σ^c and a node in Σ^r are the ones in K_S and K_T. So if $E(K_S) \cup E(K_T)$ is not a 2-join of G separating Σ^c from Σ^r, then Σ^c and Σ^r are joined by a direct connection in $G \setminus (E(K_S) \cup E(K_T))$.

Claim 1 *No node of $G \setminus \Sigma$ is adjacent to a node of Σ^c and a node of Σ^r.*

Proof of Claim 1: Let v be a node contradicting the claim. Let $\Sigma_{ijkl} = CS(P_i^c, P_j^c; P_k^r, P_l^r)$ be connected squares contained in Σ such that v has neighbors in P_i^c and in P_k^r. Then, by Lemma 7.2, node v is of Type a[7.2] with respect to Σ_{ijkl}. Now by Lemma 7.10, node v belongs to Σ, a contradiction. This proves Claim 1.

Claim 2 *If $E(K_S) \cup E(K_T)$ is not a 2-join of G separating Σ^c from Σ^r, and neither K_S nor K_T is a biclique cutset of G, then there exists a path $P = x_1, x_2, \ldots, x_n$, $n > 1$, with at least one of the following properties:*

- *The path P is a direct connection between $\Sigma^c \setminus S^c$ and $\Sigma^r \setminus T^r$, avoiding $S^c \cup T^r$, such that no node x_h, $1 < h < n$, is adjacent to a node in T^r.*
- *The path P is a direct connection between $\Sigma^c \setminus S^c$ and $\Sigma^r \setminus T^r$, avoiding $S^c \cup T^r$, such that no node x_h, $1 < h < n$, is adjacent to a node in S^c.*
- *The path P is a direct connection between $\Sigma^c \setminus T^c$ and $\Sigma^r \setminus S^r$, avoiding $T^c \cup S^r$, such that no node x_h, $1 < h < n$, is adjacent to a node in T^c.*
- *The path P is a direct connection between $\Sigma^c \setminus T^c$ and $\Sigma^r \setminus S^r$, avoiding $T^c \cup S^r$, such that no node x_h, $1 < h < n$, is adjacent to a node in S^r.*

Proof of Claim 2: If $E(K_S) \cup E(K_T)$ is not a 2-join of G, then $G \setminus (E(K_S) \cup E(K_T))$ contains a direct connection $P = x_1, x_2, \ldots, x_n$ between Σ^c and Σ^r, where x_1 is adjacent to a node in Σ^c and x_n is adjacent to a node in Σ^r. (Note that by Claim 1, $n > 1$).

If $(N(x_1) \cup N(x_n)) \cap V(\Sigma) \not\subseteq S$ and $(N(x_1) \cup N(x_n)) \cap V(\Sigma) \not\subseteq T$, then P belongs to at least one of the above four families of direct connections and we are done. So assume w.l.o.g. that $(N(x_1) \cup N(x_n)) \cap V(\Sigma) \subseteq S$, that is, the set $N(x_1) \cap V(\Sigma)$ is contained in S^c and the set $N(x_n) \cap V(\Sigma)$ is contained in S^r.

Since K_S is not a biclique cutset, separating P from $\Sigma \setminus S$, G contains a direct connection $Q = y_1, y_2, \ldots, y_m$ between $V(P)$ and $V(\Sigma) \setminus S$ and avoiding S, where y_1 is adjacent to a node of P and y_m is adjacent to a node of $\Sigma \setminus S$. Note that for all $1 < i < m$, we have that $N(y_i) \cap V(\Sigma) \subset S$ and by Claim 1, we can assume w.l.o.g. that $N(y_m) \cap V(\Sigma) \subseteq V(\Sigma^r)$.

If some intermediate node of Q is adjacent to a node in S^c, let $y_i \neq y_m$ be such a node with highest index. Then $Q_{y_i y_m}$ is a direct connection between $\Sigma^c \setminus T^c$ and $\Sigma^r \setminus S^r$, avoiding $T^c \cup S^r$. Note that by construction, an intermediate node of such subpath cannot be adjacent to a node in T^c.

If no intermediate node in Q is adjacent to a node in S^c, let x_j be the node of lowest index in P, adjacent to y_1. Then the path $R = x_1, P_{x_1 x_j}, x_j, y_1, Q, y_m$ is a direct connection between $\Sigma^c \setminus T^c$ and $\Sigma^r \setminus S^r$, avoiding $T^c \cup S^r$, such that no intermediate node in R is adjacent to a node in T^c. This completes the proof of Claim 2.

Let $P = x_1, x_2, \ldots, x_n$ be a direct connection between $\Sigma^c \setminus T^c$ and $\Sigma^r \setminus S^r$, avoiding $T^c \cup S^r$ such that no node x_h, $1 < h < n$, is adjacent to a node in S^r. So the intermediate nodes of P have no neighbors in $V(\Sigma) \setminus T^c$. By Claim 1, $n > 1$, so x_1 has no neighbor in Σ^r and x_n has no neighbor in Σ^c. There exist i and k such that x_1 has a neighbor in $G_i \setminus T^c$ and x_n has a neighbor in $G_k \setminus S^r$. Let P_i^c be an ST-path of G_i such that x_1 has a neighbor in P_i^c distinct from its endnode t_i^c in T_i and let P_k^r be an ST-path of G_k such that x_n has a neighbor in P_k^r distinct from its endnode s_k^r in S_k. For any ST-path of Σ^c in G_j, $j \neq i$, say P_j^c, and any ST-path of Σ^r in G_l, $l \neq k$, say P_l^r, the connected squares $\Sigma_{ijkl} = CS(P_i^c, P_j^c; P_k^r, P_l^r)$ have the following properties: x_1 has a neighbor in P_i^c distinct from t_i^c and possibly a neighbor in P_j^c, x_n has a neighbor in P_k^r distinct from s_k^r and possibly a neighbor in P_l^r, and intermediate nodes of P may be adjacent to t_i^c and to the endnode t_j^c of P_j^c in T_j but to no other node of Σ_{ijkl}.

Claim 3 *Neither x_1 nor x_n are strongly adjacent nodes of Type c[7.2] with respect to Σ_{ijkl}.*

Proof of Claim 3: Assume that x_1 is Type c[7.2] with respect to Σ_{ijkl}. Let s_i^c, s_j^c and s_k^r denote the endnodes of P_i^c, P_j^c and P_k^r in S_i, S_j and S_k respectively. Then s_i^c, s_j^c and x_2 are intermediate nodes in a $3PC(x_1, s_k^r)$. The proof for x_n is identical.

We now distinguish two cases:

Case 1 *Either $N(x_1) \cap (V(\Sigma) \setminus (S^c \cup T^c)) \neq \emptyset$ or $N(x_1) \cap (V(\Sigma) \setminus T^c) \subseteq S_i$.*

Claim 4 *$N(x_1) \cap V(\Sigma) \subseteq V(G_i)$ and x_1 has a unique neighbor, say x_0, in P_i^c.*

Proof of Claim 4: Assume first $N(x_1) \cap (V(\Sigma) \setminus T^c) \subseteq S_i$. If $N(x_1) \cap V(\Sigma) \nsubseteq V(G_i)$, node x_1 is adjacent to $t_q^c \in T_q$, for some $q \neq i$. Choose an ST-path P_q^c in G_q containing t_q^c as endnode. Now x_1 is a strongly adjacent node, violating Lemma 7.2 in $CS(P_i^c, P_q^c; P_k^r, P_l^r)$. So $N(x_1) \cap V(\Sigma) \subseteq V(G_i)$. If x_1 has more than one neighbor in connected squares Σ_{ijkl}, (i.e. x_1 is adjacent to t_i^c and s_i^c), then since $N(x_1) \cap V(\Sigma) \subseteq V(G_i)$ and Σ is extreme, $x_1 \in V(\Sigma)$, a contradiction.

Assume now that $N(x_1) \cap (V(\Sigma) \setminus (S^c \cup T^c)) \neq \emptyset$. We assume w.l.o.g. that $N(x_1) \cap (V(G_i) \setminus (S^c \cup T^c)) \neq \emptyset$ and we choose an ST-path Q_i^c in G_i so that x_1 has a neighbor x_0 that is an intermediate node of Q_i^c. Then $N(x_1) \cap V(\Sigma) \subseteq V(G_i)$, for otherwise we can choose an ST-path P_q^c in G_q containing a neighbor of x_1, for some $q \neq i$. By Claim 1, x_1 cannot be of Type a[7.2] in $CS(Q_i^c, P_q^c; P_k^r, P_l^r)$ and by Claim 3, x_1 cannot be of Type c[7.2]. So x_1 violates Lemma 7.2 in $CS(Q_i^c, P_q^c; P_k^r, P_l^r)$. Now x_1 can have only one neighbor in P_i^c, for otherwise $N(x_1) \cap V(\Sigma) \subseteq V(G_i)$ and the fact that Σ is extreme would imply that $x_1 \in V(\Sigma)$, a contradiction. This completes the proof of Claim 4.

Claim 5 *No intermediate node of P is adjacent to a node in T^c.*

Proof of Claim 5: Let x_m be a node of P with lowest index $m > 1$, adjacent to a node in T^c. If x_m is adjacent to $t_q^c \in T_q$, for some $q \neq i$, let $CS(P_i^c, P_q^c; P_k^r, P_l^r)$, be connected squares such that $t_q^c \in P_q^c$. If x_m is adjacent to both t_i^c, t_q^c, then x_m is a Type b[7.2] node with respect to $CS(P_i^c, P_q^c; P_k^r, P_l^r)$ and the direct connection $P_{x_1, x_{m-1}}$ violates Lemma 7.6. (Note that, since by Claim 4, x_0 is the unique neighbor of x_1 in $CS(P_i^c, P_q^c; P_k^r, P_l^r)$, then x_m is attached in $CS(P_i^c, P_q^c; P_k^r, P_l^r)$). Node x_m cannot be adjacent to t_q^c only for, if $x_0 \in V^r$, we have a $3PC(x_0, t_q^c)$ and if $x_0 \in V^c$, we have a $3PC(x_0, t_k^r)$. So x_m has no neighbor in $T^c \setminus T_i$.

Let $t_i' \in T_i$ be a neighbor of x_m, let P' be the $x_0 s_i^c$-subpath of P_i^c and Q be the chordless path made up by the concatenation of P' and $P_{x_0 x_m}$. Then x_m is the unique neighbor of t_i' in Q, for otherwise t_i' has a neighbor in P' and (C, t_i') is a wheel, where C is defined as follows:

-If P contains an intermediate node adjacent to t_q^c, for $q \neq i$, let x_h, $m < h$, be such a node with lowest index. Let $C = s_i^c, Q, x_m, P_{x_m x_h}, x_h, t_q^c, t_k^r, P_k^r, s_k^r, s_i^c$.

-If P contains no intermediate node adjacent to t_q^c, for $q \neq i$, let x_{n+1} be the neighbor of x_n, closest to t_k^r in P_k^r. Since $x_{n+1} \neq s_k^r$, then by Lemma 7.2, x_n is not adjacent to s_l^r. Let $C = s_i^c, Q, x_m, P_{x_m x_n}, x_n, x_{n+1}, (P_k^r)_{x_{n+1} t_k^r}, t_k^r, t_q^c, P_q^c, s_q^c, s_l^r, s_i^c$.

Since $N(x_1) \cap V(\Sigma) \subseteq V(G_i)$ by Claim 4, $N(x_m) \cap V(\Sigma) \subseteq T_i$ and x_m is the unique neighbor of t_i' in Q, then $V(Q) \subset V(\Sigma)$ and, since Σ is extreme, this contradicts $x_1 \notin V(\Sigma)$. This completes the proof of Claim 5.

We show that $N(x_n) \cap V(\Sigma) \subseteq V(G_k)$. For, if not, we can choose P_k^r and P_l^r so that x_n has neighbors in both. Now by Claim 1, x_n cannot be of Type a[7.2] in Σ_{ijkl} and by Claim 3, x_n cannot be of Type c[7.2]. Finally x_n cannot be of Type b[7.2], for by Claims 4 and 5, $P_{x_1 x_{n-1}}$ shows that x_n is an attached Type b[7.2] node, a contradiction to Lemma 7.11.

Finally, x_n has a unique neighbor, say x_{n+1}, in P_k^r for, if not, since $N(x_n) \cap V(\Sigma) \subseteq V(G_k)$, then $x_n \in V(\Sigma)$. Since $x_0 \neq t_i^c$ and $x_{n+1} \neq s_k^r$, there is a $3PC(t_j^c, s_l^r)$. So Case 1 cannot happen.

Case 2 $N(x_1) \cap (\Sigma \setminus T^c) \subseteq S^c$, $N(x_1) \cap S_i^c \neq \emptyset$ *and* $N(x_1) \cap S_j^c \neq \emptyset$.

Choose P_i^c and P_j^c so that x_1 is a Type b[7.2] node in Σ_{ijkl}, adjacent to s_i^c and s_j^c. Now, since $x_1 \notin V(\Sigma)$, Lemma 7.11 shows that x_1 is not attached in Σ_{ijkl}. So no subpath of P is an attached direct connection for x_1 in Σ_{ijkl}. Since $\mathcal{P}_{x_1}(\Sigma_{ijkl}) \neq \emptyset$, P contains a node x_m such that $P_{x_2 x_m}$ is a detached direct connection for x_1 in Σ_{ijkl}. So, by Lemma 7.7, x_m is adjacent to both t_i^c, t_j^c and no intermediate node of $P_{x_1 x_m}$ is adjacent to t_i^c or t_j^c.

Claim 6 $N(x_1) \cap V(\Sigma) = S^c$, $N(x_m) \cap V(\Sigma) = T^c$ *and no intermediate node of* $P_{x_1 x_m}$ *is adjacent to a node of* Σ.

Proof of Claim 6: We first show $N(x_1) \cap V(\Sigma) = S^c$. If not, choose P_q^c not containing a neighbor of x_1 and by symmetry we can assume $q \neq i$. Then in $\Sigma_{iqkl} = CS(P_i^c, P_q^c; P_k^r, P_l^r)$, s_i^c is the unique neighbor of x_1. Now x_1 satisfies Case 1 in the subgraph G' of G induced by $V(\Sigma_{iqkl}) \cup V(P)$ and Σ_{iqkl} is extreme in G'. This is not possible since Case 1 cannot happen.

Assume now that a node x_h, $1 < h < m$, is adjacent to t_q^c (we assume that h is the lowest such index) or x_m is not adjacent to t_q^c. We assume w.l.o.g. that $q \neq i$ and we choose P_q^c containing t_q^c. Then since $N(x_1) \cap V(\Sigma) = S^c$, node x_1 is a Type b[7.2] adjacent to s_i^c, s_q^c in $\Sigma_{iqkl} = CS(P_i^c, P_q^c; P_k^r, P_l^r)$ and $P_{x_1 x_m}$ in the first case, $P_{x_1 x_h}$ in the second case, shows that x_1 is attached in Σ_{iqkl}. By Lemma 7.11, $x_1 \in V(\Sigma)$ and this completes Claim 6.

Let Σ' be the graph induced by $V(\Sigma) \cup V(P_{x_1 x_m})$. Define $S' = S \cup \{x_m\}$ and $T' = T \cup \{x_1\}$. By Claim 6, S' and T' induce bicliques and $E(K_{S'}) \cup E(K_{T'})$ is a 2-join of Σ'. The connected components of $\Sigma' \setminus (E(K_{S'}) \cup E(K_{T'}))$ satisfy the conditions of Definition 7.8 and every node of Σ' belongs to an $S'T'$-path of Σ'. Now $V(\Sigma) \subset V(\Sigma')$ contradicts the choice of Σ as extreme connected squares. $\square$

8 Goggles

8.1 Introduction

Goggles are formed by a parachute with short top, long sides and long middle, together with a direct connection from bottom to top of Type j[6.4]. In this section, we assume that G is a weakly balanced graph that contains goggles $\Gamma = Go(P, Q, R, S, T)$, see Figure 23. We use the following notation: $P = x, \ldots, i, h$, $Q = a, \ldots, v$, $R = b, \ldots, v$, $S = u, \ldots, j, h$ and $T = h, k, \ldots, v$.

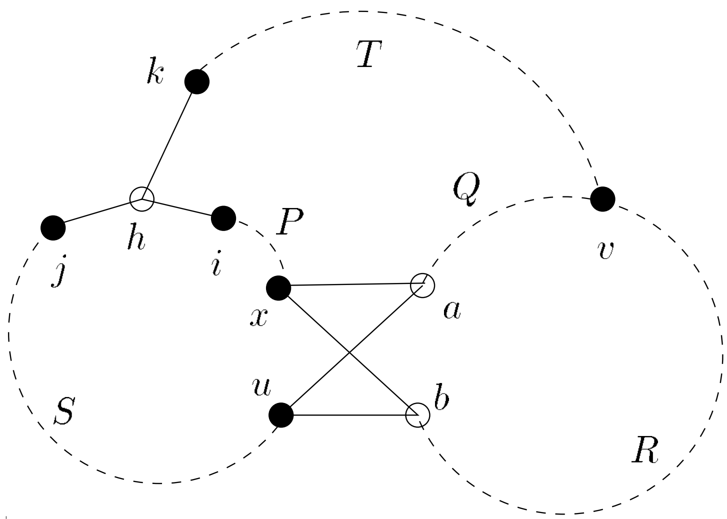


Figure 23: Goggles

The paths P, Q, R, S have length greater than 1, but the path T may be of length 1 in which case $h \in N(v)$ and $k = v$. It will be convenient to denote the length of T by $|T|$, since we will often distinguish the cases $|T| = 1$ and $|T| > 1$. Assume w.l.o.g. that $a \in V^r$. Then $x, u, v \in V^c$ and $b, h \in V^r$. Further, we assume that G contains

- no connected squares,
- no extended star cutset.

Since G does not contain an extended star cutset, it follows from Sections 3 and 4 that G contains

- no wheel,
- no parachute with long sides and long top,
- no parachute with long sides, short top and short middle.

The main result of this section is that G has a 2-join.

This is achieved by considering goggles Γ with shortest possible top path T (among all goggles in G) and, subject to this, it is assumed that Γ has the fewest number of nodes. Throughout this section, this assumption is made about Γ.

We show how certain 2-joins of Γ extend to the full graph G. This requires an understanding of the possible paths connecting nodes of Γ. In order to identify these paths, we first list the strongly adjacent nodes to Γ. This is done in the next subsection. In Subsection 8.3, we describe paths connecting nodes of Γ, starting from a strongly adjacent node, and in Subsection 8.4, those starting from node h. In Subection 8.5, we identify candidate bicliques for a 2-join of G. Finally, in Subsections 8.6 and 8.7, we prove the 2-join theorem.

Our assumptions on G imply the following results about parachutes:

Corollary 8.1 *If Π is a parachute with long sides, long middle, short top v_1, t, v_2 and center node v then, in G, every direct connection between the bottom of Π and the top node t avoiding $N(v) \cup (N(v_1) \cap N(v_2)) \setminus \{t\}$ is of Type j[6.4] or j1[6.5]. Furthermore, Π has at least one such direct connection.*

Proof: This is a consequence of Theorem 6.17 and the assumption that G contains no extended star cutset and no connected squares. $\Box$

Lemma 8.2 *Let Π be a parachute with long sides, long middle, short top v_1, t, v_2 and center node v. There exists no chordless path $x_1, \ldots, x_m$ such that*

(i) *$m \geq 2$ and $x_p \in V(G) \setminus V(\Pi)$, $1 \leq p \leq m$,*

(ii) *nodes x_1 and x_m each have a unique neighbor in Π, and these two neighbors are distinct nodes in the set $\{t, v, v_1, v_2\}$,*

(iii) *for $2 \leq p \leq m-1$, node x_p has no neighbor in Π.*

Proof: Assume such a path exists.

Case 1: Node x_1 is adjacent to v_1, v_2 or v, and x_m is adjacent to t.

Since G has no extended star cutset, there exists a direct connection $Y = y_1, \ldots, y_n$ between the bottom of Π and $\{x_1, \ldots, x_m\}$ avoiding $N(v) \cup (N(v_1) \cap N(v_2))$. This implies a direct connection $W = w_1, \ldots, w_l$ from the bottom of Π to $\{t\}$ avoiding $N(v) \cup (N(v_1) \cap N(v_2)) \setminus \{t\}$. By Corollary 8.1, the direct connection W is of Type j[6.4] or j1[6.5].

If W is of Type j[6.4], let $s \in M$ be the neighbor of y_1. Since s is not adjacent to v, there is a $3PC(v, s)$ irrespective of whether Y contains neighbors of v_1, v_2 or t.

If W is of Type j1[6.5], there is a $3PC(v, y_1)$.

Case 2: Node x_1 is adjacent to either v_1 or v_2, say v_1, and x_m is adjacent to v or v_2.

If x_m is adjacent to v_2, there is a parachute with long sides and long top $x_1, \ldots, x_m$, a contradiction. Consider now the case where x_m is adjacent to v. By Corollary 8.1, there exists a direct connection $Y = y_1, \ldots, y_n$ of Type j[6.4] or j1[6.5] between the bottom of Π and t. Let $s \in M$ be the neighbor of y_1 closest to the bottom node z of Π. No node of Y is coincident with or adjacent to one of the nodes $x_1, \ldots, x_m$, for otherwise there is a $3PC(v, s)$ or $3PC(v, y_1)$. Consider the parachute with top path v_1, v, v_2, same side paths as Π and middle path $t, y_n, \ldots, y_1, s, M_{sz}, z$. Now the result follows from Case 1. $\Box$

8.2 Strongly Adjacent Nodes

Lemma 8.3 *Let $w \in V(G) \setminus V(\Gamma)$ be a strongly adjacent node to Γ. Then, one of the following holds:*

(i) *Node w is a twin of a node of Γ, relative to Γ.*

(ii) *Node w is of one of the following types, see Figure 24.*

Type a *Node w has exactly two neighbors in Γ and w is adjacent to x and u (a and b resp.).*

Type b *Node w has exactly two neighbors in Γ and is adjacent to the two neighbors of h (v resp.) in P, S (Q, R resp.).*

Type c *Node w has exactly two neighbors in Γ, one of them is the neighbor of h (v resp.) in T and the other is the neighbor of h (v resp.) in either P or S (Q or R resp.).*

Type d *$w \in V^c$ ($w \in V^r$ resp.) has exactly two neighbors in Γ, one of them in $P \setminus \{h\}$ and the other in $S \setminus \{h\}$ ($Q \setminus \{v\}$ and $R \setminus \{v\}$ resp.).*

Proof: We consider first the case where w has two neighbors in Γ and then the case where w has three or more neighbors.

Case 1 Node w has two neighbors in Γ, say α and β.

If α and β belong to T, then w must be a twin, otherwise there are goggles with shorter top. If α and β belong to the same path P, Q, R, S, then w must be a twin, otherwise there are goggles with fewer nodes and same top. Now assume no path P, Q, R, S, T contains both α and β. Because of the symmetry between paths P, Q, R and S, we can assume w.l.o.g. that $\alpha \in V(P)$.

Assume first that $\beta \in V(S)$. If $w \in V^c$, then w is of Type d. Suppose now $w \in V^r$. If $\alpha = x$, then $\beta = u$ for otherwise we have a $3PC(x, h)$. Thus node w is of Type a. Suppose now $\alpha \neq x$. By symmetry, $\beta \neq u$. Now, if α (β resp.) is not adjacent to h, there is a $3PC(h, \alpha)$ ($3PC(h, \beta)$ resp.). Hence both α and β are neighbors of h. Thus node w is of Type b.

If $\beta \in V(T)$, then $w \in V^r$, otherwise there are goggles with shorter top path $T_{v\beta}$. If α (β resp.) is not adjacent to h, there is $3PC(h, \alpha)$ ($3PC(h, \beta)$ resp.). Hence, both α and β are neighbors of h. Thus node w is of Type c.

Finally, if $\beta \in V(Q) \cup V(R)$, because of symmetry, we can assume that $\beta \in V(Q)$ and $w \in V^c$. If β is not adjacent to v, there is a $3PC(v, \beta)$. Hence $\beta \neq a$. Now if $\alpha \neq h$ or if $|T| > 1$, there is a $3PC(x, \beta)$. Hence it follows that $\alpha = h$ and $\beta, h \in N(v)$. Then w is of Type c.

Case 2 Node w has three or more neighbors in Γ.

Clearly w has at most one neighbor in each of the paths P, Q, R, S, T, for otherwise there is a wheel. We now consider two cases depending upon whether $|N(w) \cap T| = 0$ or 1.

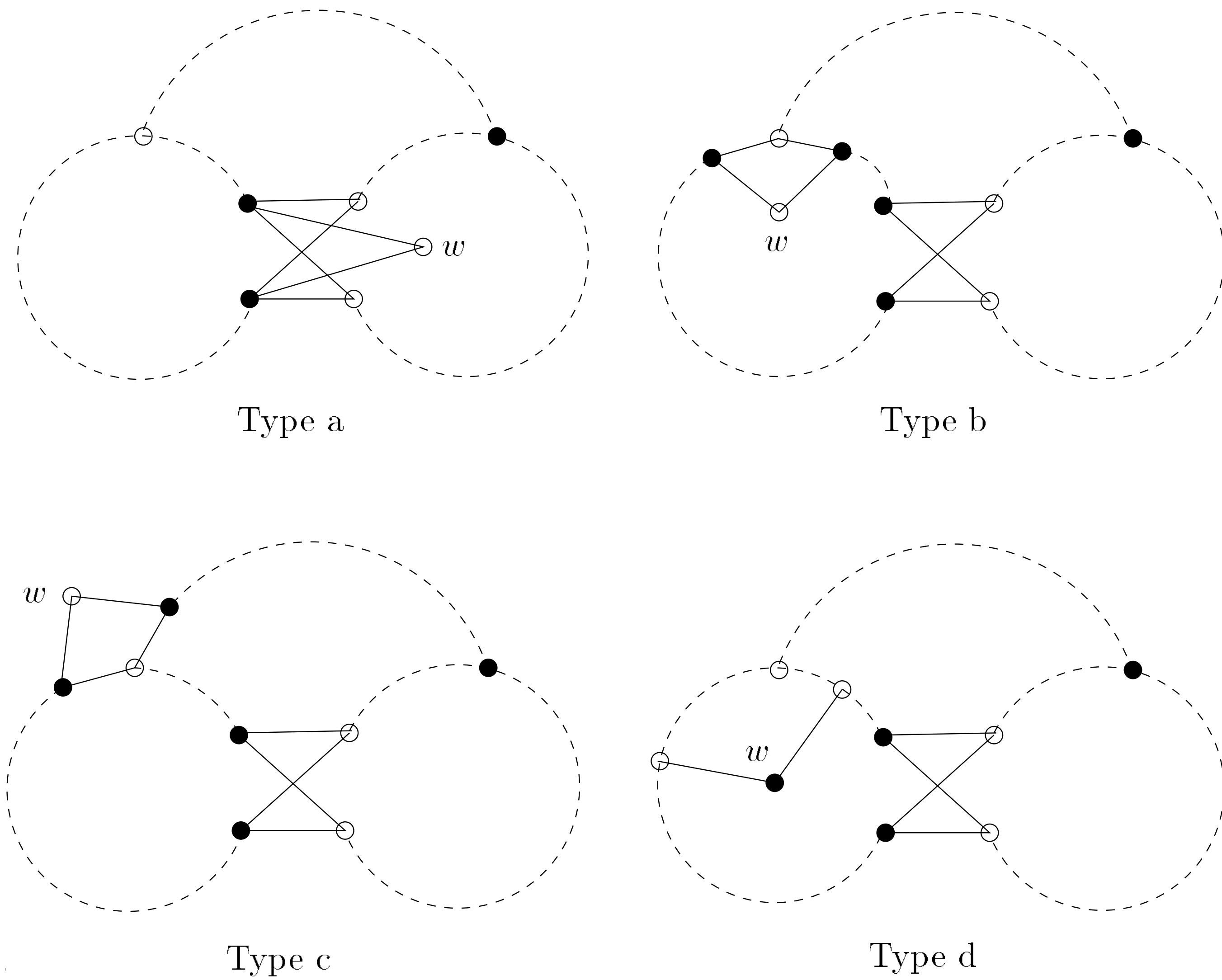


Figure 24: Strongly adjacent nodes

Case 2.1 $|N(w) \cap V(T)| = 1$.

Now w has neighbors in at least two different paths P, Q, R, S. Because of symmetry, we assume that w has exactly one neighbor in $V(P) \setminus \{h\}$. It follows that w is not adjacent to h nor to $(V(Q) \cup V(R)) \setminus \{v\}$ for otherwise there is a wheel. This implies that w has exactly one neighbor in $V(S) \setminus \{h\}$ and three neighbors in Γ. If $w \in V^c$, there is a $3PC(w, h)$. Hence $w \in V^r$. Let α, β, γ be the neighbors of w in P, S and T respectively. We now consider the following two subcases.

Assume first α or β is a neighbor of h, say $\alpha \in N(h)$. If $\beta = u$, there is a parachute with short middle path w, u, a, where w is the center node and a the bottom node. Suppose now $\beta \neq u$. If $\gamma \notin N(h)$, there are goggles with a shorter top path, and if $\gamma \in N(h)$ and $\beta \notin N(h)$, there are goggles with fewer nodes but a top path of the same length as T. So w must be a twin of h.

Assume now that neither α nor β is a neighbor of h. If $\alpha \neq x$ or $\beta \neq u$, say $\alpha \neq x$, we have a parachute with long top $\gamma, T_{\gamma h}, hP_{h\alpha}, \alpha$ and long sides, with center node w and bottom node a. If $\alpha = x$ and $\beta = u$, consider the parachute with side paths P and S, center node w, middle path $w, \gamma, T_{\gamma h}, h$ and top path x, b, u. If $\gamma = k$, this parachute has long sides and short middle path. So $|T| \geq 2$ and $\gamma \neq k$. Now, by Corollary 8.1, there exists a direct connection Y of Type j[6.4] or j1[6.5] between b and $T_{\gamma h} \setminus \{\gamma, h\}$. This parachute and the path Y induce goggles with a shorter top path than T.

Case 2.2 $|N(w) \cap V(T)| = 0$.

Clearly, $h, v \notin N(w)$. Suppose w has four neighbors in Γ, one in each of the paths P, Q, R, S. Because of symmetry, we can assume w.l.o.g. that $w \in V^c$. This implies a $3PC(w, h)$. Consequently we can assume w.l.o.g. that w has no neighbor in Q and has exactly one neighbor in P, R and S. If $w \in V^c$, there is a $3PC(w, h)$. Hence $w \in V^r$. Let α, β and γ be the neighbors of w in P, S and R respectively. If $\alpha = x$ and $\beta = u$, then either w is a twin of a or b or there are goggles with fewer nodes than Γ but same top. Suppose now $\alpha \neq x$ or $\beta \neq u$, say $\alpha \neq x$. If $\beta \notin N(h)$, there is a $3PC(a, \alpha)$. If $\beta \in N(h)$ and $\alpha \notin N(h)$, there is a $3PC(a, \beta)$. Hence $\alpha, \beta \in N(h)$. Now there are connected squares, which is a contradiction.

$\square$

Remark 8.4 *There cannot exist nodes w and z of Type b[8.3] or Type c[8.3] having exactly one common neighbor in Γ.*

Proof: Otherwise there is a wheel with center h or v. $\square$

Remark 8.5 *When $|T| = 1$, if w and z are Type c[8.3] nodes adjacent to h and v respectively, then w and z are adjacent.*

Proof: If w and z are not adjacent, there is a violation of Lemma 8.2, as follows. W.l.o.g. assume w is adjacent to the neighbor of v in Q, say t, and assume that z is adjacent to i. The parachute has top h, w, t, side paths S and t, Q_{ta}, a, u, center node v and middle path v, R, b, u. The extra path is h, i, z, v. $\square$

8.3 Direct Connections from Strongly Adjacent Nodes of Type a, b and c

Lemma 8.6 *Let w be a Type a[8.3] node adjacent to a and b. Let W be the node set consisting of the twins of a and b and Type a[8.3] nodes, but not nodes a, b and w. In $G \backslash \{wa, wb\}$, every direct connection $X = x_1, \ldots, x_n$ between w and $V(\Gamma)$ avoiding W is one of the following types, see Figure 25.*

Type 1 *$n = 1$ and node x_1 is adjacent to u or x but not strongly adjacent to Γ, or $n = 2$ and x_2 is a twin of u or x.*

Type 2 *Node x_n has a unique neighbor $t \in V^r$ in Γ and $t \in V(P) \cup V(S)$, or node $x_n \in V^r$ is a twin of a node in $V(P) \cup V(S)$.*

Furthermore, in $G \setminus \{wa, wb\}$, there exists a direct connection of Type 2 between w and $V(\Gamma)$.

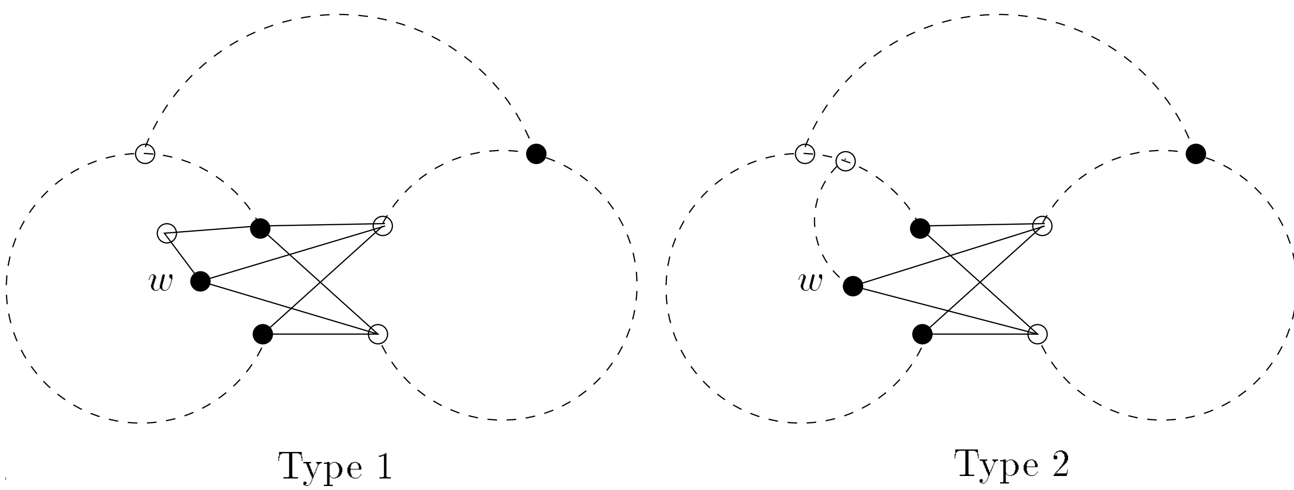


Figure 25: Direct connections from a Type a node

Proof: Let Π_x be the parachute with top path a, w, b, side paths Q and R, and middle path x, P, h, T, v. Similarly, let Π_u be the parachute with top path a, w, b, side paths Q and R, and middle path induced by u, S, h, T, v.

Assume first that x_n is adjacent to u or x, say x. Since X avoids W, x_n is not a twin of a or b nor a Type a[8.3] node. So either x_n is a twin of the neighbor of x in P, and then X is of Type 2, or x is the unique neighbor of x_n in Γ, and then X is of Type 1, since $n > 1$ would violate Lemma 8.2 in Π_x.

Assume next that x_n is a twin of u or x, say x. Then X is of Type 1, since $n > 2$ would imply a violation of Lemma 8.2 in the parachute obtained from Π_x by substituting x_n for x.

Assume now that x_n is not adjacent to u or x, and is not a twin of u or x. If the neighbors of x_n in Γ all lie in S or all lie in R, then the path X is of Type 2. If x_n has a or b as unique neighbor in Γ, there is a violation of Lemma 8.2 in Π_x. So x_n has a neighbor in Γ distinct from a, b, x and the neighbor of x in P. This implies that X is a direct connection from the bottom of Π_x to the top avoiding $N(x) \cup ((N(a) \cap N(b)) \setminus \{w\})$. Therefore, by Corollary 8.1, X is of Type j[6.4] or j1[6.5] in the parachute Π_x. Similarly, X is of Type j[6.4] or j1[6.5] in the parachute Π_u. So x_n has its neighbors in $V(P) \cup V(S) \cup V(T)$. Furthermore, if x_n has two neighbors in $V(P) \cup V(T)$ or in $V(S) \cup V(T)$, then $x_n \in V^r$. And, if x_n has only one neighbor in $V(P) \cup V(T)$ and one in $V(S) \cup V(T)$, then $x_n \in V^c$. It follows that, if the only neighbors of x_n are in $V(T) \setminus \{h\}$, there are goggles with shorter top. It also follows that x_n cannot be of Type b or of Type c[8.3]. If x_n is of Type d[8.3], there is a $3PC(x_n, h)$. Thus path X is of Type 2.

To complete the proof of the theorem, note that, by applying Corollary 8.1 to Π_x, a path X of Type 2 must exist. □

Lemma 8.7 *Let w be a Type b[8.3] node adjacent to i and j. Let W be the node set consisting of the twins of i and j and Type b[8.3] nodes adjacent to i and j, but not nodes i, j and w. Then the top path T has length greater than 1 and, in $G \setminus \{wi, wj\}$, every direct connection X between w and $V(\Gamma)$, avoiding W is one of the following types, see Figure 26.*

Type 1 *$n = 1$ and node x_1 is adjacent to h but not strongly adjacent to Γ, or $n = 2$ and x_2 is a twin of h.*

Type 2 *Either x_n has a unique neighbor $t \in V^c$ in Γ and t is in T or $x_n \in V^c$ is a twin of a node in T.*

Furthermore, in $G \setminus \{wi, wj\}$, there exists a direct connection of Type 2 such that x_n is not adjacent to k and x_n is not a twin of k.

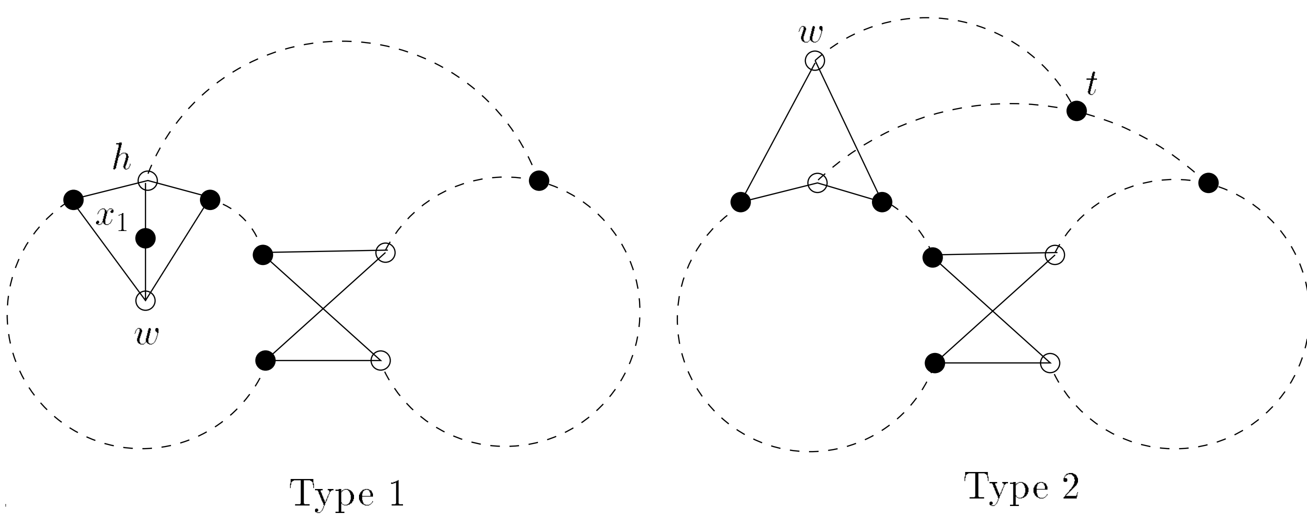


Figure 26: Direct connections from a Type b node

Proof: Let Π_a be the parachute with top path i, w, j, side paths i, P_{ix}, x, a and j, S_{ju}, u, a, center node h and middle path h, T, v, Q, a. Π_b is defined similarly, with bottom node b and middle path h, T, v, R, b.

Assume first that x_n is adjacent to h. Since X avoids W, x_n is not a twin of i or j. So there are three possibilities. When h is the unique neighbor of x_n in Γ, the path X is of Type 1, since $n > 1$ would violate Lemma 8.2 in Π_a. When x_n is a twin of k, the path X is of Type 2. Finally, when $|T| = 1$ and x_n is of Type c[8.3], there is an odd wheel with center h.

Assume next that x_n is a twin of h. Then X is of Type 1, since $n > 2$ would imply a violation of Lemma 8.2 in the parachute obtained from Π_a by substituting x_n for h.

Assume now that x_n is not adjacent to h and is not a twin of h. If x_n has k as unique neighbor in Γ, path X is of Type 2. If x_n has i or j as unique neighbor in Γ, there is a violation of Lemma 8.2 in Π_a. By Remark 8.4, x_n is not of Type c[8.3] with neighbors i and j. So, x_n has a neighbor in Γ distinct from h, i, j, k. This implies that X is a direct connection from the bottom of Π_a (and Π_b) to the top, avoiding $N(h) \cup ((N(i) \cap N(j)) \setminus \{w\})$. Therefore, by Corollary 8.1, X is of Type j[6.4] or j1[6.5] in Π_a and in Π_b. x_n cannot be of Type a[8.3] by Lemma 8.6, it cannot of Type d[8.3] since there would be a $3PC(x_n, v)$, and it cannot be of Type b or c[8.3] since $x_n \in V^c$ would violate Corollary 8.1 in Π_a or Π_b. If x_n is adjacent to $(V(Q) \cup V(R)) \setminus \{v\}$, there are connected squares. So x_n must have all its neighbors in T. Thus path X is of Type 2.

Finally, by Corollary 8.1 applied to Π_a, a path X of Type 2 must exist such that x_n is not adjacent to k and x_n is not a twin of k. This implies that T is of length greater than 1. $\square$

Lemma 8.8 *Let w be a Type c[8.3] node adjacent to i and k. Let W be the node set consisting of the twins of i and k and Type c[8.3] nodes adjacent to i and k, but not nodes i, k and w. In $G \setminus \{wi, wk\}$, every direct connection X between w and $V(\Gamma)$, avoiding W is one of the following types, see Figure 27.*

Type 1 *Node x_1 is adjacent to h but not strongly adjacent to Γ, or $n = 2$ and x_2 is a twin of h.*

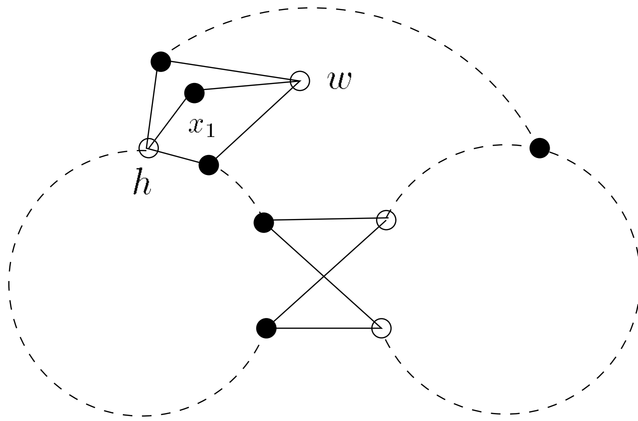
Type 2 *Either x_n has a unique neighbor $t \in V^c$ in Γ and t is in S or $x_n \in V^c$ is a twin of a node in S.*

Type 3 *Node x_1 is a Type c[8.3] node. The top path T of Γ is of length 1. Node x_1 is adjacent to h and to the node in $Q \cap N(v)$ or to the node in $R \cap N(v)$.*

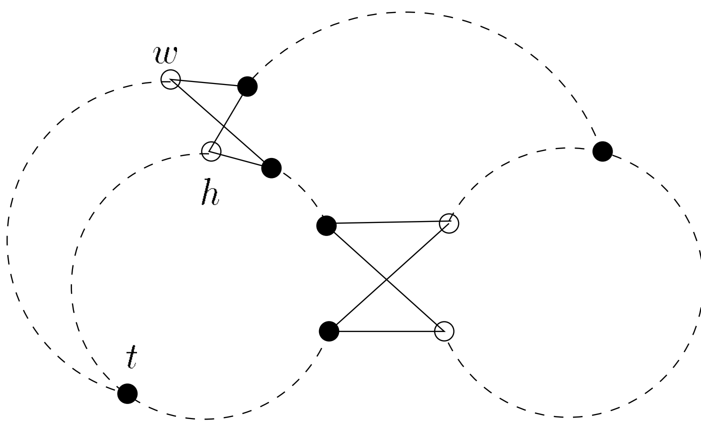
Furthermore, in $G \setminus \{wi, wj\}$, there exists a direct connection of Type 2 such that x_n is not adjacent to j and x_n is not a twin of j.

Proof: Let Π_a be the parachute with top path i, w, k, side paths i, P_{ix}, x, a and k, T_{kv}, v, Q, a, center node h and middle path h, S, u, a. Π_b is defined similarly, with bottom node b.

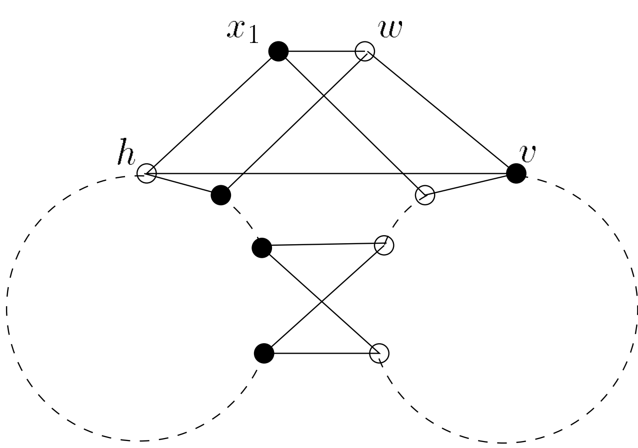
Assume first that x_n is adjacent to h. Since X avoids W, x_n is not a twin of i or k. So, there are three possibilities. When h is the unique neighbor of x_n in Γ, the path X is of Type 1, since $n > 1$ would violate Lemma 8.2 in Π_a. When x_n is a twin of j, the path X is of Type 2. Finally, when $|T| = 1$ and x_n is of Type c[8.3], $n = 1$ by Remark 8.5 and the path X is of Type 3.



Type 1



Type 2



Type 3

Figure 27: Direct connections from a Type c node

Assume next that x_n is a twin of h. Then X is of Type 1, since $n > 2$ would imply a violation of Lemma 8.2 in the parachute obtained from Π_a by substituting x_n for h.

Assume now that x_n is not adjacent to h and is not a twin of h. If x_n has j as unique neighbor in Γ, we have a path X of Type 2. If x_n has i or k as unique neighbor in Γ, there is a violation of Lemma 8.3 in Π_a. So x_n has a neighbor in Γ distinct from h, i, j, k. This implies that X is a direct connection from the bottom of Π_a (and Π_b) to the top, avoiding $N(h) \cup ((N(i) \cap N(j)) \setminus \{w\}$. Therefore, by Corollary 8.1, X is of Type j[6.4] or j1[6.5] in Π_a and in Π_b. x_n cannot be of Type a[8.3] by Lemma 8.6. Thus path X is of Type 2.

Finally, by Corollary 8.1 applied to Π_a, a path X of Type 2 must exist such that x_n is not adjacent to j and x_n is not a twin of j. □

8.4 Partition of the Neighbors of h

Let $Z(h)$ comprise the nodes of $N(h)$ together with the nodes with at least two neighbors in $\{i, j, k\}$. By Remark 8.4, $Z(h)$ is an extended star with center h.

Let $H(h)$ be the set of nodes of $G \setminus V(\Gamma)$ that have h as unique neighbor in Γ.

Lemma 8.9 *Suppose $|T| > 1$ and let $y \in H(h)$. Every direct connection $Y = y_1, \ldots, y_n$ from y to $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h) \setminus \{y\}$ is one of the following types, see Figure 28.*

Type 1 *Node y_n either has a unique neighbor in $V(P) \cup V(S) \setminus \{h, i, j\}$ or is a twin of a node in $V(P) \cup V(S) \setminus \{h, i, j\}$. Furthermore, if y_n has a neighbor in P, then no node of Y is adjacent to j or k, and if y_n has a neighbor in S, then no node of Y is adjacent to i or k.*

Type 2 *Node y_n is of Type a[8.3], adjacent to a and b and no node of Y is adjacent to i, j or k.*

Type 3 *Node y_n is adjacent to $p \in V(T) \setminus \{h, k\}$ but is not of Type c[8.3], or y_n is a Type b[8.3] node adjacent to the neighbors of v in Q and R. If y_n is adjacent to $p \in V(T) \setminus \{h, k\}$, then no node of Y is adjacent to i or j. If y_n is a Type b[8.3] node, then no node of Y is adjacent to i, j, k.*

Proof: There are two cases to consider.

Case 1 Node y_n has only one neighbor p in $V(\Gamma) \setminus \{h, i, j, k\}$.

Suppose $p \in (P \cup S) \setminus \{h, i, j\}$. W.l.o.g. assume $p \in P \setminus \{h, i\}$. If any of the nodes in Y is adjacent to j or k, there is a $3PC(a, j)$ or $3PC(a, k)$. Thus path Y is of Type 1.

Suppose $p \in T \setminus \{h, k\}$. If Y has a node adjacent to i or j, there is a $3PC(a, i)$ or a $3PC(a, j)$. Hence path Y is of Type 3.

Suppose $p \in (Q \cup R) \setminus \{v\}$. W.l.o.g. assume $p \in Q \setminus \{v\}$. If Y has a node adjacent to i or j, there is a $3PC(a, i)$ or a $3PC(a, j)$. If none of the nodes in Y is adjacent to k, there is a $3PC(h, v)$ if $p \neq a$ and a $3PC(a, v)$ if $p = a$. So, let t be the largest index such that y_t is adjacent to k. If $p \in V^r$, there is a $3PC(k, p)$. So p is not adjacent to v and there are goggles with a shorter top path h, k obtained from Γ by replacing Q by $a, Q_{ap}, p, y_n, Y_{y_n y_t}, y_t, k$ and R by b, R, v, T_{vk}, k.

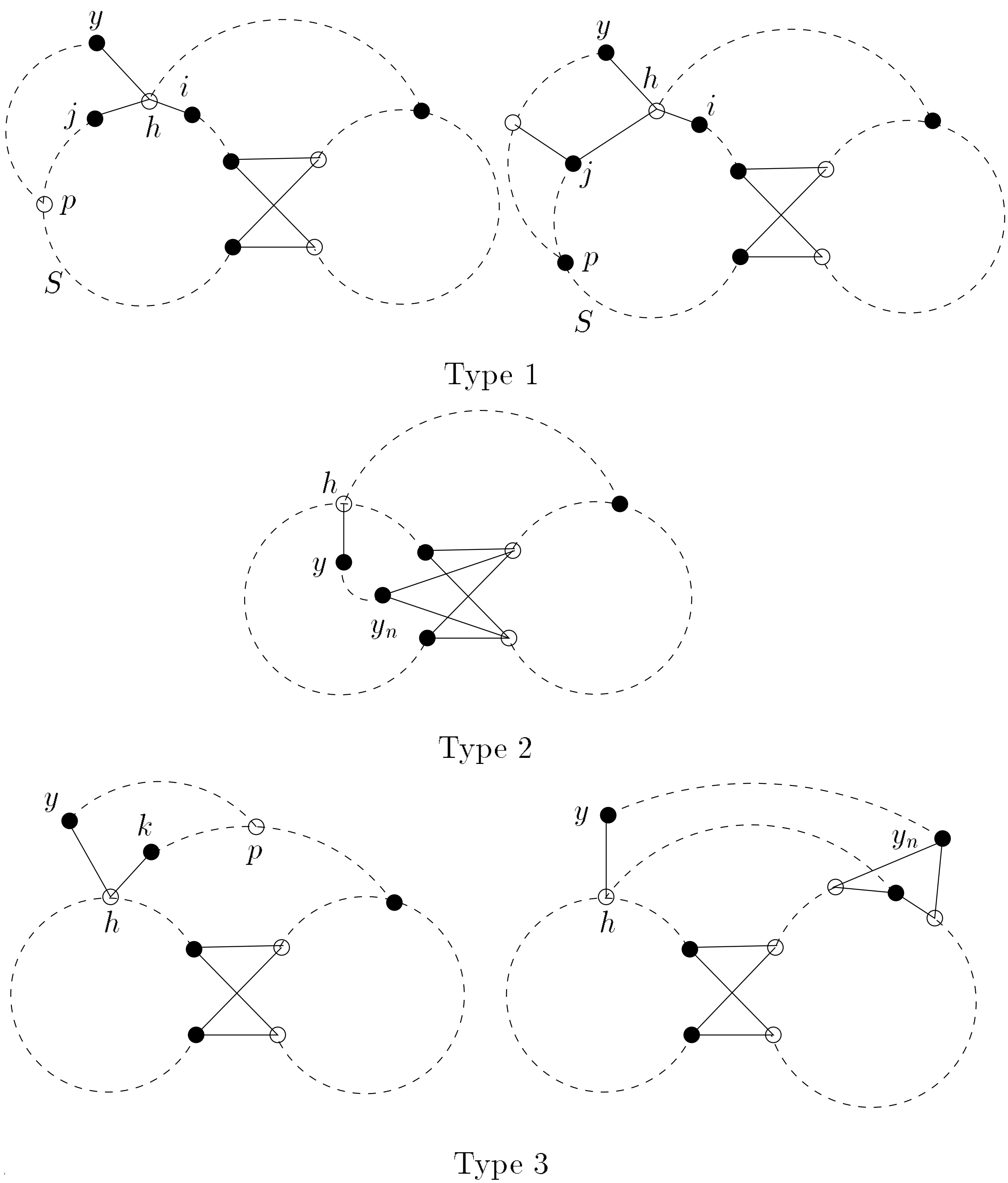


Figure 28: Direct connections from $y \in H(h)$ when $|T| > 1$

Case 2 Node y_n has at least two neighbors in $V(\Gamma) \setminus \{h, i, j, k\}$.

Assume first that y_n is a twin of a node d of Γ. Let Γ' be the goggles obtained from Γ by substituting y_n for d. If $n = 1$, node y_1 must be adjacent to i, j or k, for otherwise y is a strongly adjacent node violating Lemma 8.3 in Γ'. If y_1 is adjacent to k, we get a path of Type 3, otherwise we get a path of Type 1. Now assume $n \geq 2$. If y_{n-1} has a unique neighbor in Γ', applying Case 1 to $Y' = y_1, \ldots, y_{n-1}$ and Γ', we get that Y is of Type 1 or 3 in Γ. If y_{n-1} is strongly adjacent to Γ', then y_{n-1} is adjacent to i, j or k and, by Lemma 8.3 applied to Γ', y_{n-1} is a twin of a node d' in Γ'. Now, either the path $y_1, \ldots, y_{n-1}$ is reduced to a single node, in which case Y is of Type 1 or 3 in Γ, or y_{n-2} has a unique neighbor in the goggles Γ'' obtained from Γ' by replacing d' by y_{n-1} and again, by applying Case 1 to $Y'' = y_1, \ldots, y_{n-2}$ and Γ'', we obtain that Y is of Type 1 or 3 in Γ.

Assume now node y_n is a Type a[8.3] node. Suppose y_n is adjacent to x and u. There is a contradiction to Lemma 8.6, irrespective of whether any of the nodes in Y is adjacent to i, j or k. Suppose y_n is adjacent to a and b. If any of the nodes in Y is adjacent to i, j or k, we have a violation of Lemma 8.6. Otherwise we have a Type 2 path.

Assume node y_n is a Type b[8.3] node adjacent to the neighbors of v in Q and R, say q and r. Then no node in Y is adjacent to i or j, otherwise there is a $3PC(a, i)$ or $3PC(a, j)$. Now no node in Y is adjacent to k, otherwise there is a $3PC(q, k)$. Thus path Y is of Type 3.

Assume node y_n is a Type c[8.3] node adjacent to the neighbors of v in Q and T. This contradicts Lemma 8.8, irrespective of whether or not a node of Y is adjacent to i, j or k.

Finally, assume node y_n is a Type d[8.3] node. Irrespective of whether or not a node of Y is adjacent to i, j or k, there is a $3PC(y_n, x)$ if the neighbors of y_n are in $V(Q) \cup V(R)$, and there is a $3PC(a, y_n)$ if they are in $V(P) \cup V(S)$.

$\square$

Lemma 8.10 *Suppose $|T| = 1$ and let $y \in H(h)$. Every direct connection $Y = y_1, \ldots, y_n$ from y to $V(\Gamma) \setminus \{h, i, j, v\}$ avoiding $Z(h) \setminus \{y\}$ is one of the following types, see Figure 29.*

Type 1 *Node y_n either has a unique neighbor in $(V(P) \cup V(S)) \setminus \{h, i, j\}$ or is a twin of a node in $(V(P) \cup V(S)) \setminus \{h, i, j\}$. Furthermore, if y_n has a neighbor in P, then no node of Y is adjacent to j or v, and if y_n has a neighbor in S, then no node of Y is adjacent to i or v.*

Type 2 *Node y_n is of Type a[8.3], adjacent to a and b, and no node of Y is adjacent to i, j or v.*

Type 3 *Node y_n is of Type a[8.3], adjacent to x and u, and no node of Y is adjacent to i, j. Node y_1 is the unique node of Y adjacent to v.*

Proof: There are two cases to consider.

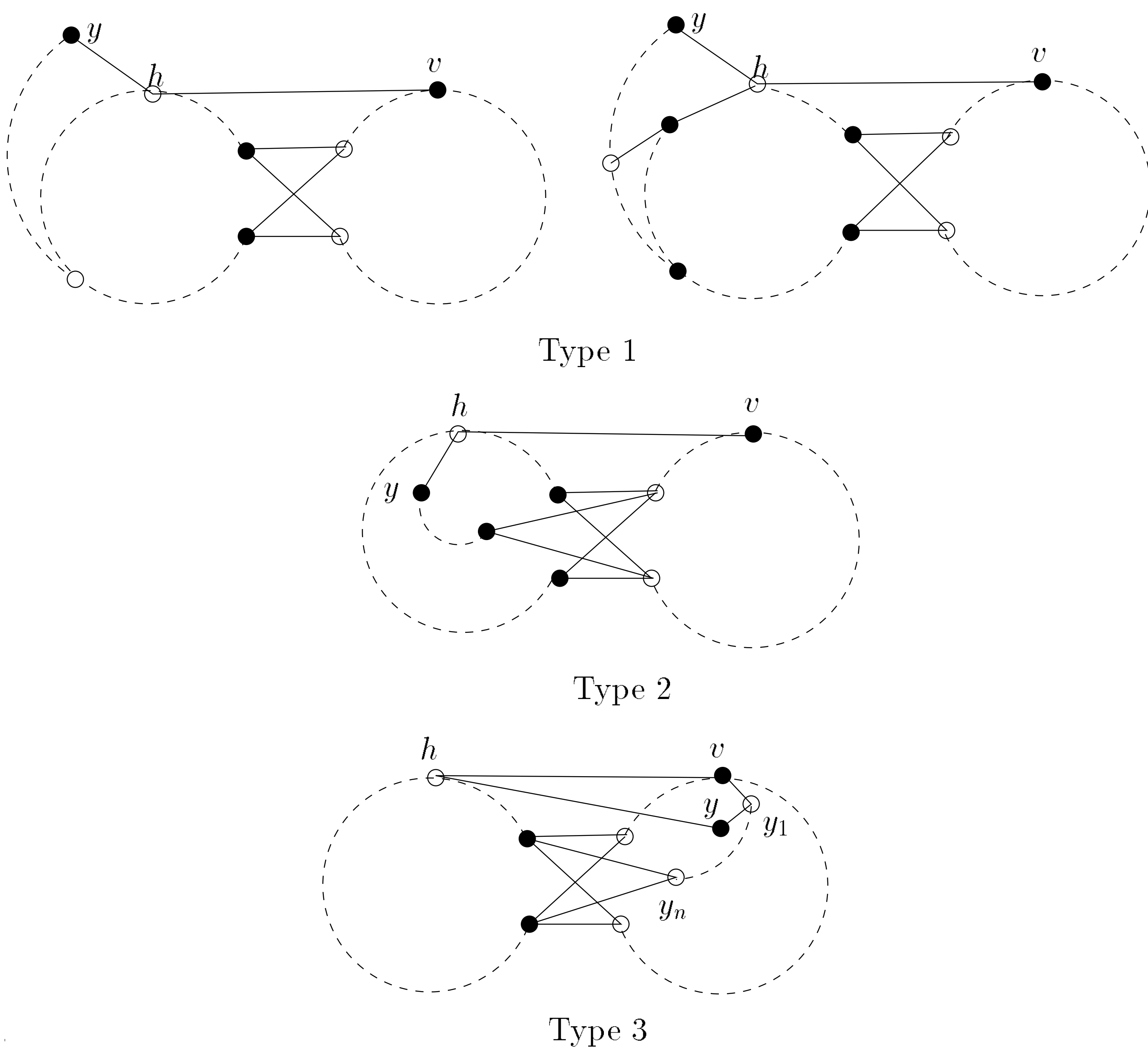


Figure 29: Direct connections from $y \in H(h)$ when $|T| = 1$

Case 1 Node y_n has only one neighbor p in $V(\Gamma) \setminus \{h, i, j, v\}$.

Suppose $p \in (P \cup S) \setminus \{h, i, j\}$. W.l.o.g. assume $p \in P \setminus \{h, i\}$. If any of the nodes in Y is adjacent to j or v, there is a $3PC(a, j)$ or $3PC(a, v)$. Thus path Y is of Type 1.

Suppose $p \in (Q \cup R) \setminus \{v\}$. W.l.o.g. assume $p \in Q \setminus \{v\}$. If Y has a node adjacent to i or j, there is a $3PC(a, i)$ or a $3PC(a, j)$. At least one node of Y is adjacent to v, otherwise there is a parachute with long top if p is adjacent to v and a $3PC(h, x)$ if not. Now, exactly one node of Y is adjacent to v, say y_t, and p is not adjacent to v since, otherwise, there would be a wheel with center v. Consider the parachute Π_1, with top $h, y, y_1, Y_{y_1 y_t}, y_t$, side paths P and $y_t, Y_{y_t y_n}, y_n, p, Q_{pa}, a, x$, center node v and middle path v, R, b, x. If $t \neq 1$, Π_1 has long top. If $t = 1$, applying Corollary 8.1, we have that G contains a path $X = x_1, \ldots, x_m$ of Type j[6.4] or j1[6.5] relative to Π_1 where x_m is adjacent to a node of R distinct from v and its neighbor. No node of X can be adjacent to a node of S for otherwise there is a violation of Corollary 8.1 applied to the parachute Π_2 obtained from Π_1 by replacing the side path P by S and the bottom node x by u. Now there is a $3PC(h, x)$.

Case 2 Node y_n has at least two neighbors in $V(\Gamma) \setminus \{h, i, j, v\}$.

Assume first that node y_n is a twin of a node d of Γ. Let Γ' be the goggles obtained from Γ by substituting y_n for d. If $n = 1$, node y_1 must be adjacent to i, j or v, for otherwise y is a strongly adjacent node violating Lemma 8.3 in Γ'. If y_1 is adjacent to i or j, we have a Type 1 path. If y_1 is adjacent to v, assume w.l.o.g that $d \in Q$. Node y is of Type c[8.3] in Γ' and, by Lemma 8.8 applied to Γ', there exists a path $X = x_1, \ldots, x_m$ of Type 2[8.8] from y to R such that x_n has a neighbor s in R that is distinct from the neighbor of v in R. Now if d has no neighbor in X let $H = v, Q, a, u, b, R_{bs}, s, x_n, X_{x_n x_1}, x_1, y, h, v$ and (H, y_1) is a odd wheel. If d has at least one neighbor in X, there is a $3PC(d, u)$. If $n \geq 2$, we can apply Case 1 or 2 to Γ', so Y is of Type 1.

Assume now node y_n is a Type a[8.3] node. Suppose y_n is adjacent to x and u. No node of Y is adjacent to i or j for otherwise there is $3PC(y_n, i)$ or a $3PC(y_n, j)$. Now a node of Y must be adjacent to v otherwise there is a $3PC(h, x)$. Furthermore, since there is no wheel, node v has a unique neighbor, say y_t, in Y. If $t \neq 1$, there is a parachute with long top $h, y, y_1, Y_{y_1 y_t}, y_t$, side paths P and $y_t, Y_{y_t y_n}, y_n, x$, center node v and middle path v, R, b, x. Hence $t = 1$. Thus path Y is of Type 3. Suppose y_n is adjacent to a and b. If any of the nodes in Y is adjacent to i, j or v, we have a violation of Lemma 8.6. Otherwise we have a Type 2 path.

By Lemma 8.7, node y_n cannot be of Type b[8.3] since $|T| = 1$.

Node y_n cannot be of Type c[8.3] since such nodes belong to $Z(h)$.

Assume node y_n is a Type d[8.3] node. Irrespective of whether or not a node of Y is adjacent to i, j or v, there is a $3PC(y_n, x)$ when the neighbors of y_n are in $Q \cup R$ and there is a $3PC(a, y_n)$ when they are in $P \cup S$.

□

Definition 8.11 *If* $|T| > 1$, *let*

$H_{PS}(h) = \{y \in H(h)$: *there is a Type 1 or Type 2[8.9] path from* y *to* $V(\Gamma)\}$,

$H_{QR}(h) = \{y \in H(h)$: there is a Type 3[8.9] path from y to $V(\Gamma)\}$.

If $|T| = 1$, let

$H_{PS}(h) = \{y \in H(h)$: there is a Type 1[8.10] path from y to $V(\Gamma)\} \cup \{y \in H$: there is a Type 2[8.10] but no Type 3[8.10] path from y to $V(\Gamma)\}$,

$H_{QR}(h) = \{y \in H(h)$: there is a Type 3[8.10] path from y to $V(\Gamma)\}$.

Lemma 8.12 *$H_{PS}(h) \cap H_{QR}(h) = \emptyset$.*

Proof: We consider two cases depending on the length of T.

Case 1 $|T| > 1$.

Suppose the lemma is false, i.e. there exists y in $H_{PS}(h) \cap H_{QR}(h)$. Let $X = x_1, \dots, x_n$ be a Type 1 or 2[8.9] path and $Y = y_1, \dots, y_m$ a Type 3[8.9] path.

In fact, X is a Type 1[8.9] path for, otherwise, there is a violation of Lemma 8.6 if a node of X is coincident with or adjacent to a node of Y, and there is a $3PC(a, y)$ if not.

W.l.o.g. assume that x_n is adjacent to $p \in V(S) \setminus \{h, j\}$. Note that by Lemma 8.9, no node in $V(X) \cup V(Y)$ is adjacent to i.

A node of X is coincident with or adjacent to a node of Y, otherwise there is a $3PC(a, y)$. If y_m is a Type b[8.3] node adjacent to the neighbors of v in Q and R, there are connected squares since by Lemma 8.9, no node of $V(X) \cup V(Y)$ is adjacent to k. So y_m is adjacent to $V(T) \setminus \{h, k\}$. If y_m is a twin of node d, then y_m has no neighbor in X and $m > 1$, else there is a wheel with center y_m. Therefore $m \geq 2$ and we can replace $d \in V(T)$ by y_m and Y by the shorter path $y_1, \dots, y_{m-1}$. By repeating this shortening argument if necessary, we can assume w.l.o.g. that y_m has a unique neighbor t in Γ and $t \in V(T) \setminus \{h, k\}$. Now, if node k is not adjacent to any node in Y, then there is a $3PC(h, t)$ if $t \in V^c$ and there are goggles with shorter top than Γ if $t \in V^r$. So k has a neighbor in Y. If some node of $Y \setminus \{y_m\}$ is coincident with or adjacent to a node of X, there is a $3PC(k, a)$. Otherwise, y_m is adjacent to a node of X but no other node of Y is, therefore there is a $3PC(k, y_m)$ or a $3PC(a, y_m)$.

Case 2 $|T| = 1$

Suppose the lemma is false. By Definition 8.11, both a Type 1[8.10] direct connection X and a Type 3[8.10] direct connection Y exist, from y. Let $X = x_1, \dots, x_n$ where x_n is adjacent to P and let $Y = y_1, \dots, y_m$ where y_m is adjacent to x and u. If no node of X is adjacent to or coincident with a node of Y, there is a $3PC(y_m, y)$. If y_1 is the only node of Y adjacent to a node of X, then there is a violation of Lemma 8.10, since $y_1 \in N(v)$. So, some node of X is coincident with or adjacent to a node of $V(Y) \setminus \{y_1\}$. By Lemma 8.6, x_n is a twin of x and it is the only node of X with a neighbor in $V(Y) \setminus \{y_1\}$. Furthermore, x_n is not adjacent to y_m (else there is a violation of Lemma 8.6 in the goggles obtained from Γ by substituting x_n for x). Now there is a $3PC(x_n, y_m)$.

□

8.5 Bicliques

Lemma 8.13 *(i) Nodes u, x, a, b, their twins and the Type a[8.3] nodes adjacent to u and x form a biclique.*

(ii) Nodes u, x, a, b, their twins and the Type a[8.3] nodes adjacent to a and b form a biclique.

(iii) When $|T| > 1$, nodes u, x, a, b, their twins and all Type a[8.3] nodes form a biclique.

Proof: (i) If a twin of x is not adjacent to a twin of a or b, there is a $3PC(h, u)$. Let w be a Type a[8.3] node adjacent to u and x and let Y be a path of Type 2[8.6]. Suppose that w is not adjacent to a twin x^* of x. If x^* has no neighbor in Y, there is a $3PC(h, u)$ and otherwise there is a $3PC(h, x^*)$.

(ii) follows by symmetry.

(iii) Let w be a Type a[8.3] node adjacent to a and b, and let y be a Type a[8.3] node adjacent to u and x. Suppose w and y are not adjacent. By Lemma 8.6, there exist a direct connection $X = x_1, \ldots, x_n$ of Type 2[8.6] from w to P or S, say P, and a direct connection $Y = y_1, \ldots, y_m$ of Type 2[8.6] from y to Q or R, say Q. No node of Y is adjacent to a node of X, otherwise there is a $3PC(h, v)$. If w is adjacent to a node in Y, there is a $3PC(w, h)$. If y is adjacent to X, there is a $3PC(y, v)$. So no node of $Y \cup \{y\}$ is adjacent to a node of $X \cup \{w\}$. But now there is a $3PC(u, h)$. $\square$

Lemma 8.14 *(i) Nodes h, i, j, k and their twins form a biclique.*

(ii) Nodes h, i, j, their twins and the Type b[8.3] nodes adjacent to i and j form a biclique.

(iii) Nodes h, i, k, their twins and the Type c[8.3] nodes adjacent to i and k form a biclique.

(iv) When $|T| = 1$, the nodes h, v, their twins and all Type c[8.3] nodes form a biclique.

Proof: (i) Suppose a twin h^* of h is not adjacent to a twin i^* of i. Node h^* is of Type c[8.3] in the goggles Γ^* obtained from Γ by substituting i^* for i. So, by Lemma 8.8, there exists a path $Y = y_1, \ldots, y_n$ of Type 2[8.8] from h^* to P such that y_n is not adjacent to i^* and is not a twin of i^*. Now there is a wheel with center i. By symmetry, h^* is also adjacent to the twins of j. Now suppose h^* is not adjacent to a twin k^* of k. Node h^* is of Type b[8.3] in the goggles Γ^* obtained from Γ by substituting k^* for k. So, by Lemma 8.7, there exists a path $Y = y_1, \ldots, y_n$ of Type 2[8.7] from h^* to T such that y_n is not adjacent to k^* and is not a twin of k^*. Now there is a wheel with center k.

To prove (ii), we show that any Type b[8.3] node w adjacent to i and j is also adjacent to all the twins of i and j. Let $Y = y_1, \ldots, y_n$ be a path of Type 2[8.7] from w to T such that y_n is not adjacent to k and is not a twin of k. Suppose that w is not adjacent to a twin i^* of i. If i^* has no neighbor in Y, there is a wheel with center i, and otherwise there is a $3PC(i^*, a)$. By symmetry, w is also adjacent to the twins of j.

To prove (iii), consider a Type c[8.3] node w adjacent to i and k, and let $Y = y_1, \ldots, y_n$ be a path of Type 2[8.8] from w to S such that y_n is not adjacent to j and is not a twin of j. If w is not adjacent to a twin i^* of i, there is a wheel with center i if i^* has no neighbor in Y, and a $3PC(i^*, a)$ otherwise. Similarly, if w is not adjacent to a twin k^* of k, there is a wheel with center k if k^* has no neighbor on Y, and a $3PC(k^*, a)$ otherwise.

Finally, (iv) follows from (iii) and Remark 8.5. $\square$

The next result shows that, if $|T| > 1$, then node h and its twins form a biclique with i, j, their twins and the nodes in $H_{PS}(h)$, or with k, its twins and the nodes in $H_{QR}(h)$.

For a twin h' of h, the node sets $H_{PS}(h')$ and $H_{QR}(h')$ are subsets of $H(h')$ defined as in Definition 8.11, but relative to the goggles Γ' obtained from Γ by substituting h' for h.

Lemma 8.15 *If $|T| > 1$, then either $H_{PS}(h') = H_{PS}(h)$ for every twin h' of h, or $H_{QR}(h') = H_{QR}(h)$ for every twin h' of h, or both.*

Proof: Assume that h has a twin h^* such that $H_{PS}(h^*) \neq H_{PS}(h)$. To prove the lemma, we will show that $H_{QR}(h') = H_{QR}(h)$ for every twin h' of h.

Claim 1: *There exist a node $y \in H_{PS}(h) \setminus H_{PS}(h^*)$ and a direct connection Y of Type 1 or 2[8.9] from y to Γ such that h^* has no neighbor in Y and there exist a node $z \in H_{PS}(h^*) \setminus H_{PS}(h)$ and a direct connection Z of Type 1 or 2[8.9] from z to Γ such that h has no neighbor in Z.*

Proof: Among all possible choices of $y \in H_{PS}(h) \Delta H_{PS}(h^*)$ and of direct connection Y of Type 1 or 2[8.9] from y to Γ, choose y and $Y = y_1, \ldots, y_m$ such that Y is shortest. We assume w.l.o.g. that $y \in H_{PS}(h) \setminus H_{PS}(h^*)$. Next we show that h^* has no neighbor in $V(Y) \cup \{y\}$. Assume otherwise. Then h^* has a unique neighbor y^* in $Y \cup \{y\}$, else there is a wheel with center h^*. Since $y \notin H_{PS}(h^*)$, we have that $y^* \neq y$. If $y^* = y_m$, then by applying Lemma 8.3 to the goggles Γ^* obtained from Γ by substituting h^* for h, we get that node y^* must be a twin of i or j in Γ^*. Now, by Lemma 8.14, y^* is adjacent to h, a contradiction. So $y^* = y_j$, $j < m$, and $Y_{y_j y_m}$ is shorter than Y, a contradiction to our choice of y. So, h^* has no neighbor in $V(Y) \cup \{y\}$.

When Y is of Type 1[8.9], assume w.l.o.g. that y_m has its neighbors in P. Let $X = h, y, \ldots, a$ be a shortest path from h to a, whose intermediate nodes are the ones in Y and possibly part of P. Consider the parachute Π whose top is j, h^*, k, middle path X and side paths k, T_{kv}, v, Q, a and j, S, u, a. Applying Corollary 8.1 to Π, we obtain a path $Z^* = z, z_1, \ldots, z_n$ from h^* to X. Let Π' be the parachute obtained from Π by substituting a with b and Q with R. By applying Corollary 8.1 to Π', we obtain that Z^* has no neighbor in R. So the only neighbors of $V(Z^*) \setminus \{z_n\}$ in Γ may be in $V(P) \setminus (V(X) \cup \{h\})$ and the only neighbors of z_n in $V(Y) \cup V(\Gamma)$ are in $V(Y) \cup V(P) \setminus \{h\}$. So $z \in H_{PS}(h^*) \setminus H_{PS}(h)$ and a subpath Z of $V(Z^*) \cup V(Y)$ is the required direct connection. This proves Claim 1.

Claim 2: $H_{QR}(h^*) = H_{QR}(h)$.

Proof: Among all possible choices of $w \in H_{QR}(h) \Delta H_{QR}(h^*)$ and of direct connection W of Type 3[8.9] from w to Γ, choose w and $W = w_1, \ldots, w_p$ such that W is shortest. We assume w.l.o.g. $w \in H_{QR}(h) \setminus H_{QR}(h^*)$. Next we show that h^* has no neighbor in $W \cup \{w\}$. Assume otherwise. Then h^* has a unique neighbor w^* in $W \cup \{w\}$, else there is a wheel with center h^*. Since $w \notin H_{QR}(h^*)$, we have that $w^* \neq w$. If $w^* = w_p$, then by applying Lemma 8.3 to the goggles Γ^* obtained from Γ by substituting h^* for h, we get that node w^* must be a twin of k in Γ^*. Now, by Lemma 8.14, w^* is adjacent to h, a contradiction. So $w^* = w_j$, $j < p$, and $W_{w_j w_p}$ is shorter than W, a contradiction to our choice of w. So, h^* has no neighbor in $V(W) \cup \{w\}$.

By Claim 1, there exist $z \in H_{PS}(h^*) \setminus H_{PS}(h)$ and a direct connection $Z = z_1, \ldots, z_n$ of Type 1 or 2[8.9] from z to Γ such that h has no neighbor in Z. No node of $V(Z) \cup \{z\}$ is adjacent to a node of $V(W) \cup \{w\}$ for the following reason.

- If some node of $(V(Z) \setminus \{z_n\}) \cup \{z\}$ has a neighbor in W, then $z \in H_{PS}(h^*) \cap H_{QR}(h^*)$, a contradiction to Lemma 8.12.
- If some node of $(V(W) \setminus \{w_p\}) \cup \{w\}$ has a neighbor in Z, then $w \in H_{PS}(h) \cap H_{QR}(h)$, a contradiction to Lemma 8.12.
- If z_n is adjacent to w_p, node z_n cannot be of Type a[8.3], for otherwise there would be a violation of Lemma 8.6. So Z is of Type 1[8.9] and at least one neighbor of z_n is in $V(P) \setminus \{h, i\}$. If w_p is a Type b[8.3] node, then there is a $3PC(w_p, a)$. So all the neighbors of w_p are in T. Node w_p has a unique neighbor π in T, else there is a wheel with center w_p. Node π is distinct from v, else there is a $3PC(v, a)$. Now, there are goggles with shorter top $T_{\pi v}$ in $V(\Gamma) \cup \{w_p, z_n\}$, contradicting our choice of Γ.

Now there is a $3PC(j, a)$ with the following paths.

$$j, S_{ju}, u, a \qquad j, h, w, W, \ldots, Q, a \qquad j, h^*, z, Z, \ldots, a.$$

This completes the proof of Claim 2.

Suppose now that $H_{QR}(h') \neq H_{QR}(h)$ for some twin h' of h. This implies $H_{PS}(h') = H_{PS}(h)$ by Claim 2. In addition $H_{QR}(h') \neq H_{QR}(h^*)$ implies $H_{PS}(h') = H_{PS}(h^*)$. But this contradicts $H_{PS}(h^*) \neq H_{PS}(h)$. □

Lemma 8.16 *(i) If there exists a Type b[8.3] node adjacent to i and j, denote by D the node set comprising i, j, their twins and $H_{PS}(h)$ and denote by F the node set comprising h, its twins and the Type b[8.3] nodes adjacent to i and j. Then $D \cup F$ induces a biclique.*

(ii) If there exists a Type c[8.3] node adjacent to i and k or if $|T| = 1$, denote by D the node set comprising h, its twins and the Type c[8.3] nodes adjacent to i and k and denote by F the node set comprising k, its twins, $H_{QR}(h)$ and, when $T = 1$, the Type c[8.3] nodes adjacent to h. Then $D \cup F$ induces a biclique.

Proof: (i) By Lemma 8.14(ii), it suffices to show that $y \in H_{PS}(h)$ is adjacent to any twin h^* of h and to any Type b[8.3] node w adjacent to i and j.

By Lemma 8.7, $|T| > 1$ and there exists a Type 2[8.7] path $X = x_1, \ldots, x_m$ from w to $V(\Gamma) \setminus \{h, i, j, k\}$. By Definition 8.11, there is a Type 1 or Type 2[8.9] path $Y = y_1, \ldots, y_n$ from y to $V(\Gamma) \setminus \{h, i, j, k\}$. If Y is of Type 2[8.9], y_n is adjacent to a and b, nodes y_n and x_m are not adjacent, otherwise there is a violation of Lemma 8.6. If Y is of Type 1[8.9], assume w.l.o.g. that y_n is adjacent to $p \in V(S) \setminus \{h, j\}$. Again, nodes y_n and x_m are not adjacent, otherwise there is a wheel with center x_m if x_m is a strongly adjacent node, or a $3PC(t, a)$ if x_m has a unique neighbor $t \in V^c$ in Γ. In both cases, no node of $Y \cup \{y\}$ is adjacent to a node of X and no node of Y is adjacent to w, otherwise there is a violation of Lemma 8.12 or Lemma 8.7. Now there is a $3PC(a, i)$ unless w and y are adjacent.

Suppose now that y is not adjacent to a twin h^* of h. No node of X and at most one node of Y is adjacent to h^*, otherwise there is a wheel with center h^*. Now there is a $3PC(i, a)$

whether or not h^* has a neighbor in Y. Indeed, when h^* has no neighbor in Y, the three paths are

$$i, P_{ia}, x, a \quad i, h^*, k, T_{kv}, v, Q, a \quad i, w, y, Y, \ldots, u, a$$

and when h^* has a neighbor y_j in Y, the three paths are

$$i, P_{ia}, x, a \quad i, w, X, \ldots, v, Q, a \quad i, h^*, y_j, Y_{y_j y_n}, y_n, \ldots, u, a.$$

(ii) **Case 1** $|T| > 1$ and there is a Type c[8.3] node w adjacent to i and k.

By Lemma 8.14(iii), it suffices to show that $y \in H_{QR}(h)$ is adjacent to any twin h^* of h and to any Type c[8.3] node w adjacent to i and k.

By Lemma 8.8, there exists a Type 2[8.8] path $X = x_1, \ldots, x_m$ from w to $V(\Gamma) \setminus \{h, i, j, k\}$. By Definition 8.11, there is a Type 3[8.9] path $Y = y_1, \ldots, y_n$ from y to $V(\Gamma) \setminus \{h, i, j, k\}$. If y_n has a unique neighbor $p \in V(T) \setminus \{h, k\}$, nodes y_n and x_m are not adjacent for, otherwise, there are goggles with shorter top if $p \in V^r$ and there is a $3PC(a, p)$ if $p \in V^c$. If y_n is a Type b[8.3] node, y_n and x_m are not adjacent, otherwise there is a violation of Lemma 8.7. In both cases, no node of $V(Y) \cup \{y\}$ is adjacent to a node of X and no node of Y is adjacent to w, otherwise there is a violation of Lemma 8.12 or Lemma 8.8. Now there is a $3PC(a, i)$ unless w and y are adjacent.

Suppose now that y is not adjacent to h^*. No node of X is adjacent to h^*, otherwise there is a wheel with center h^*. Now there is a $3PC(a, i)$ whether or not h^* is adjacent to a node of Y.

Case 2 $|T| = 1$.

By Lemma 8.14(iv), it suffices to show that $y \in H_{QR}(h)$ is adjacent to any twin h^* of h and to any Type c[8.3] node w adjacent to i and v, if such a node exists.

By Definition 8.11, there is a Type 3[8.10] path $Y = y_1, \ldots, y_n$ from y to $V(\Gamma) \setminus \{h, i, j, v\}$. Consider the parachute with top path h, y, y_1, side paths P and y_1, Y, y_n, x, center node v and middle path v, Q, a, x. By Corollary 8.1, there must be a Type j[6.4] or Type j1[6.5] direct connection $X = x_1, \ldots, x_m$ from node y to a node of $V(Q) \setminus \{v\}$. No node of X is adjacent to a node of Y or of $V(P) \setminus \{x\}$. Moreover, no node of X is adjacent to a node of $V(S) \setminus \{u\}$, for otherwise a direct connection contradicting Corollary 8.1 would exist. Finally, using the middle path v, R, b, x instead of v, Q, a, x above, and by Lemma 8.3, x_m must be a Type a[8.3] node adjacent to a and b.

No node of Y is adjacent to h^*, for otherwise y_n would violate Lemma 8.6. Node x_m is not adjacent to h^*, else there is a violation of Lemma 8.3 in the goggles Γ^* obtained from Γ by replacing h with h^*. Now, suppose y is not adjacent to h^*. Then no node of X is adjacent to h^*. For, otherwise, let x^* be the neighbor of h^* in X, closest to x_m. Consider the hole $H = y, y_1, Y, y_n, x, a, x_m, X_{x_m x^*}, x^*, h^*, i, h, y$. Node v has three neighbors in H, namely y_1, h and h^* and (H, v) is an odd wheel. But then the nodes in $X \cup \{y, y_1\}$ induce a direct connection from a Type a[8.3] node that violates Lemma 8.6 in Γ^*. So y is adjacent to h^*.

Suppose y is not adjacent to a Type c[8.3] node w. By Lemma 8.8, w has no neighbor on Y. Consider the parachute with the top path h, y, y_1, side paths h, S, u and y_1, Y, y_n, u and middle path v, Q, a, u. Then the path v, w, i, h contradicts Lemma 8.2 with respect to that parachute. □

8.6 One More Lemma

Lemma 8.17 *Let Z be the node set comprising nodes i, j, their twins and the nodes in $H_{PS}(h)$. Let W be the node set comprising node k, its twins, the nodes in $H_{QR}(h)$ and, when $|T| = 1$, the Type c[8.3] nodes adjacent to h. Finally, let $S = (V(\Gamma) \cup N(V(\Gamma))) \setminus (Z \cup W)$. Then there exists no direct connection from Z to W avoiding S.*

Proof: Suppose the contrary. Then there exists a direct connection $Y = y_1, \ldots, y_m$, from $z \in Z$ to $w \in W$ avoiding S.

Case 1 $z \in H_{PS}(h)$ and $w \in H_{QR}(h)$.

Since no node of Y belongs to the extended star $Z(h)$ (the nodes of $N(h)$, together with the nodes with at least two neighbors in $\{i, j, k\}$), there must be a direct connection X from Y to $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h)$. Now $Y \cup X$ contains either a direct connection establishing $z \in H_{QR}(h)$ or a direct connection establishing $w \in H_{PS}(h)$, both contradicting Lemma 8.12.

Case 2 $z = i$ or j or a twin of i or j, and $w = k$ or a twin of k or, when $|T| = 1$, a Type c[8.3] node adjacent to h.

If y_m is adjacent to a Type c[8.3] node adjacent to h, then path Y contradicts Lemma 8.8. Now we claim that $m \geq 2$. When $z = i$ or j, and $w = k$, this follows from the definition of S. When z is a twin of i or j, and $w = k$, the claim follows from Lemma 8.14(iii) applied to the goggles obtained from Γ by replacing i by z. Similarly, the claim follows when w is a twin of k. Now, $m \geq 2$ implies that there is a parachute with long top Y and long sides, a contradiction.

Case 3 $z \in H_{PS}(h)$ and $w = k$ or a twin of k or, when $|T| = 1$, a Type c[8.3] node adjacent to h.

It follows from Case 2 that Y contains no node adjacent to i or j. Since no node of Y belongs to the extended star $Z(h)$, there must be a direct connection X from Y to $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h)$. By Lemma 8.9 or 8.10 and by Lemma 8.12, $Y \cup X$ contains a direct connection of Type 1 or Type 2[8.9 or 8.10] for z. Now there is a $3PC(w, a)$.

Case 4 $z = i$ or j or a twin of i or j, and $w \in H_{QR}(h)$.

It follows from Case 2 that Y contains no node adjacent to k. Since no node of Y belongs to the extended star $Z(h)$, there must be a direct connection X from Y to $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h)$. By Lemma 8.9 or 8.10 and by Lemma 8.12, $Y \cup X$ contains a direct connection of Type 3[8.9 or 8.10] for w. Now, there is a $3PC(z, a)$.

□

8.7 2-Join Theorem

Now we prove the main result of this section.

Theorem 8.18 *Suppose G is a weakly balanced graph that contains goggles but no wheel, no connected squares and no extended star cutset. Then G contains a 2-join.*

Proof: Among the goggles of G, let Γ be one with shortest top path T and, subject to this condition, with the fewest number of nodes. By Lemmas 8.15 and 8.16(ii), $H_{PS}(h') = H_{PS}(h)$ or $H_{QR}(h') = H_{QR}(h)$ for every twin h' of h. The proof will distinguish between these two cases. Note that, if there is a Type b[8.3] node adjacent to i and j, then the first case always occurs. Indeed, by Lemma 8.16(i), each node in $H_{PS}(h)$ is adjacent to each twin h' of h. Hence $H_{PS}(h) \subseteq H_{PS}(h')$ for each twin h' of h. Replacing h by h' in the goggles yields the reverse inclusion. Similar arguments can be used to show that the second case always occurs when $|T| = 1$ or when there is a Type c[8.3] node adjacent to k. In both cases we define six disjoint sets A, B, D, F, M and N such that the nodes in $A \cup B$ induce a biclique K_{AB} and the nodes in $D \cup F$ induce a biclique K_{DF}. We then prove that $E(K_{AB}) \cup E(K_{DF})$ is a 2-join of the graph, separating the nodes in $A \cup D \cup M$ from the nodes in $B \cup F \cup N$.

Let $U_{ab} = \{w|\ w$ is a Type a[8.3] node adjacent to a and $b\}$. Similarly, let $U_{ux} = \{w|\ w$ is a Type a[8.3] node adjacent to u and $x\}$. Let U_1 be the nodes in U_{ab} that are adjacent to all nodes in U_{ux} and let $U_2 = U_{ab} \setminus U_1$.

- The set A comprises x, u, their twins and U_1.
- The set B comprises a, b, their twins and U_{ux}.

By Lemma 8.13(i, ii), the nodes in $A \cup B$ induce a biclique K_{AB}. In addition, by Lemma 8.13(iii), $U_{ab} = U_1$ when $|T| > 1$.

Case 1 $H_{PS}(h') = H_{PS}(h)$ for every twin h' of h (if h' exists). Furthermore, $|T| > 1$ and there exists no Type c[8.3] node adjacent to k.

- The set D comprises i, j, their twins and $H_{PS}(h)$.
- The set F comprises h, its twins and the Type b[8.3] nodes adjacent to i and j.
- The set M comprises the nodes in $P \cup S \setminus \{h, i, j, u, x\}$.
- The set N comprises the nodes in $Q \cup R \cup T \setminus \{h, a, b\}$.

$D \cup F$ induces a biclique K_{DF}. Indeed, if there is a Type b[8.3] node, this follows from Lemma 8.16(i). Else, it follows from Lemma 8.14(i) and the fact that $H_{PS}(h') = H_{PS}(h)$ for every twin h' of h.

Case 2 $H_{QR}(h') = H_{QR}(h)$ for every twin h' of h (if h' exists). Furthermore, there exists no Type b[8.3] node adjacent to i, j.

By Remark 8.4, we may assume that no Type c[8.3] node is adjacent to j.

- The set D comprises h, its twins and the Type c[8.3] nodes adjacent to i and k.
- The set F comprises k, its twins, $H_{QR}(h)$ and, when $|T| = 1$, the Type c[8.3] node adjacent to h.
- The set M comprises the nodes in $P \cup S \setminus \{h, u, x\}$.

- The set N comprises the nodes in $Q \cup R \cup T \setminus \{h, k, a, b\}$.

  $D \cup F$ induces a biclique K_{DF}. Indeed, if there is a Type c[8.3] node adjacent to k or if $|T| = 1$, this follows from Lemma 8.16(ii). Else, it follows from Lemma 8.14(i) and the fact that $H_{QR}(h') = H_{QR}(h)$ for every twin h' of h.

  Lemmas 8.15 and 8.16 show that the above two cases exhaust all possibilities.

If the theorem is false, there must be a direct connection $Y = y_1, \ldots, y_m$ between $A \cup D \cup M$ and $B \cup F \cup N$ in the partial graph $G \setminus (E(K_{AB}) \cup E(K_{DF}))$. Let y_0 be a neighbor of y_1 in $A \cup D \cup M$. W.l.o.g. choose y_0 in M if possible, and if y_1 has no neighbor in M, choose y_0 in $V(\Gamma) \cap (A \cup D)$ if possible. Similarly, let y_{m+1} be a neighbor of y_m in $B \cup F \cup N$ and choose $y_{m+1} \in N$ if possible or, when y_m has no neighbor in N, choose y_{m+1} in $V(\Gamma) \cap (B \cup F)$ if possible. We show that such a direct connection Y violates one of the lemmas proved in the earlier subsections.

Claim 1: $m \geq 2$

Proof: Assume $m = 1$. If $y_0 \in M$ and $y_2 \in N$, then y_1 violates Lemma 8.3. If $y_0 \in A$ or $y_2 \in B$, then y_1 violates Lemma 8.3 or Y violates of Lemma 8.6. So, $y_0 \in D$ or $y_2 \in F$. Now again, y_1 violates Lemma 8.3 unless y_0 is of Type c[8.3] (in Case 2) or in $H_{PS}(h)$ (in Case 1) or y_2 is of Type b[8.3] (in Case 1) or in $H_{QR}(h)$ (in Case 2) or of Type c[8.3] adjacent to h (in Case 2 and $|T| = 1$).

If y_0 or y_2 is of Type c[8.3], Y contradicts Lemma 8.8. In particular, Y is not of Type 3[8.8] since $|T| = 1$ implies that Case 2 occurs and, in this case, the Type c[8.3] nodes adjacent to h belong to F, the Type c[8.3] nodes adjacent to v belong to D and all the edges between them are removed in the 2-join.

If y_2 is of Type b[8.3], Y contradicts Lemma 8.7.

Finally, if $y_0 \in H_{PS}(h)$ or $y_2 \in H_{QR}(h)$, Y contradicts Lemma 8.12. This proves Claim 1.

Claim 2: *y_1 has at most one neighbor in Γ or is a twin of a node in M, and y_m has at most one neighbor in Γ or is a twin of a node in N.*

Proof: Suppose not. Then by Lemma 8.3, y_1 or y_m is a node of Type a, b, c or d[8.3].

For y_1, the only possibility is Type d[8.3], by the definition of sets A, B, D, F. In this case, if y_{m+1} is a twin of h or a Type b[8.3] node adjacent to i and j, there is a $3PC(y_1, y_{m+1})$. Otherwise, there is a $3PC(y_1, h)$.

There are four possibilities for y_m.

If y_m is of Type a[8.3], by Lemma 8.13(iii), $|T| = 1$, $y_m \in U_2$ and Case 2 applies. Furthermore, by Lemma 8.6, either $V(Y) \setminus \{y_m\}$ induces a Type 1 or 2[8.6] direct connection from y_m to $V(\Gamma)$, or y_0 is a Type a[8.3] node in A. Let z be a Type a[8.3] node adjacent to u and x but not to y_m. By Lemma 8.6, there exists a direct connection $Z = z_1, \ldots, z_p$ of Type 2[8.6] from z to $V(Q) \cup V(R)$. Consider first the case where $V(Y) \setminus \{y_m\}$ induces a path of Type 2[8.6] from y_m to $V(P) \cup V(S)$. Let $y^* = y_0$ or y_1, whichever belongs to V^r. If no node of Y is adjacent to a node of Z, then there is a $3PC(x, y^*)$. On the other hand, by Lemma 8.6 applied to z and to y_m, the only adjacency between Y and Z is between y_1 and z_p. So y_1 and z_p are adjacent. If $z_p \in V^c$, let $z^* = z_p$, and if $z_p \in V^r$, let z^* be the unique neighbor of z_p in $V(Q) \cup V(R)$. Now $y_0 = h$ or a twin of h, and $z^* = v$, otherwise there is a $3PC(y^*, z^*)$. But this implies $y_1 \in H_{QR}(h)$, a contradiction. Consider now the case where either y_0 is a Type a[8.3] node in A or $V(Y) \setminus \{y_m\}$ induces a direct connection of Type 1[8.6] from y_m to u or x or a twin of u or x, say $y_0 = x$ or a twin of x. No node of Z is adjacent to a node of Y, otherwise there is a $3PC(y_0, h)$ or there is a direct connection from y_m to $V(\Gamma)$ violating Lemma 8.6. Assume w.l.o.g. that z_p has a neighbor in R. Hence we have a violation of Lemma 8.2 for the parachute with top path a, y_0, z, middle node u, bottom node v and the extra path is y_0, y_1, Y, y_m, a or a subpath of it, in case Y contains a neighbor of z. So y_m is not of Type a[8.3].

If y_m is of Type b[8.3], then its neighbors in Γ are the neighbors of v in Q and R. By Lemma 8.7, $|T| > 1$ and y_1 is adjacent to a node y_0 in D. So $y_0 \in H_{PS}(h)$ (in Case 1) or $y_0 = h$ or a twin of h (in Case 2). Now, in Case 1, the path Y shows that $y_0 \in H_{QR}(h)$, a contradiction to Lemma 8.12. In Case 2, the path Y shows that $y_1 \in H_{QR}(h)$ and therefore $y_1 \in F$, a contradiction.

If y_m is of Type c[8.3], path Y contradicts Lemma 8.8.

If y_m is of Type d[8.3], there is a $3PC(y_m, v)$. This completes the proof of Claim 2.

Claim 3: *$y_0 \in A$ and $y_{m+1} \in B$ cannot occur.*

Proof: Suppose $y_0 \in A$ and $y_{m+1} \in B$. If y_0 is u or x, say x, then by Claim 2, y_1 is not adjacent to u. Similarly, if y_0 is a twin of u or x, say a twin of x, then y_1 cannot be adjacent to u by our choice of y_0. Now, the path Y contradicts Lemma 8.2 in a parachute with side paths P, S, bottom node h and top node y_{m+1}. So y_0 cannot be x, u or one of their twins. Such a contradiction also occurs when y_{m+1} is a, b or one of their twins. So $y_0 \in U_{ab}$ and $y_{m+1} \in U_{ux}$. By Lemma 8.6, there exists a path $Z = z_1, \ldots, z_p$ of Type 2[8.6] from y_{m+1} to Q or R, say Q. Now no node of $V(Y) \setminus \{y_m\}$ is adjacent to a node of Z, otherwise there is a direct connection from y_0 to Γ violating Lemma 8.6. Now we have a violation of Lemma 8.2 for the following parachute. The top path is b, y_0, y_{m+1}, the side paths are R and $y_{m+1}, Z, z_p, \ldots, v$, the center node is u and the middle path is u, S, h, T, v. The extra path is Y. This completes the proof of Claim 3.

Claim 4: *In Case 1, $y_0 \in D$ and $y_{m+1} \in F$ cannot occur.*

Proof: Assume first y_{m+1} is node h or a twin of h. Since $y_m \notin H_{PS}(h)$ and $H_{PS}(h) = H_{PS}(y_{m+1})$, it follows that $y_m \in H_{QR}(y_{m+1})$. Now, Lemma 8.17 applied to y_0 and y_m in the goggles Γ^* obtained from Γ by replacing h by y_{m+1} (when y_{m+1} is a twin of h) contradicts the existence of Y. Indeed, by Lemma 8.14, y_0 belongs to the set Z relative to the goggles Γ^*.

Now assume y_{m+1} is of Type b[8.3]. By Lemma 8.7, y_0 is node i, j or one of their twins. W.l.o.g. $y_0 = j$ or one of its twins. Then there is a violation of Lemma 8.2 in parachute with center node h, top path i, y_{m+1}, y_0, where Y is the extra path. This completes the proof of Claim 4.

Claim 5: *In Case 2, $y_0 \in D$ and $y_{m+1} \in F$ cannot occur.*

Proof: Assume first y_0 is node h or a twin of h. Since $y_1 \notin H_{QR}(h)$ and $H_{QR}(h) = H_{QR}(y_0)$, it follows that $y_1 \in H_{PS}(y_0)$. Now, Y contradicts Lemma 8.17 applied to y_1 and

y_{m+1} in the goggles Γ^* obtained from Γ by replacing h by y_0 (when y_0 is a twin of h). Indeed, by Lemma 8.14, y_{m+1} belongs to the set W relative to the goggles Γ^*.

Now assume y_0 is of Type c[8.3]. By Lemma 8.8, y_{m+1} is node k or one of its twins, or a Type c[8.3] node adjacent to h. Then there is a violation of Lemma 8.2 in parachute with center node h, top path i, y_0, y_{m+1}, where Y is the extra path. This completes the proof of Claim 5.

Claim 6: $y_0 \notin A$.

Proof: Assume $y_0 \in A$. By Claim 3, $y_{m+1} \in F \cup N$. If $y_0 = u, x$ or one of their twins, there is a $3PC(y_0, h)$ if y_{m+1} is not a twin of h and a $3PC(y_0, y_{m+1})$ if y_{m+1} is a twin of h. So consider $y_0 \in U_1$. In Case 1, by Lemma 8.6, the only possibility is that $y_{m+1} = h$ or a twin of h. So $y_m \in H_{PS}(y_{m+1})$, since Y is a path of Type 2[8.9]. But then $y_m \in D$, a contradiction. In Case 2, by Lemma 8.6, the only possibility is that $y_{m+1} \in H_{QR}(h)$. Now, Y is a Type 2[8.9 or 8.10] path from y_{m+1}. So, when $|T| > 1$, $y_{m+1} \in H_{PS}(h)$ and this contradicts Lemma 8.12. Now consider $|T| = 1$. Since $y_{m+1} \in H_{QR}(h)$, there exists a Type 3[8.10] path $X = x_1, \ldots, x_n$ from y_{m+1} where x_n is adjacent to x and u. There is no adjacency between $V(Y) \cup \{y_0\}$ and $V(X) \setminus \{x_n\}$, else we get a violation of Lemma 8.6 for y_0. Now, nodes x_n and y_0 are not adjacent, else there is a $3PC(x_n, y_{m+1})$. This contradicts the assumption $y_0 \in U_1$. This completes the proof of Claim 6.

Claim 7: $y_0 \notin D$.

Proof: Assume $y_0 \in D$. By Claims 4 and 5, $y_{m+1} \in B \cup N$. In Case 1, $y_0 \in H_{PS}(h)$ or y_0 coincides with i or j or with a twin of i or j. If y_m is a twin of k or y_m has k as its unique neighbor in Γ, the path Y contradicts Lemma 8.17. Otherwise, if $y_0 = i$ or j or a twin of i or j, there is a $3PC(y_0, a)$, and if $y_0 \in H_{PS}(h)$, there is a violation of Lemma 8.12.

In Case 2, if y_0 is a Type c[8.3] node adjacent to i and k, there is a violation of Lemma 8.8. Now, suppose y_0 is h or one of its twins. Then $y_1 \notin F$ implies that $y_1 \in H_{PS}(h)$. $V(Y) \setminus \{y_1\}$ induces a direct connection from y to $V(\Gamma) \setminus \{h, i, j, k\}$ satisfying Lemma 8.9 or 8.10. By Lemma 8.12, this path cannot be of Type 3[8.9] or [8.10]. Therefore, it must be of Type 2[8.9] or [8.10] since $y_{m+1} \in B \cup N$. In fact, the only possibility is $y_m \in U_2$, but this was excluded in Claim 2. This completes the proof of Claim 7.

Claim 8: $y_0 \notin M$.

Proof: Suppose y_1 is a twin of $p \in M$ or has a unique neighbor $y_0 = p \in M$. W.l.o.g. $y_0 \in P \setminus \{x, h\}$.

If y_{m+1} is a, b or one of their twins, there is a $3PC(y_{m+1}, v)$.

If $y_{m+1} \in U_{ux}$, there is a violation of Lemma 8.6.

In Case 1, if y_{m+1} is h or a twin of h, then $y_m \in H_{QR}(y_{m+1})$ since $y_m \notin D$. But $y_m \in H_{PS}(y_{m+1})$ since $V(Y) \setminus \{y_m\}$ is of Type 1[8.9] for y_m. This contradicts Lemma 8.12. If y_{m+1} is of Type b[8.3], there is a contradiction of Lemma 8.7.

In Case 2, if y_{m+1} is k, a twin of k, $y_{m+1} \in H_{QR}(h)$ or a Type c[8.3] node adjacent to h, then Lemma 8.17 is violated when $p = i$, or there is a $3PC(y_{m+1}, a)$ otherwise.

Now, consider the case when y_m is a twin of a node in N (in this case, let $q = y_m$) or has a unique neighbor $y_{m+1} \in N$ (in this case, let $q = y_{m+1}$). If $q \in T$, there is a $3PC(q, a)$

if $q \in V^c$, there are goggles with a shorter top if $q \in V^r$ and $p \neq i$ and j, and there is a $3PC(p, a)$ or $3PC(y_1, a)$ if $q \in V^r$ and $p = i$ or j. If $q \in Q \cup R \setminus \{v\}$, there is a $3PC(u, h)$ if q is not adjacent to v or a $3PC(x, q)$ otherwise.

This completes the proof of the theorem. □

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