Edge-Coloring Almost Bipartite Multigraphs

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Abstract

Bipartite multigraphs have chromatic index equal to the largest degree d. We consider multigraphs obtained by inserting k vertices in edges of a connected bipartite multigraph, and show that the chromatic index may increase to at most $d+1$. We further show that (1) the chromatic index always increases if $k=1$ when the original bipartite multigraph is regular, but does not increase for $k=1$ if the bipartite multigraph is not regular; (2) the chromatic index does not increase if $k=2$ (regular or not); (3) the chromatic index may increase if $k=3$ when the original bipartite multigraph is regular, but does not increase for $k=3$ if the bipartite multigraph is not regular; (4) the chromatic index does not increase if $k=4$ (regular or not); (5) the chromatic index increases for infinitely many 3-regular bipartite graphs and each $k \geq 3$ odd, and (6) the chromatic index does not increase for infinitely many 3-regular bipartite graphs and each $k \geq 3$ (even or odd).

We finally study the list chromatic index of such multigraphs and show that it does not increase past $d+1$ either; in fact in some sense it stays almost d. All the results presented here render polynomial-time algorithms, since the proofs are based on bipartite graph matching and stable marriage.

1 Introduction

Let G be a multigraph of maximum degree d. The *chromatic index* of G is the minimum number of colors that can be assigned to the edges to obtain a valid *edge coloring*, where adjacent edges have different colors. If G is bipartite, the chromatic index of G is d. If G is a graph (no parallel edges), then by Vizing's theorem (e.g., see Berge [1]) the chromatic index of G is either d or $d+1$. If G is a cubic graph (regular with $d=3$), then determining whether the chromatic index of G is 3 was shown by Holyer [9] to be NP-complete.

Let now G be a connected bipartite multigraph of maximum degree $d \geq 2$ (such a G has chromatic index d). We consider graphs G' obtained from G by replacing the edges e with paths p_e of length $\ell_e \geq 1$, thus inserting a total of $k = \sum_e (\ell_e - 1)$ new vertices in G of degree 2 in edges of G to obtain G' (note that G' may not be bipartite).

If G is d-regular, then for $k=1$ we get G' of chromatic index $d+1$, while for $k=2$ or $k=4$ we get G' of chromatic index d. If G is not d-regular, then for $k=1,2,3$ or 4 we get G' of chromatic index still d.

On the other hand, for $G = K_{3,3}$ with $d=3$, if we let $k=3$ by replacing the edges e of a matching M with paths of length $\ell_e = 2$, the chromatic index of G' is $d+1=4$; this almost bipartite graph is the Petersen graph minus a vertex. By contrast, Sanders and Thomas, in their study of apex graphs (the graph minus a vertex is planar) and Robertson, Seymour and Thomas, in their study of Petersen minor-free graphs [11], showed that if G is a planar cubic biconnected

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multigraph and we insert $k = 2$ or $k = 3$ vertices of degree 2 in the edges of G to obtain G', then G' still has chromatic index $d = 3$ (if $k = 1$ the chromatic index increases to $d + 1 = 4$). Tait showed that the Four Color theorem is equivalent to the statement that planar cubic multigraphs have chromatic index $d = 3$.

Finally, the chromatic index of G' is $d + 1 = 4$ for infinitely many 3-regular bipartite graphs G and each $k \geq 3$ odd, and the chromatic index of G' is $d = 3$ for infinitely many 3-regular bipartite graphs G and each $k \geq 3$ (even or odd).

Briefly, the chromatic index of the multigraph G' obtained from a bipartite multigraph G of maximum degree d by inserting k vertices on edges is at most $d + 1$. More generally, we consider the *list chromatic index* of G' so obtained from G, where the available colors given at each edge may be different. It is known that G has list chromatic index equal to the maximum degree d. We show that the list chromatic index increases to at most $d+1$ for such G'. In fact, it stays $(d, d+1)$ in the sense that edges with lists of size $d + 1$ are only needed for one of the two edges replacing e with $\ell_e = 2$. A stronger result shows that we may use lists of size only d, provided that any two edges replacing e with $\ell_e = 2$ have different lists (of size d).

Besides our study of almost bipartite multigraphs, nearly bipartite multigraphs have been studied for both edge-coloring and list edge-coloring [10, 2], where G is nearly bipartite if the graph minus a vertex is bipartite.

2 Negative Results for Almost Bipartite Multigraphs

We first consider the case $k = 1$.

Theorem 1 *Let G be a d-regular multigraph with an even number of vertices (e.g., if d is odd or G is bipartite, then G has an even number of vertices). Let G' be obtained from G by inserting one vertex of degree two in an edge of G (that is by replacing an edge xy in G with two edges xv and vy, where v is the new vertex) to obtain G'. Then G' has chromatic index at least $d + 1$.*

Proof. If G' were d-edge-colorable, then each of the d colors would match the even number of vertices of degree d and, therefore, none of them could involve the vertex v of degree two, a contradiction. So G is not d-edge-colorable. □

We next consider the case $k = 3$.

Theorem 2 *If $G = (X, Y, E) = K_{3,3}$, with $d = 3$, and G' is obtained from G by inserting three vertices v_i in the three edges x_iy_i of a perfect matching of G, $x_i \in X$, $y_i \in Y$, then G' has chromatic index $d + 1 = 4$.*

Proof. If G' were 3-edge-colorable, then since G' has 9 vertices and 12 edges, the graph G' would be decomposed into three matchings M_i, each with 4 edges and including the vertices x_i, y_i. No matching can include both x_1v_1 and x_2v_2, so to cover the 6 edges involving the v_i we must include say x_1v_1 and v_2y_2 in M_1, and the other two edges in M_1 will be x_3y_1 and x_2y_3. But then M_2 and M_3 would have to cover the odd 5-cycle $x_3v_3y_3x_1y_2$, which is not possible. □

Theorem 2 generalizes as follows.

Theorem 3 *The chromatic index is $d + 1 = 4$ for infinitely many 3-regular connected bipartite graphs and each $k \geq 3$ odd, and the chromatic index is $d = 3$ for infinitely many 3-regular connected bipartite graphs and each $k \geq 3$ (even or odd).*

Proof. For $i = 1, 2$, let G_i be a 3-regular connected bipartite graph into which k_i vertices of degree two are inserted to obtain G'_i, and let $e_i = x_i y_i$ be an edge of G_i into which no vertices were inserted. We can combine the two G_i into a new graph 3-regular connected bipartite graph G by removing the edges e_i from the G_i and adding the edges x_1y_2, x_2y_1, and obtain similarly G' from the G'_i with $k_1 + k_2$ vertices of degree 2.

If both G'_i have chromatic index 3, then so will G'. We can choose any G'_i will chromatic index 3, and require G'_2 have $k_2 = 0$ or $k_2 = 2$ to either not increase or increase by two from k_1 to $k = k_1 + k_2$. Repeatedly performing this operation generates infinitely many graphs G with chromatic index 3, provided we verify that the base case $k = 3$ has at least one graph G' with chromatic index 3 (e.g., if G is a planar biconnected cubic graph, see [11]).

Suppose instead that G'_1 has chromatic index 4, and that if we add an additional vertex of degree two to G'_2 on $e_2 = x_2y_2$, we obtain G''_2 with $k_2 + 1$ added vertices that has chromatic index 4 (e.g., $K_{3,3}$ with $k_2 + 1 = 3$ from theorem 2 gives $k_2 = 2$, and with $k_2 + 1 = 1$ from theorem 1 gives $k_2 = 0$). Then if G' were 3-edge colorable, either x_1y_2, x_2y_1 would get the same color (contrary to G'_1 not having chromatic index 3), or x_1y_2, x_2y_1 would get different colors (contrary to G''_2 not having chromatic index 3). Repeatedly performing this operation generates infinitely many graphs G with chromatic index 4 and each $k \geq 3$ odd, the base case $k = 3$ provided by theorem 2. $\square$

3 Positive Results for Almost Bipartite Multigraphs

We first consider the case $k = 2$.

Theorem 4 *Let G be a d-regular connected bipartite multigraph with $d \geq 2$ (therefore with chromatic index d). If G' is obtained from G by inserting a total of two vertices of degree two on edges e of G, then G' has chromatic index d.*

Proof. The result is trivial for $d = 2$ (G and G' are even cycles), so we consider $d \geq 3$.

We assume that only one vertex has been inserted in an edge e. Otherwise we have paths $q = xyzt$, with y, z of degree two in G'. Such a path q can be replaced by the edge xt, returning to G which has chromatic index d. The color assigned to xt in G will be used for xy and zt in G', and any other color for yz.

We also assume that the edges e that had a vertex inserted do not share endpoints. Otherwise we have paths $q' = xyztu$, with y, t of degree two in G'. Such a path q' can be replaced by the path xzu, returning to G which has chromatic index d. The colors $i \neq j$ assigned to xz and zu respectively in G will be used for xy, zt (color i) and yz, tu (color j) in G'.

We shall show that such a G' has a perfect matching M. After removing the edges of M from G', the multigraph graph $G'' = G - M$ has vertices of degrees 1 and $d-1$ (instead of degrees 2 and d for G'), and G'' is bipartite, thus $(d-1)$-edge-colorable, and we may use the remaining color for M, completing the proof.

Let $G = (X, Y, E)$, and consider the disjoint edges x_iy_i that are replaced by paths $x_iv_iy_i$ in G', with $1 \leq i \leq 2$, $x_i \in X$, $y_i \in Y$. To obtain the perfect matching M, select for M the edges x_1v_1 and v_2y_2.

Let $G'' = (X', Y', E')$ be the subgraph of G induced by $V(G)$ minus the vertices x_1 and y_2. If G'' has a perfect matching M', then combining M' with the selected edges involving v_i gives the perfect matching M. Suppose G'' does not have a perfect matching. Then by Hall's theorem there exist subsets $S \leq X'$ and $T \leq Y'$ with $s = |S| > t = |T|$ such that all the neighbors of S in G'' are in T.

Let $T' = T \cup \{y_2\}$ in G. Then $t' = |T'| = |T| + 1 = t + 1 \leq s$. Since the neighbors of S in G are all in $T' \cup \{v_2\}$, and each vertex of S has d edges coming out, the number of edges from S to T' is $d|S| = ds$. The number of edges from S to T' is at most $d|T'| = dt'$. Therefore $ds \leq dt'$, so $s \leq t'$. As $t' \leq s$, we have that $t' = s$, and the subgraph H' of G induced by $S \cup T'$ is d-regular, therefore H' is a component of G not containing x_1, contrary to G being connected. □

In the following result, G' is regular.

Corollary 1 *Let G be a d-regular connected bipartite multigraph with $d \geq 2$ (therefore with chromatic index d). If G' is obtained from G by inserting two vertices v_1, v_2 of degree two in edges e of G, and then joining v_1, v_2 with $d-2$ parallel edges, then G' is d-regular and has chromatic index d.*

Proof. Each of the d edge colors in theorem 4 must match all the vertices of degree d in $X \cup Y$ to some other vertex, so if x_1 matches v_1, then y_2 matches v_2 (color i), and if y_1 matches v_1, then x_2 matches v_2 (color $j \neq i$). The $d-2$ parallel edges joining v_1, v_2 can be given the colors other than i, j. □

In the following result, G' is non-regular.

Corollary 2 *Let G be a connected bipartite multigraph of maximum degree d that is not d-regular. If G' is obtained from G by inserting either one or two vertices, in one or two edges, then G' has chromatic index d.*

Proof. The multigraph G can be made a connected bipartite d-regular multigraph G'' by possibly adding vertices and adding at least one edge e. The result follows by applying theorem 4 to G'', using for the case of adding just one vertex an added edge e with an inserted vertex. □

We next consider the case $k = 4$.

Theorem 5 *Let G be a d-regular connected bipartite multigraph with $d \geq 2$ (therefore with chromatic index d). If G' is obtained from G by inserting a total of four vertices of degree two on edges e of G, then G' has chromatic index d as well.*

Proof. The result is trivial for $d = 2$ (G and G' are even cycles), so we consider $d \geq 3$.

We assume that only one vertex has been inserted in an edge e. Otherwise we have paths $q = xyzt$, with y, z of degree two in G'. Such a path q can be replaced by the edge xt, returning to G with only two inserted vertices, which has chromatic index d by theorem 4. The color assigned to xt in G will be used for xy and zt in G', and any other color for yz.

We also assume that the edges e that had a vertex inserted do not share endpoints. Otherwise we have paths $q' = xyztu$, with y, t of degree two in G'. Such a path q' can be replaced by the path xzu, returning to G with only two inserted vertices, which has chromatic index d by theorem 4 The colors $i \neq j$ assigned to xz and zu respectively in G will be used for xy, zt (color i) and yz, tu (color j) in G'.

We shall show that such a G' has a perfect matching M, except for one case to be described later. After removing the edges of M from G', the multigraph graph $G'' = G - M$ has vertices of degrees 1 and $d-1$ (instead of degrees 2 and d for G'), and G'' is bipartite, thus $(d-1)$-edge-colorable, and we may use the remaining color for M, completing the proof except for the one case.

Let $G = (X, Y, E)$, and consider the disjoint edges $x_i y_i$ that are replaced by paths $x_i v_i y_i$ in G', with $1 \leq i \leq 4$, $x_i \in X$, $y_i \in Y$. To obtain the perfect matching M, select for M the edges $x_1 v_1$, $x_2 v_2$, $v_3 y_3$, and $v_4 y_4$.

Let $G'' = (X', Y', E')$ be the subgraph of G induced by $V(G)$ minus the vertices x_1, x_2, y_3 and y_4. If G'' has a perfect matching M', then combining M' with the selected edges involving v_i gives the perfect matching M. Suppose G'' does not have a perfect matching. Then by Hall's theorem there exist subsets $S \leq X'$ and $T \leq Y'$ with $s = |S| > t = |T|$ such that all the neighbors of S in G'' are in T.

Let $T' = T \cup \{y_3, y_4\}$ in G. Then $t' = |T'| = |T| + 1 = t + 2 \leq s + 1$. Suppose $t' \leq s$. Since the neighbors of S in G are all in $T' \cup \{v_2\}$, and each vertex of S has d edges coming out, the number of edges from S to T' is $d|S| = ds$. The number of edges from S to T' is at most $d|T'| = dt'$. Therefore $ds \leq dt'$, so $s \leq t'$. As $t' \leq s$, we have that $t' = s$, and the subgraph H' of G induced by $S \cup T'$ is d-regular, therefore H' is a component of G not containing x_1, contrary to G being connected.

In the remaining one case we have $t' = s + 1$. In this case the degrees in G of T' and S differ by $dt' - ds = d$, so we have d edges e_i from T' to $X \setminus S$. Consider again the subgraph H' of G induced by $S \cup T'$. Notice that v_1, v_2 are not inserted in H' to obtain G' (since x_1, x_2 are in $X \setminus S$), so only v_3, v_4 may be inserted in H' to obtain G'. If none or both of v_3, v_4 are inserted in H', then we separate G' into two multigraphs G_1 and G_2. The multigraph G_1 is obtained by considering H' and identifying the endpoints not in H' of the d edges e_i to a new vertex a. The multigraph G_2 is obtained by considering the subgraph H'' induced by vertices of G not in H' and identifying the endpoints in H' of the d edges e_i to a new vertex b. We then insert the four vertices of degree two to obtain G'_1 and G'_2, where the vertices inserted in some of the d edges e_i are inserted in G'_2. Both G'_j can be d-edge colored by induction, since they each have zero, two, or four vertices inserted. Each of the d colors selects one of the d edges e_i^1 from e_i attached at a in G'_1, and one of the d edges e_i^2 from e_i attached at b in G'_2, so the two colorings can be combined to d-edge color G'.

Finally if exactly one of v_3, v_4 is inserted in H', say v_3, then v_4 is inserted in an edge joining that joins at y_4 the subgraphs H' and H'' defined as before. If none or both of v_1, v_2 are inserted in H'', then we proceed as in the previous case to separate the subgraphs G_1 and G_2 of G with vertices a, b, this time inserting the vertices inserted in edges e_i in G'_1 (instead of G'_2), and proceed similarly as before by symmetry. Otherwise say only v_1 is insertex in H'', only v_3 is inserted in H', and v_2, v_4 are inserted in edges joining H' to H''. We then form G_1 and G_2 with added vertices s, t as before, but with v_2 in G'_1 and v_4 in G'_2. Both G'_j can be d-edge colored, since they each have two vertices inserted. Each of the d colors selects one of the d edges e_i^j from e_i in both G'_j, so the two colorings can be combined. □

In the following result, G' is non-regular.

Corollary 3 *Let G be a connected bipartite multigraph of maximum degree d that is not d-regular. If G' is obtained from G by inserting either three or four vertices of degree two on edges, then G' has chromatic index d.*

Proof. The multigraph G can be made a connected bipartite d-regular multigraph G'' by possibly adding vertices and at least one edge e. The result follows by applying theorem 5 to G'', and for the case of adding just three vertices, adding a vertex to an added edge e. □

We finally show the following case.

Theorem 6 *Let $G = (X, Y, E)$ be a d-regular bipartite multigraph, and consider a perfect matching M in G consisting of edges x_iy_i, $1 \leq i \leq n$. Insert $k = n$ vertices v_i of degree two, one in each edge x_iy_i of M, to obtain G'. Let G'' be the n-vertex graph obtained from G by shrinking the edges in M, so that G'' is $2(d-1)$-regular and not necessarily bipartite. If G'' has a perfect matching M', then G' has a perfect matching M'', and G' has chromatic index d.*

Proof. If an edge e of M' in G'' comes from an edge x_iy_j of G, $i \neq j$, then M'' can choose the edges x_iy_j, v_iy_i, x_jv_j, and M'' will be a perfect matching in G'. The multigraph $G' - M''$ is bipartite of degrees 1 and $d-1$, so $G' - M''$ can be $(d-1)$-edge colored, and an additional color for M'' gives the d-edge coloring for G'. □

4 List-Edge-Coloring Almost Bipartite Multigraphs

The *list-edge-coloring problem* for a multigraph G is the edge-coloring problem for G where each edge is given a list of available colors. The *List-Edge-Coloring Conjecture* states that if a multigraph G can be edge-colored with d colors, then G can be list-edge-colored with arbitrary lists of size d at the edges, i.e., G is *d-edge-choosable*. In other words, the conjecture states that the *choosing number* of a graph is the same as its chromatic index.

Examples of multigraphs where the List-Edge-Coloring Conjecture has been proven are bipartite multigraphs [6, 12, 10], some nearly bipartite multigraphs (other than our almost bipartite multigraphs) [10, 2], d-regular planar multigraphs of chromatic index d [3, 13], and cliques of size n for n odd [8].

Theorem 7 *Suppose G is a multigraph that can be list-edge-colored with lists of size $d \geq 3$. Suppose we insert one or more vertices of degree two on some of the edges of G to obtain G'. Then G' can be list-edge-colored with lists of sizes d and $d+1$, where the only lists of size $d+1$ occur as exactly one of the two edges where only one vertex of degree two was inserted.*

Proof. If an edge e in G was replaced by a path p in G', put the list of the first edge in p of size d on e and list-edge-color G. Back on G', we use the chosen color for e at the first edge of p, and then color the remaining edges of p from the end backwards. At most d available colors suffice at each step of going back through p, since each edge has at most $d-1$ previously colored adjacent edges in this process, unless p had just two edges in which case the last edge of p will need $d+1$ available colors. □

The following lemma and theorem are known.

Lemma 1 *[6, 12, 10] Let $G = (X, Y, E)$ be a bipartite multigraph of maximum degree d that has the edges colored $1, 2, \ldots, d$. Then G has a matching M (not necessarily a perfect matching) such that if $e \in E \setminus M$ and $e = (x, y)$, then either there is some $f = (x, y') \in M$ with the color of f smaller than the color of e, or there is some $g = (x', y) \in M$ with the color of g greater than the color of e.*

Proof. Consider the stable marriage problem with X the set of men and Y the set of women. The preference list of men has the edges incident to $x \in X$ listed in preference of increasing order of color (as they list corresponding mates $y \in Y$), and the preference list of women has the edges incident to $y \in Y$ listed in preference of decreasing order of color (as they list corresponding mates $x \in X$. Then M will always exist as a stable marriage M by the result of Gale and Shapley [5, 7, 4].

□

The following result sometimes allows lists smaller than d.

Theorem 8 *[6, 12, 10] Let $G = (X, Y, E)$ be a bipartite multigraph of maximum degree d that has the edges colored $1, 2, \ldots, d$. Let $F(e)$ be the sum $r_e + s_e + 1 \leq d$ for an edge $e = (x, y)$ of color i, where $r_e \leq i - 1$ is the number of edges with endpoint x other than e of smaller color than i, and $s_e \leq d - i$ is the number of edges with endpoint y other than e of color greater than i. If we assign to each such e a list $L(e)$ with $|L(e)| \geq F(e)$, then a coloring with colors from $L(e)$ for each e exists.*

Proof. Let c be such that c appears in at least one $L(e)$, and let $G_c = (X, Y, E_c)$ be the subgraph of G with $c \in L(e)$ for each $e \in E_c$. Find a stable marriage M_c in G_c as in Lemma 1, assign c to each $e \in M_c$, and remove c from $L(e)$ for each c $E_c \setminus M_c$. Each time the size of $L(e)$ is reduced by one in this last case, we have that $r_e + s_e + 1$ is reduced by one as well in going from G to $G' = (X, Y, E - M_c)$ by the stable marriage condition, thus preserving $|L(e)| \geq F(e)$. Eventually we go through the various c appearing in the $L(e)$ and get the solution desired. □

As a corollary to theorem 8, we obtain the following result.

Theorem 9 *Let $G = (X, Y, E)$ be a biparite multigraph of maximum degree d, and let G' be obtained from G by inserting k vertices on the edges of G. Suppose each $f \in E(G')$ is given $L(f)$ with $|L(f)| = d$, but in such a way that if e with $\ell_e = 2$ was replaced by g, h, then $L(g) \neq L(h)$. Then a list edge coloring with such lists of size d exists.*

Proof. Color the edges of G with $1, 2, \ldots, d$, and if $e = xy$ with $f(e) = r_e + s_e + 1 \leq d$ was replaced by g, h with $g = xz$ and $h = zy$, replace g, h by splitting z, that is, $g' = xz'$ and $h' = z''y$. Then $F(g') = r_e + 1$ and $F(h') = s_e + 1$ with $F(g') + F(h') \leq d + 1$, and we can alway replace the lists $L(g) \neq L(h)$ having $|L(g)| = |L(h)| = d$, with lists $L'(g'), L'(h')$ having $L'(g') \cap L'(h') = \emptyset$, $L'(g') \subseteq L(g)$, $L'(h') \subseteq L(h)$, $|L'(g')| = r_e + 1$ and $|L'(h')| = s_e + 1$, so that $|L(g')| + |L(h')| \leq d + 1$. The key is to select the most elements from the disjoint parts of $L(g), L(h)$ to obtain $L'(g'), L'(h')$ of the right size and disjoint, which is possible since $|L(g) \cup L(h)| \geq d + 1$. Once $L'(g'), L'(h')$ are disjoint we can solve the problem as in theorem 8, since the multigraph is now bipartite, and joining z', z'' at z later colors g, h. (The case of inserting two or more vertices in an edge is handled without restrictions as in theorem 7.) □

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