The Structure of Bi-Arc Trees

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Abstract

Bi-arc graphs generalize (reflexive) interval graphs and those (irreflexive) bipartite graphs whose complements are circular arc graphs. They are relevant for the so-called list homomorphism problem: when H is a bi-arc graph, the problem is polynomial time solvable, otherwise it is NP-complete. Bi-arc graphs have a forbidden structure characterization, and can be recognized in polynomial time. More importantly for this paper, bi-arc graphs can be characterized by the existence of a conservative majority function. (This function plays an important role in proving the correctness of a polynomial time list homomorphism algorithm.)

The forbidden structure theorem for bi-arc graphs is quite complex, and the existence of a conservative majority function is proved without giving an explicit description of it.

In this note we focus on bi-arc graphs that are trees (with loops allowed). We describe the structure of bi-arc trees, and give a simple forbidden subtree characterization. Based on this structure theorem, we are able to explicitely describe the conservative majority functions.

1 Introduction

We consider graphs with loops allowed (cf. [8]). A graph in which every vertex has a loop is called *reflexive*; a graph in which no vertex has a loop called *irreflexive*. A bipartite graph is, by definition, irreflexive. A *tree* is a connected graph without cycles of length greater than one. (Thus trees may contain loops.)

A *homomorphism* f of a graph G to a graph H is a mapping $f : V(G) \to V(H)$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. Note the role of vertices with loops: firstly, if a vertex u has a loop in G, then $f(u)$ must also have a loop in H. Secondly, if u

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and v are adjacent in G and $f(u) = f(v)$, then the vertex $f(u)$ of G must have a loop. Homomorphisms to graphs with loops are allowed are of interest in statistical physics, cf. [2, 3, 8].

Given graphs G, H, and lists $L(v) \subseteq V(H), v \in V(G)$, a *list homomorphism* of G to H with respect to the lists L is a homomorphism f of G to H, such that $f(v) \in L(v)$ for all $v \in V(G)$. For a fixed graph H, the *list homomorphism problem* LIST-HOMH asks whether or not an input graph G with lists L admits a list homomorphism to H with respect to L.

In [6], we have given the following complete classification of the complexity of list homomorphism problems LIST-HOMH.

Theorem 1.1 *[6] The list homomorphism problem* LIST-HOMH *is polynomial time solvable if H is a bi-arc graph, and is NP-complete otherwise.*

Bi-arc graphs are defined as follows. Let C be a circle with two specified points p and q on C. A *bi-arc* is an ordered pair of arcs (N, S) on C such that N contains p but not q, and S contains q but not p. A graph H is a *bi-arc graph* if there is a family of bi-arcs $\{(N_x, S_x) : x \in V(H)\}$ such that, for any $x, y \in V(H)$, not necessarily distinct, the following hold:

- if x and y are adjacent, then neither N_x intersects S_y nor N_y intersects S_x;
- if x and y are not adjacent, then both N_x intersects S_y and N_y intersects S_x.

The family of bi-arcs $\{(N_x, S_x) : x \in V(H)\}$ is called a *bi-arc representation* of H. Note that a bi-arc representation can not contain bi-arcs $(N, S), (N', S')$ such that N intersects S' but S does not intersect N' or vice versa.

Bi-arc graphs unite two well-known families of graphs: the interval graphs, and the circular arc graphs of clique covering number two (or, more precisely, their complements). We say that a reflexive graph H is an *interval graph* if each vertex x of H can be associated with a real interval I_x, so that two vertices x and y are adjacent in H if and only if I_x and I_y intersect. The family of intervals $\{I_x : x \in V(H)\}$ is called an *interval representation* of H. It turns out that every interval graph is a bi-arc graph, and every reflexive bi-arc graph is an interval graph [6]. We say that an irreflexive graph H is a *circular arc graph* if each vertex x of H can be associated with an arc on a circle, so that two vertices are adjacent if and only if the associated arcs intersect. Again, the family of associated arcs is called a *circular arc representation* of H. We have also shown that every bipartite (hence irreflexive) graph whose complement is a circular arc graph is a bi-arc graph, and that every irreflexive bi-arc graph is a bipartite graph whose complement is a circular arc graph [6]. (We note that we are only taking complements of irreflexive graphs in this paper, and these complements are by definition also irreflexive; in other words, loops never need to be complemented.) In particular, when H is a reflexive graph, the problem LIST-HOMH is polynomial time solvable if H is an interval graph, and is NP-complete otherwise, cf. [4]. When H is an irreflexive graph, the problem LIST-HOMH is polynomial time solvable if H is a bipartite graph whose complement is a circular arc graph, and is NP-complete otherwise, cf. [5].

For a graph H, the *associated bipartite graph* H^* is defined as follows. The vertex set of H^* consists of sets $A_H = \{x'; x \in V(H)\}$ and $B_H = \{x''; x \in V(H)\}$, forming a bipartition of H^*. The edge set of H^* consists of all edges $x'y''$ such that xy is an edge of H. Note that there are two kinds of edges in H^*, namely edges $x'x''$ where x has a loop in H, and edges $x'y''$ where $x \neq y$ and $xy \in E(H)$, which go in pairs, i.e., both $x'y''$ and $x''y'$ are present in H^*. A simple yet useful observation is that a bi-arc representation of H corresponds precisely to a circular arc representation of the complement of H^*. Thus we obtain the following fact.

Proposition 1.2 *H is a bi-arc graph if and only if the complement of H^* is a circular arc graph.*

This fact will aid us in describing a forbidden structure characterization of bi-arc graphs. It is possible to describe the forbidden structres in terms of the graph H, but we obtain a neater description if we state it in terms of the associated bipartite graph H^*. In particular, the characterization using H^* becomes reminiscent of the celebrated characterization of interval graphs by Lekkerkerker and Boland [9].

An *edge-asteroid* in a bipartite graph, with the bipartition (X, Y), is a set of $2k+1$ edges $u_0v_0, u_1v_1, \ldots, u_{2k}v_{2k}$ ($k \geq 1$ and each $u_i \in X$ and $v_i \in Y$), and $2k+1$ paths, $P_{0,1}, P_{1,2}, \ldots, P_{2k,0}$, where each $P_{i,i+1}$ joins u_i to u_{i+1}, such that for each $i = 0, 1, \ldots, 2k$ there is no edge between $\{u_i, v_i\}$ and $\{v_{i+k}, v_{i+k+1}\} \cup V(P_{i+k,i+k+1})$. (Subscripts are modulo $2k+1$.) We refer to the (odd) integer $2k+1$ as the *order* of the edge-asteroid.

We have the following forbidden substructure characterization of bi-arc graphs.

Theorem 1.3 *[5] H is a bi-arc graph if and only if H^* contains no edge-asteroid, and no induced cycle of length greater than four.*

Finally, we relate bi-arc graphs to a certain class of functions, useful in analyzing the complexity of list homomorphism problems. Let H be an arbitrary graph. A function $g : V(H) \times V(H) \times V(H) \to V(H)$ is called a *conservative majority function* (also called a *majority choice function* [6]) if it satisfies the following properties:

1. $g(x, y, z) \in \{x, y, z\}$,
2. $g(x, w, w) = g(w, y, w) = g(w, w, z) = w$,
3. $g(x, y, z)g(x', y', z') \in E(H)$ whenever $xx', yy', zz' \in E(H)$.

Feder and Vardi [7] proved that if H admits a conservative majority function, then LIST-HOMH is polynomial time solvable, cf. [8]. The algorithm itself does not use the conservative majority function; its existence is however needed to establish the correctness of the algorithm.

In [6] we used this algorithm to solve (in polynomial time) the problem LIST-HOMH for a bi-arc graph H. We were able to prove [6] that a bi-arc graph H always admits conservative majority function. However, we were not able to explicitely describe the function in terms of the graph H. (On the other hand, there is a polynomial time algorithm to find it for any concrete graph H, [7].)

In fact, it turns out that bi-arc graphs are precisely the graphs which admit conservative majority functions. This further underscores the importance of this natural class of graphs.

Theorem 1.4 *[1] H is a bi-arc graph if and only if it admits a conservative majority function.*

In this paper we focus on bi-arc graphs that are trees. We prove a structure theorem giving a complete description (including a characterization in terms of forbidden induced subgraphs) of all bi-arc trees. Using this structure theorem we are then able to explicitely describe conservative majority functions for all bi-arc trees H.

2 The Structure Theorem

For a graph H, we use H^o to denote the subgraph of H induced by vertices with loops. We say that H^o is *convex*, if every induced path P of H joining two vertices of H^o satisfies $V(P) \subseteq V(H^o)$.

Lemma 2.1 *If H is a bi-arc graph, then H^o is convex.*

Proof: Suppose that H^o is not convex. Then there is an induced path $x_1x_2\ldots x_k$ ($k \geq 3$), such that $x_i \in V(H^o)$ if and only if $i = 1$ or k. Thus in H^* the vertices $x'_1, x''_1, x'_2, x''_2, \ldots, x'_k, x''_k$ form an induced cycle of length greater than four. By Theorem 1.3, the complement of H^* is not a circular arc graph and hence H is not a bi-arc graph. $\square$

Lemma 2.2 *Let H be a graph. Suppose that there exists a representation of H by a family of intervals $\{I_v : v \in V(H)\}$ such that*

- *for $u \neq v$, I_u and I_v intersect if and only if u and v adjacent;*
- *for each $u \in V(H)$ and $v \in V(H) - V(H^o)$, either I_u contains I_v, or I_u and I_v do not intersect.*

Then H is a bi-arc graph.

Proof: We first describe how to transform the given interval representation $\{I_v : v \in V(H)\}$ to a bi-arc representation of the reflexive graph obtained from H by adding all missing loops. (This is similar to the technique explained in [6], where it is described in more detail and with illustrations.) Let C be a circle with two specified diamterically opposed points p and q. We first represent each vertex by an arc on the open segment extending from p to q clockwise. That is, each vertex v corresponds to an arc I_v from a point a_v to a point b_v clockwise in such a way that two *distinct* vertices v and v' are adjacent in H if and only if I_v intersects $I_{v'}$. We obtain a second set of arcs J_v by rotating the first one by 180 degrees. In the second set of arcs, each vertex v corresponds to an arc J_v from a point c_v to a point d_v clockwise in the open segment from q to p. Now our bi-arc representation of H will consist of pairs (N_v, S_v), with N_v being the arc from d_v to a_v, and S_v being the arc from b_v to c_v. Clearly, each N_v contains p but not q and each S_v contains q but not p. It is easy to verify that this is indeed a bi-arc representation of H. Observe that for each $v \in V(H) - V(H^o)$, the bi-arc (N_v, S_v) contains the endpoints of every other arc. In other words, no other arc has its endpoints contained in the two 'gaps' created by N_v and S_v. Thus it is possible to extend N_v and S_v so that they intersect each other. Doing this for all vertices in $V(H) - V(H^o)$, we obtain a bi-arc representation of the given graph H. $\square$

Let H be an irreflexive bipartite graph, with a bipartition (A, B). Suppose that there is a representation of H by a family $\{J_v : v \in V(H)\}$ of arcs on a circle with two specified diamterically opposite points p and q, such that each J_a with $a \in A$ contains p but not q, and each J_b with $b \in B$ contains q but not p. Suppose that two vertices a and b are adjacent in H if and only if J_a and J_b do not intersect. Let S be a subset of A, such that for each $s \in S$, the clockwise endpoint of the arc J_s is not contained in any J_b, with $b \in B$. If $|S| \geq 2$, we further assume that there is at most one vertex $w \in B$ such that for each $b \in B - \{w\}$ and each $s \in S$, the arc J_b contains the counterclockwise endpoint of J_s. Then we say that $\{J_s : s \in S\}$ is *in a good position* in the circular arc representation. (A similar definition can be given when S is a subset of B.) In the case when $S = \{s\}$ consists of a single vertex, we also say that J_s is in a good position (in the representation).

Let H^1 be the graph obtained from H by deleting all loops in H.

Lemma 2.3 *Let H^1 be a bipartite graph and (A, B) be a bipartition of H^1. Suppose that H^o is a star. Suppose that there is a circular arc representation $\{J_v : v \in V(H^1)\}$ of H^1 such that both $\{J_a : a \in V(H^o) \cap A\}$ and $\{J_b : b \in V(H^o) \cap B\}$ are in good positions. Then H is a bi-arc graph.*

Proof: We first obtain a bi-arc representation of H^1 as follows. (This is again similar to the technique from [6].) Let $\mathcal{F}_1$ be the above family of arcs $J_v, v \in V(H)$. It is easy to see that we may assume that there are two other diameterically opposite points r and s on the circle, with the clockwise order p, r, q, s (we think of p at 12, r at 3, q at 6, and s at 9), where each $J_a, a \in A$ contains both p and r, but neither q nor s, and each

$J_b, b \in B$ contains both q and s, but neither p nor r. Let $\mathcal{F}_2$ be a second family of arcs, obtained from $\mathcal{F}_1$ by reflecting all arcs of $\mathcal{F}_1$ with respect to the line passing through r and s. Thus, in $\mathcal{F}_2$, each arc J_a with $a \in A$ contains q and s, but neither p nor r, and each arc J_b with $b \in B$ contains p and r, but neither q nor s. Further, two vertices $a \in A$ and $b \in B$ are adjacent in H^1 if and only if J_a and J_b do not intersect. Then these two families $\mathcal{F}_1$ and $\mathcal{F}_2$ together form a bi-arc representation of H^1. A bi-arc representation of the given graph H can now be obtained by 'shortening' the two arcs of each bi-arc with $v \in V(H^o)$, to make sure that they are no longer overlapping. This is possible, as both $\{J_v : v \in V(H^o) \cap A\}$ and $\{J_v : v \in V(H^o) \cap B\}$ are in good position. $\square$

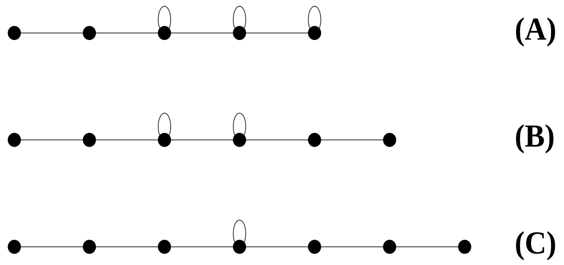

Figure 1:

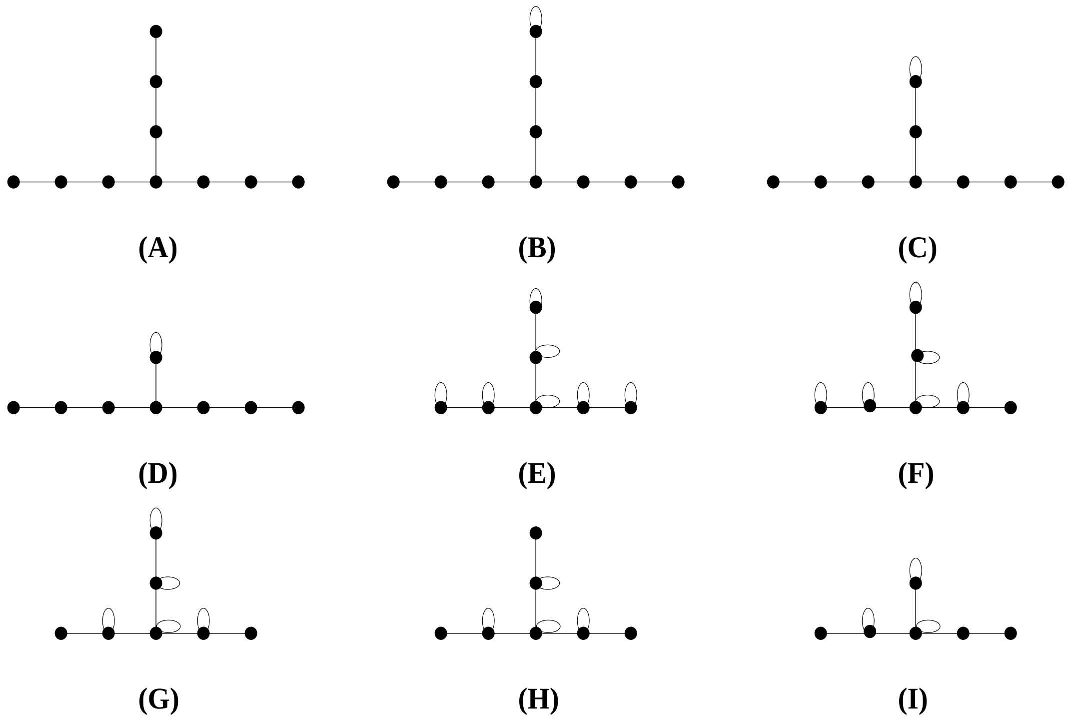

Figure 2:

Lemma 2.4 *If H contains any graph from Figure 1 or from Figure 2 as an induced subgraph, then H is not a bi-arc graph.*

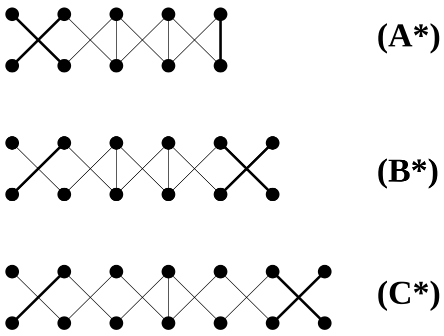

Figure 3:

Proof: It suffices to show that if H is one of graphs in Figure 1 or 2, then H^* contains an edge-asteroid. This is indeed the case, as can be seen in Figures 3 and 4, which contain the associated bipartite graphs of the graphs in Figures 1 and 2. An edge-asteroid can be found in each, by taking the bold edges together with suitable paths. (All the asteroids are of order three, except the one in Figure 4(I^*), which is of order five.) $\square$

When H is a reflexive tree, H is an interval graph if and only if it does not contain the graph in Figure 2(E), [9]. Such a tree is also known as a *reflexive caterpillar.* Thus it follows that in the case when H is reflexive, LIST-HOMH is polynomial time solvable if H is a caterpillar, and is NP-complete otherwise. We shall refer to a reflexive caterpillar also as a *reflexive polynomial tree.* When H is an irreflexive tree, it follows from Theorem 1.3 that the complement of H is a circular arc graph if and only if H does not contain the graph in Figure 2(A) (since H does not contain any cycle and if it contains an edge-asteroid then it must contain the graph in Figure 2(A)). Hence, when H is irreflexive, LIST-HOMH is polynomial time solvable if H does not contain the graph in Figure 2(A), and is NP-complete otherwise. We shall refer to an irreflexive tree which does not contain the graph in Figure 2(A) as an *irreflexive polynomial tree.*

We define a vertex v in a tree to be a *good vertex* if there does not exist a path P of length six, with the middle vertex u, such that v is joined to u by a path (possibly of length zero) which is internally vertex-disjoint from P. For instance, every vertex of the graph in Figure 1(A) is a good vertex, and so is every vertex of the graph in Figure 1(B). All vertices except the center vertex of the graph in Figure 1(C) are good vertices. The graph in Figure 2(A) has no good vertices.

Our main result is the following structural description of bi-arc trees:

Theorem 2.5 *Let H be a tree such that H^o is a subtree of H. Then the following three statements are equivalent.*

(i) H is a bi-arc graph;

(ii) H does not contain as an induced subgraph any graph from Figure 1 or from Figure 2;

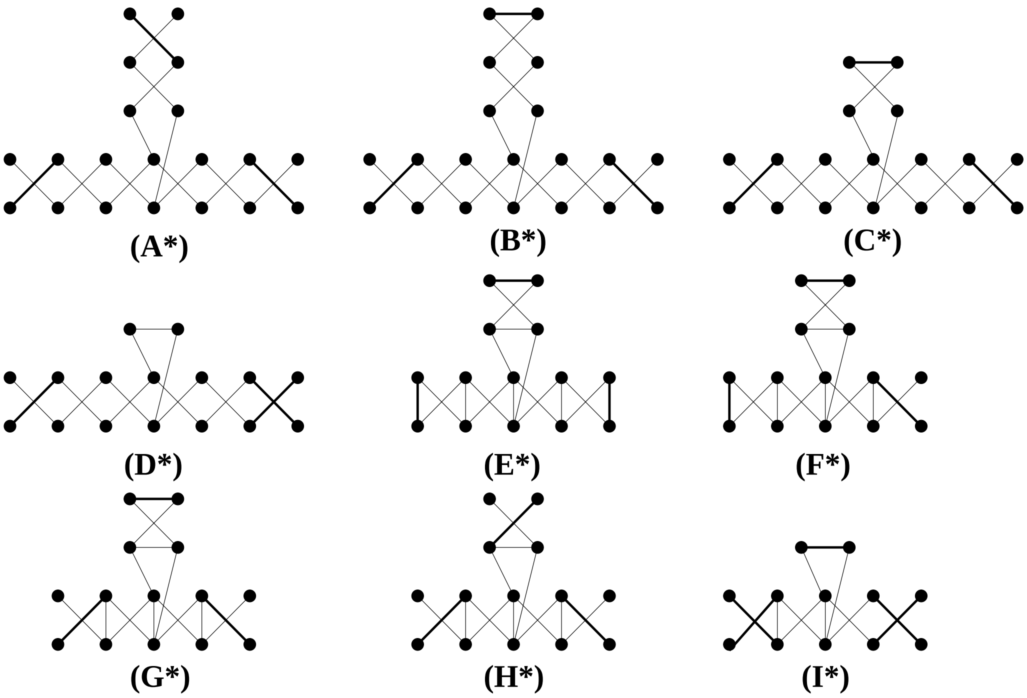

Figure 4:

(iii) H is obtained either

from a reflexive polynomial tree by deleting the loops at a subset (possibly empty) of leaves, or

from an irreflexive polynomial tree by performing one of the following three operations:

- *adding no loops, or adding a loop on a good vertex v,*
- *adding a loop on a good vertex v and on a neighbour w of v, such that each neighbour of w other than v is a leaf,*
- *adding a loop on a good vertex v, and loops on any number of neighbours of v which are leaves.*

The cases are illustrated in Figures 5 and 6.

Proof: $(i) \Rightarrow (ii)$: This follows from Lemma 2.4.

$(ii) \Rightarrow (iii)$: Suppose that H does not contain as an induced subgraph any graph from Figure 1 or from Figure 2. Consider first the case when H^o contains a path, say Q, of length at least three. In this case, each vertex of $H - V(H^o)$ must be a leaf of H. If not, then $H - V(H^o)$ contains two adjacent vertices x, y. Following the unique path from $\{x, y\}$ to Q, we see that H contains the graph in Figure 1(A), contradicting the assumption. Since H does not contain any graph in Figure 2(E)-(H) as an induced subgraph, H^1 is a caterpillar. Hence H is obtained from a reflexive polynomial tree by deleting the loops at some leaves. Thus we may now assume that H^o contains no path of length at least three. This implies that either H is irreflexive or H^o is a star. When H is irreflexive, H is an irreflexive polynomial tree because it does not contain the graph in Figure 2(A) as an induced subgraph. So we assume that H^o is a star. Since H does not contain as an induced subgraph any graph in Figure 1(A)-(C) or Figure 2(A)-(D), H^1 is an irreflexive polynomial tree. We claim that every vertex of H^o is a good vertex of H. Suppose that some $y \in V(H^o)$ is not a good vertex of H. Then there exists a path $P = x_1x_2x_3x_4x_5x_6x_7$ and a path $R = y \dots x_4$ (possibly of length zero) which is internally disjoint from P. If $x_4 \notin V(H^o)$, then H must contain one of the graphs in Figure 2(A)-(D), a contradiction. If both x_4 and x_6 are in $V(H^o)$, then $x_5 \in V(H^o)$ and $\{x_1, x_2, \dots, x_6\}$ induces a copy of the graph in Figure 1(A), a contradiction; similarly, if both x_4 and x_2 are in $V(H^o)$, then $\{x_2, x_3, \dots, x_6\}$ induces a copy of the graph in Figure 1(A), a contradiction. So x_4 is in $V(H^o)$ but neither x_2 nor x_6 is in $V(H^o)$. It is now easy to see that the subgraph induced by $\{x_1, x_2, \dots, x_7\}$ contains one of the graphs in Figure 1(A)-(C), which is again a contradiction. Therefore we have shown that every vertex of H^o is indeed a good vertex of H.

If H^o has exactly one vertex, then H is obtained from an irreflexive polynomial tree by adding a loop on a good vertex. Suppose that H^o has exactly two vertices, say u and v. If each of u and v has a neighbour which is not in $\{u, v\}$ and which is not a leaf, then it is easy to see that H contains the graph in Figure 1(B), a contradiction. Hence H is obtained from an irreflexive polynomial tree by performing the second operation. Suppose that H^o has at least three vertices. Let $V(H^o) = \{v, v_1, \dots, v_k\}$ with $k \geq 2$, where v is the center of the star H^o. If v_i is a leaf of H for each $i = 1, 2, \dots, k$, then H is obtained from an irreflexive polynomial tree by performing the third operation. So assume that at least one v_j is not a leaf. Using the fact that H does not contain any of the two graphs in Figure 1(A) and Figure 2(H)(I), we see that each vertex of $H - V(H^o)$ must be a leaf of H. Hence H is obtained from a reflexive polynomial tree by deleting loops at some leaves.

(iii) $\Rightarrow$ *(i)*: Suppose that H is obtained from a reflexive polynomial tree by deleting the loops at some leaves. Then H^1 is an interval graph and each vertex of $H - V(H^o)$ is a leaf of H. Thus there is a representation of H by a family of intervals satisfying the properties of Lemma 2.2, and hence H is a bi-arc graph. So suppose next that H is obtained from an irreflexive polynomial tree by performing one of the three operations listed in the theorem. Let (A, B) be a bipartition of H. We may assume that $V(H^o) \neq \emptyset$. It is easy to see that every vertex of $V(H^o)$ is a good vertex. We will show by induction on $|V(H)|$ that there is a representation of H by a family of circular arcs, such that the two arc sets corresponding to $V(H^o) \cap A$ and $V(H^o) \cap B$ are both in good positions.

We first consider the case when $V(H^o) = \{v\}$. This means that H is obtained from an irreflexive polynomial tree by adding a loop on a good vertex v. Without loss of generality, assume that $v \in A$. Suppose that v is a leaf and is adjacent to x in H. Then x is a good vertex of $H - v$, and hence there is a circular arc representation of $H - v$ such that the arc corresponding to x is in a good position. Add to the family a new arc which contains all the clockwise endpoints of the arcs corresponding to the vertices of $B - \{x\}$, and which does not contain the counterclockwise endpoint of any arc corresponding to a vertex of B. This is possible since in the existing family the arc corresponding to x is in a good position. The resulting family is a circular arc representation of H, in which the added arc corresponding to v is in a good position. Suppose that v is adjacent to a leaf z. Represent $H - z$ by a family of circular arcs in such a way that the arc corresponding to v is in a good position. It is possible to add an arc corresponding to z, to obtain a circular arc representation of H in which the arc corresponding to v is still in a good position. So we may assume that no vertex in $N(v) \cup \{v\}$ is a leaf. Denote $N(v) = \{u_0, u_1, \ldots, u_k\}$. Then $k \geq 1$ and $N(u_i) - \{v\} \neq \emptyset$ for each $i = 0, 1, \ldots, k$. Since v is a good vertex, we may assume without loss generality that all vertices in $N(u_i) - \{v\}$ are leaves, for each $i = 1, 2, \ldots, k$. Represent $H - \cup_{i=1}^{k}(N(u_i) \cup \{u_i\} - \{v\})$ by a family of circular arcs so that the arc corresponding to v is in a good position. It is now possible to add new arcs corresponding to the vertices of $\cup_{i=1}^{k}(N(u_i) \cup \{u_i\} - \{v\})$ to obtain a circular arc representation of H in which the arc corresponding to v is still in a good position.

Suppose next that $V(H^o) = \{v, w\}$. Assume that all neighbours of w other than v are leaves. According to the analysis above, it is possible to represent $H - (N(w) \cup \{w\} - \{v\})$ by a family of circular arcs so that the arc corresponding to v is in a good position. Now a circular arc representation can be obtained as follows: Add an arc corresponding to w so that it is in a good position and then add arcs corresponding to the vertices of $N(w) - \{v\}$. It is again easy to see that this can be done.

Finally, suppose that $V(H^o) = \{v, w_1, \ldots, w_p\}$ induces a star with center v, where each w_i is a leaf in H. In this case, first represent $H - \{w_1, \ldots, w_p\}$ by a family of circular arcs so that the arc corresponding to v is in a good position. Then add arcs corresponding to $w_1, \ldots, w_p$, so that the arc corresponding to v is in a good position and also the arc set corresponding to $\{w_1, \ldots, w_p\}$ is in a good position.

We have shown that there is a representation of H by a family of circular arcs so that the two arc sets corresponding to $V(H^o) \cap A$ and $V(H^o) \cap B$ are both in a good position. By Lemma 2.3, H is a bi-arc graph. $\square$

3 The Conservative Majority Functions

Recall that we have no explicit construction of the conservative majority function for an arbitrary bi-arc graph (even though [6] guarantees that one exists). Our structure theorem allows us to explicitly construct these conservative majority functions for bi-arc trees, according to the two classes described in (iii), Theorem 2.5.

Suppose first that H is a bi-arc tree obtained from a reflexive polynomial tree by deleting the loops at a subset of leaves. We may assume that $V(H^o) \neq \emptyset$. Let $P : v_1 v_2 \ldots v_l$ be a longest path in H^o. Since H is a caterpillar, so is H^o. Each vertex of $H^o - V(P)$ and hence of $H - V(P)$ is a leaf of H. Let S_i be the set of vertices of $H - V(P)$ adjacent to $v_i \in V(P)$ for each $i = 1, 2, \ldots, l$. Thus for each $x \in V(H)$ there exists a unique $\pi(x)$ such that $x \in S_{\pi(x)} \cup \{v_{\pi(x)}\}$. For each i, $i = 1, 2, \ldots, l$, we let O_i be an ordering of the vertices of $S_i \cup \{v_i\}$, in which v_i is the first vertex.

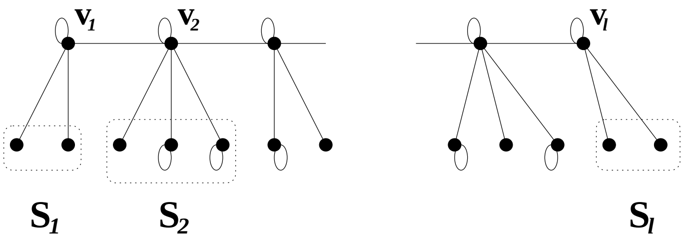

Figure 5:

We define a function $f : V(H) \times V(H) \times V(H) \to V(H)$ as follows: Let x, y, z be three vertices of H. If any two of x, y, z are the same vertex, say w, then $f(x, y, z) = w$; otherwise let m be the median of $\pi(x), \pi(y), \pi(z)$ and set $f(x, y, z)$ to be the first vertex of $\{x, y, z\}$ in the ordering O_m.

Proposition 3.1 *[6] The function f is a conservative majority function.*

Suppose next that H is obtained from an irreflexive polynomial tree by performing one of the three operations listed in Theorem 2.5. We may again assume that H contains at least one loop. Let $Q : u_1 u_2 \ldots u_r$ be a longest path in H such that u_1 is the vertex v in $V^o(H)$ described in Theorem 2.5, and it is the only vertex in Q that has a loop. Note that the vertices of $H - V(Q)$ can again be partitioned into sets $T_1, T_2, \ldots, T_r$ such that $T_i \cup \{u_i\}$ induces a subtree for each $i = 1, 2, \ldots, r$. (Note that the maximality of Q assures that $T_r = \emptyset$.) Thus for each $x \in V(H)$ there is again a unique $\sigma(x)$ such that $x \in T_{\sigma(x)} \cup \{u_{\sigma(x)}\}$. Note that $\sigma(x) = 1$ for all $x \in V(H^o)$. Since v is a good vertex and Q is a longest path, the distance between u_i and any other vertex in $T_i \cup \{u_i\}$ is either one or two.

We order the vertices of H as follows: First for each $i = 1, 2, \ldots, r$ we obtain an ordering O_i of the vertices of $T_i \cup \{u_i\}$, using the depth-first search starting at u_i. When $i = 1$, the search is in favour of vertices of H^o (note that $V(H^o) \subseteq T_1 \cup \{u_1\}$). Then we combine these orderings into a vertex ordering O of H by concatenating them in the order $O_1, O_2, \ldots, O_r$. Suppose that O is the ordering $v_1, v_2, \ldots, v_n$. Next, we colour the vertices of H with two colours so that two distinct adjacent vertices receive different colours (a proper two-colouring of H^1). Call each of the two sets of vertices of the same colour a *colour class*.

We now define a function $g : V(H) \times V(H) \times V(H) \to V(H)$ as follows: Let x, y, z be three vertices of H.

(a) If any two of x, y, z are the same vertex, say w, then $g(x, y, z) = w$.

(b) Suppose that x, y, z are distinct and in the same colour class. Let m be the median of $\sigma(x)$, $\sigma(y)$, $\sigma(z)$. We define $g(x, y, z)$ to be the *first* vertex among x, y, z in the ordering O_m, except in the following four exceptional situations:

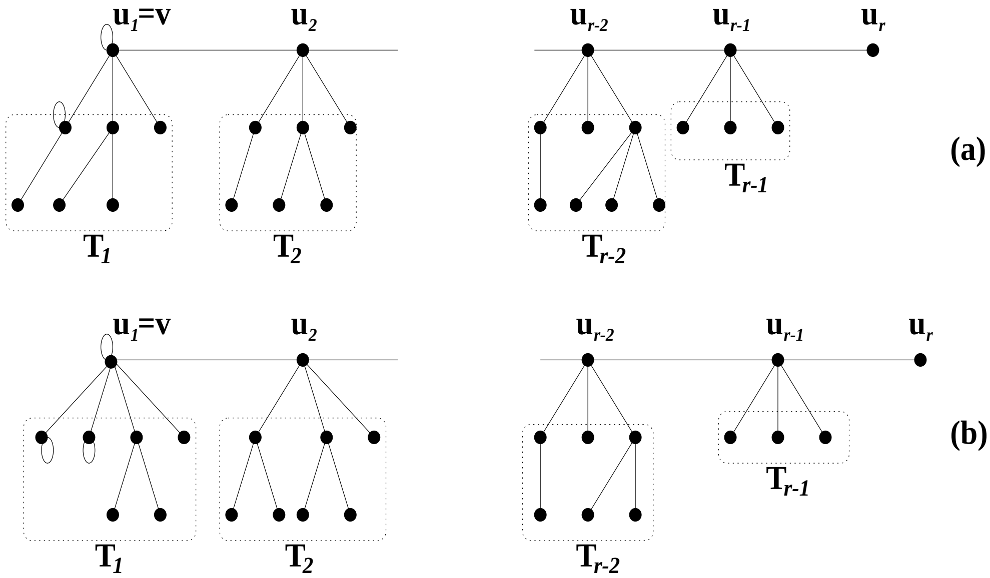

Figure 6:

- All three vertices x, y, z lie in T_m with $m \geq 2$;
- All three vertices x, y, z lie in T_1 and at most one of them has a loop;
- Exactly two of x, y, z lie in T_1 and exactly one of them has a loop;
- Exactly two of x, y, z lie in T_1, neither has a loop, exactly one of them is adjacent to the unique neighbour of u_1 with a loop, and the third vertex of x, y, z is not u_1. (This last case only applies in the situation in Figure 6(a).)

In these situations, we define $g(x, y, z)$ be the *second* vertex among x, y, z in the ordering O_m.

(c) Suppose that x, y, z are distinct but not all in the same colour class. We define $g(x, y, z)$ to be the first vertex in O of the two vertices in the same colour class, except when $\{x, y, z\}$ contains u_1 and at least one of its leaf neighbours with a loop, in which case we define $g(x, y, z) = u_1$. (This exceptional case only applies in the situation in Figure 6(b).)

Proposition 3.2 *[6] The function g is a conservative majority function.*

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