Line Graphs of Cliques and Clique Minors

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January 9, 2010

1 Introduction

Let K_n be the clique with vertices $0, 1, 2, \ldots, n-1$. The *line graph* L_n of K_n has vertices ij with $0 \leq i < j < n$ and edges $(ij, i'j')$ whenever $\{i, j\}$ and $\{i', j'\}$ intersect in exactly one element.

A *minor* of a graph G is a graph H obtained from G by removing and contracting edges. We show that L_n has a minor which is a clique with $\Theta(n^{1.5})$ vertices, and this bound is optimal up to constant factors.

A *special minor* of a graph G is a graph H obtained from G by replacing paths which are disjoint except at the end vertices with single edges. We show that L_n has a special minor which is a clique with $\frac{4}{3}n - \Theta(1)$ vertices, and this bound is optimal up to constant factors.

2 Clique Minors of Clique Linegraphs

Theorem 1 *Let $n = 2d + 1 + r$ with $d \geq 2$ and $0 \leq r \leq 1$. The line graph L_n has a clique minor K_m for $m = 3d + 1 + r = \lfloor \frac{3}{2}(n-1) \rfloor$.*

PROOF: The vertices are $0i$ for $1 \leq i < n$ and $(2j-1)(2j)$ for $1 \leq j \leq d$.

To join $(2j-1)(2j)$ and $(2j'-1)(2j')$ with each other and with $0(2j-1), 0(2j), 0(2j'-1), 0(2j')$ for $1 \leq j < j' \leq d$, contract with $(2j-1)(2j)$ the vertices $(2j-1)(2j'-1)$, $(2j)(2j')$, and contract with $(2j'-1)(2j')$ the vertices $(2j)(2j'-1)$, $(2j-1)(2j')$.

If $r = 1$, contract with $0(n-1)$ all the vertices $i(n-1)$ for $1 \leq i < n-1$. ■

Theorem 2 *Let $n = 3r = 3k^2$. Then the line graph L_n has a complete bipartite graph minor $K_{m,m}$ and a clique minor K_{m+1} with $m = rk = (\frac{n}{3})^{1.5}$.*

PROOF: Divide the n cliques into one group of cliques x of size r, another group of cliques y of size r, and another group of cliques (z, t) of size $r = k^2$ with k choices for each of z, t. Compress in one side of the bipartite graph the k vertices $x_0(z_0, t)$ for each of the rk choices of x_0, z_0, and in the other side of the bipartite graph compress the k vertices $y_0(z, t_0)$

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for each of the rk choices of y_0, t_0. The adjacency between the two sides of size $m = rk$ is between $x_0(z_0, t_0)$ and $y_0(z_0, t_0)$, giving the minor $K_{m,m}$. The minor K_{m+1} is obtained from $K_{m,m}$ by compressing all but one edge of a perfect matching of $K_{m,m}$. ■

Theorem 3 *If the line graph L_n has a complete bipartite graph minor $K_{m,m}$ then $m \leq (n(n-1)(n-2)/2)^{0.5}$, and if L_n has a clique minor K_{m+1} then $m \leq (n(n-1)(n-2))^{0.5}$.*

PROOF: The number of edges (xy, xz) in L_n is

$$n\binom{n-1}{2} = n(n-1)(n-2)/2$$

given by n choices for x and $\binom{n-1}{2}$ choices for y, z. But for $K_{m,m}$ we must have at least m^2 edges, proving the first bound, and for K_{m+1} we must have at least $\binom{m+1}{2} = m(m+1)/2$ edges, proving the second bound. ■

3 Special Clique Minors of Clique Linegraphs

Theorem 4 *Let $n = 6d + r$ with $d \geq 1$ and $0 \leq r \leq 5$. The line graph L_n has a special clique minor K_m for $m = 8d - 1 + r$ if $0 \leq r \leq 3$ and $m = 8d + r$ if $4 \leq r \leq 5$. This gives $m = \frac{4}{3}n - Theta(1)$.*

PROOF: The vertices are $0i$ for $1 \leq i < n$ and $(2j-1)(2j)$ for $1 \leq j \leq 2d$ if $0 \leq r \leq 3$ or for $1 \leq j \leq 2d + 1$ if $4 \leq r \leq 5$.

To join $(2j-1)(2j)$ and $(2j'-1)(2j')$ with $0(2j-1), 0(2j), 0(2j'-1), 0(2j')$ for $j < j'$, extend $(2j-1)(2j)$ with the vertices $(2j-1)(2j'-1)$, $(2j)(2j')$, and extend $(2j'-1)(2j')$ with the vertices $(2j)(2j'-1)$, $(2j-1)(2j')$.

To join $(2j-1)(2j)$ with $0i$ with $2(2d) < i < n$ or $2(2d+1) < i < n$ in the two cases use vertex $(2j-1)i$.

To join the vertices $(2j-1)(2j)$ among themselves, consider a clique Q on these $2d$ or $2d + 1$ vertices whose edges are decomposed into $2d - 1$ perfect matchings in the even $2d$ case or $2d + 1$ matchings of all but one vertex in the odd $2d + 1$ case. The matchings use vertices $(2j)i$ with $2(2d) < i < n$ or $2(2d+1) < i < n$ in the two cases, each i corresponding to a matching. ■

Theorem 5 *If the line graph L_n has a special clique minor K_m, then $m \leq 2n - 3$.*

PROOF: This follows from the fact that the degree of each vertex in L_n is $2n - 4$. ■