# Homomorphic Image $d$-Cubes of $d+1$-Cubes are Retracts

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June 26, 2009

**Theorem 1** *Suppose a $d$-cube $H_d$ is a homomorphic image of a $d+1$-cube $H_{d+1}$ (with loops included). Then $H_{d+1}$ contains a $d$-subcube $H'_d$ such that the homomorphism restricted to $H'_d$ is an isomorphism to $H_d$.*

Proof: If $d=0$ then any vertex $H'_d$ maps to the vertex $H_d$ by an isomorphism. If $d=1$ then any edge $H'_d$ that maps to the edge that is not a loop in $H_d$ has $H'_d$ mapping to $H_d$ by an isomorphism. So assume $d\geq 2$.

If some vertex $v$ in $H_d$ is the image of just one vertex $w$ in $H_{d+1}$, then the $d+1$ edges incident to $w$ must map to at least the $d$ edges incident to $v$. Choose $d$ edges incident to $w$ that map to the $d$ such distinct edges incident to $v$ other than the loop. We may assume $w$ is all 0s, and that these $d$ edges leave fixed $d$ distinct vertices adjacent to $w$ that have exactly one 1. Then the vertex adjacent to two such vertices with exactly one 1, having exactly two 1s, is also fixed, so as the fixed points of an endomorphism form a 2-satisfiability instance, and the 2-satisfiability instance has no clauses, we have that $H'_d$ involving $w$ and all these $d$ dimensions has all vertices fixed, so if we set $H_d=H'_d$ then the homomorphism is a retraction mapping $H'_d$ to $H_d$ by an isomorphism, as desired.

Otherwise every vertex in $H_d$ has at least two vertices in $H_{d+1}$ that map to it, thus exactly two. Suppose $v,v'$ are antipodal in $H_d$, thus at distance $d$, and have preimages in $H_{d+1}$ given by $w_1,w_2$ and $w'_1,w'_d$, where $w_i$ and $w'_i$ are at distance $d$ or $d+1$ from each other. Then $w_1$ and $w_2$ are at distance 1 or 2 from each other.

Say $w_1$ and $w_2$ are at distance 2 from each other. Ignoring every dimension other than the first three (such dimensions are set to 0), we may assume $w_1=000$ and $w_2=110$. Consider the vertices 001 and 111. The vertex 011 is adjacent to $001,111,010$ so two of these three must be mapped to the same vertex. Similary the vertex 101 is adjacent to $001,111,100$ so two of these tree must be mapped to the same vertex. So either 001 is identified with 111, or say 001 with 010 and 111 with 100. Since dimension 3 was chosen arbitrarily, we can assume that the second choice happens for at most one choice of dimension 3, so for all but at most one choice of dimension 3, we have that 001 is identified with 111. But then 000 has at least $d$ neighbors that map to different neighbors, without choosing both 010 and 100, so

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as in the proof above we obtain a 2-satisfiability instance without clauses, giving an $H'_d$ that maps to $H_d$ by an isomorphism.

Finally if $w_1, w_2$ are at distance 1 from each other, then changing any other dimension from both $w_1$ and $w_2$ we obtain again two adjacent vertices that map to the same vertex, so as before we can choose $d$ edges coming out of $w_1$ that map to different edges that are not the loop, and we get a 2-satisfiability instance without clauses as in the proof above, giving an $H'_d$ that maps to $H_d$ by an isomorphism, completing the proof. ■

**Theorem 2** *A $d+1$-cube can retract to a $d$-cube in exactly $d(d+1)/2+2d$ non-isomorphic manners for $d \geq 2$.*

Proof: Let $f(d)$ be the number of such retractions, up to isomorphism. Let $H_0, H_1$ be the two $d$-cubes, such that $H_0$ stays fixed and $H_1$ maps to $H_0$. Let $f(d)$ be the number of retractions. If $H_1$ is mapped by an isomorphism to $H_0$, one can show that there are exactly three mappings, namely fix the cube, flip an edge, or rotate a square (this last one for $d \geq 2$). Otherwise $H_1$ decomposes into $H_{10}$ and $H_{11}$, with $H_{11}$ mapping by projection to $H_{10}$. If $H_{10}$ never comes back to $H_{11}$, there are $f(d-1)$ possibilities for $H_{10}$. Otherwise, we proceed further to project $H_{111}$ to $H_{110}$ and so on, for $2 \leq i \leq d$ steps, and the resulting subcube goes back to $H_{11}$, in $d-1$ possible manners. We thus have $f(1) = 3$, $f(d) = f(d-1)+3+d-1$, giving $f(d) = d(d+1)/2+2d$. ■