On Graphs with Many Minimum Cuts

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1 Introduction

2 Results

Let G be a graph with positive edge weights and minimum cut of weight 2.

Lemma 2.1 *If the vertices of G are partitioned into four nonempty parts $A_{11}, A_{12}, A_{21}, A_{22}$ so that the edges separating A_{11}, A_{12} from A_{21}, A_{22} form a minimum cut of weight 1 and the edges separating A_{11}, A_{21} from A_{12}, A_{22} form a minimum cut, then there are no edges joining A_{11} to A_{22}, or A_{12} to A_{21}, and we have weight 1 joining A_{11} to A_{12}, joining A_{21} to A_{22}, joining A_{11} to A_{21}, and joining A_{12} to A_{22}.*

Proof. We may assume $w_{11,12} \leq w_{21,22}$ and $w_{11,21} \leq w_{21,22}$ by symmetry. Then the cut separating A_{11} from the rest of the graph has weight

$$2 \leq w_{11,12} + w_{11,21} + w_{11,22}$$

$$\leq 2(2 - w_{21,22} - w_{11,22})/2 + w_{11,22}$$

$$= 2 - w_{21,22}$$

, proving that $w_{21,22} = 0$, $w_{11,12} = w_{21,22}$ and $w_{11,21} = w_{21,22}$. By symmetry, $w_{11,22} = 0$ and $w_{11,12} = w_{21,22} = w_{11,21} = w_{21,22} = 1$. □

A *non-vertex cut* is a cut that does not isolate a vertex.

Lemma 2.2 *If every edge participates in a non-vertex minimum cut of weight 2, then there is a minimum cut that separates two vertices x, y from the rest of the graph, and the weight of the edge xy is 1.*

Proof. Consider a minimum cut C having fewest vertices $r \geq 2$ in one side. We claim $r = 2$. If $r \geq 3$, then by assumption these r vertices can be separated by a minimum non-vertex cut, and by Lemma 2.1 we could combine the two cuts and decrease r. So $r = 2$ and C separates two vertices x, y from the rest of the graph. A non-vertex cut separating x, y will by Lemma 2.1 give weight 1 to xy. □

An *integral cut* is a cut consisting of two edges of weight 1. A *special integral cut* is an integral cut whose edges share an endpoint and have endpoints whose incident edges form a minimum cut of weight 2.

Theorem 2.3 *If every edge participates in a non-vertex minimum cut of weight 2, then there are at least two special integral cuts.*

Proof. Consider a minimum cut C that separates two vertices x, y from the rest of the graph as in Lemma 2.2, and consider a non-vertex minimum cut C' that separates x and y. If C' also has two vertices in one side, say x and z, then by Lemmas 2.1 and 2.2 we have that xy, xz have weight 1 and x, y, z have incident edges of total weight 2, forming a special integral cut.

If C' has more than two vertices in each side, then we may shrink the edge xy without obtaining parallel edges by Lemma 2.1, any non-vertex minimum cut that used to go through xy can instead be made to go through the other edges incident to x, and the cut C that now separates just one vertex after shrinking xy can be combined with C' to obtain a non-vertex cut. So the assumptions of the Theorem hold after shrinking and we proceed by induction to obtain a special integral cut after repeated shrinking. The edes xy, xz in the shrunk graph can then be distanced when we add the edges that were shrunk, forming a path of edges of weight 1 joining x' and x'' in $x'y$ and $x''z$. Clearly this path and the edges $x'y$ and $x''z$ can be used to obtain a special integral cut.

To obtain a second special integral cut, we proceed as in Lemma refsecond to shrink the side of C that does not contain x, y to just two vertices, and there by the argument of shrinking edges will give the other special integral cut. □

We show that the results of Theorem 2.3 are tight. The graph G is called *regular* if the edges coming out of each vertex form a minimum cut of weigth 2.

Theorem 2.4 *There exist infinitely many regular examples where each edge belongs to at least one non-vertex minimum cut and there are just two special integral cuts.*

Proof. For $n \geq 1$, let the vertices be $s, t, v_0, \ldots, v_n, w_0, \ldots, w_n$, edges sv_0, sw_0, sv_n, sw_n, of weight 1, and edges $v_iv_{i+1}, v_iw_{i+1}, w_iv_{i+1}, w_iw_{i+1}$, for $0 \leq i < n$, of weight 1/2. □

Conjecture 2.5 *If every edge belongs two a non-vertex minimum cut of weight 2, then every edge of weight 1 belongs to an integral cut.*

Theorem 2.6 *If every edge participates in at least $k \geq 1$ non-vertex minimum cuts, then there exist at least $k + 1$ special integral cuts.*

Proof. Find a special integral cut by Theorem refthird, and shrink one of the two edges of weight 1 in the cut. This will decrease k by at most 1, so we may find k special integral cuts in the shrunk graph, therefore $k + 1$ by reversing the shrinking. □

The simplest cases of graphs like in Theorem 2.6 with integral weights (edges of weight 1) are the $k + 3$-cycle and any 2-edge-connected graph whose edges are replaced by paths of length $k + 3$.

Conjecture 2.7 *If every edge participates in at least $k \geq 1$ non-vertex cuts, then there exists a path of length $k+1$ whose any two consecutive edges form a special integral cut.*

Theorem 2.8 *If every edge participates in at least $k \geq 1$ non-vertex cuts, and the graph is regular, then there are $n(1 - O(1/k))$ edges of weight 1.*

We can thus strengthen Theorem 2.6 in the regular case for k large.

Corollary 2.9 *If every edge participates in at least $k \geq 1$ non-vertex cuts, and the graph is regular, then there are $n(1 - O(1/k))$ special integral cuts.*

We can thus also prove a weaker version of Conjecture 2.7

Corollary 2.10 *If every edge participates in at least $k \geq 1$ non-vertex cuts, and the graph is regular, then there exists a path of length $\Omega(k)$ whose any two consecutive edges form a special integral cut.*

The bound in Theorem 2.8 is essentially tight.

Theorem 2.11 *For each $k \geq 1$, there are infinitely many regular examples with k non-vertex cuts for each edge and $n(1 - 1/(k+2))$ edges of weight 1.*

Proof. For $n \geq k$, consider $n+1$ paths P_i of length $k+1$ with edges of weigth 1, $0 \leq i \leq n$, and connect the endpoints of P_i to the endpoints of P_{i+1}, where $i+1$ is modulo $n+1$, with four edges of weigth 1/4. □

References