Orthogonal Matching Decompositions of Matching and Hamilton Decompositions

Tomás Feder* Carlos Subi

Abstract

We study graphs that can be decomposed into disjoint perfect matchings or disjoint Hamilton circuits, and find in such graphs matchings that intersect in such a decomposition each object in exactly one edge. We call such matchings orthogonal to the decomposition, and look for a decomposition into orthogonal matchings as well.

1 Introduction

We examine four settings for the question in the abstract. The first one is complete bipartite graphs decomposed into perfect matchings, and we look for orthogonal matchings and orthogonal matching decompositions so that each orthogonal matching intersects each matching in the decomposition in exactly one edge. The second setting examines arbitrary regular bipartite graphs decomposed into perfect matchings, and we look for orthogonal matchings that interesect each matching in the decomposition in exactly one edge. The third setting is complete bipartite graphs with an even number of vertices in each side decomposed into Hamilton cycles, and we look for orthogonal matchings and orthogonal matching decompositions so that each orthogonal matching intersects each Hamilton cycle in the decomposition in exactly two edges. The fourth setting is complete graphs with an odd number of vertices decomposed into Hamilton cycles, and we look for orthogonal matchings and orthogonal matching decompositions so that each orthogonal matching intersects each Hamilton cycle in the decomposition in exactly one edge.

The work presented here continues the studyof the following problem which was posed by Brian Alspach [2].

Let $F_1, F_2, \ldots, F_d$ be any 2-factorization of a $2d$-regular graph G. Is it possible to find a d-matching M in G such that M contains precisely one edge from each of $F_1, F_2, \ldots, F_d$?

Clearly G is of order $n \geq 2d+1$ and an easy counting argument shows that the answer is yes if $n \geq 4d-3$. This bound was slightly improved to $4d-5$ by G. Liu (see [3]), and further to $n \geq 3.23d$ by Kouider and Sotteau [4].

2 Orthogonal Matchings for Matchings in Complete Bipartite Graphs

Let $G = (U, V, E)$ be a complete bipartite graph with $|U| = |V| = n$, and we write U and V as $\{1, 2, \ldots, n\}$. We define a canonical matching decomposition of G into n perfect matchings $M_0, M_1, \ldots, M_{n-1}$ by letting M_i have edges $(j, j+i)$, where addition is modulo n, for $1 \leq j \leq n$.

Theorem 1 *If n is odd, then the canonical matching decompostion has an orthogonal matching decomposition.*

*268 Waverley Street, Palo Alto, CA 94301. Email: `tomas@theory.stanford.edu`

Proof. We define orthogonal matchings $N_0, N_1, \ldots, N_{n-1}$ by letting N_i have edges $(i+j, i+2j)$, where addition is modulo n, for $1 \leq j \leq n$. Since $(i+2j)-(i+j)=j$, each N_i meets each M_j in exactly one edge. $\square$

We may conjecture, for n odd, that each matching decomposition has an orthogonal matching decomposition, or at least one orthogonal matching. However, for n even, we show that the canonical matching decomposition is a counter example.

Theorem 2 *If n is even, then the canonical matching decomposition does not have an orthogonal matching.*

Proof. Suppose N intersects each M_i in exactly one edge. Then the edges of N are (x_i, x_i+i) for some choice of x_i. The sum of the quantities $(x_i+i)-x_i=i$ for $0 \leq i < n$ is $\binom{n}{2}=n(n-1)/2$, which is $n/2$ modulo n. However, this sum should be 0 modulo n as the $\{x_i+i\}$ are a permutation of the $\{x_i\}$, a contradiction. $\square$

Corollary 1 *If $n=2$, then no matching decomposition has an orthogonal matching.*

Proof. There is only one matching decomposition, namely the canonical matching decomposition, so the result follows from Theorem 2. $\square$

Theorem 3 *If $n=2k \geq 4$ is even, there exists a matching decomposition that has an orthogonal matching.*

Proof. Consider the canonical matching decomposition for $n-1=2k-1$, which has an orthogonal matching decomposition by Theorem 1. Add one additional vertex n to each side to get from $n-1$ to n, with (n,n) matched with the additional matching M_{n-1}, and for each edge $(i,2i)$ modulo $n-1$ in M_i put $(i,2i)$ in M_{n-1} instead to form the new matching M_{n-1}, and put the edges (i,n) and $(n,2i)$ in M_i to complete the new M_i, for $0 \leq i \leq n-2$. The orhogonal matching can then be taken to be $(1+i, 1+2i)$ modulo $n-1$ plus (n,n). $\square$

We do not know, if $n=2(2k+1)$ is twice an odd number with $2k+1 \geq 3$, if an orthogonal matching decomposition can exist for some matching decomposition. However, for $n=4k$, the remaining case, this indeed holds. The smallest open problem is thus finding a matching decomposition that has an orthogonal matching decomposition for $n=6$.

Theorem 4 *If $n=4k$, there exists a matching decomposition that has an orthogonal matching decomposition.*

Lemma 1 *If $n=4$, there exists a matching decomposition that has an orthogonal matching decomposition.*

Proof. Let M_1 be $11, 22, 33, 44$, let M_2 be $12, 21, 34, 43$, let M_3 be $13, 24, 31, 42$, and let M_4 be $14, 23, 32, 41$. The orthogonal matching decomposition is obtained if we let N_1 be $11, 34, 42, 23$, let N_2 be $22, 43, 31, 14$, let N_3 be $33, 12, 24, 41$, and let N_4 be $44, 21, 13, 32$. $\square$

Lemma 2 *If $n = 8$, there exists a matching decomposition that has an orthogonal matching decomposition.*

Proof. We begin with the construction for $n = 4$, and replace each vertex i with two vertices i, i', where $i'' = i$ by convention. For each edge jk of M_i, we include in M_i' the edges $jk, j'k'$ and in M_i'' the edges $jk', j'k$. This gives the eight perfect matchings M_i', M_i''.

We combine N_1 and N_2 to obtain N' given by $31', 1'1, 14', 3'4, 42, 2'2', 23, 43'$. We combine N_3 and N_4 to obtain N'' given by $24, 4'4', 41, 2'1', 13', 3'3, 32', 1'2$. Finally, if N' has edges ij, we obtain three additional matchings with edges $i'j, ij', i'j'$ respectively, and similarly If N'' has edges ij, we obtain three additional matchings with edges $i'j, ij', i'j'$ respectively. These eight matchings (four coming from N' and four from N'') form an orthogonal matching decomposition. □

Lemma 3 *If there exists a matching decomposition that has an orthogonal matching decomposition for both n' and n'', then this is also the case for $n = n'n''$.*

Proof. Decompose the n vertices in each side into n' groups S_i each having n'' vertices. If in the matchings M_j' and orthogonal matchings N_j' for n' we had edges ab we shall join S_a to S_b, in such a way as to have these two groups of size n'' joined as in the matchings M_k'' and orthogonal matchings N_k'' for n''. Thus we obain matchings for pairs M_j', M_k'' and $N_{j'}', N_{k'}''$ that meet at the edge where M_j' and $N_{j'}'$ meet, as well as M_k'' and $N_{k'}''$ meet. □

Proof. (of Theorem 4) Write $n = 4k$ as $n = 4k'k''$ for k' a power of two and k'' odd. By Lemma 3 it suffices to prove the result for k'' and for $4k'$. The result for k'' follows from Theorem 1, while $4k'$ can be decomposed into repeated products of the numbers 4 and 8, which follow from Lemmas 1 and 2, and can be combined by Lemma 3. □

As said before, finding an orthogonal matching decomposition for some matching decomposition if $n = 2(2k + 1)$ and $2k + 1 \geq 3$ remains open, with $n = 6$ the smallest open problem.

3 Orthogonal Matchings for Matchings in Regular Bipartite Graphs

Let $G = (U, V, E)$ be a d-regular bipartite graph with $|U| = |V| = n$ decomposed into perfect matchings. We look for conditions on n and d that will guarantee the existence of an orthogonal matching. Note that $n \geq d$ always, and if d is even, then $n \geq d + 1$ is needed in general by Theorem 2.

Theorem 5 *If $n \geq 2d - 3$ and $n \geq 3$, then an orthogonal matching that intersects all perfect matchings in the decomposition in one edge exists.*

Proof. The proof is by induction on d with n fixed. Suppose the result holds for $d - 1$, and we have found a matching N that intersects the first $d - 1$ of the d matchings $M_1, \ldots, M_{d-1}, M_d$ in exactly one edge. Then N has $d - 1$ edges $e_i = (x_i, y_i)$, with e_i in M_i, $1 \leq i \leq d - 1$.

If $n \geq 2d - 1$, then one of the n edges of M_d, say e, does not contain any of the $n - 1 = 2(d - 1)$ endpoints x_i, y_i of e_i, and adding e to N completes the proof.

If $n = 2d - 2$, $d \geq 3$, then the result follows by a similar argument that finds an uncovered edge of M_d, unless the $n = 2(d - 1)$ vertices x_i, y_i cover the n edges of M_d. In this case let the edges of M_d be (x_i, z_i) and (t_i, y_i). Since z_i has d neighbors, and there are only $d - 1$ possible neighbors

x_i, there must be an edge $f = (z_i, t_j)$, say in M_k for some $1 \leq k \leq d-1$. We can switch the edge $e_k = (x_k, y_k)$ with the edge f in N, and then add either $e' = (x_k, z_k)$ or $e'' = (t_k, y_k)$ to N to complete the proof, unless $f = (z_k, t_k)$. In this last case, the edges of the form $f = (z_i, t_j)$ are precisely the edges $f_k = (z_k, t_k)$, so each z_k is adjacent to all x_i, and each t_k is adjacent to all y_i, to achieve degree d. But this situation of two joined complete bipartite graphs plus a matching does not arise after swithing e_k and f as before, so the result follows.

In the last case $n = 2d-3$, $d \geq 3$. If $d = 3$ we have the canonical decomposition of Theorem 1 and the result follows, so assume $d \geq 4$. As for the case $n = 2d-2$, the result follows if some edge of M_d is not covered by x_i, y_i, otherwise M_d has edges (x_i, z_i) and (t_i, y_i), this time say with $(x_{d-1}, z_{d-1}) = (t_1, y_1)$, and all other such edges distinct. By an argument similar to the case $n = 2d-2$, a switch of edges in N suffices unless the only edges joining z_i with $1 \leq i \leq d-2$ and t_j with $2 \leq j \leq d-1$, are the edges $f_k = (t_k, z_k)$ in M_k for $2 \leq k \leq d-2$, an edge $(t_{j'}, z_1)$ in M_1, and an edge $(t_{d-1}, z_{i'})$ in M_{d-1}, with at least one such edge for each z_i, t_j. Again this situation does not arise after the switch, say for some z_i of degree one to the t_j with edge (t_j, z_i), after switching some (x_k, y_k) with $k \neq i$ we will be able to switch the edge (x_k, z_i). □

4 Orthogonal Matchings for Hamilton Cycles in Bipartite Graphs

Suppose n is even for complete bipartite graphs, and we obtain the canonical decomposition into perfect matchings M_i, $0 \leq i < n$, as in Section 2. Combining matchings as M_{2i}, M_{2i+1} into Hamilton cycles H_i gives a Hamilton decomposition into $n/2$ cycles H_i, $0 \leq i < n/2$. Call this the canonical Hamilton decomposition.

Theorem 6 *If n is even, then the canonical Hamilton decompostion has an orthogonal matching decomposition where each matching meets each Hamilton cycle in exactly two edges.*

Proof. Consider the matchings N_i with edges $(i+j, i-j)$, and the matchings N'_i with edges $(i+1+j, i-j)$ where the operations are modulo n for $0 \leq i < n/2$ and $-n/2 \leq j < n/2$. The matchings N_i have $(i+j)-(i-j) = 2j$, and each even such difference occurs exactly twice, meeting each M_{2j} exactly twice. The matchings N'_i have $(i+1+j)-(i-j) = 2j+1$, and each odd such difference occurs exactly twice, meeting each M_{2j+1} exactly twice. Combining M_{2j} and M_{2j+1} into H_j gives the result that each H_j is met exactly twice by each N_i and each N'_i. □

5 Orthogonal Matchings for Hamilton Cycles in Complete Graphs

Let G be a complete graph with n vertices, where n is odd. It is known that G can be decomposed into $(n-1)/2$ Hamilton cycles, with a construction that we shall give later. Suppose first n is an odd prime. Then the cycles H_i can be taken to have edges $(j, j+i)$ for $0 \leq j < n$ and $1 \leq i \leq (n-1)/2$. The orthogonal matchings will match all but one vertex, and meet each H_i in exactly one edge. We call this the canonical prime Hamilton decomposition.

Theorem 7 *If n is an odd prime, then the canonical prime Hamilton decompostion has an orthogonal matching decomposition.*

Proof. Let the matchings N_i have edges $(i-j, i+j)$ for $0 \leq i < n$ and $1 \leq j \leq (n-1)/2$, leaving i not matched. Then the lengths $(i+j)-(i-j) = 2j$ cover all lengths modulo n, between 1 and

$(n-1)/2$, and therefore each N_i intersects each H_j at exactly one edge. $\square$

We now give Walecki's construction for an arbitrary odd $n = 2k+1$ [1, 5]. Label the vertices of K_{2k+1} as $u_0, u_1, u_2, \ldots, u_{2k}$ and let σ be the permutation whose disjoint cycle decomposition is

$$(u_0)(u_1 u_2 \ldots u_{2k}).$$

Let H_1 be the Hamilton cycle

$$H_1 = u_0 u_1 u_2 u_{2k} u_3 u_{2k-1} \cdots u_n u_{k+2} u_{k+1} u_0.$$

Then let $H_i = \sigma^{i-1}(H_1)$ for $1 \leq i \leq n$. Then the H_i form a Hamilton decomposition.

Theorem 8 *If n is odd, then the Walecki Hamilton decompostion has an orthogonal matching.*

Proof. The matching N we choose has edges $(i, i+k)$, where the addition is modulo $2k$ and avoids the value 0. Thus vertex 0 is not matched, and each H_i is met at exactly one edge by N. $\square$

We do not know if the Walecki Hamilton decomposition has an orthogonal matching decomposition. It is also open whether some matching decomposition for the composite number $n = 9$ has an orthogonal matching decomposition.

References

[1] Brian Alspach, The wonderful Walecki construction, April 10, 2006.

[2] B. Alspach, Problem 89, Discrete Mathematics 69 (1988), 106.

[3] B. Alspach, K. Heinrich, and G. Liu, Orthogonal factorization of graphs, preprint.

[4] M. Kouider and Sotteau, On the existence of a matching orthogonal to a 2-factorization, Discrete Math. 73 (1989), 301–304.

[5] E. Lucas, Récréations Mathématiques, four volumes, Gautheirs-Villars, Paris, 1882–1894.