Brooks Type Theorems for Pair-List Colourings and List Homomorphisms

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Abstract

Brooks proved that every connected graph other than a clique or odd cycle can be coloured with Δ colours. Erdős, Rubin, and Taylor (and independently Vizing) generalized the theorem of Brooks to list colourings, describing all uncolourable connected graphs in which no vertex has a list smaller than its degree. Other authors have extended this to list T-colourings and their generalizations. We further extend to model to pair-list colourings. In addition to including all of the previous situations, pair-list colourings also generalize list homomorphisms (also known as list H-colourings). In the general context of pair-list colourings, we prove a Brooks type theorem which extends many (but not all) of the existing results. Our result applies to both graphs and digraphs, with or without loops. We discuss several applications of the result, including a polynomial test for the existence of balanced list homomorphisms and retractions.

1 Background

The theorem of Brooks states that a connected graph G can be coloured by $\Delta(G)$ (the maximum degree) colours, except for cliques and odd cycles, which can be coloured by $\Delta(G) + 1$ colours. The theorem of Erdős, Rubin, and Taylor (and independently Vizing) [6, 21] generalizes this result to list colourings. In a list colouring problem, each vertex v of G is equipped with a list $L(v)$ of allowed colours that can be used as a colour of v, while at the same time seeking a proper colouring in the usual sense (adjacent vertices

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are to obtain distinct colours). Instead of a condition relating the number of colours to the maximum degree $\Delta(G)$, it is now natural to assume a condition relating the number of colours available at a vertex v to its degree $deg_G(v)$. We say that G with lists L is *balanced* if each vertex v has $|L(v)| = deg_G(v)$. Erdős, Rubin, and Taylor described all balanced problems that do not admit a list colourings (cf. Corollary 2). Further extension of this idea to list T-colourings and their generalizations was explored in [13, 22]. Let T be a set of nonnegative integers (usually including 0). A T-colouring of G is a mapping $f : V(G) \to \mathbf{N}$ such that $|f(u) - f(v)| \notin T$ whenever uv is an edge of G. (Note that when $T = \{0\}$, a T-colouring is just a usual colouring.) Given a graph G with lists $L(v) \subseteq \mathbf{N}$, a list T-colouring of G is a T-colouring f of G with all $f(v) \in L(v)$.

List colourings and list T-colourings are naturally generalized to list homomorphisms, also known as list H-colourings. Let H be a fixed graph (or digraph), and let G be a graph (respectively digraph) together with lists $L(v) \subseteq V(H), v \in V(G)$. A *list H-colouring* of G, with respect to lists L, is a homomorphism (edge-preserving mapping of the vertices) f of G to H such that all $f(v) \in L(v)$. The *list H-colouring problem* asks whether or not an instance graph G with lists L admits a list H-colouring. The problem was investigated, and its complexity classified, in [7, 8, 9], cf. also [17]. When all lists are equal to $V(H)$, the list H-colouring problem is known as the *H-colouring problem* [15]. We explicitly note that when H is a complete graph (without loops), then list H-colourings become just list colourings. On the other hand, when H is the graph with vertex set $\mathbf{N}$ and with edges ij such that $|i - j| \notin T$, then H-colourings become exactly T-colourings, and list H-colourings become exactly list T-colourings. (Note that if $0 \in T$, then the graph H has no loops.)

In [13], a more general model for list T-colourings was introduced. An instance of the *generalized list T-colouring problem* is a graph G in which each edge uv has its own constraints on the difference of colours of its endpoints, i.e., its own set $T(u, v)$ of integers, requiring that $|f(u) - f(v)| \notin T(u, v)$. The authors gave a result for these colourings which generalizes all of the earlier Brooks type results [13].

In the same spirit, we introduce the following *pair-list colouring problem*. Let G be a graph, and C a set of *colours*. For each edge $uv \in E(G)$, let the *forbidden set* $T(u, v)$ be a subset of $C \times C$. For consistency, we shall assume that $T(u, v) = \{(a, b) : (b, a) \in T(v, u)\}$. We shall extend the definition of $T(u, v)$ to all ordered pairs of distinct vertices, where $T(u, v) = \emptyset$ if uv is not an edge. Since edges $uv \in E(G)$ having $T(u, v) = \emptyset$ don't restrict our search for a colouring (see below), we will in the sequel always assume that

$T(u,v) \neq \emptyset$ if (and only if) $uv \in E(G)$. Given a graph G with forbidden sets $T(u,v)$, a *pair-list colouring* is a mapping $f : V(G) \to C$ such that $uv \in E(G)$ implies $f(u)f(v) \notin T(u,v)$. The *pair-list colouring problem* asks whether or not a given instance G, T has a pair-list colouring.

We shall also assume that each vertex v is equipped with a *list* $L(v) \subseteq C$ in the usual sense, and pair-list colourings f must additionally have each $f(v) \in L(v)$. If these lists were not specifically given, we may set $L(v)$ equal to the set of all colours v is not explicitly forbidden by some set $T(v,w), w \in V(G)$. Note that this is not a more general model, as the lists can be eliminated by adjusting the forbidden sets. (We can remove a list $L(v)$ and add to each $T(v,w)$ all pairs (a,b) with $a \notin L(v)$, without changing the problem.) In the rest of the paper, we shall consider an instance of the problem to be a graph G, with lists $L(v), v \in V(G)$, and forbidden sets $T(v,w), v, w \in V(G)$, such that $T(v,w) \neq \emptyset$ if and only if vw is an edge of G. We shall often refer to an instance G, L, T only by its graph G; for example, we may say the instance is *connected* if the graph G is connected.

This concept is a common extension of (generalized) list T-colourings and list H-colourings of graphs (and digraphs). When $C = \mathbf{N}$ and each set $T(u,v)$ is described by listing the forbidden set D of differences, i.e., $T(u,v) = \{(a,b) : |a-b| \in D\}$, the pair-list colouring problem becomes the list T-colouring problem from [1, 22] (if the set D is fixed), or the generalized list T-colouring problem (if D depends on the edge uv).

Now to explain the connection of pair-list colourings to list H-colourings, let H be a graph. When the forbidden set for each edge $uv \in E(G)$ is defined as $T(u,v) = \{(a,b) : a \in L(u), b \in L(v), ab \notin E(H)\}$, the pair-list colouring problem becomes the list H-colouring problem. The same transformation works also when H is a digraph. In that case, the instance G is also a digraph; however, the directions of its arcs can be ignored by adjusting the forbidden sets so that only homomorphisms preserving the arc directions are allowed by the forbidden sets. (Note that the consistency assumption that $T(u,v) = \{(a,b) : (b,a) \in T(v,u)\}$ does not prevent us from modeling digraphs in this way.) At this point it is convenient to introduce a product graph $G \times T$ with the vertex set $V(G) \times C$, in which $(u,a)(v,b)$ is an edge whenever $(a,b) \in T(u,v)$. In fact, we may restrict our attention to vertices (v,a) with $a \in L(v)$. This product conveniently captures the structure of forbidden colour pairs. We depict its edges by dashed lines to emphasize that these are the *forbidden* pairs. Thus a pair list colouring f of G corresponds to a set of vertices $(v, f(v)), v \in V(G)$, one from each set $\{v\} \times L(v)$, which does not include an edge of $G \times T$.

The inclusion relations amongst the various versions of these colouring problems are illustrated in Figure 1.

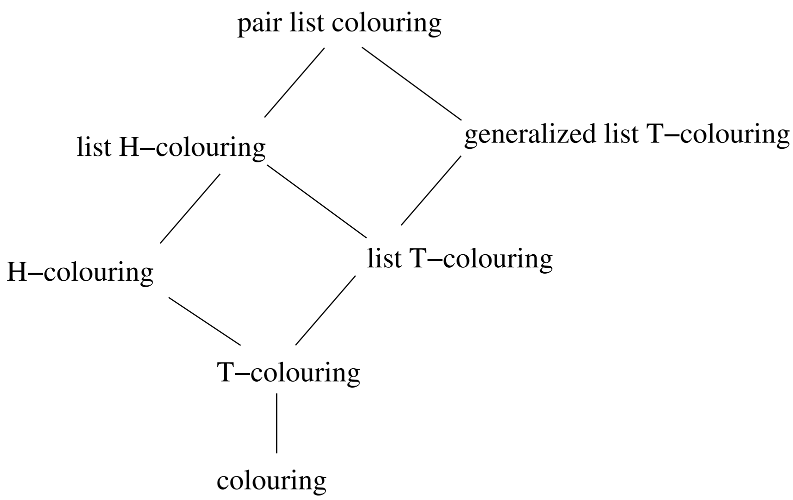

Figure 1: Versions of colouring problems

Our main result is a Brooks-like characterization of balanced instances which do not admit a pair list colourings. It implies the theorems of Brooks and Erdős, Rubin, Taylor, but only a weaker version of the results in [13, 22]. On the other hand, our result also applies to (generalized) list T-colourings in which $0 \notin T$, providing some answers to the questions raised in [13] concerning this case. It seems possible to extend our context to forbidden triples, and higher order tuples, to generalize also the hypergraph Brooks theorems of [18], but we will keep our focus on forbidden pairs.

We shall consider both graphs and digraphs, and allow loops, unless specifically excluded. Graphs and digraphs without loops are called *irreflexive*. Graphs and digraphs in which every vertex has a loop are called *reflexive*. An *oriented graph* is a digraph without opposite arcs uv, vu. Graphs and digraphs have different definitions, and various concepts such as degree are applied differently to them; however, if convenient, we may view graphs as symmetric digraphs (digraphs in which $uv \in E(G)$ if and only if $vu \in E(G)$), as the homomorphism properties under this definition are the same [17]. In this context, it is important to consider a loop to contribute one to the degree of a vertex in the case of graphs, and to both the indegree and the outdegree of a vertex in the case of digraphs.

2 The General Result

In this section we deal with the problem in full generality. Note that loops in G have no effect on the problem, and we shall assume that the graph G is irreflexive. Consider an instance of the pair-list colouring problem, i.e., a graph G with lists L and forbidden sets T. Let $N(v)$ denote the set of neighbours of the vertex v in G. We say that v is *pair-balanced* if it satisfies the following two assumptions.

(i) For each $u \in N(v)$ and each assignment $f(x) \in L(x)$, $x \in N(v) - u$, there exists a colour $a \in L(v)$ with all $(a, f(x)) \notin T(v, x)$. (We say that the assignment f can be *compatibly extended* to v.)

(ii) If an assignment $f(x) \in L(x)$ for all $x \in N(v)$ cannot be compatibly extended to v, then for each colour $a \in L(v)$ there exists a unique $u \in N(v)$ with $(f(u), a) \in T(u, v)$.

Recall the product $G \times T$ defined in the introduction. The *T-weight* of an ordered pair (u, v) of vertices of G is the maximum number of neighbours of any one (v, b) amongst all $(u, a), a \in L(u)$, in $G \times T$. Note that if $uv \notin E(G)$ the weight of both pairs $(u, v), (v, u)$ is 0, and that in general the weight of (u, v) is not the same as that of (v, u). The T-weight of (u, v) represents the maximum number of possible colours for u that can be forbidden by one colour of v. The *T-degree* of u, denoted by $deg_T(u)$, is the sum over all $v, v \neq u$ of the T-weights of ordered pairs (u, v). It is easy to see that the following fact holds.

Proposition 1 *A vertex v with $|L(v)| \geq deg_T(v)$ is pair-balanced.* $\square$

A graph G with lists L and forbidden sets T is *pair-balanced* if all its vertices are pair-balanced; it is *pair-overbalanced* if all vertices are pair-balanced, and for at least one vertex v the assumption (ii) can be replaced by the following stronger assumption.

(ii') Every assignment $f(x) \in L(x), x \in N(v)$, can be compatibly extended to v.

The easy part of Brooks' theorem states that any graph with maximum degree Δ admits a colouring with $\Delta + 1$ colours. The counterpart of this result in our context is as follows.

Theorem 1 *Every connected pair-overbalanced instance of the pair-list problem has a solution.*

Proof. If (ii') holds for vertex v, we can colour G greedily in the order of decreasing distance to v. When colouring a vertex u other than the special vertex v we can use (i), such a vertex u has at least one uncoloured neighbour closer to u; when colouring the special vertex v we can use (ii'). □

There are pair-balanced instances of the pair-list problem which do not have a solution. A connected instance G, L, T of the pair-list problem is a *Gallai configuration* if

1. each block B of G is a clique or a cycle,

2. each list $L(v), v \in V$, can be partitioned into nonempty subsets $L_B(v)$, over all blocks B of G with $v \in B$, so that

3. if block B is a clique with $q+1$ vertices, then each list $L_B(v)$ can be further partitioned into q nonempty subsets $L_B^1(v), \ldots, L_B^q(v)$, such that for $u, v \in B$ with $u \neq v$ we have
$$T(u,v) = \{(a,b) : a \in L_B^i(u), b \in L_B^i(v), i = 1, 2, \ldots, q\};$$

4. if block B is a cycle, then each list $L_B(v)$ can be further partitioned into two nonempty subsets $L_B^1(v), L_B^2(v)$, so that for one edge uv of the cycle B we have
$$T(u,v) = \{(a,b) : a \in L_B^i(u), b \in L_B^i(v), i = 1, 2\},$$
and for all other edges uv of B we have
$$T(u,v) = \{(a,b) : a \in L_B^i(u), b \in L_B^{3-i}(v), i = 1, 2\};$$

5. if B, B' are distinct blocks, and $u \in B, v \in B \cap B'$ are distinct vertices, then for any $a \in L_B(u)$, $b \in L_{B'}(v)$, we have $(a,b) \notin T(u,v)$.

These conditions imply a particularly simple structure of the product $G \times T$, illustrated in Figure 2: the vertices $(v,a), a \in L(v)$ are always grouped close together; as mentioned earlier, the edges of $G \times T$ are dashed. In both examples, all parts $L_B^i(v)$ are singletons; if a part has more vertices the adjacencies are just repeated. As mentioned earlier, a pair-list colouring needs to choose one vertex from each group, and not include a dashed edge; clearly this is impossible in both these cases.

Our main result states that Gallai configurations are the only pair-balanced instances which do not admit a pair-list colouring.

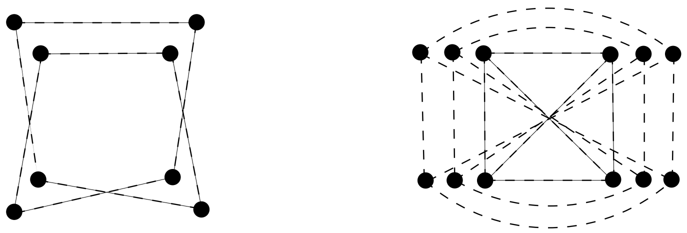

Figure 2: The product $G \times T$ for a four-cycle and a four-clique block G

Theorem 2 *Let G be a connected instance of the pair-list problem. Then the following statements are equivalent.*

- *The instance is pair-balanced but has no solution.*
- *The instance is a Gallai configuration.*

PROOF. We first prove that a Gallai configuration is pair-balanced and has no solution. For a vertex which belongs to a unique block of a Gallai configuration it is easy to verify the assumptions (i) and (ii), by considering the above descriptions of the various types of blocks; for a vertex v which is a cutpoint, i.e., belongs to several blocks, we use condition 4 to focus on individual blocks containing v. To show that a Gallai configuration admits no solution, consider the block-cutpoint tree of G: starting with a leaf block B, it is easy to check (in each of the types of blocks described above) that no solution exists using only colours from $L_B(v), v \in B$. (Compare with Figure 2.) Thus the unique cutpoint u of G contained in B must be assigned a colour not from $L_B(u)$, in any solution. Since condition 5 ensures that any other colour it may be assigned is compatible with other choices made within B, we may remove the entire block B and delete from $L(u)$ the colours of $L_B(u)$, and proceed recursively. A contradiction is reached when only one block remains, since no assignment from within the block is possible.

To prove the converse, we shall use the following three simple lemmas.

Lemma 1 *Let G, L, T be a pair-balanced instance which has no pair-list colouring. Let u be a vertex of G such that $G - u$ is connected. Then for every neighbour v of u and each $a \in L(u)$ there exists some $b \in L(v)$ with $(a, b) \in T(u, v)$.*

PROOF. Suppose v is a vertex of G such that $(a, b) \notin T(u, v)$ for all $b \in L(v)$. Then a pair-list colouring of G exists. We first colour u by colour

a; since $G - u$ is connected, we can colour $G - u$ greedily in the order of decreasing distance to v, taking into account the colour of u. When colouring v, we can disregard u, by our assumption. □

Lemma 2 *Let G, L, T be a pair-balanced instance which has no pair-list colouring. Let u, v be two vertices of G such that $G - u - v$ is connected. Then for every vertex w adjacent to both u and v and all $a \in L(u), b \in L(v), c \in L(w)$ with $(a, c) \in T(u, w)$, and $(b, c) \in T(v, w)$, we must also have $(a, b) \in T(u, v)$.*

PROOF. If $(a, b) \notin T(u, v)$, then there again is a pair-list colouring of G: colour u by a and v by b, and then proceed as in Lemma 1, colouring $G - u - v$ greedily in the order of decreasing distance to w. At the end, w can be coloured by (ii), since its neighbours u, v forbid the same colour c. □

Lemma 3 *Let G, L, T be a pair-balanced instance which has no pair-list colouring. Let v be a vertex of G with precisely two neighbours, u and w, and assume that $G - v - w$ is connected. Then for all $a \in L(u), b, b' \in L(v)$, and $c \in L(w)$ with $(b, c) \notin T(v, w), (b', c) \notin T(v, w)$, and $(a, b) \notin T(u, v)$, we must also have $(a, b') \notin T(u, v)$.*

PROOF. If $(a, b') \in T(u, v)$, then a pair-list colouring of G can be found as follows. Colour w by c and v by b', and then proceed as in Lemma 1, colouring $G - v - w$ greedily in the order of decreasing distance to u, using (i), until reaching the vertex u. If u cannot be coloured then by assumption (ii) the colour a is only forbidden by the colour b' on v. Re-colouring v by b doesn't create a problem for w, and solves the problem for u; since the degree of v is two, this is a pair-list colouring. □

We now continue proving the theorem; thus assume that we have a connected pair-balanced instance G, L, T which has no solution.

Assume first that G is a block, that is, has no cutpoint. If G has only two vertices, then it is an edge uv with $T(u, v) = L(u) \times L(v)$; this is a Gallai configuration of type 3 with $q = 1$. Thus we assume in that G has at least three vertices. Assume further that G has no vertices of degree two. In this case, we first note that there must exist a vertex v such that $G - v$ is also a block. (Indeed, according to a theorem of Kaugars, every critical block other than an edge contains a vertex of degree two, cf. [2], page 49.) We now show that v is adjacent to every vertex of $G - v$. Otherwise, there would be two vertices u, w in $G - v$ such that v is adjacent to w, w is adjacent to u, but v is not adjacent to u. Then there exist $b \in L(v)$ and $c \in L(w)$ so that

$(b,c) \in T(v,w)$, since vw is an edge of G. By Lemma 1, there also exists a colour $a \in L(u)$ such that $(a,c) \in T(u,w)$. Since uv is not an edge of G, $(a,b) \notin T(u,v)$, contrary to Lemma 2. This contradiction shows that v is adjacent to all other vertices of G. Since $G-v$ is a block, $G-v'$ must be a block for every other vertex v' in G, and so the preceding argument applies to v' as well, i.e., G is a clique.

We have shown that if G is a block without vertices of degree two, then G is a clique. We now prove that if G is a clique then it is a Gallai configuration of type 3, i.e., that the lists $L(v)$ can be partitioned as claimed. Consider any three vertices u,v,w of G and $a \in L(u)$, $b \in L(v)$, $c \in L(w)$: if $(a,b) \in T(u,v)$ and $(b,c) \in T(v,w)$, then by Lemma 2 we also have $(a,c) \in T(u,w)$. For $a \in L(u)$, we let $S(v)$ be the set of elements $b \in L(v)$ such that $(a,b) \in T(u,v)$, and let $S(w)$ be the set of elements $c \in L(w)$ such that $(a,c) \in T(u,w)$. Then $(b,c) \in T(v,w)$ if $b \in S(v), c \in S(w)$, and by symmetry $(b,c) \notin T(v,w)$ if $b \in S(v)$ and $c \in L(w) - S(w)$, or if $b \in L(v) - S(v)$ and $c \in S(w)$. Thus the lists $L(u)$ can each be partitioned into some number q of nonempty subsets $L^1(u), \ldots, L^q(u)$ such that $(a,b) \in T(u,v)$ for $a \in L^i(u)$, $b \in L^j(v)$ if and only if $i=j$. Since the graph has no pair-list colouring, it must have at least $q+1$ vertices. On the other hand, G cannot have more than $q+1$ vertices, since there is no pair-list colouring, yet all vertices are pair-balanced. Thus G has exactly $q+1$ vertices, as claimed.

Now we claim that if G (a block pair-balanced instance without a solution) has a vertex v of degree two, then it is a Gallai configuration of type 4. This is clear if G is a three-cycle, by the above argument for cliques. (Note that in this case the partitions of lists in cases 3 and 4 can be relabeled to coincide.) Otherwise, let v be a vertex of degree two, with neighbours u,w in G. We construct a new instance G', L', T' as follows: the graph G' is obtained from $G-v$ by adding the edge uw if it was not present (it will turn out below that it was not present); the lists L' are the lists L restricted to the vertices of G', and the forbidden sets T' are the same as T, except for $T'(u,w)$ which is obtained from $T(u,w)$ by adding the set $T^*(u,w)$ consisting of all pairs (a,b) such that $a \in L(u), b \in L(v)$ and each $c \in L(v)$ has $(a,c) \in T(u,v)$ or $(b,c) \in T(w,v)$. It is easy to observe that the instance G', L', T' has no solution, since the definition of $T'(u,w)$ would allow it to be extended to a solution of G, L, T. It is also clear that G' is a block. Moreover, it can be argued that G', L', T' is also pair-balanced. (For instance, it follows from assumption (i) that for every $c \in L(w)$ there exists a $b \in L(v)$ such that $(b,c) \notin T(v,w)$; therefore if $(a,c) \in T^*(u,w)$, then $(a,b) \in T(u,v)$. Thus when proving that u is pair-balanced in G', we can use b instead of c and appeal to the fact that u is pair-balanced in G.)

Suppose first that G' does not contain a vertex of degree two, and so is a clique (with at least three vertices). Suppose two vertices $b, b' \in L(v)$ have a vertex $c \in L(w)$ with $(b,c) \notin T(v,w), (b',c) \notin T(v,w)$. Then, by Lemma 3, for any $a \in L(u)$, we have $(a,b) \notin T(u,v)$ if and only if $(a,b') \notin T(u,v)$. There exists an $a \in L(u)$ with $(a,b) \notin T(u,v)$ by assumption (i), and thus also $(a,b') \notin T(u,v)$. By a symmetric argument, for any $c \in L(w)$ we have $(b,c) \notin T(v,w)$ if and only if $(b',c) \notin T(v,w)$. It follows from this that there exists a $p \geq 2$ such that each of the lists $L(x), x = u, v, w$, can be partitioned into nonempty disjoint subsets $L^1(x), \ldots, L^p(x)$ such that for $a \in L^i(u), b \in L^j(v), c \in L^\ell(w)$, we have $(a,b) \in T(u,v)$ if and only if $i \neq j$, $(b,c) \in T(v,w)$ if and only if $j \neq \ell$, and hence $(a,c) \in T^*(u,w)$ if and only if $i \neq \ell$. This means that the subgraph of $G' \times T'$ induced by $\{u\} \times L(u) \cup \{w\} \times L(w)$ has at least all edges (a,c) where $a \in L^i(u), c \in L^\ell(w), i \neq \ell$ (maybe more if uw was an edge of G), and hence is connected unless $p = 2$. On the other hand, since G' is a clique, $G' \times T$ has the form dictated by condition 3, in which the subgraph induced by $\{u\} \times L(u) \cup \{w\} \times L(w)$ has $q \geq 2$ components. Thus $p = 2$, and condition 3 implies that also $q = 2$, whence G' is a triangle. We now argue that G could not have had an edge uw: indeed, if $T'(u,w)$ contained more pairs than $T^*(u,w)$, then the subgraph of $G' \times T'$ induced by $\{u\} \times L(u) \cup \{w\} \times L(w)$ would again be connected, contradicting the form of $G' \times T'$ from condition 3, as above. On the other hand, if $T'(u,w) = T(u,w) \cup T^*(u,w) = T^*(u,w)$, then condition (i) would imply that G' admits a pair-list colouring, which is not the case. It is now clear that G is a four-cycle, with the lists partitioned as described in condition 4.

If $G' = G_1$ does contain a vertex of degree two, we continue removing such vertices, until we arrive at a graph G_t, which is a clique. The previous argument shows that G_{t-1} is a four-cycle, of the form in condition 4, and the same logic can be applied backwards to $G_{t-2}, G_{t-3}, \ldots, G_1 = G'$, and finally G. This completes the proof in the case when G is a block.

Finally, if G is not a block, then G has a leaf block B with cutpoint $w \in B$. In B, we may assign colours $f(v) \in L(v)$ to $v \neq w$ in the order of decreasing distance from w by our assumption (i). If it is not possible to assign a colour $f(w)$, then we let $L_B(w)$ be the set of colours for w forbidden by assigned colours to neighbors v of w in B, and we let $L'(w)$ be the set of colours for w forbidden by assigned colours to neighbors u of w in $G - (B - w)$. By assumption (ii), the sets $L_B(w)$ and $L'(w)$ are disjoint. Both the problem on B with $L(w)$ replaced by $L_B(w)$ and the problem on $G - (B - w)$ with $L(w)$ replaced by $L'(w)$ satisfy our assumptions (i) and (ii), and the problem on G admits an assignment if and only if either of these two subproblems on B and $G-(B-w)$ has an assignment. Proceeding inductively to decompose the subproblem on $G-(B-w)$, either the problem on G has an assignment, or it decomposes into subproblems for the various blocks B_i, with corresponding decompositions of the lists $L(v)$, so that the original problem has a solution if and only if at least one subproblem for a block B_i has a solution. This proves all the claims, except for the last one, for arbitrary G. For the last claim, if v and w are adjacent in G, with $v \in B$ and $w \in B \cap B'$ for distinct blocks B and B', and there exist $b \in L_B(v)$ and $c \in L_{B'}(w)$ such that $(b, c) \in T(v, w)$, we proceed as follows. Consider B with lists $L_B(x)$, set $f(v) = b$, and assign $f(x) \in L_B(x)$ for $x \neq w$ in B in order of decreasing distance from w. For each block B'' containing w other than B, assign $f(x) \in L_{B''}(x)$ for $x \neq w$ from B'' in order of decreasing distance from w. The resulting assignments for blocks B'' other than B forbid at least all of $L_{B''}(w)$ for w, and the assignment for block B forbids at least all of $L_B(w)$ as well as $f(w) = c$ for w. Thus there is no possible colour for w. Yet $f(w) = c$ is forbidden both by $f(v) = b$ in B and by some $f(u)$ in B', since $c \in L_{B'}(w)$ and all of $L_{B'}(w)$ is forbidden by neighbors of w in B'. This contradicts assumption (ii). Therefore, there is no $b \in L_B(v)$ and $c \in L_{B'}(w)$ such that $(b, c) \in T(v, w)$. This completes the proof. □

Since checking the block structure of an instance, and seeking the partitions of the simple kind described above can be accomplished in polynomial time, this represents a polynomial time algorithm to solve the pair-list problem for pair-balanced instances.

3 Balanced List H-colouring Problems

Recall that in the list H-colouring problem we have a fixed graph (or digraph) H, and an instance consists of a graph (respectively digraph) G together with lists $L(v) \subseteq V(H), v \in V(G)$. A solution is a list H-colouring f of G, i.e., a homomorphism of G to H in which all $f(v) \in L(v)$. While we will allow loops in H, it will be simpler to assume that G is irreflexive. (Note that if G is not irreflexive, we can make it so by eliminating a loop at vertex v and at the same time deleting from $L(v)$ any vertices of H which do not have a loop.)

Our Theorem 2 can be applied to list H-colourings by taking each forbidden set $T(u, v)$ to be the set $L(u) \times L(v) - E(H)$. Thus there are many different applications possible. We shall single out just two very simple contexts to illustrate the applications, one for graphs and one for digraphs. In both cases we focus on instances with $deg_G(v) = |L(v)|$, in analogy with

[6]. We shall call such instances *balanced*. (For digraphs, $deg_G(v)$ is to be interpreted as the total degree of v in G, i.e., the sum of in- and out-degrees of v in G.) A balanced instance G, L, is called *overbalanced* if it contains a vertex with degree, respectively total degree, $deg_G(v)$, which satisfies $deg_G(v) > |L(v)|$. The *balanced list H-colouring problem* for a graph H, is the restriction of the list H-colouring problem to balanced graphs G with lists L. The *balanced list H-colouring problem* for a digraph H, is the restriction of the list H-colouring problem to balanced *oriented* graphs G with lists L. (Recall that an oriented graph is a digraph without a pair of opposite arcs.) We note that our notion of balanced vertices is not the same as that of [13].

In this situation we naturally focus on graphs (respectively digraphs) H that are *nearly complete*, in the sense that at each vertex v, they have $vw \in E(H)$ for all $w \in V(H)$ except possibly one vertex w, which could be equal to v (respectively $vw \in E(H)$ for all $w \in V(H)$ except possibly one w, and $uv \in E(H)$ for all $u \in V(H)$ except possibly one u, where $u = v = w$ is possible.) Thus in a nearly complete graph, each vertex has at most one edge missing, and that edge could be the loop; and in a nearly complete digraph, each vertex has at most one in-edge and at most one out-edge missing, and if it has the loop missing, it counts as both the missing in-edge and the missing out-edge. The *complement* of H has edges and loops precisely where H does not have edges and loops. (Normally complements are defined without reference to loops.) Note that the usual irreflexive complete graph is nearly complete in this sense, as it is missing the loop at each vertex. Some other nearly complete graphs and digraphs are depicted in Figure 3. recall that when we model the list H-colouring problem as a pair-list colouring problem, the forbidden sets $T(u, v)$ for $uv \in E(G)$ are those pairs $(a, b), a \in L(u), b \in L(v)$ which have $ab \notin E(H)$.

Recall the definition of the T-degree of a vertex v. Since the T-weight of any pair (u, v) with $uv \in E(G)$ (or, in the case of digraphs, $uv \in E(G)$ or $vu \in E(G)$) is at most 1, the T-degree of v is at most $deg_G(v)$. Therefore Proposition 1 ensures that a balanced graph (or oriented graph) G with lists L is pair-balanced, and hence Theorems 1 and 2 apply to guarantee that connected overbalanced instances always have a solution (list H-colouring), and to describe all connected unsolvable balanced instances, as Gallai configurations. It remains to consider which of the Gallai configurations are actually possible. It is immediately obvious that all parts $L_B^i(v)$ must have a single element, since otherwise there would exist two distinct pairs $(a, b), (a', b) \in T(v, w)$.

First, we focus on the balanced list H-colouring problem for nearly com-

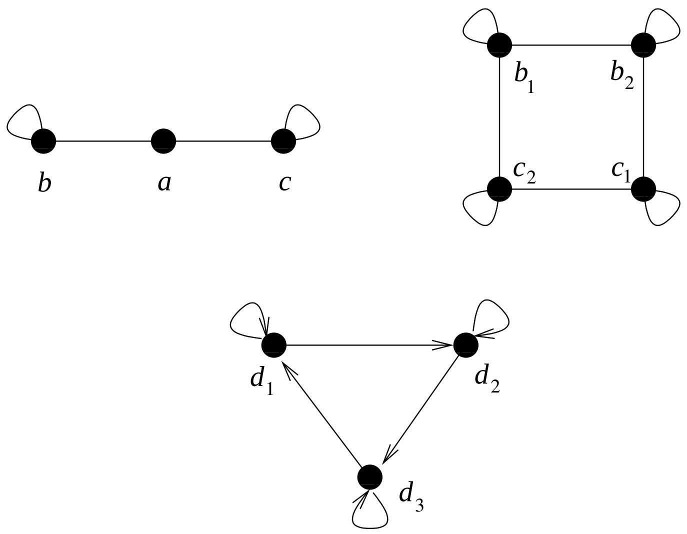

Figure 3: Some example nearly complete graphs and digraphs

plete graphs H. Note that in this situation colours $a, b \in V(H)$ either satisfy $(a, b) \notin T(u, v)$ for all edges $uv \in E(H)$ or satisfy $(a, b) \in T(u, v)$ for all edges $uv \in E(H)$. Consider a block B of a Gallai configuration, which is a clique of $q + 1 \geq 3$ vertices, each with a list of q colours. Suppose u, v, w are distinct vertices with $L^i_B(u) = \{a\}, L^i_B(v) = \{b\}, L^i_B(w) = \{c\}$. Then $(a, b) \in T(u, v), (a, c) \in T(u, w)$, i.e., $ab, ac \notin E(H)$, whence $b = c$ since H is nearly complete. The same argument shows that all sets $L_B(v), v \in B$ are the same set, L_B, of q vertices of H. Consider a block B which is a cycle $v_1, v_2, \ldots, v_k$, and suppose the list of each v_i consists of $a_i \in L^1_B(v_i)$ and $b_i \in L^2_B(v_i)$, where $(a_i, a_{i+1}) \in T(v_i, v_{i+1})$ and $(b_i, b_{i+1}) \in T(v_i, v_{i+1})$ for $i = 1, 2, \ldots, k-1$, and $(a_k, b_1) \in T(v_k, v_1), (b_k, a_1) \in T(v_k, v_1)$. We again apply the fact that H is nearly complete to deduce that $a_1 = a_3 = a_5 = \ldots$, $b_1 = b_3 = b_5 = \ldots, a_2 = a_4 = \ldots, b_2 = b_4 = \ldots$. If k is even, $a_k = a_2, b_k = b_2$ and so both (a_1, a_2) and (a_1, b_2) are in $T(v_1, v_2)$, which is impossible in a Gallai configuration. Hence k is odd, and therefore all lists consist of the same pair of vertices in H. Thus, for nearly complete graphs H, a block of size at least three has the lists of all its vertices equal to the same set L_B; and all cycle blocks are odd. The vertices of H which appear in these blocks must be missing only their loops. On the other hand, the blocks that have two vertices could have equal lists, corresponding to missing loops in H, or they could have different lists, corresponding to missing non-loop edges in

H. Summarizing this additional information, we have the following result for graphs H (with loops allowed).

Corollary 1 *Suppose H is a nearly complete graph, whose complement has isolated loops $a_i,\ i = 1, 2, \ldots, k$, and isolated edges $b_i c_i,\ i = 1, 2, \ldots, \ell$.*

If G is a balanced connected irreflexive graph with lists L, then either G has a list H-colouring, or it is a Gallai configuration in which

- *each block B that is a clique with $q + 1 \geq 3$ vertices has all lists $L_B(v),\ v \in B$ equal to the same set of q vertices a_i,*
- *each block B that is a cycle has odd length and all the lists $L_B(v),\ v \in B$ are equal to the same pair of vertices a_i, a_j,*
- *each block B that is an edge $uv \in E(B)$ (clique with two vertices) has either $L_B(u) = L_B(v) = \{a_i\}$ for some $i = 1, 2, \ldots k$, or has $L_B(u) = \{b_j\},\ L_B(v) = \{c_j\}$ for some $j = 1, 2, \ldots, \ell$.*

Note that vertices of H that have no missing incident edges play no role in the result.

We single out two important special cases of the Corollary. When H is irreflexive, we have $\ell = 0$, i.e., H is an irreflexive complete graph, and we obtain the following theorem of Erdős, Rubin, Taylor [6, 21] (for other proofs see [4, 13, 18]).

Corollary 2 *If G is a connected balanced irreflexive graph with lists $L(v),\ v \in V(G)$, then either G admits a list colouring, or it is a Gallai configuration in which each block is a clique with $q + 1$ vertices all with the same list of size q, or is an odd cycle with all vertices having the same list of size two.*

The only difference from the above is that the third item on cliques of size $q + 1$ also applies to $q + 1 = 2$, i.e., the last item is not needed. ($L_B(u)$ and $L_B(v)$ are equal.) A Gallai configuration for this case is illustrated in Figure 4.

When H is reflexive, we have $k = 0$, and we obtain the following new result.

Corollary 3 *Suppose H is a reflexive graph whose complement has isolated edges $b_i c_i,\ i = 1, 2, \ldots, \ell$.*

If G is a connected balanced irreflexive graph with lists $L(v) \subseteq V(H),\ v \in V(G)$, then either G admits a list H-colouring, or it is a Gallai configuration that is a tree, in which each edge block $B = uv$ has $L_B(u),\ L_B(v)$ equal to $\{b_j\}, \{c_j\}$ for some j.

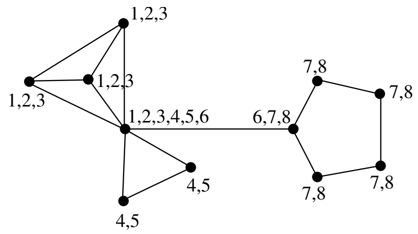

Figure 4: A Gallai configuration for the irreflexive complete graph H

We illustrate a Gallai configuration for this case in Figure 5. The graph H is the reflexive four-cycle from Figure 3. Each edge uv of the tree is a block of the Gallai configuration and has $L_B(u)$, $L_B(v)$ chosen either b_1, c_1, or b_2, c_2 in some order, labelled on the edge with small letters. The lists of the leaves are marked, all other vertices have the full lists $\{a_1, a_2, b_1, b_2\}$. It is instructive to see that there is no list H-colouring.

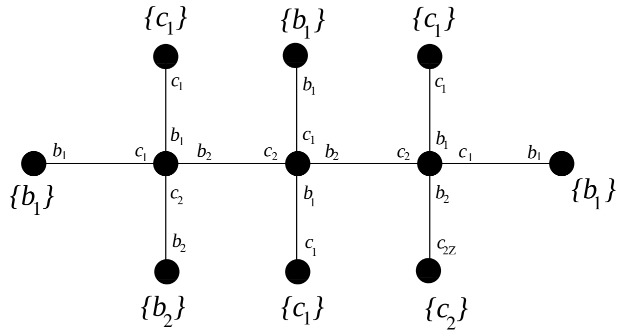

Figure 5: A Gallai configuration for the reflexive four-cycle H

We now briefly turn to the balanced list H-colouring problem for nearly complete digraphs H. In this situation colours $a, b \in V(H)$ may have $(a, b) \notin T(u, v)$ if uv is an edge, but not if vu is an edge, of H. However, it is still the case that (a, b) is either in $T(u, v)$ for all edges $uv \in E(H)$ or not in $T(u, v)$ for all edges $uv \in E(H)$. The instance of the balanced H-colouring problem consists of an oriented graph G with lists $L(v), v \in V(H)$, and total degrees $deg_G(v) = |L(v)|$ for all $v \in V(H)$. Recall that to model this by a pair-list colouring problem, we remove the orientations of the edges of G, and set, for each edge $uv \in E(H)$, the forbidden set $T(u, v)$ to consist of those pairs $(a, b), a \in L(u), b \in L(v)$ for which $ab \notin E(H)$. Thus the orientations of the edges of G can be ignored because they are encoded in the pair lists.

Consider a block B of a Gallai configuration, containing two edges uv, uw with $v \neq w$. Since H is nearly complete, it follows as above that the lists $L_B(v)$ and $L_B(w)$ must be the same. This is easily seen to imply that for a block B that is a clique with $q + 1 \geq 4$ vertices, each vertex has the same list q colours, and each colour is a vertex of H that has no loop. A block B that is a clique with three vertices, either has each vertex v with same lists $L_B(v)$ consisting of two vertices of H without loops, or B is a directed three-cycle with distinct lists corresponding to two distinct directed three-cycles in the complement of H. Consider a block B that is a cycle of G; it may be a directed k-cycle with k even, with lists that correspond to a directed $2k$-cycle in H; or it may be a directed k-cycle with k odd, with lists that correspond to two disjoint k-cycles in H. The cycle B could also be oriented in more complex ways, implying certain equalities of lists; once two consecutive vertices of the cycle B are to have the same lists, all vertices of B must have the same lists. It is not hard to see that this can only occur when k is odd. Instead of stating all these necessary and sufficient conditions in such great detail, we only summarize a few necessary conditions.

Corollary 4 *Suppose H is a nearly complete digraph, whose complement $\overline{H}$ consists of disjoint directed cycles $D_1, D_2, \ldots, D_k$ (some of which may have length one).*

If G is a balanced connected irreflexive oriented graph with lists L which does not admit a list H-colouring, then it is a Gallai configuration in which

- *each block B that is a clique with $q + 1 \geq 4$ vertices has all lists $L_B(v), v \in B$ equal to the same set of q vertices lying in cycles of length one in $\overline{H}$,*
- *each block B that is a clique with 3 vertices either has all lists $L_B(v), v \in B$ equal to the same set of 2 vertices lying in cycles of length one in $\overline{H}$, or is a directed three-cycle of G, and its vertices have lists formed from two three-cycles of $\overline{H}$,*
- *each block B that is an oriented cycle has equal lists $L_B(u) = L_B(v)$ on vertices u, v that admit a $w \in B$ with $uw, vw \in E(B)$ or with $wu, wv \in E(B)$.*

Thus if H is a nearly complete graphs or digraph, we have solved the balanced list H-colouring problem in polynomial time.

4 Extension Problems

In a future paper [11], we shall apply our results to extension problems. Let H be a fixed graph, or digraph. Given a graph, respectively oriented graph, G and a partial assignment f of some vertices of G to vertices of H, a homomorphism F of G to H is an *extension* of f, if $F(v) = f(v)$ for all vertices for which $f(v)$ is defined. The *H-extension problem* asks whether or not an instance G, f has an extension. This problem is also known as the *precolouring extension problem*, the *one-or-all list H-colouring problem*, or the *H-retraction problem* [17]. To classify the complexity of H-extension problems appears to be much harder than for list H-colouring problems [9, 12].

It is easy to see that the H-extension problem is a restriction of the list H-colouring problem to instances G, L in which each $L(v)$ is either equal to $V(H)$, or consists of a single colour (vertex of H). If moreover the degrees of G are at most $|V(H)|$, we have basically a balanced instance, since a vertex v with $deg_G(v) = k$ and $L(v) = \{a\}$ can be replaced by k vertices of degree one, with the same list, without changing the problem. Let us say that a graph is *k-bounded* if its maximum degree is at most k. Thus if H is a nearly complete graph or digraph with k vertices, then the H-extension problem for k-bounded graphs, respectively oriented graphs, can be solved in polynomial time.

In [11], we shall also prove the following fact.

Theorem 3 *Let H be a graph (with loops allowed) such that any two vertices a, b have a common neighbor c in H. Then the H-extension problem for 3-bounded irreflexive graphs can be solved in polynomial time.*

If H is a reflexive graph of diameter at most two, then it satisfies the condition. In particular, this implies that the reflexive five-cycle H has a polynomial time solvable H-extension problem for 3-bounded graphs. For the reflexive four-cycle H (which is a nearly complete graph, see Figure 3), Corollary 3 implies that the H-extension problem is polynomial time solvable for any 4-bounded graph. Finally, for the reflexive three-cycle H, any mapping of any graph G to H is a homomorphism, so the H-extension problem is trivial for all graphs. We shall prove in [11] that for all other reflexive cycles H, the H-extension problem is NP-complete, even for 3-bounded graphs; we shall also show that for the reflexive five-cycle H the H-extension problem is NP-complete. Summarizing this information, we conclude that for the reflexive k-cycle H, the H-extension problem in general is only polynomial when $k = 3$ (unless P=NP), but also becomes polynomial when $k = 4$ for 4-bounded graphs, and additionally when $k = 5$ for 3-bounded graphs. Similarly, it will turn out that for the irreflexive k-cycle H, the H-extension problem in general is only polynomial when $k = 4$ or $k = 6$ (unless P=NP), but also becomes polynomial when $k = 3$ for 3-bounded graphs [11]. This kind of sensitivity to degree constraints is unusual for list H-colouring problems in general; in [10] we have proved that the complexity classification of list H-colouring problems (as NP-complete or polynomial time solvable) does not change when the instances are restricted to k-bounded graphs for any $k \geq 3$.

Consider now instances in which the graph G is allowed to have loops. For reflexive graphs H of diameter at most two, our polynomial time algorithm will still apply. The condition in Theorem 3 is also satisfied if H is an irreflexive graph of diameter at most two, all of whose edges belong to triangles; and the algorithm also applies to graphs G with loops allowed. For general graphs H (with loops allowed), we shall require, in addition to having, for any two vertices a, b, a common neighbor c, also that each vertex $a \in V(H)$ has a neighbour in H that has a loop. Then the H-extension problem for 3-bounded graphs G with loops allowed also has a polynomial time algorithm [11]. In all cases, the algorithms proceed by exploring some Gallai type configurations that may or may not turn out to be uncolourable [11].

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