# Mapping Large Odd Girth Cubic Planar Graphs to Odd Cycles

Tomás Feder* Carlos Subi†

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**Abstract**

A version of a conjecture of Jaeger's is that every planar graph of odd girth at least $4k+1$ admits a homomorphism to a cycle of length $2k+1$, for all $k \geq 1$. This was shown for $k = 1$ by Grotszch, i.e., every triangle free planar graph is 3-colorable. A lower bound of $4k+1$ was shown by DeVos for all $k \geq 1$.

In this paper, we conjecture that the correct bound on odd girth is at least $4k-1$ for $k \geq 2$ if the planar graph is both cubic and biconnected. A lower bound of $4k-1$ is shown for all $k \geq 2$, for this case. We examine two special cases where only the lower bound of $4k-3$ seems to hold.

## 1 Introduction

The girth of a graph $G$ is the length of the shortest cycle in $G$, if $G$ is not a forest. The odd girth of a graph $G$ is the length of the shortest odd cycle in $G$, if $G$ is not bipartite. A homomorphism from a graph $G$ to a graph $H$ is a mapping from vertices of $G$ to vertices of $H$ such that adjacent vertices in $G$ map to adjacent vertices in $H$. In this paper we study the following.

**Conjecture 1** *Every planar graph of odd girth at least $4k+1$ admits a homomorphism to a cycle of length $2k+1$, for each $k \geq 1$.*

The original conjecture of Jaeger's is slightly weaker, and assumes the the planar graph has girth at least $4k$ [4, 11, 14]. The two versions are equivalent if $k = 1$, and this case was shown by Grötzsch [10] namely, every triangle-free planar graph has a 3-coloring. For $k = 2$, the existence of a homomorphism to the 5-cycle was shown by planar graphs of odd girth at least 11, by DeVos and Deckelbaum [4]. The conjecture can be shown for planar graph of girth at least $10k-4$ [7, 12], which is improved to odd girth at least $8k-3$ [14], and for planar graphs of girth at least $(20k-2)/3$ [2]. A lower bound of $4k+1$ on odd girth needed in planar graphs to guarantee a homomorphism to a cycle of length $2k+1$ was also shown

*268 Waverley St., Palo Alto, CA 94301. E-mail: tomas@theory.stanford.edu.
†E-mail: carlos.subi@hotmail.com.

by DeVos [4]. The assumption of planarity is crucial to this conjecture, as Erdős [5] showed that there exist graphs of simultaneously arbitrarily large girth and chromatic number.

In this paper we consider subcubic planar graphs (degrees at most 3), and show a lower bound of $4k-1$ for subcubic planar graphs, and show also that this lower bound holds for biconnected cubic planar graphs. The lower bound of $4k+1=9$ for $k=2$ is shown for subcubic planar graphs, but we do not know how to extend this improved lower bound to biconnected cubic planar graphs. Note that to map $G$ to a cycle of length $2k+1$, it suffices to consider biconnected components of $G$ and map them.

Nešetřil asked whether a sufficiently high girth cubic graph maps to the pentagon. We extend a theorem of Brooks in this direction.

We then consider Barnette graphs, which are 3-connected cubic bipartite planar graphs. Barnette conjectured that every such graph is Hamiltonian [1, 8, 9]. The faces of such graphs can be colored with three colors, say red, blue, green faces, so that adjacent faces have different colors. We replace the edges between green and blue by paths, so that red faces stay even, and show a lower bound of $2k-3$. We obtain upper bounds in some cases.

Given a planar graph $G$, the dual graph $G^d$ has vertices corresponding to faces of $G$, and an edge in $G^d$ joins two vertices for each edge separating the two corresponding faces in $G$. The chopped graph $G^c$ is obtained from $G^d$ by removing each vertex of $G^d$ of some degree $D$ and connecting the corresponding $D$ endpoints left with a $D$-gon. Thus the chopped graph $G^c$ is cubic. We show again a lower bound of $2k-3$ for chopped graphs. We obtain upper bounds in some cases as well. Chopped graphs apeared in the search for Hamiltonian circuits as well [6].

We show the lower bound of $4k+1$ on odd girth again for planar graphs of degree at most 5, or for graphs embeddable in the projective plane, the torus, or the Klein bottle, of degree at most 4.

Jaeger's original conjecture was also restated by duality as the statement that every $4k$-edge connected planar graph can have its edges oriented so that at each vertex the difference between both directions is a multiple of $2k+1$. This is called the modularity conjecture, and here Jaeger then drops the assumption of planarity from this dual graph. Indeed, the dual of a cycle of length $l$ is a cut of $l$ edges, and each face maps to a cycle of length $2k+1$ if and only if the vertex corresponding to the face in the dual satisfies the modulo $2k+1$ property. This allows us to state the above stronger conjecture by duality as well.

# 2 Lower Bounds

We first show the lower bound of $4k+1$ on the odd girth of planar graphs, so we must exhibit a planar graph of odd girth $4k-1$ that does not map to the cycle of length $2k+1$ [11]. We present the result in more generality, for results to be obtained later.

Consider integers $c, d, r, \ell \geq 0$ so that $c+d$ is odd, and $c+d+r \leq 2k$. Let $C$ be a cycle of length $c\ell$, and let $D$ be a cycle of length $d\ell$, with $l \geq 3$ odd. Connect for each $0 \leq i < l$ the vertex in position $ci$ on $C$ with the vertex in position $di$ on $D$, by a path of length $r$.

If we consider the cycles of length $c+d+2r \leq 4k-1$ and odd, these must map one time around the cycle of length $2k+1$, either clockwise or counter clockwise. Whichever of these directions is chosen, it must be the majority along the paths of length $r$, because otherwise we advance at most $c+d+r \leq 2k$ and not go one time around the cycle when we map.

But then if the majority is clockwise in one such cycle, it will be counter clockwise in the adjacent cycle sharing a path of length $r$ and viceversa, counter clockwise to clockwise. Since $\ell \geq 3$, after shifting cycles $\ell$ times we reach a contradiction, getting back to the original cycle, a contradiction.

The original result of DeVos's has $c=0, d=1, r=2k-1, \ell=4k-1$, so the odd girth is indeed $4k-1$, and the lower bound on odd girth is $4k+1$ for mapping to a cycle of length $2k+1$.

**Theorem 1** *The class of biconnected cubic graphs has a lower bound of* $4k-1$ *for mapping to a cycle of length* $2k+1$.

Proof: We first build a subcubic graph $G$ of odd girth $2k-3$, namely we let $c=1, d=2, r=2k-3, \ell=2k-3$. By the preceding remarks, this graph does not map to a cycle of length $2k+1$. To make the graph cubic, we increase $\ell$ to a large odd number multiple of 3, group the vertices of degree 2 on $D$ into groups of three consecutive joined to a new vertex of degree 3, and join along the paths of length $r$ consecutive pairs to two adjacent vertices in a cube after removing that joining edge. The graph will then be biconnected and cubic. ■

We now construct a subcubic graph $G$. Let $R$ be the dodecahedron. Let $v_i, 1 \leq i \leq 4$ be four vertices in $R$ that do not pairwise share a face. Obtain $G$ by inserting in $R$ a vertex in each of the three edges coming out of each $v_i$. As a result, all the faces have length 7, and $G$ has odd girth 7.

**Theorem 2** $G$ *does not map to the cycle of length* $2k+1=5$, *thus we have a lower bound of* $4k+1=9$ *for the odd girth of subcubic graphs that all map to the* 5*-cycle.*

Proof: Orient the faces in the direction they would map to the cycle of length 5, with one backward edge. The faces with the same orientation (clockwise or counter clockwise) create regions delimited by cycles. If edge $e$ joins two faces in the same region, it will be the reverse edge for one of these two faces, and we make this face point to the face across $e$. These pointing digraphs of outdegree one form trees leading into a cycle.

No region can be bounded by three of more cycles, since the dual graph would contain in such a region more than one cycle. So we can assume all delimiting cycles are cycles contained in one another, from the outermost to the innermost. For each $v_i$, one vertex that was inserted is in the edge joining two faces with same orientation, so both edges around the inserted vertex are the chosen backward edges for the two faces, so this digon is the cycle that the trees lead into. But this digon cannot be the cycle in the dual of the intermediate regions bounded by two cycles, and the outer and inner region can contain just one digon each for a total of two, instead of four for the four $v_i$, a contradiction. ■

The graph $G$ can be made cubic by attaching a gadget to each vertex of degree 2, but it will no longer be biconnected. We can make $G$ both cubic and biconnected by increasing the genus to 2.

**Theorem 3** *There is a lower bound of* $4k+1$ *for the odd girth of planar graphs of degree at most 5 that all map to the cycle of length* $2k+1$.

Proof: Let $C$ be a cycle of length $\ell+1$, and let $D$ be a cycle of length $\ell$. For $0 \le i \le \ell-1$, connect the vertex in position $i$ in $C$ to the vertex in position $i$ in $D$ by a path of length $2k-1$, and connect the vertex in position $i+1$ in $C$ to the vertex in position $i$ in $D$ by a path of length $2k-1$. Finally connect the vertex in position 0 in $C$ to the vertex in postion $\ell-1$ in $D$.

Thus the vertex in position $\ell$ in $C$ has degree 3, the vertex in position $\ell-1$ in $D$ has degree 5, and all other vertices have degrees 2 or 4. Letting $\ell = 4k-1$ gives odd girth $4k-1$, and the argument that the majority in each path of length $2k-1$ gives the orientation in each triangle region as in the beginning of this section follows. ■

**Theorem 4** *There is a lower bound of $4k+1$ for the odd girth of graphs embedded in the projective plane, the torus, or the Klein bottle, of degree at most 4, that all map to the cycle of length $2k+1$.*

Proof: Let $C$ be a path of length $\ell$, and let $D$ be a path of length $\ell-1$. For $0 \le i \le \ell-1$, connect the vertex in position $i$ in $C$ to the vertex in position $i$ in $D$ by a path of length $2k-1$, and connect the vertex in position $i+1$ in $C$ to the vertex in position $i$ in $D$ by a path of length $2k-1$. Finally identify the vertex in position 0 in $C$ with the vertex in position $\ell-1$ in $D$, and identify the vertex in position 0 in $D$ with the vertex in position $\ell$ in $C$.

This gives degrees 2 and 4 and the result follows as in Theorem 3. To embed in the torus, just cross position 0 in $C$ with position 0 in $D$ with a handle. To embed in the projective plane or the Klein bottle, just view $C, D$ as the boundaries of a Möbius strip. ■

The following will be in the next section the only counterexample in an extension to a theorem of Brooks.

**Theorem 5** *Let $G$ be obtained from $K_4$ by replacing each edge with a path of length $k$. Then $G$ does not map to a cycle of length $2k+1$.*

Proof: As in the beginning of this section, setting $c = 0, d = k, r = k, \ell = 3$. ■

**Theorem 6** *There are both a chopped graph, and a Barnette based graph, that give a lower bound of $4k-3$ on the odd girth for mapping to the cycle of length $2k+1$.*

Proof: As in the beginning of this section, we set $c = 0, d = 1, r = 2k-3, \ell = 4k-5$ in $G$, then take the chopped $G^c$. If we ignore the face in $G^c$ related to $D$ in $G$, then $G^c$ becomes a Barnette based graph. In $G^c$, we have $c' = 1, d' = 2, r' = 2k-4$, except that the paths of length $r'$ are separated by squares. The majority argument still works, as in shifting from one path of length $r'$ to the other the total path length mapped to will differ by at most two, and therefore the directions will differ by at most one, preserving majority. ■

If we instead set $2k+1 = 5, c = 0, d = 1, r = 2k-2, \ell = 4k-3$, the chopped graph does map to the cycle of length $2k+1 = 5$.

Let $R$ be the octahedron, let $A$ and $B$ be two disjoint faces in $R$. Insert a vertex in each of the edges of $A$ and $B$. Then split the vertices of degree 4 into squares to obtain a Barnette based graph $G$ with blue and green faces of length 7 and 9, and red faces that are squares.

**Theorem 7** *This $G$ does not map to the cycle of length 7.*

Proof: Mapping the six faces of length 7 forces an orientation without reverse edges on each face of length 9, a contradiction. ■

## 3 General Upper Bounds

In this section we study a variety of graphs and map them to odd cycles.

**Theorem 8** *Let $R$ be an $r$-colorable graph, possibly with parallel edges. Let the graph $G$ be obtained from $R$ by replacing each edge $e$ in $R$ with a path of length at least $\ell = 2k - \lfloor \frac{2k+1}{r} \rfloor$. Then $G$ admits a homomorphism to the cycle of length $2k+1$.*

PROOF: Place the $r$ colors on vertices of the cycle of length $2k+1$, so that any two colors are at distance at least $d = \lfloor \frac{2k+1}{r} \rfloor$. A path of length $\ell$ or more can be mapped to length $\ell$ or $\ell+1$. With length $\ell$, we can reach $d+1, d+3, d+5, \dots$, by going around, and $d, d+2, d+4, \dots$ by not going around. With length $\ell+1$, we can reach $d, d+2, d+4, \dots$, by going around, and $d+1, d+3, d+5, \dots$ by not going around.

So any path joining two different colors assigned to vertices from $R$ in $G$ can be mapped the corresponding distance at least $d$. ■

Note that by the 4-color theorem, the above can be applied to planar graphs with $r = 4$.

A graph $R$, possibly with parallel edges, is said to have greedy degree $d$, if $R$ contains a vertex $v$ of degree at most $d$, and after removing $v$ from $R$ the resulting graph $R - v$ has greedy degree $d$, and so on, until we reach the empty graph that also has greedy degree $d$.

**Theorem 9** *Let $R$ be a graph of greedy degree $d$, possibly with parallel edges. Let the graph $G$ be obtained from $R$ by replacing each edge $e$ in $R$ with a path of length at least $\ell = 2k + 1 - \lceil \frac{2k+1}{d} \rceil$. Then $G$ admits a homomorphism to the cycle of length $2k+1$.*

PROOF: Suppose $R - v$ has already been mapped to the cycle of length $2k+1$. If $v$ is joined to at most $d$ vertices $v_i$ in $R$ by paths of length $\ell_i \geq \ell$, we note that each path will reach $\ell_i + 1$ possible vertices, so will not reach $\lceil \frac{2k+1}{d} \rceil - 1$ possible vertices for $v$. Multiplying this by $d$ gives $d(\lceil \frac{2k+1}{d} \rceil - 1) < 2k+1$ vertices that can not be reached, so at least one vertex is reachable by all $d$ paths, completing the induction. ■

We now extend Brooks' result [3] that every graph $G$ of maximum degree $d \geq 3$ is $d$-colorable unless $G = K_{d+1}$.

**Theorem 10** *Let $R$ be a graph of maximum degree $d + 1 \geq 3$, possibly with parallel edges. Let the graph $G$ be obtained from $R$ by replacing each edge $e$ in $R$ with a path of length at least $\ell = 2k - p$, $p = \lceil \frac{2k+1}{d} \rceil - 1$. Then $G$ admits a homomorphism to the cycle of length $2k+1$, with the exception of the cases (1) $R = K_{d+2}$, $d = 2k$, paths of length exactly $\ell = 2k - 1$ and (2)$R = K_4$, $d = 2$, paths of length exactly $\ell = k$, for $k$ odd (from Theorem 5).*

PROOF: If $R$ has at least one vertex $v$ of degree $d$ or less, then $R - v$ will have the neighbors of $v$ of degree $d$ or less, and so on, so $R$ has greedy degree $d$ and the result follows as before. Otherwise $R$ is $d+1$-regular. If some vertex $v$ disconnects $R$, then each component involving $v$ has greedy degree $d$ since $v$ will have degree at most $d$ in each component, so one can solve each component and glue them together at $v$. Otherwise $G$ is biconnected. Brooks observed that if $R$ is biconnected (without parallel edges), and $R$ is neither a clique nor a cycle, then $R$ has a vertex $u$ with two neighbors $v, w$ not adjacent to eachother, and such that $R - \{v, w\}$ is connected. In that case assign a vertex in the cycle of length $2k+1$

to $v, w$ and to vertices along the paths $uv, uw$ that are no closer than $\ell$ to $u$, so that the two vertices at distance $\ell$ from $u$ in these two paths get the same vertex in the cycle of length $2k+1$. We can then proceed to the graph $R-u$ since $u$ acts now as degree $d$, because of the coincidence along $uv, uw$. We continue with $R-u$ to repeatedly remove vertices of degree at most $d$ other than $v, w$, completing the construction.

For the counterexamples, if $R$ is $d+1$-regular, and $R$ is a clique $K_t$ with $t \leq d+2$ plus parallel edges, then either (1) $t = d+2$, or (2) $t = d+1$ and the parallel edges must form a perfect matching, or (3) $t = d$ which can be solved as above for $r$-colorable $R$ with $r = d$. If $R$ is a cycle, then $R$ is 3-colorable which solves again the problem for $d \geq 3$, and for $d = 2$ the cubic $R$ is a cycle that alternates parallel and non-parallel edges, thus an even length cycle that is 2-colorable.

In particular, the only exceptions for $d+1 = 3$ are $R = K_4$ and paths of lengths at least $k$ joining four vertices $u, v, w, x$. If the path $uv$ is even, then either (4) $wx$ is even and we may assign $u, v$, to 0 and $w, x$ to $k$ along the cycle of length $2k+1$, or (5) $wx$ is odd and we may assign $u, v$ to 0 and $w, x$ to $k, k+1$ respectively. In the remaining case, all six paths are odd. If $k$ is even, then all lengths are odd at least $k+1$, and we assign $u, v, w, x$ to $0, 1, k, k+1$. If $k$ is odd, and say $vw$ is of length at least $k+2$, we assign $u, v, w, x$ to $0, 1, k, k+1$. This leaves us with the case $k$ odd and all paths of length $k$, which is the counter example.

For case (2), either some two vertices $u, v$ of $R$ are joined in $G$ only by an even length path, in which case we may identify $u, v$ and solve as in (3), or some $u, v$ of $R$ are joined in $G$ only by an odd length path, in which case we map $u$ to 0, $v$ to 1, and the remaining vertices of $R$ to $p+1, 2p+1, \ldots, (d-1)p+1$, since $dp+1 \leq 2k+1$.

For case (1), either we can identify two vertices of $R$ joined by an even length path in $G$ and we may proceed as in case (2), or all such paths have odd length. If $\ell$ is even, then all the paths have odd length at least $\ell+1$, and we may place the vertices at $0, 1, p, p+1, 2p+1, \ldots, (d-1)p+1$; this placement also works if $\ell$ is odd and some path has length at least $\ell+2$ (this path joins 1 and $p$). Finally, all paths are of odd length $\ell$. We write $2k+1 = dp+q$, where $1 \leq q \leq d$. If $q \geq 2$, then we can map to $0, 1, p+1, p+2, \ldots, (d-1)p+2$. So $q = 1$, $2k = dp$, $p$ is odd and $d$ is even. We can map to positions $i(p+1), i(p+1)+1$ for $0 \leq i < \frac{d}{2}+1$, provided $(\frac{d}{2}+1)(p+1) \leq 2k+1 = dp+1$. This reduces to $\frac{d}{2}p \geq \frac{d}{2}+p$, which holds if both $\frac{d}{2}$ and $p$ are at least 2. Since we already examined $d = 2$, we have $\frac{d}{2} \geq 2$ and $p = 1$. Thus $d = 2k$ and $\ell = 2k-1$. Of the $2k+2$ vertices in $R = K_{d+2}$, two must map to the same vertex in the cycle of length $2k+1$, which is not possible if the vertices are joined by a path of length $2k-1$. ■

**Theorem 11** *Let $G$ be a planar graph of odd girth at least $\ell = 2\ell' + 1$, and let $F$ be a face of $G$, if any, that is of some length $m \neq \ell$. Then $F$ has two consecutive edges $e, f$ that can be identified while keeping the odd girth at least $\ell$ (giving $F'$ of length $m-2$). In particular, repeatedly applying this operation makes all faces of length $\ell$ while keeping the odd girth at least $\ell$.*

PROOF: Suppose we cannot identify $e, f$ with $e = xz$ and $f = zy$ by letting $x = y$. Then $x, y$ are joined by a path of length $\ell-2$ in $G$. More generally, we shall show that if we can go distance $d \geq 1$ in the boundary of $F$ from $x$ to $y$, then $x, y$ are joined by a path of length $\ell - d$.

The base cases $d = 1, 2$ are proven. Suppose the claim holds up to $d$. Suppose we go distance $d+1$ from $x$ to $y$ by following the edge $xz$, then a path $zt$ of length $d-1$, then the edge $ty$. Consider the two paths $xt$ and $zy$ of length $\ell - d$. These two paths must meet at some vertex $p$, giving $xp, zp, tp, yp$ paths.

If $d$ is even, then $xp, tp$ have different parity, and $zp, yp$ have different parity. Thus either (1) $xp, zp$ have the same parity, and $tp, yp$ have the same parity, in which case $|xp| + |zp| \geq \ell - 1$, $|tp| + |yp| \geq \ell - 1$, $|xp| + |tp| = \ell - d$ and $|zp| + |yp| = \ell - d$, a contradiction; or (2) $zp, tp$ have the same parity, and $xp, yp$ have the same parity, in which case $|zp| + |tp| \geq \ell - d + 1$, $|xp| + |tp| = \ell - d$ and $|zp| + |yp| = \ell - d$, giving $|xp| + |yp| \leq 2(\ell - d) - (\ell - d + 1) = \ell - d - 1$, thus $|xp| + |yp| = \ell - d - 1$, completing the induction for $d$ even.

If $d$ is odd, then $xp, tp$ have same parity, and $zp, yp$ have same parity. Thus either (1) $xp, zp$ have the same parity, and $tp, yp$ have the same parity, in which case $|xp| + |zp| \geq \ell - 1$, $|tp| + |yp| \geq \ell - 1$, $|xp| + |tp| = \ell - d$ and $|zp| + |yp| = \ell - d$, a contradiction; or (2) $zp, tp$ have different parity, and $xp, yp$ have different parity, in which case $|zp| + |tp| \geq \ell - d + 1$, $|xp| + |tp| = \ell - d$ and $|zp| + |yp| = \ell - d$, giving $|xp| + |yp| \leq 2(\ell - d) - (\ell - d + 1) = \ell - d - 1$, thus $|xp| + |yp| = \ell - d - 1$, completing the induction for $d$ odd.

If the length $m$ of $F$ satisfies $m > \ell$, then setting $d = \ell$ gives $|xy| = \ell - d = 0$, so $x = y$ at distance $\ell$ on $F$, a contradition. If the length $m$ of $F$ satisfies $m < \ell$, with $m \geq 4$ even, then setting $d = m - 1$ gives $|xy| = \ell + 1 - m$ with $x, y$ adjacent, giving a cycle through $x, y$ of length $\ell + 2 - m$, contrary to odd girth at least $\ell$. ■

**Theorem 12** *An assignment of directions to the edges of a planar graph $G$ gives a homomorphism from $G$ to the cycle of length $2k+1$ if and only if it gives a homomorphism for each face.*

PROOF: The only if direction is obvious. For the if direction, let $T$ be a spanning tree of $G$. The edges of $T$ give a mapping to the $2k+1$-cycle. To check the consistency of an edge $e$ of $G$ not in $T$, we must check the consistency of the cycle $C$ of $e$ with $T$. The cycle $C$ contains faces $F_i$, and We let $t_i$ be the difference between clockwise and counter clockwise edges in $F_i$. Each $t_i$ is divisible by $2k+1$, so the sum of these $t_i$ is also divisible by $2k+1$. This sum cancels for edges inside $C$, so the sum counts the difference in orientations for $C$. ■

**Theorem 13** *Suppose $G$ is a planar graph with faces of length $2k+1$. Then $G$ maps to the cycle of length $2k+1$ if and only if all vertices of $G$ have even degree.*

PROOF: If all vertices of $G$ have even degree, then the dual graph $G^d$ has faces of even length, so $G^d$ is bipartite. We may consider both sides of the bipartition, and orient the corresponding faces in one side clockwise and in the other side counter clockwise. This orientation of the edges maps $G$ to the cycle of length $2k+1$.

If $v$ has odd degree in $G$, then around $v$ we must alternate clockwise and counter-clockwise for the various faces, contrary to odd degree. ■

Thus the conjecture asks whether a planar graph $G$ with faces of length $4k+1$ and odd girth $4k+1$ can have its edges pairwise joined to achieve face length $2k+1$, but with even degree.

**Theorem 14** *Let $R$ be a triangulated planar graph with even degrees, and replace each edge in $R$ by a path of length at least $\ell = \lceil \frac{2k+1}{3} \rceil$ to obtain $G$, so that each face of $G$ has odd len Then $G$ maps to a cycle of length $2k+1$.*

PROOF: Since the dual $G^d$ is bipartite, we can orient faces clockwise and counter clockwise so as to orient the edges of $G$ consistently. We consistently choose $2k+1$ edges per face by selecting the $\ell$ edges in $G$ per edge in $R$. The dual $G^d$ has degree $3\ell \geq 2k+1$, so by bipartite matching in $G^d$ we may select $2k+1$ edges per face. The remaining edges in $G^d$ form an Eulerian graph. An Eulerian orientation inverts exactly half of the remaining edges, so the difference in favor of the original orientation is exactly $2k+1$, thus mapping to the cycle of length $2k+1$. ■

**Theorem 15** *Let $R$ be a triangulated planar graph, and replace each edge $e$ in $R$ with two parallel paths of lengths $1 \leq a_e \leq b_e$, with $a_e + b_e = 2k+1$ in $G$. Whether $G$ maps to the cycle of length $2k+1$ depends on evenness of degrees as in Theorem 13*

PROOF: A triangle in $R$ gives something that maps, with corresponding $a_1 \leq a_2 \leq a_3$, if and only if $a_1 + a_2 = a_3$ (equivalently $a_1 + a_2 + b_3 = 2k+1$) or $a_1 + a_2 + a_3 = 2k+1$. Choosing the corresponding three paths to form a cycle of length $2k+1$. This will choose one or both parallel paths from the two faces corresponding to triangles in $R$, and it suffices to check whether all degrees are even as before. ■

# 4 Barnette-Like Graphs

Recall that a Barnette graph is a 3-connected cubic bipartite planar graph $R$. The faces of $R$ can be colored red, blue, green so that adjacent faces have different colors. We shall replace the edges joining blue and green faces by paths to obtain $G$. We often apply the following operation to all red faces: Go around each red face clockwise, identify the vertices after a blue face and before a green face all together. This replaces the red face of length $2t$ with a vertex of degree $2t$, and adds length one to each path replacing a blue-green edge of $R$.

**Corollary 1** *Suppose the green faces of $R$ are all squares, the red faces of $G$ are squares or hexagons, the paths constructed in going from $R$ to $G$ are of length at least $k-1$, and the odd girth is at least $4k-1$. Then $G$ maps to the cycle of length $2k+1$.*

PROOF: Apply the operation to all red faces. Then the paths will have length at least $k$, and since the odd girth is at least $4k-1$, we can remove one of two parallel paths around green faces. The result for the resulting subcubic graph follows from Theorem 10. ■

A similar application of Theorem 13 after applying the operation to all red faces gives the following.

**Corollary 2** *Suppose the blue and green faces of $G$ all have length $2k+1$. Then $G$ maps to the cycle of length $2k+1$.*

A similar application of Theorem 14 after applying the operation to all red faces gives the following.

**Corollary 3** *Suppose the blue and green faces of $R$ are all hexagons, and the inserted paths are of length at least $\lceil\frac{2k+1}{3}\rceil - 1$ to obtain $G$. Then $G$ maps to a cycle of length $2k+1$.*

Suppose $R$ only has squares and hexagons (exactly six squares). Insert a single vertex on various edges, so that all faces will have length 7. We would like to know whether this graph maps to the 5-cycle, by the technique of Theorem 2, so for example one or more cycles separate regions with trees leading to cycles. In some cases we found a Hamiltonian cycle that goes through all edges with inserted vertices except for two (see [6, 8, 9] for Hamiltonian cycles.)

Now we consider another example that seems to map to the 5-cycle. Let $R$ be a triangulated planar graph, with all vertices of degree at least 3. Apply the following two operations to $R$ to obtain $G$: (1) replace each edge of $R$ with two parallel edges, one with an inserted vertex; (2) replace each vertex of degree $2d$ at least 6 in the resulting graph with a polygon having $2d$ length. The resulting $G$ will be subcubic. Notice that $G$ is a Barnette type graph, with even length red faces created in step (2) green faces of length 5 created in step (1), and blue faces of length 6,7,8, or 9, coming from the triangles of $R$. The inserted vertices are between green and blue faces. We orient the faces coming from triangles of $R$ clockwise, those created in step (1) counter clockwise, and the even length red faces are therefore oriented alternatingly.

The average length of a blue face is 7.5, since the parallel edges have average length 1.5 (1 on one side, 2 on the other side). Then if $F_i$ denotes the number of blue faces of lengh $i$, we have

$$6F_6 + 7F_7 + 8F_8 + 9F_9 = 7.5(F_6 + F_7 + F_8 + F_9),$$

$$3F_6 + F_7 = F_8 + 3F_9.$$

**Theorem 16** *If all blue faces of $G$ have 6 or 9 sides, with $F_6 = F_9$, then $G$ has a homomorphism to the 5-cycle.*

Proof: The dual of $R$ (without the parallel edges) is a bipartite cubic graph with edges joining faces with 6 sides and faces with 9 sides, which can then be matched by a perfect matching. If $f_6$ and $f_9$ are two faces so matched, we invert the 5-face in the green face in between to clockwise, and invert the two edges going to the corresponding red faces joined by this 5-face from the face $f_6$. This inverts three edges for $f_6$, with $3 - 3 = 0$ difference, and inverts two edges for $f_9$, with $7 - 2 = 5$ difference, as desired. ■

**Theorem 17** *If $R = K_4$, the tetrahedron, then all cases of $G$ map the the 5-cycle, that is, for (1) $F_6 = F_7 = F_8 = F_9 = 1$, (2) $F_7 = F_8 = 2$, (3) $F_6 = 1, F_8 = 3$, and (4) $F_9 = 1, F_7 = 3$.*

Proof: For (1), match faces $f_6$ and $f_9$ as in the previous theorem, and match faces $f_7$ and $f_8$ by inverting the 5-face in between as before, and inverting the two edges meeting the two red faces by the 5-cycle from $f_8$, thus inverting edges giving for $f_7$ the difference $6 - 1 = 5$, and for $f_8$ the difference $4 - 4 = 0$.

For (2), match two pairs $f_7, F_8$ and proceed as in (1).

For (3), match $f_6$ and the three faces $f_8$, inverting the 3 edges for $f_6$, and two edges for each $f_8$, plus two extra edges between $f_8$ and red faces, giving $3 - 3 = 0$ and $4 - 4 = 0$ as desired.

Finally for (4), invert the 5-face between $f_9$ and one $f_7$, and for the other two $f_7'$ invert in the red face that meets the 5-face. This gives again $7-2=5$ and $6-1=5$. ■

We say that $R$ is a double-wheel $W_n, n \geq 3$ if $R$ consists of a cycle $C$ of length $n$, plus two vertices $u, v$ each connected to a vertex in $C$. Thus the vertices in $C$ have degree 4, and $u, v$ have degree $n$.

**Theorem 18** *If $R = W_n$, then each of the cases having $G$ with only two cycle lengths maps to the 5-cycle, namely (1) $F_6 = F_9 = n$ even, (2) $F_7 = F_8 = n$, (3) $3F_6 = F_8 = 3n/2$, and (4) $3F_9 = F_7 = 3n/2$.*

PROOF: (Sketch.) The case (1) was solved above.

For (2) we attempt to match as before $f_7$ and $f_8$, and we match otherwise across other vertices of $R$ corresponding to red faces.

For (3) we divide the problem into cases of a vertex of $C$ of degree 4 surrounded by three $f_8$ and one $f_6$, and do the corresponding inversions.

Finally for (4) we decompose again into 4 faces, one $f_9$ sharing one edge with one $f_7$ and at the two endpoints or red faces sharing with the other two $f_7'$. ■

# 5 Chopped Graphs

Recall that given a planar graph $G$ and its dual $G^d$, the chopped graph $G^c$ is obtained from $G^d$ by removing each vertex of $G^d$ of some degree $D$ and connecting the corresponding $D$ endpoints left with a $D$-gon. Thus the chopped graph $G^c$ is cubic.

**Theorem 19** *Suppose $G$ is a planar biconnected graph that maps to the cycle of length $2k+1$. Then the chopped graph $G^c$ also maps to the cycle of length $2k+1$.*

PROOF: Orient the edges of $G$ according to the mapping to the cycle of length $2k+1$. Rotate each edge $e$ of $G$ clockwise from the center of $e$ to obtain the corresponding oriented edge $e^c$ of $G^c$. Rotate $e^c$ clockwise from each end to obtain edges corrresponding to the faces of $G$ in $G^c$. These faces work since we just have a translation of the corresponding face in $G$ to $G^c$, and the faces of even length around vertices of $G$ in $G^c$ have by definition consecutive pairs in reverse direction. ■

**Theorem 20** *Suppose $G$ is a planar biconnected subcubic graph with faces of length 5. Then $G^c$ maps to the 5-cycle.*

PROOF: (Sketch.) Skip vertices of degree 2 in $G$, and you are left with a skeleton with $F_i$ faces of length $i = 3, 4, 5$ and no other face length. Furthermore $3F_3 + 2Ff_4 + F_5 = 12$ by Euler's formula. We examined all such planar graphs, and the only ones that lead to a valid $G$ are the tetrahedron ($F_3 = 4$), the cube ($F_4 = 6$), and the dodecahedron ($F_5 = 12$).

The orientation of the faces of $G^c$ corresponding to faces of $G$ can be clockwise or counter clockwise. The orientation of the edges of $G^c$ can be viewed as orientations of edges of $G$ turned clockwise. Then if $v$ in $G$ has $a$ outgoing edges, $b$ incoming edges, $a'$ surrounding clockwise faces, and $b'$ surrounding counter clockwise faces, it is necessary and sufficient to have (*) $a = a'$ and $b = b'$. We can skip vertices of degree 2 in $G$ provided the two faces surrounding are one clockwise and one counter clockwise.

For the tetrahedron, there are two cases with the 4-cycle $xyzt$ having either one vertex inserted in each edge of the cycle, or two vertices in $xy$ and $zt$. One side of the 4-cycle is clockwise, namely the side containing $xz$, and the other side is counter clockwise, namely the side containing $yt$. We orient the edges as $xz, zy, zt, yt, xy, tx$, verifying (*).

For the cube, we add one vertex on alternatining edges of the 6-cycle avoiding 000 and 111. We orient clockwise the faces containing 000, and orient counter clockwise the faces containing 111, on either side of the 6-cycle. Then we orient all edges in the direction of shortest paths from 000 to 111, verifying (*).

For the dodecahedron, we select two opposite faces $F, F'$, orient $F$ counter clockwise, orient $F'$ clockwise, orient the 5 faces adjacent to $F$ clockwise, and orient the 5 facees adjacent to $F'$ counter clockwise. We orient the edges along shortest paths from $F$ to $F'$, plus the edges around $F$ in one direction and around $F'$ in one direction, verifying (*). ■

Let $R$ be a triangulated planar graph. The dual $R^d$ is cubic and has a perfect matching $M$ that matches pairs of faces $x_1x_2x_3, x_1'x_2x_3$ in $R$. To obtain $G$, we perform one of two operations in each such pair of faces in $R$ matched by $M$: (1) insert two vertices in $x_2x_3$, or (2) insert vertex $t$ with paths $ty_ix_i, 1 \leq i \leq 3$ in the first face, and similarly vertex $t'$ with paths $t'y_1'x_1', t'y_i'x_i, 2 \leq i \leq 3$ in the second face. In the end, all faces of $G$ have length 5, and we consider the chopped $G^c$.

**Theorem 21** *This $G^c$ maps to the 5-cycle.*

PROOF: We use property (*) from the previous theorem. In case (1), say we orient face $x_1x_2x_3$ clockwise and orient face $x_1'x_2x_3$ counter clockwise. Then the path $x_2x_3$ with two vertices inserted simplifies to a single edge $x_2x_3$, since the orientation of faces cancels at $x_2, x_3$, while the imbalance at $x_1, x_1'$ can be viewed as a single edge $x_1x_1'$.

In case (2), orient $x_1x_2t, x_2x_3t$ clockwise and orient $x_3x_1t$ counter clockwise. This gives orientations $y_2x_2, y_2t, ty_1x_1, ty_3x_3$. Reversing the orientations for the second face after mapping $x_1$ to $x_1'$ and $t$ to $t'$, cancels the orientations at $x_2, x_3$, while the imbalance at $x_1, x_1'$ can be viewed as a single edge $x_1x_1'$.

If the degree at $v$ without the edges $x_2x_3, x_1x_1'$ in all pairs is $d$, then after adding this pair of edges the degree will be $2d$, so all vertices of this simplified graph obtained from $G$ have even degree and the simplified graph is Eulerian, so an Eulerian orientation suffices. ■

Let $R$ be a biconnected subcubic graph. Define G by considering each face $F = x_1x_2 \cdots x_r$ of $R$, inserting a vertex $v$ in $F$, and joining $v$ to each $x_i$ by a path of length 2. Thus all faces of $G$ have length 5.

We shall show that $G$ maps to the 5-cycle if $R^d$ has odd degree vertices $v_i$ joined in pairs by disjoint paths (going through even degree vertices). In particular, if all vertices of $R^d$ have odd degree, these paths form a perfect matching. For these paths to exist, it suffices that $R^d$ be Hamiltonian, and for this in turn it suffices that $R^d$ be 4-vertex-connected, as shown by Tutte [13].

In the other direction, if $R^d$ is not 4-vertex-connected, such a perfect matching may not exist. Consider here a double wheel $W_n$, add to $W_n$ parallel edges forming a perfect matching to make all degrees odd, and finally insert in each triangle $T$ a vertex $v$ adjacent to all vertices of $T$ to obtain $R^d$ Then the $2n$ vertices inserted in triangles $T$ cannot be matched to the remaining $n+2$ vertices. Note that the 3 vertices of such $T$ disconnect $R^d$.

**Theorem 22** *In this construction, if $R^d$ is Hamiltonian (in particular if $R^d$ is 4-vertex-connected) then the chopped $G^c$ maps to the 5-cycle.*

PROOF: Traverse the Hamiltonian cycle $C$ of $R^d$ starting at a vertex of odd degree to the next vertex of odd degree, then skip the next such path, then select the next such path, and so on. This gives us vertex disjoint paths joining vertices of odd degree in pairs, with the intermediate vertices of even degree. We shorten these paths if two consecutive edges can be replaced by a direct edge.

These paths can be viewed as sequences $F_1F_2\cdots F_r$ of faces in $R$, where $F_1, F_r$ have odd length, $F_2,\ldots,F_{r-1}$ have even length, and $F_i, F_{i+1}$ share an edge $e_i$. In fact, $e_{i-1}, e_i$ will not be consecutive edges in $F_i$, else we would skip $F_i$ by shortening the path in $R^d$.

Let $x_1, x_2, x_3, \ldots$ be consecutive vertices in $F_1$ such that $e_1 = x_1x_2$. Thus $x_1, x_2, x_3', x_4', \ldots$ are consecutive vertices in $F_2$. Let $t$ be the vertex that was added to $F_1$ in $G$ to form paths of length 2 joining to the boundary of $F$, and let $t'$ be the corresponding vertex in $F_2$. We orient the two faces $tx_1x_2$ and $tx_2x_3$ clockwise, and orient the two faces $t'x_1x_2$ and $t'x_2x_3$ counter clockwise. We orient forward the path from $t$ to $x_1$, the edges on the path from $t$ to $x_2$ away from the middle point, forwward the path from $x_1$ to $t'$, the edges on the path from $x_2$ to $t'$ towards the middle point.

Orient the faces around $t$ in $F_1$ alternatingly clockwise and counter clockwise, except for the two consecutive clockwise faces described above, so there will be one more clockwise than counter clockwise faces around $t$, and to satisfy property (*) one more edge will come out of $t$ than go into $t$.

If $r=2$, so that $F_2$ is of odd length, proceed to orienting alternatingly the faces around $t'$ in $F_2$, except for the two consecutive counter clockwise faces described above, so there will be one more counter clockwise than clockwise faces around $t$, and to satisfy property (*) one more edge will go into $t$ than come out of $t$. Remove the paths already oriented joining $tx_1t'$ and $tx_2t'$, and join $tt'$ by an edge (losing planarity).

If $r \geq 3$, so that $F_2$ is of even length, proceed to select $x'_{2i+1}x'_{2i+2}$ and $x'_{2i+2}x'_{2i+3}$, so that one of these two edges is $e_2$. The two corresponding faces with $t'$ will be oriented clockwise if we orient otherwise alternatingly. Remove the paths already oriented joining $tx_1t'$ and $tx_2t'$, and proceed similarly to $F_3$.

In the end we reach $F_r$ with $t_r$, and after the process for the face $F_r$ of odd length we join $tt_r$ with an edge.

Once this is done with all paths $F_1,\ldots,F_r$ we end up with an Eulerian subgraph of $G$ that be assigned an Eulerian orientation. ■

Let $R$ be a triangulated planar graph. The dual $R^d$ is cubic and is 3-edge colorable, so the edges of $R$ can be 3-edge colored so that each triangle that is a face uses all three colors. To obtain $G$, insert a vertex in blue and green edges while leaving red edges intact. In the end, all faces of $G$ have length 5, and we consider the chopped $G^c$.

**Theorem 23** *This $G^c$ maps to the 5-cycle.*

Proof: Insert a vertex $v$ in each face of $G$, of length 5, and connect $v$ to the five vertices of the face. This gives the dual $G^{cd}$, and we can apply the modular characterization of the problem from the introduction. If we repeatedly go across blue and green edges of $R^d$, we obtain a decomposition into even length cycles, so we can label the edges at $v$ of degree 5 alternatingly as outgoing or incoming. The edge going to the middle of the green or blue edge from $v$ and across to $v'$ can be removed, and the middle vertex skipped, leaving $v$ of degree 3. Similarly, the edges from $v$ to $v'$ through the endpoints of the green edge can be removed, leaving $v$ of degree 1 and connecting to an endpoint of a blue edge $e$. Then we can go across $e$ from $v$ to $v''$, possibly traversing $e$.

This eliminates all the added vertices $v$, the change is even in degree at all vertices other that such $v$, and these vertices had even degree already in $G^{cd}$, so an Eulerian orientation proves the modular property. ■

We now define a family of triangulated graphs $I_r$, $r \geq 2$. The graph $I_r$ has two vertices $u, v$ of degree $r$ and the remaining vertices $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_r$ each of degree 5. The vertex $u$ has an edge to each $x_i$, the vertex $v$ has an edge to each $y_i$, and the remaining edges are $x_i y_i$, $x_{i+1} y_i$, $1 \leq i \leq r$, where $x_{r+1} = x_1$.

Thus $I_r$ has $2r+2$ vertices and $4r$ faces (triangles), and in particular $I_5$ is the icosahedron. We now define a graph $J_r$ from $I_r$ by inserting a vertex in each edge $ux_i$, $vy_j$, $x_i y_j$. Thus the faces of $J_r$ have length 5.

**Theorem 24** *If $G = J_r$, then the chopped $G^c$ maps to the 5-cycle.*

Proof: If $r$ is even, then the result follows from Theorem 23. Assume thus that $r \geq 3$ is odd.

Orient the faces $ux_i x_{i+1}$ alternatingly clockwise and counter clockwise, with the exception of faces $ux_1x_2$ and $ux_2x_3$ that are both oriented clockwise. Orient the paths $ux_3, ux_4, \ldots, ux_r$ in alternation, starting with outward, then inward, and so on, with the last one oriented outward. Orient the path $ux_1$ outward, and orient $ux_2$ away from its middle vertex.

Consider the difference $D_w$ between the number of incoming edges at $w$ plus clockwise oriented faces at $w$, and the number of outgoing edges at $w$ plus counter clockwise oriented faces at $w$. We must establish $D_w = 0$ at all vertices $w$ as before for (*).

So far we have $D_w = 0$ at $u$ and at the middle vertices of the paths $ux_i$. The sequence $s = D_{x_1}, \ldots, D_{x_r}$ is $1, 3, (1, -1)^m, 1$, with $r = 2m + 3$. We now correspond $u$ to $v$ and $x_i$ to $y_{r+2-i}$ (with $y_{r+1} = y_1$), while inverting orientations of faces and edges in this correspondence. The sequence $t = D_{y_1}, \ldots, D_{y_r}$ is $-1, -1(1, -1)^m, 3$.

Now orient the triangles $x_1y_1x_2, y_1x_2y_2, x_2y_2x_3$ counter clockwise. This changes $D_{x_2}$ from 3 to 0. Similarly, orient the triangles $y_{r-1}x_ry_r, x_ry_rx_1, y_rx_1y_2$ clockwise. This changes $D_{y_r}$ from $-3$ to 0. Orient the triangles $y_{i-1}x_iy_i$ clockwise and the triangles $x_iy_ix_{i+1}$ counter clockwise, for $3 \leq i \leq r-1$. The sequences then become $s = 2, 0, (0, -2)^m, 2$ and $t = -2, -2, (2, 0)^m, 0$.

For $r = 3$ we have $m = 0$ and sequences $s = (2, 0), 2$ and $t = -2, (-2, 0)$. Matching the two groups in the two sequences gives two triangles $2, 0, -2$ each of which gives two paths from the 2 to the $-2$, which in turns sets $D_w = 0$ for all $w$. The remaining unoriented edges form an Eulerian graph that can be given an Eulerian orientation.

For $r \geq 5$ we have $m \geq 1$ $s = (2, 0), 0, (-2, 0)^{m-1}, (-2, 2)$ $t = -2, (-2, 2), (0, 2)^{m-1}, 0, 0$. Ignoring the last 0 in $t$, we can match the $m + 2$ groups in $s$ and $t$ to form triangles and squares with one 2, one $-2$, and one or two 0s per match, each of which gives two paths from the 2 to the $-2$, which in turns sets $D_w = 0$ for all $w$. The remaining unoriented edges form an Eulerian graph that can be given an Eulerian orientation. ■

Let $R$ be a triangulated planar graph. The dual graph $R^d$ is cubic and contains a perfect matching $M$. The edges of $M$ correspond to a set of edges $P$ of $R$. We insert a vertex in each edge of $E(R) \setminus P$ to obtain a graph $G$ whose faces have length 5.

**Theorem 25** *If $G$ has vertex degrees at most 5, then the chopped $G^c$ maps to the 5-cycle.*

Proof: We first show that the set of edges $P$ forms a subgraph of $R$ with degrees at most 2. Assume otherwise that $P$ contains three edges $tx, ty, tz$. Since $txy$ cannot be a face of $R$ (only one edge per face is in $P$), there must be an edge $tx'$ in between $tx$ and $ty$, and similarly an edge $ty'$ in between $ty$ and $tz$, and an edge $tz'$ in between $tz$ and $tx$. But then $t$ would have degree at least 6.

The set of edges of $E(R^d) \setminus M$ forms a subgraph of $M$ with degrees exactly 2, thus a union of cycles $C_i$. Each $C_i$ corresponds to a set of edges $D_i$ of $R$, and the edges of $P$ that complete faces (that are triangles in $R$) with pairs of edges of $D_i$ form two connected components $P_i^1, P_i^2$ of the subgraph of $R$ formed by the edges of $P$, one for each side of the edges of $D_i$.

Suppose for some $i$, both $P_i^1$ and $P_i^2$ are cycles. Then no vertex $v$ in the subgraph of $R$ formed by $D_i$ can have degree 3 or more. Otherwise say $v$ is in $P_i^1$, and then $v$ has degree 5 or more in the subgraph of $R$ formed by $D_i \cup P_i^1$. As before there must be an edge $vt$ on the other side from $D_i$ of $P_i^1$, else the two edges of $P_i^1$ joined at $v$ would form a triangular face on the other side (only one edge of $P$ per face). This would give degree at least 6 to $v$.

Furthermore, no vertex $w$ in the subgraph of $R$ formed by $D_i$ can have degree 1, since the neighbor $v$ of $w$ in $D_i$ would have degree at least 3 in $D_i$ to complete triangular faces, contrary to the previous claim.

So if both $P_i^1$ and $P_i^2$ are cycles, then $D_i$ forms a cycle of length $2r$ for some $r$, with each of $P_i^1, P_i^2$ cycles of length $r$. Each $v$ in $P_i^1$ has degree 4 in $P_i^1 \cup D_i$, so exactly one additional edge $vt$ exists on the other side of $P_i^1$ from $D_i$. This implies that $t$ must be the same for all such $v$ in $P_i^1$. The same argument gives a $t'$ on the other side of $P_i^2$ from $D_i$. But then $R = I_r$ and $G = J_r$, which was shown to map to the 5-cycle in Theorem 24.

If both $P_i^1$ and $P_i^2$ are paths, then there is only one $D_i$ (there is no other side of such $P_i^j$), and since $R$ has an even number of triangles, the result follows from Theorem 23.

The last case has exactly two $D_i$, for $i = 1, 2$, with $P^1 = P_1^1 = P_1^2$ a cycle and $P_2^1, P_2^2$ paths. If each of the two sides of $P^1$ has an even number of triangles, the result follows from Theorem 23 again, so we may instead assume an odd number of triangles in each side of $P^1$.

We show that $P^1$ has a vertex $v$ of degree 4 in $R$, with two edges in $P^1$, one edge in each of $D_1$ and $D_2$. For this, notice that if the path $P_j^2$ has length $\ell_j$, then as we go around $P_j^2$ we obtain each edge twice, so there are $2\ell_j$ vertices of degree two in $D_j$ from $P^1$ (forming triangles with each traversal of an edge in $P_j^2$). The odd number of triangles formed with $D_j$ is then $k_j + 4\ell_j$, where $k_j$ is the odd number of vertices of degree one in $D_j$ from $P^1$. But then no $v$ can have 6 edges, 2 in $P^1$, 2 in $D_1$, and 2 in $D_2$, so $k_1 \geq 2\ell_2$, and by parity $k_1 \geq 2\ell_2 + 1$, thus one of the $k_1$ vertices is one of the $k_2$ vertices, proving the claim.

So let $v$ be in $P^1$ of degree 4 in $R$, with two edges in $P^1$, one edge in each of $D_1$ and $D_2$. Let $uvw$ be the corresponding path in $P^1$, and let $s$ be the common neighbor of $u, v, w$ in $P_1^2$, with $t$ the common neighbor of $u, v, w$ in $P_2^2$.

We proceed as in Theorem 23. Insert a vertex $x$ in each face of $G$, of length 5, and connect $x$ to the five vertices of the face. This gives the dual $G^{cd}$, and we can apply the modular characterization of the problem from the introduction. This time we label all edges of $D_1 \setminus \{vs\}$ (and of $D_2 \setminus \{vt\}$) alternatingly blue and green. We can label the edges at $x$ of degree 5 alternatingly as outgoing or incoming, except for both outgoing inside $uvs$ and $vws$, and both incoming inside $uvt$ and $vwt$. We orient the edges out of the middle vertex of $vs$ and into the middle vertex of $vt$. The edge going to the middle of the green or blue edge from $x$ and across to $x$ can be removed, and the middle vertex skipped in the remaining cases. Similarly, the edges from $x$ to $x'$ through the endpoints of the green edge can be removed. We are left with $x$ and connecting to an endpoint of a blue edge $e$. Then we can go across $e$ from $x$ to $x''$, possibly traversing $e$.

Say $uvs$ contains $x$, and $vws$ contains $x'$, with $us$ green and $ws$ blue. This process leaves us with $x, x'$ each having two outgoing edges, one to the middle of $vs$ and one to $v$, and with $x'$ having a third outgoing edge to either $w$ or $s$. In the latter case we orient $sw$ forward. The extra incoming edge into $s$ from the middle of $sv$ is disposed via orienting $su$ forward.

We are thus left with 3 extra incoming edges at $v$, 1 extra at each of $u, w$. On the side of $t$, on the other hand, we are left with 3 extra outgoing edges at $v$, 1 extra at each of $u, w$, so the two sides of $s$ and $t$ cancel out.

This eliminates all the added vertices $x$, the change is even in degree at all vertices other than such $x$, and these vertices had even degree already in $G^{cd}$, so an Eulerian orientation proves the modular property. ■

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