Maximum Gap Labelings of Graphs

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Abstract

Given a graph, we define a base set to be a set of integers of size equal to the number of vertices in the graph. Given a graph and a base set, a labeling of the graph from the base set is an assignment of distinct integers from the base set to the vertices of the graph. The gap of an edge in a labeled graph is the absolute value of the difference between the labels of its endpoints. The gap of a labeled graph is the sum of the gaps of its edges.

The maximum gap graph labeling problem takes as input a graph and a base set and maximizes the gap of the graph over all possible labelings from the base set. We show that this problem is NP-complete even when the base set is restricted to consecutive integers. We also show that this restricted case has polynomial time approximations that achieve a factor of 2/3 for trees, of 1/2 for bipartite graphs, and of 1/4 for general graphs, with a deterministic algorithm, while an expected factor of 1/3 for general graphs is achieved with a randomized algorithm. The case of general base sets is approximated within an expected factor of 1/16 for general graphs with a randomized polynomial time algorithm.

We finally give a polynomial time algorithm that solves the maiximum gap graph labeling problem for a graph that has bounded degree and bounded treewidth. The maximum graph labeling problem shows connections with the graceful tree conjecture.

1 Introduction

Given a graph G with vertex set $V(G) = \{v_0, v_1, \ldots, v_n\}$ and edge set $E(G)$ with m edges, we define a *base set* to be a set $B = \{b_0, b_1, \ldots, b_n\}$ of $n+1$ distinct integers, that is, with as many integers as vertices in G. The base set is said to be *uniform* if $b_i = i$, that is, if $B = \{0, 1, \ldots, n\}$.

Given a graph G and a base set B, a *labeling* L of the graph from the base set is a one-to-one assignment of integers $\ell(v)$ from the base set B to the vertices v of the graph G. The *gap of an edge* vw in a labeled graph is the absolute difference $|\ell(v) - \ell(w)|$. The *gap of a labeled graph* G is the sum $\sum_{vw \in E(G)} |\ell(v) - \ell(w)|$ of the gaps of its edges.

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The *maximum graph labeling problem* takes as input a graph G and a base set B and produces a labeling of G with maximum gap. We show that the maximum graph labeling problem is NP-complete when the base set is uniform. We also show that this restricted case has a deterministic polynomial time approximation that achieves a factor of the maximum of 2/3 for trees, 1/2 for bipartite graphs, and 1/4 for general graphs and a randomized polynomial time algorithm that achieves an expected factor of 1/3 for general graphs. The case of general base sets B is approximated within an expected factor of 1/16 for general graphs with a randomized polynomial time algorithm. We finally show a polynomial time algorithm that solves the problem when the graph has bounded degree and bounded treewidth.

There are many interesting problems connected to the maximum gap labeling problem. For instance one can ask to minimize instead of maximize the gap of a labeling. This is also similar to the study of the minimum linear arrangement problem. A problem that has been extensively studied is the graceful tree conjecture [1]. A labeling of a graph with $n+1$ vertices and $m = n$ edges is *graceful* if the base set is uniform, and the gaps of the edges are the distinct integers $1, 2, \ldots, n$. The *graceful tree conjecture* states that every tree is graceful and remains open. Notice that graceful labelings have gap $\binom{n+1}{2}$, which we call the *graceful gap*. In the last section we show connections between the maximum graph labeling problem for trees and the graceful tree conjecture. In particular, the maximum gap in a tree is always at least the graceful gap.

2 Preliminaries

Let G be a graph and let B be a base set. A *sign assignment* S of G is an assignment of signs $+$ or $-$ to the vertices of G so that adjacent vertices have different signs, that is, for $uv \in E(G)$, we have that either u has $+$ sign and v has $-$ sign, or u has $-$ sign and v has $+$ sign. Notice that a graph G has a sign assignment if and only if G is bipartite, and that if G is bipartite with k connected components, then G has 2^k possible sign assignments.

Let G be a bipartite graph, and let B and S be a base set and a sign assignment of G. A labeling L of G from B is *alternating* for S if $\ell(v) < \ell(w)$ whenever $vw \in E(G)$, the vertex v has a $-$ sign, and the vertex w has a $+$ sign. Note that the gap of an alternating labeling L is $\sum_{v \in V(G)} s(v)d(v)\ell(v)$, where $s(v), d(v), \ell(v)$ are the sign, the degree, and the label of v respectively.

A labeling L of G from B is *bipartite* for S if $\ell(v) < \ell(w)$ whenever the vertex v has a $-$ sign, and the vertex w has a $+$ sign. A bipartite labeling L of G from B for S is *stretched* if whenever $\ell(v) < \ell(w)$ with v and w both having $-$ signs, we have degrees of v and w satisfying $d(v) \geq d(w)$, and whenever $\ell(v) < \ell(w)$ with v and w both having $+$ signs, we have degrees of v and w satisfying $d(v) \leq d(w)$.

Theorem 1 *Given a bipartite graph G with base set B and sign assignment S, the alternating labelings of maximum gap are the labelings of L that are stretched bipartite labelings up to change of sign of isolated vertices.*

Proof: If L is not bipartite, then $\ell(v) < \ell(w)$ for some v with $+$ sign and some w with $-$ sign. Exchanging the labels of v and w while leaving the other labels the same, we obtain a new labeling L' that is still an alternating labeling. Furthermore if v and w are not both isolated then L' gives larger gap to G than L, while if v and w are both isolated then exchanging the signs of v and w in L gives a new labeling L'' that is still an alternating. When it is no longer possible to obtain a new L' or L'' from L by the corresponding exchanges, the resulting labeling will be bipartite.

If L is bipartite but not stretched, then either $\ell(v) < \ell(w)$ with v, w having $-$ signs and $d(v) < d(w)$, or $\ell(v) < \ell(w)$ with v, w having $+$ signs and $d(v) > d(w)$. In either case, exchanging the labels of v and w increases the gap of the bipartite labeling. When it is no longer possible to obtain a new bipartite labeling by such exchanges, the resulting labeling will be stretched bipartite.

Thus a maximum gap alternating labeling L is a stretched bipartite labeling. Notice finally that the values of the terms in the gaps $\sum_{v\in V(G)} s(v)d(v)\ell(v)$ are the same for all stretched bipartite labelings, so all stretched bipartite labelings achieve the maximum gap over alternating labelings L. ■

Say that a connected bipartite graph K is *symmetric* if K is an isolated vertex or there is an isomorphism of K exchanging the two sides of the bipartition.

Theorem 2 *Let G be a bipartite graph with base set B. Given a sign assignment S, a bipartite stretched labeling L can be computed in $O(m + n \log n)$ time. If S is not given and G has k connected components K such that all other connected components are known to be symmetric, then a sign assignment S and a bipartite stretched labeling L of maximum gap can be computed in $O(m + 2^k n \log n)$ time.*

Proof: Given S, one can determine degrees and sort the $+$ vertices and the $-$ vertices by degrees in $O(m + n \log n)$ time, proving the first claim. If S is not given, we can still determine degrees in $O(m)$ time, and then every connected component K that is not known to be symmetric has two possible sign assignments, giving a total of 2^k sign assignments for such K. For all symmetric components K the two sign assigments are equivalent because of the isomorphism exchanging the two sides of K, or because the sign of an isolated vertex does not matter as it has no incident edges. Sorting in $O(n \log n)$ time for these 2^k cases finally gives an $O(m + 2^k n \log n)$ algorithm, proving the second claim. ■

We say that a graph G *can be modified to* G' if there exist distinct vertices $u, v, w \in V(G)$ and edges $uv, vw \in E(G)$ so that G' is obtained from G by removing the eges uv, vw and adding the edge uw. Notice that this operation may add parallel edges. Furthermore, this operation reduces the degree of v by two while keeping all other degrees the same. We say that a sequence of graphs $G_0, \ldots, G_r$ is *obtained by modifying* G if $G = G_0$, G_i can be modified to G_{i+1} for all $0 \le i < r$, and G_r is bipartite. Notice again that the G_i may have parallel edges. Note that G_r is not necessarily the first bipartite graph in the sequence of G_i, and we may have $r = 0$.

Theorem 3 *Let G be a graph and B a base set. Then a maximum gap labeling L can be described by giving a sequence of graphs $G_0, \ldots, G_r$ obtained by modifying G and finding any stretched bipartite labeling of G_r.*

Proof: Let L be a maximum graph labeling for $G = G_0$ with base set B. Suppose $uv, vw \in E(G)$ with $\ell(u) < \ell(v) < \ell(w)$. Then removing edges uv, vw and adding edge uw

gives a labeled graph G_1 with gap equal to the gap of G'. We may repeat this operation for G_r until no longer possible to obtain a final bipartite graph G_r. Now obtain a sign assignment S for G_r as follows: Assign $-$ to vertices v whose neighbors have labels larger than $\ell(v)$, and assign $+$ to vertices w whose neighbors have labels smaller than $\ell(v)$. Then L is a stretched bipartite labeling of G_r with B, S and any such stretched bipartite labeling achieves the same gap as L by Theorem 1. ■

3 NP-completeness with Uniform Base Sets

The *minimum edge deletion bipartition* problem takes as input a graph G and outputs the minimum number of edges of G such that their removal leads to a bipartite graph G'. The *not-all-equal satisfiability* problem takes as input a collection of triples xyz of variables and outputs an assignment of $+$ or $-$ to each variable so that no triple is assigned three $+$s or three $-$s. Both problems are known to be NP-complete [2, 3]. We give a particular reduction of the second problem to the first problem that will be used to prove the main result of this section.

Theorem 4 *The minimum edge deletion bipartition problem is NP-complete even if all vertices have degrees 2 or 3.*

PROOF: Consider an instance I of the not-all-equal satisfiability problem. We construct an instance I'' of the minimum edge deletion problem as follows (see Figure 1). If a variable x occurs in k_x triples of variables in I, make k_x copies of x, one for each occurrence of x. Represent each triple xyz in I by a triangle in I' whose vertices are chosen copies of the corresponding variables. Thus I' consists of disjoint triangles, one for each triple in I. Now if x has k_x copies, we construct a tree T_x of maximum degree 3 with k_x leaves at even distance from each other in the tree T_x. We finally identify for each variable x the leaves of T_x with the k_x occurrences of x in disjoint triangles to obtain the instance I''. Notice that all vertices in I'' have degrees 2 or 3.

```
          t ---- * ---- t
         / \           / \
        /   \         /   \
       y --- x       x --- z
       |      \     /      |
       |       \   /       |
       |        \ /        |
       *         *         *
        \        |        /
         \       |       /
          \      |      /
           \     x     /
            \   / \   /
             \ /   \ /
              y --- z
```

Figure 1: Graph I'' induced by the instance I with triples xyz, xzt, xty.

Suppose the instance I has a solution for the not-all-equal satisfiability problem. Then each triple xyz in I has either two +s and a −, or two −s and a +. We remove the edge of the corresponding triangle joining the two +s or the two −s respectively, and use the fact that all leaves of each T_x are at even distance from each other to label adjacent vertices of T_x with different signs + or −. This turns I'' into a graph G such that adjacent vertices have different signs + or −, that is, G is bipartite. Furthermore, as we have only removed one edge per triangle, which was necessary to make I'' bipartite, this gives an optimal solution to the minimum edge deletion problem.

In the other direction, suppose I'' has a solution that removes only one edge per triangle to obtain a bipartite graph G. Label the two sides of the bipartition of G by + and − respectively. Then a triangle xyz with the edge xz removed will have signs $+-+$ or $-+-$, and the k_x copies of x at even distance from each other in T_x will have the same sign + or −, solving I. ■

We use the construction in Theorem 4 to infer the following.

Theorem 5 *The maximum gap graph labeling problem is NP-complete with a uniform base set and vertices of degrees 2 or 4.*

Proof: Let I be an instance of the not-all-equal satisfiability problem. Construct the instance I' of the minimum edge deletion bipartition problem as in Theorem 4. Make two copies I'_1 and I'_2 of I, and join corresponding vertices of I'_1 and I'_2 that have degree 3 by an edge, to obtain I''. Finally obtain the instance J of the maximum gap graph labeling problem by replacing each edge xt of I'' by a path $xyzt$.

Suppose I has a solution that assigns +s and −s. This corresponds to removing an edge xt from each triangle in I' to obtain an optimal solution in Theorem 4. This in turn corresponds to considering the 9-cycle containing a path $xyzt$ and replacing the two edges xy and yz by the edge xz in J, for the occurrences of xt in both I'_1 and I'_2. We have thus replaced for each triangle in I' two degree 2 vertices by two degree 0 vertices to obtain G. By the symmetry between I'_1 and I'_2, this gives the same number of degree 2 vertices with − and with +, and the same number of degree 4 vertices with − and with +, which gives a maximum stretched bipartite solution with the given number of degree 2 and degree 4 vertices. If instead of replacing a degree 2 by a degree 0 in the 9-cycle we replace a degree 4 by a degree 2 in the 9-cycle, we obtain a solution of smaller value for J. If we do any other additional such substitution, we also obtain a solution of smaller value for J. Thus the solution of maximum possible value for J is obtained in this form. Furthermore, if I has no solution then additional substitutions will be needed, thus reducing the value. Therefore the optimum solution as in Theorem 3 for J will reach the maximum value only if we do one modification per 9-cycle, which happens only if I has a solution. ■

4 Approximation Algorithms

The first result of this section gives a lower bound on the gap.

Theorem 6 *Let G be a graph with $n+1$ vertices and m edges, and consider the uniform base set. If G is bipartite and is assigned signs S, then the expected gap over a random permutation of $+$ vertices and a random permutation of $-$ vertices in a bipartite labeling satisfying S is $m(n+1)/2$. Thus the maximum bipartite labeling of Theorem 2 guarantees a factor of $1/2$ from the upper bound mn in deterministic time $O(m+n\log n)$, for bipartite graphs. If G is arbitrary, then we may assign signs S so that at least $m/2$ edges have endpoints of different signs, in deterministic time $O(m)$. We thus obtain for general graphs an approximation of $1/4$ in deterministic $O(m+n\log n)$ time. If G is arbitrary, then a uniformly random assignment of labels to vertices has expected gap $m(n+2)/3$, giving in $O(m)$ randomized time an approximation of $1/3$ from the upper bound mn.*

Proof: Suppose G is bipartite and the vertices have been assigned signs, say s signs $-$ and t signs $+$, with $s+t=n+1$. If we permute the $-$ vertices at random, with labels from 0 to $s-1$, the expected value of a $-$ vertex x will be $(s-1)/2$. Similarly, if we permute the $+$ vertices at random, with labels from s to $s+t-1$, the expected value of a $+$ vertex y will be $s+(t-1)/2$. Thus the expected value for the edge xy will be $s+(t-1)/2-(s-1)/2=(s+t)/2=(n+1)/2$, for a total of $m(n+1)/2$ over all the edges, at least $1/2$ of the upper bound mn.

If G is arbitrary, we may consider the vertices x of G one at a time and assign them signs $-$ or $+$, depending on whether the majority of the neighbors of x assigned earlier have signs $+$ or $-$ respectively. Thus at least half of the edges will join vertices with different signs. This takes time $O(m)$, and after removing the edges with same sign endpoints we obtained a bipartite graph with at least $m/2$ edges, giving by the bipartite result a bound $m(n+1)/4$, at least $1/4$ of the upper bound mn.

If G is arbitrary, permute the labels $0,1,\ldots,n$ at random among the $n+1$ vertices. An edge xy can be assigned $\binom{n+1}{2}$ possible pairs of values, and a pair of gap i will occur with probability $(n-i+1)/\binom{n+1}{2}$. The expected gap of an edge will thus be

$$\sum_{1\le i\le n} i(n-i+1)/\binom{n+1}{2}$$

$$=(n\sum_{1\le i\le n} i-2\sum_{1\le i\le n}\binom{i}{2})/\binom{n+1}{2}$$

$$=(n\binom{n+1}{2}-2\binom{n+1}{3})/\binom{n+1}{2}$$

$$=(n+2)/3,$$

for a total of $m(n+2)/3$, giving in $O(m)$ time the $1/3$ approximation with respect to the upper bound mn. ■

A weaker bound is now obtained when the base set is not necessarily uniform.

Theorem 7 *The maximum gap labeling problem on arbitrary graphs G and arbitrary base sets B has an expected factor $1/16$ randomized $O(m+n\log n)$ time algorithm.*

PROOF: The algorithm assigns to each vertex a $+$ or a $-$ at random with probability $1/2$, then removes edges whose endpoints have the same sign, then finds a bipartite stretched labeling G' by Theorem 2 on the resulting graph.

Consider the optimal solution on $G = G_0$, and consider the signed bipartite graph G_r obtain in Theorem 3. Now change the signs of the vertices in G_r to the signs obtained at random above in G', while keeping the labels as assigned in G_r, then repeatedly exchange $+$ and $-$ vertices until all $-$ vertices precede the $+$ vertices, to obtain G''. The sum of the differences of $+$ and $-$ vertices in G' is at least as large as that sum for the labels in G'' is no larger than the sum in the stretched labelling G' which only requires further exchanging $+$ vertices or $-$ vertices.

It remains to observe how much of the sum of differences of $+$ and $-$ vertices in G_r is preserved in G''. First of all, the signs for such a difference are preserved with probability $(1/2)(1/2) = 1/4$, as signs are chosen at random with probability $1/2$. Let R be the set of vertices that have the same sign in G_r as in the signed solution. Let xy be such an edge in G_r that keeps signs $-+$. If xy is an edge of $G = G_0$, it will get at least the gap in G_r, giving an expected factor of $1/4$ for this edge xy. Otherwise xy corresponds to a path $xz \cdots ty$. If $z \neq t$, then z will get $+$ and t will get $-$ in the algorithm with probability $(1/2)(1/2) = 1/4$, for a total probability $(1/4)(1/4) = 1/16$, and in this case we have in G'' that $\ell(z) - \ell(x) + \ell(y) - \ell(t) \geq \ell(y) - \ell(x)$ as desired, as t precedes z in the ordering by signs in G''. Finally if $z = t$, then the path is xzt, and we examine what happens before we do the exchanges of $-$ and $+$ to obtain G'' We have a gap $\max(\ell(z) - \ell(x), \ell(y) - \ell(z)) \geq (\ell(y) - \ell(x))/2$ with probability at least $1/2$, since this is the probability of xz being $-+$ and of zy being $-+$. This gives expected gap at least $(1/4)(1/2)(1/2) = 1/16$ of the gap in G_r. ■

5 Exact Solution with Bounded Degree and Treewidth

Given a graph G, a *tree decomposition* of G is a tree T with subtrees T_v assigned to the vertices $v \in V(G)$ such that if $vw \in E(G)$ then T_v and T_w have at least one vertex in common. The *treewidth* of G in this decomposition is the least k such that at most $k+1$ trees T_v meet at a given vertex of T.

Theorem 8 *Let c, d, k be constants. Then the maximum gap labeling problem for graphs G with a given tree decomposition and with arbitrary base sets B can be solved exactly in polynomial time $O(n^{O(c+d+k)})$ over graphs with at most c vertices of degree greater than d and with treewidth at most k.*

PROOF: Consider a solution as in Theorem 3 giving $G = G_0$ and G_r signs $-$ and $+$. All that matters to evaluate such a G_r is the number of vertices of each degree with $-$ and with $+$ signs, so we can order the vertices by degree in the solution as in Theorem 3. The degrees of the c vertices of arbitrary degrees have n possible values and such vertices have two possible signs, giving for such vertices $n^{O(c)}$ possibilities. The remaining vertices have degree at most d, and there are n possible values for the number of vertices of each sign and each degree at most d, giving for such vertices $n^{O(d)}$. Thus the total number of final cases to be evaluated is $n^{O(c+d)}$, thus polynomial.

It remains to do a determination of all possible such cases by looking at the tree decomposition T of treewidth k We may traverse the tree decomposition T from the leaves to the roots, while keeping at each vertex v of T the information about the subtree below g of degrees and signs, which will still be bounded by $n^{O(c+d)}$ as calculated above. We must also keep for the at most $k+1$ vertices at g of the number of signs at the endpoints of edges coming out to the graph above g in T. If we traverse the children of g from left to right, we will keep track for the vertices corresponding to g of how many neighbors of these vertices in G_r have been encountered with which signs, giving at most $n^{O(k)}$ choices. The total number of choices to keep track of is thus $n^{O(c+d+k)}$, hence polynomial in n for c, d, k constant. ■

6 The Case of Trees

Trees have treewidth 1, so by Theorem 8 they can be solved exactly in polynomial time if they have at most c vertices of degree greater than d for constants c, d.

Theorem 9 *Let T be a tree. If T is a path or a star, then the maximum gap labeling with arbitrary base set B is bipartite. On the other hand, if T is obtained by joining two stars of the same size at a leaf, then the maximum bipartite labeling has gap $(n^2/2)+n-1$ while the maximum general labeling has gap $(3n^2/4)-(n/2)$, with uniform base set B.*

PROOF: Suppose T is a tree. If a maximum labeling has three consecutive vertices u, v, w in the path with $\ell(u) < \ell(v) < \ell(w)$, then exchanging the labels of v and w can only increase the gap for the path. Thus the optimal labeling is alternating and by Theorem 1 stretched bipartite.

Suppose T is a star, the root has been assigned label ℓ, and the leafs attached at the root have been assigned s labels $\ell'_i < \ell$, ℓ'_0 being the smallest, and t labels $\ell''_j > \ell$, ℓ''_0 being the largest. If we exchange ℓ with ℓ'_0 we lose at most $(\ell-\ell'_0)(s-1)$ and gain at least $(\ell-\ell'_0)t$. If we exchange ℓ with ℓ''_0 we lose at most $(\ell''_0-\ell)(t-1)$ and gain at least $(\ell''_0-\ell)s$. If we do not gain in either case then $(\ell-\ell'_0)(s-1-t) \geq 0$ and $(\ell''_0-\ell)(t-1-s) \geq 0$, which is not possible.

Suppose n is even and T is obtained by joining two stars T_1 and T_2 with $n/2$ leaves at a leaf. The maximum bipartite labeling assigns 0 to the root of T_1, 1 to the root of T_2, n to the common leaf, and $2, 3, \ldots, n-1$ to the remaining leaves. The resulting gap is easily calculated as $(n^2/2)+n-1$. A different labelling assigns 0 to the root of T_1, n to the root of T_2, $n/2$ to the common leaf, $(n/2)+1, \ldots, n-1$ to the remaining leaves of T_1, and $1, \ldots, (n/2)-1$ to the remaining leaves of T_2. The resulting gap is easily calculated as $(3n^2/4)-(n/2)$. ■

Thus bipartite labelings for uniform base sets cannot guarantee an approximation factor better than 2/3. We show how to achieve such a bound. A labeling of a tree is *graceful* if the base set is uniform, and the gaps of the edges are the distinct integers $1, 2, \ldots, n$. The *graceful tree conjecture* states that every tree is graceful and remains open [1]. Notice that graceful labelings have gap $\binom{n+1}{2}$, which we call the *graceful gap*.

Theorem 10 *Let T be a tree with uniform base set B. Then the expected gap of a bipartite labeling of T is the graceful gap $\binom{n+1}{2}$. The maximum possible gap is $3n^2/4$ for n even and*

$(3n^2+1)/4$ *for* n *odd. Thus the algorithm of Theorem 2 achieves an approximation of* $2/3$ *for trees with uniform base set in* $O(m + n \log n)$ *time.*

PROOF: The first result follows by Theorem 6. For the second result, after labeling, root the tree at the vertex with label n, and orient all edges towards the root. The edge coming out of labels between 0 and $n/2$ reach the maximum by being joined to n, while the edges coming out of labels between $n/2$ and $n-1$ reach the maximum by being joined to 0. This gives the maximum gap as $\sum_{0 \le i \le n/2} n - i + \sum_{n/2 < i < n} i$, which is $3n^2/4$ for n even and $(3n^2+1)/4$ for n odd. As the algorithm achieves at least graceful gap $(n^2+n)/2$, the bound of $2/3$ follows. ■

The graceful gap itself cannot always be achieved by bipartite labelings. Let T be a tree with root v and k paths vx_iy_i coming out of v. If we seek a bipartite labeling, we must assign $k+1, \ldots, 2k$ to the x_i and assign $0, 1, \ldots, k$ to v and the y_i, so the gap is only affected by the choice of $0 \le z \le k$ assigned to v. The gap is then easily calculated to be the graceful gap plus $(k-1)((k/2)-z)$. Since $z = k/2$ can only be achieved for k even, it follows that the graceful gap can be achieved if and only if $k = 1$ or k is even by bipartite labelings. We note that of these only $k = 1, 2$ actually give a graceful bipartite labeling.

A caterpillar is a tree whose vertices of degree at least 2 form a path. It remains open to find a polynomial time algorithm that computes the maximum gap labeling for caterpillars. As mentioned at the beginning of this section, we know how to do this only if the caterpillar has a bounded number of vertices of degrees exceeding some given bound.

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