On Monochromatic Paths and Rainbow Triangles in Tournaments

Tomás Feder* Carlos Subi

Abstract

It has been conjectured that any tournament whose edges are colored with three colors has either a directed 3-cycle whose edges use all three distinct colors or a vertex from which every other vertex can be reached by a monochromatic path. We prove results related to this conjecture.

1 Introduction

Let T be a tournament whose edges are colored with r colors. A *rainbow triangle* in T is a directed 3-cycle whose edges use three distinct colors. A *monochromatic path* in T is a directed path whose edges all have the same color. It has been conjectured that any such tournament T whose edges are colored with $r = 3$ colors has either a rainbow triangle or a vertex v from which every other vertex w can be reached by a monochromatic path [1, 2, 3].

In this paper we prove results related to this conjecture. The results are special cases if $r = 3$, but we also consider greater values of r.

2 Minimal Counter Examples

A *minimal counter example* to the conjecture is a tournament T whose edges are colored with r colors that does not have a vertex v from which every other vertex w can be reached by a monochromatic path, such that every subtournament T' of T has such a vertex v. We consider minimal counter examples for any number r of colors, instead of just $r = 3$. The following can be inferred from [4].

Theorem 1 *If T is a minimal counter example for any integer $r \geq 1$, then T contains a Hamiltonian cycle $0, 1, 2, \ldots, n-1, 0$ such that for every pair of vertices $0 \leq i, j < n$ in T, either $j = i - 1$ modulo n and j cannot be reached from i by a monochromatic path, or there exists a monochromatic path from i to j.*

Proof. If we remove vertex i from T to obtain $T - i$, then there exists a vertex $\pi(i)$ in $T - i$ from which every vertex in $T - i$ can be reached by a monochromatic path, by minimality. So the only vertex in T that cannot be reached from $\pi(i)$ is i by a monochromatic path. The permutation π decomposes into cycles, and in fact it must decompose into a single cycle, since otherwise the vertices of a cycle of π would form a smaller counter example. The Hamilonian cycle thus has $\pi(0) = n - 1, \pi(n-1) = n-2, \ldots, \pi(1) = 0$. □

*268 Waverley Street, Palo Alto, CA 94301. Email: tomas@theory.stanford.edu

3 The Need for a Third Color

The following follows as in [4].

Theorem 2 *Consider a minimal counter example for any r as in Theorem 1. Then we may assume* 01 *is red,* 12 *is blue, and there is a yellow path from* 2 *to* 0 *that does not go through* 1.

Proof. If all the edges of the cycle $0, 1, \ldots, n-1, 0$ are blue, we contradict the fact that there is no path from 1 to 0. So there must be at least two colors in this cycle, say 01 is red and 12 is blue. The path from 2 to 0 cannot be blue as there would be a blue path from 1 to 0, and cannot be red as there would be a red path from 2 to 1, so the path from 2 to 0 must be yellow, and cannot go through 1, otherwise there would be a yellow path from 2 to 1 and from 1 to 0. $\square$

4 On the Yellow Path

The following is a new result.

Theorem 3 *Let T be a tournament as in Theorem 1, for any integer r. Then the yellow path from* 2 *to* 0 *as in Theorem 2 cannot go through all the vertices* $3 \leq i < n$.

Proof. Suppose the yellow path goes in order $2, 3, 4, \ldots, n-1, 0$. Then the edges $1i$ are oriented as $1i$ for $i = 2$ and is $i1$ for $i = 0$, so there must be a $2 \leq i < n$ so that we have orientations $1i$ and $(i+1)1$, giving a directed triangle $1, i, i+1$. If $1i$ is yellow, then we have a yellow path from 1 to 0, and if $(i+1)1$ is yellow, then we have a yellow path from 2 to 1, so these two edges are not yellow, and must be the same color (say green, which could be red or blue), otherwise $1, i, i+1$ is a rainbow triangle. But then we have a green path from $i+1$ to i, contrary to assumption.

If the yellow path visits all vertices $3 \leq i < n$ not in order, then it must visit $i+1$ before it visits i for some i, so there is a yellow path from $i+1$ to i, contrary to assumption. $\square$

5 Orientation of Edges

We prove that a certain orientation cannot occur in minimal counter examples.

Theorem 4 *Let T be a minimal counterexample as in Theorem 1, for any r. Then it cannot be that all edges incident to* 0 *other than* 01 *are oriented* $i0$ *and all edges incident to* 1 *other than* 01 *are oriented* $1i$.

Proof. Consider the directed triangles $0, 1, i$, and suppose 01 is red. The edges $1i$ and $i0$ cannot be blue and yellow, or we would have rainbow triangle. They cannot also be the same color or else we would have a monochromatic path from 1 to 0. So one is red and the other one is say blue Say for some $i \geq 2$ we have $i0$ blue and $1i$ red. Then $(i+1)0$ cannot be red or we would have a red path from $i+1$ to i. So $1(i+1)$ is red. But if 20 is red then there is a red path from 2 to 1, and if $1(n-1)$ is red then there is a red path from 0 to $n-1$, a contradiction. $\square$

6 The Conjecture Holds for $n \leq 6$

.

The following counter example was known in [4], but its uniqueness was not known.

Theorem 5 *The conjecture holds for $n \leq 5$ and any r, with the exception of a single counter example and its reverse for $n = 5$ with $r = 5$ colors.*

Proof. By Theorem 1 we must have $n \geq 3$ in a counter example, and by Theorems 2 and 3 a minimal counter example for $n = 3, 4$ would have a rainbow triangle $0, 1, 2$, so assume $n = 5$. Thus by Theorems 2 and 3 we may assume 01 is red and 12 is blue, and the yellow path from 2 to 0 goes through 3 or 4. Say by symmetry of reversing edges it goes through 3. Then 23 is yellow and 30 is yellow. Then there is a green (possibly green is red) path from 3 to 1 going through 4, so 34 is green and 41 is green. Then there is a pink (possibly an earlier color) path from 4 to 2 going through 0, so 40 is pink and 02 is pink. Then there must be a red path from 0 to 3, so 13 is red. Then there must be a blue path from 1 to 4, so 24 is blue.

This completes the counter example. If green is red, then there is a red path 0134 from 0 to 4, contrary to assumption. By symmetry, a similar blue path 1240 occurs if pink is blue, and if pink is red we have a red path 4013, contrary to assumption. Thus the counter example for $n = 5$ has exactly $r = 5$ colors, each color used twice, plus the counter example obtained by reversing the edges. Thus the edges of the same color are in one case $(i, i + 1)$ and $(i + 1, i + 3)$, modulo 5, and in the other case $(i, i+2)$ and $(i+2, i+3)$, modulo 5, so in either case each color occurs twice. □

We have been able to verify the conjecture for $n = 6$ with $r = 3$ by a computer calculation that took about 15 minutes, just assuming a Hamiltionian cycle with two consecutive edges of different chosen colors as in Theorems 1 and 2.

By a computer calculation assuming Theorems 1, 2, and 3, we also determined that there are exactly two minimal counter examples to the conjecture for $n = 6$ with $r = 4$. In the two minimal counter examples, edges of color 1 are $12, 24, 45$, edges of color 2 are $13, 61, 34$, edges of color 3 are $23, 52, 36, 56$, edges of color 4 are $51, 62, 35, 46$, and edge 41 has color 3 or 4 for the two minimal counter examples.

Notice that these two minimal counter examples have a symmetry invariant, namely reverse the edges, map vertex i to vertex $5 - i$ modulo 6, and exchange colors 1 and 2. There are also minimal counter examples for $n = 6$ with $r = 5, 6, 7$, obtained by doing any number of the following three changes: (1) change 41 to color 5, (2) change 56 to color 6, (3) change 35 and 51 to color 7. We have sketched a proof indicating that there are no minimal counter examples for $n = 6$ with $r \geq 8$.

7 Sufficient Obstructions for any r

The following is known from [4].

Theorem 6 *Any case where the conjecture does not hold for any r must have a triangle xyz with edges oriented xy, yz, xz using all three different colors.*

Proof. Consider the situation of Theorem 2, and consider the yellow path $2 = v_0, v_1, \ldots, v_k = 0$. If we consider the edges $1v_i$, none of them can be yellow, because if $1v_i$ is yellow we have a yellow path from 1 to 0, and if v_i1 is yellow we have a yellow path from 2 to 1. So all $1v_i$, oriented in either direction, are not But since $1v_0$ is blue and $1v_k$ is red, we must have for some i that $1v_i$ and

$1v_{i+1}$ are two different colors other than yellow, and since the triangle is $1v_iv_{i+1}$ is not a rainbow triangle it must be of the form in the statement of the theorem. $\square$

We prove another such result.

Theorem 7 *Any case where the conjecture does not hold for any r as in Theorem 2, must have four vertices x, y, z, t and oriented edges as xy, yz blue, zx yellow, and zt, tx two different colors other than blue (up to naming of the colors). In particular, (1) some directed triangle (xyz) is not monochromatic, (2) some directed 4-cycle ($xyzt$) uses three colors with the repeated color used in consecutive edges (call this a rainbow square), and (3) some vertex v (x or z) has three edges, not all in the same direction with regards to v, that use three different colors.*

Proof. We consider the vertices v_i in the path as in Theorem 6, and shall insist that y be the vertex 1. We then proceed by induction on the number of yellow edges. Since the edges are oriented $1v_0$ and v_k1, there must be an i so that the edges are oriented $1v_i$ and $v_{i+1}1$. These two edges are not yellow as in Theorem 6, and they do not use two different colors, otherwise $1v_iv_{i+1}$ would be a rainbow triangle. Therefore these two edges are both the same color, say both blue. We then have $y = 1$, $x = v_{i+1}$, $z = v_i$. Now if we change the edge zx yellow to xz blue, we do not create any new monochromatic path, since the path xyz is already blue, but we reduce the number of yellow edges and proceed by induction, unless we then get a rainbow triangle xzt, in which case zt, tx are two different colors other than blue. $\square$

8 A Result one r-edge-colored cliques

Theorem 8 *Let $G = K_n$ be an n-vertex clique whose edges are colored with r colors. Then either G contains a triangle using three different colors, or G has a monochromatic spanning tree.*

Proof. Suppose by induction the result holds on $G - v = K_{n-1}$, and $G - v$ has a yellow spanning tree. If some edge out of v is also yellow, we can add this edge to the spanning tree of $G - v$ and obtain a yellow spanning tree of G. Otherwise all edges out of v are not yellow. If they are all red, then we obtain with these edges a red spanning tree of G, respectively. In the remaining case one edge vu_1 is red and one edge vu_k is blue, with a yellow path $u_1, \dots, u_k$ in $G - v$. Since all vu_i are not yellow, it follows that for some i we have vu_i and vu_{i+1} different colors and not yellow, and u_iu_{i+1} yellow, giving the triangle vu_iu_{i+1} that uses all three colors. $\square$

References

[1] J. Bang-Jensen and G. Gutin, Paths, trees and cycles in tournaments (survey available online).

[2] M. Devos, B. Sands, N.W. Sauer, and R.E. Woodrow, Monochromatic reachability or rainbow triangles, Open Problem Garden, `http://garden.irmacs.sfu.ca/?q=op/monochromatic reachability vs rainbow triangles`

[3] B. Sands, N. Sauer, and R. Woodrow, On monochromatic paths in edge-coloured digraphs, J. Combin. Theory Ser. B 33 (1982), 271–275.

[4] M.G. Shen, On monochromatic paths in m-coloured tournaments, J. Combin. Theory Ser. B 45 (1988), 108–111.