On Quadrangles and Convex Figures in the Plane

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Abstract

It is well known that any triangle can circumscribe any convex figure in the plane, and any triangle can be inscribed in any convex figure in the plane. We show that if a parallelogram circumscribes a circle, then the parallelogram is a rhombus; furthermore, every rhombus circumscribes every convex figure in the plane. We also show that if a parallelogram is inscribed in a circle, then the parallelogram is a rectangle; it is only known that every square can be inscribed in every convex figure in the plane. We further show that if a quadrilateral can be inscribed in every isosceles triangle, then it is a trapezoid; it follows that the quadrilaterals that can be inscribed in the circle and in all triangles are the isosceles trapezoids. We obtain a converse by showing that every isosceles trapezoid can be inscribed in every convex figure that has an axis of symmetry that meets the figure in at least one smooth point of the figure.

1 Introduction

Say that a polygon P can circumscribe a convex figure C in the plane if some polygon P' similar to P up to translation, rotation, and scaling, is such that P' contains C and every face of P' has a point in the boundary of C.

Similarly, say that a polygon P can be inscribed in a convex figure C in the plane if some polygon P' similar to P up to translation, rotation, and scaling, is such that P' is contained contains C and every vertex of P' is in the boundary of C.

It is well known that every triangle can circumscribe and be inscribed in every convex figure in the plane. We consider polygons that are parallelograms instead, and show that such polygons can circumscribe or be inscribed in every convex figure in the plane only if they are rhombuses or rectangles respectively. We show that indeed all rhombuses circumscribe every convex figure in the plane, but it is only known that squares can be inscribed in every convex figure in the plane. Finally, we show that if a quadrilateral can be inscribed in every isosceles triangle, then at least two opposite sides of the quadrilateral must be parallel, and so the quadrilateral is a trapezoid. This leaves isoceles trapezoids as the only possible quadrilaterals that can be inscribed both in the circle and in all isosceles triangles, and we show that such isosceles trapezoids can be inscribed in all convex figures having an axis of symmetry that meets the figure in at least one smooth corner.

2 Circumbscribing Rhombuses

We begin with the circle.

Theorem 2.1 *A parallelogram can circumscribe the circle if and only if it is a rhombus.*

Proof. A circumscribed parallelogram is tangent to the circle at four points p, q, p', q' in the four faces of the parallelogram, so p, p' must be diametrically opposite in the circle, and q, q' must also be diametrically opposite in the circle. Thus the parallelogram is a rhombus. Leaving p, p' fixed and rotating q, q' gives all rhombuses. □

We proceed to general convex figures.

Theorem 2.2 *A parallelogram can circumscribe every convex figure in the plane if and only if it is a rhombus.*

Proof. The only if part follows from Theorem 2.1. For the if part, Consider a rhombus R and a convex figure C. Let p, q be two points on C that are farthest apart, and let P, Q be parallel lines going through p, q respectively, with P, Q perpendicular to the line pq. If α is an angle of R, rotate P, Q clockwise by an angle α to obtain parallel lines P', Q' tangent to C. Similarly, rotate P, Q counter clockwise by an angle α to obtain parallel lines P'', Q''. The parallelograms $R' = P', P, Q', Q$ and $R'' = P, P'', Q, Q'$ will be rhombuses if and only if the distances satisfy $d(P', Q') = d(P, Q)$ and $d(P, Q) = d(P'', Q'')$ respectively, where $d(P, Q) = d(p, q)$. All we know is $d(P', Q') \leq d(P, Q)$ and $d(P, Q) \geq d(P'', Q'')$. If we proceed counter clockwise from R' to R'' gradually, we will by the mean value theorem obtain in some intermediate direction a parallelogram $R_i = P'_i, P_i, Q'_i, Q_i$ with $d(P'_i, Q'_i) = d(P_i, Q_i)$. Thus R_i is the desired rhombus. □

3 Inscribing Quadrilaterals

We begin with the circle.

Theorem 3.1 *A parallelogram can be inscribed in the circle if and only if it is a rectangle.*

Proof. As parallel opposite sides of the parallelogram have the same length and join two points in the circle, the parallelogram must be a rectangle. Any rectangle can be obtained by bringing opposite parallel equal length sides closer or farther. □

We continue with isosceles triangles.

Theorem 3.2 *If a quadrilateral can be inscribed in every isosceles triangle, then the quadrilateral has at least two opposite parallel sides and is thus a trapezoid. The only quadrilaterals that can be inscribed in the circle and in all isosceles triangles are the isosceles trapezoids, of the form* $(0, -1), (0, 1), (x, y), (x, -y)$*, and all these can be inscribed in circles and in all triangles.*

Proof. Consider an isosceles triangle T for which the angle that occurs twice is an arbitrary small angle α. As two vertices of the quadrilateral Q must be on the same side of the triangle T, it follows that either two opposite sides of Q are at an angle at most α, or two consecutive sides of Q form an angle at least $\pi - 2\alpha$. Letting α tend to zero shows that at least two opposite sides of Q must be parallel.

The only quadrilaterals with two parallel opposite sides that can be inscribed in the circle are those of the form $(0, -1), (0, 1), (x, y), (x, -y)$, and all these can be inscribed in the circle. The rectangle $(0, -1), (0, 1), (x, 1), (x, -1)$ can be inscribed in every triangle with the line $(x, 1), (x, -1)$ contained in the longest side of the triangle. This inscribes the quadrilaterals $(0, -1), (0, 1), (x, y), (x, -y)$ with $0 < y \leq 1$ as well in every triangle. □

We prove a more general converse.

Theorem 3.3 *The quadrilaterals that can be inscribed in all convex figures that have an axis of symmetry meeting the figure in at least one smooth corner are precisely the isosceles trapezoids.*

Proof. That only the isoceles trapezoids can be inscribed in all such convex figures was shown in the preceeding theorem. Let F is a convex figure with axis of symmetry s meeting F at a smooth corner p, whose only tangent t is perpendicular to s. Consider an isosceles trapezoid $(0,-1),(0,1),(x,y),(x,-y)$ with $0 < y \leq 1$. We may insert the short parallel side $(x,y),(x,-y)$ parallel to t and perpendicular to s, and very close to t. This will give us sides with corresponding proportional lengths for $(0,1),(x,y)$ and $(x,-y),(0,-1)$ giving an inscribed quadrilateral with the angles at (x,y) and $(x,-y)$ almost flat, and flatter the closer the short parallel side $(x,y),(x,-y)$ is chosen to t. If we gradually move the short parallel side away from t, then the angles at the points for $(x,y),(x,-y)$ will vary until a right angle is achieved. Thus any angle at $(x,y),(x,-y)$ is achievable, inscribing the desired isosceles trapezoid. □

We proceed to general convex figures, and only repeat the proof of the following known result [2, 3, 4, 5].

Theorem 3.4 *A square can be inscribed in every convex figure in the plane.*

Proof. For any direction D, we may find the two parallel lines P,Q tangent to the convex figure C on opposite sides of C in direction D. We may gradually move P,Q closer, thus obtaining P_i,Q_i so that if P_i meets the boundary of C at p_i,p'_i and Q_i meets the boundary of C at q_i,q'_i in the same direction, then $d(p_i,p'_i) = d(q_i,q'_i)$. Initially $d(p_i,p'_i) = d(q_i,q'_i) = 0$, and in the end $d(p_i,q_i) = d(p'_i,q'_i) = 0$, so in some intermediate i we have by the mean value theorem. $d(p_i,p'_i) = d(q_i,q'_i) = d(p_i,q_i) = d(p'_i,q'_i)$. At this intermediate value, $R(D) = R_i = (p_i,p'_i,q'_i,q_i)$ is a rhombus. If the angle $p_ip'_iq'_i$ is straight then $R(D)$ is the desired square. Say the angle $p_ip'_iq'_i$ is acute. Then the angle $p'_iq'_iq_i$ is obstuse, giving an obstuse angle $p_ip'_iq'_i$ for $R(D')$ with the direction D' given by p_iq_i. Thus every direction D gives an $R(D)$ with angle acute or obstuse, some direction is acute, and some direction is obstuse. Furthermore, the set A of acute directions is closed, as a sequence of rhombuses has at least one limit, and this limit is a rhombus as well. Similarly, the set O of obstuse directions is closed. Thus the set of all directions is the union of two closed sets A and O, so A and O intersect and contain a common direction D. The direction D gives the desired square $R(D)$. □

4 Conclusion

We have shown that rhombuses and rectangles respectively are the only possible parallelograms that can circumscribe or be inscribed in any convex figure in the plane, with the circle providing the negative example for other parallelograms. The rhombuses can indeed circumscribe every convex figure, while only the square was known to inscribe in every convex figure. We have also shown that quadrilaterals that inscribe in every isosceles triangle have two opposite parallel sides. This characterizes the quadrilaterals that can be inscribed in the circle and in every triangle as the isosceles trapezoids, which can be inscribed in every convex figure having an axis of symmetry that meets the figure at at least one smooth corner.

If we consider a closed curve in the plane that does not necessarily define a convex figure, it is known that every triangle has a similar triangle, up to translation, rotation, and scaling, with all three vertices on the closed curve; it has been conjectured that this property of closed curves still holds when the square, instead of triangles, is considered, so that every closed curve would have four points forming a square in some orientation [1].

References

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