Steiner Triple System and Near 2-Factor Decompositions

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Abstract

A Steiner triple system $STS(n)$ is a set of triples whose elements are in $[n] = \{1, 2, \ldots, n\}$, $n \geq 1$, such that $|A \cap B| \leq 1$ for distinct triples A, B in the system, and if $i, j \in [n]$, then $i, j \in C$ for some triple C in the system.

It is known that $STS(n)$s exist if and only if $n \equiv 1, 3 \pmod 6$. It remains open to determine for which n all the triples contained in $[n]$ can be decomposed into some ℓ $STS(n)$s (for $n \geq 3$, we would have $\ell = n - 2$).

In this paper we show that such a decomposition does not exist for $n = 7$, but does exist for an infinite sequence

$$\ell = n - 2 = 7, 23, 31, 47, 49, 71, 73, 79, 89, 103, 127, 151, 161, 167, 191, 199, \ldots$$

which lists all numbers $2^{2m+1} - 1$ and their divisors. This consists of all numbers $\ell = n - 2$ with prime factors $8k - 1$, and some additional prime factors $8k + 1$.

We also formulate a question on decompositions into near 2-factors for each $K_n, n \geq 4$.

1 Introduction

The chromatic index C of a hypergraph is the minimum number of sets of disjoint edges (these sets can be viewed as colors) whose union contains all the edges. The degree of a vertex is the number of edges that contain the vertex, and we let D be the maximum degree over all vertices.

Clearly $C \geq D$. Pippenger and Spencer [6, 1] showed that if a collection of hypergraphs (1) is uniform (every edge contains exactly k vertices, for some fixed k), (2) has minimum degree asymptotic to the maximum degree, and (3) has maximum codegree (the number of edges containing a pair of verticess) asymptotically negligible compared with the maximum degree, then $C = D(1 + o(1))$.

In this note, we fix integers $r > s$, and define for each integer $n \geq r$ a hypergraph H_n whose vertices are the subsets of $[n]$ of size s, and whose edges correspond to subsets of $[n]$

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of size r. A vertex is in an edge if the corresponding subset of size s is contained in the corresponding subset of size r.

Clearly the degree is uniform $D = \binom{n-s}{r-s}$, and the maximum codegree is $\binom{n-s-1}{r-s-1} = \frac{r-s}{n-s}D = o(D)$, so $C = D(1+o(1))$ by the result in [6].

We focus on the case $s = r-1$. Then $C \geq D = n-r+1$. Many authors have observed that in this case $C \leq n$. Indeed, let the color of $e \subseteq [n]$ with $|e| = r$ be $\sum_{i \in e} i$ modulo n, so that if e, e' differ in a single switch i, i', then they will yield different sums.

The Walecki case [2] is the special case $r = 2, s = r-1 = 1$. In this case $D = n-1 \leq C \leq n$, and the question is to decompose the clique K_n into the least number of matchings. Clearly $C = n$ is tight for n odd, since a matching must miss one vertex i, which will need $n-1$ more matchings to finish matching i. Furthermore, $C = n-1$ matchings suffice for $n-1$ odd, so an nth vertex can be added to the unmatched vertex in each of the $n-1$ matchings, giving $n-1$ perfect matchings and $C_n = n-1$ for n even.

We further focus on the case $r = 3, s = r-1 = 2$. In this case $D = n-2 \leq C \leq n$. Each color is a set of triples that pairwise share at most one $i \in [n]$. If all pairs $\{i, j\}$ are covered by some triple in the color, then the set of triples is a Steiner triple system, which exists precisely when n is odd and $n-2$ is not divisible by 3, that is $n = 6k+1, 6k+3$.

If $C = n-2$, then since $\frac{n-2}{3}\binom{n}{2} = \binom{n}{3}$, each color must be a Steiner triple system, so $n-1 \leq C \leq n$ for $n = 6k, 6k+2, 6k+4, 6k+5$. Furthermore, if $C_n = n-2$, then $C_{n'} = n'-1$ for $n' = n-1$, and if $C_{n'} = n'-1$ for $n' = n-1$ even, then each color avoids a matching on $[n']$, with these matchings edge-disjoint, so $C_n = n-2$ by combining each matching with the additional vertex. For $n = 6k+5$, $C_n \neq n-2$, so $C_{n'} \neq n'-1$, and therefore $C = n$ for $n = 6k+4$.

We finally show that $C = n-1 = 6$ for $n = 6k+1 = 7$, and $C = n-2$ if 2 is of odd multiplicative order modulo $n-2$, that is for

$$\ell = n-2 = 7, 23, 31, 47, 49, 71, 73, 79, 89, 103, 127, 151, 161, 167, 191, 199, \ldots$$

which lists all numbers $2^{2m+1}-1$ and their divisors. This list is generated by taking all the primes of the form $8k-1$, and only some primes of the form $8k+1$, and then taking closure under multiplication.

Note that $C_3 = 1, C_4 = 4, C_5 = 5, C_6 = 6, C_7 = 6, C_8 = 7, C_9 = 7, C_{10} = 10, 10 \leq C_{11} \leq 11, 11 \leq C_{12} \leq 12, 11 \leq C_{13} \leq 13, 13 \leq C_{14} \leq 14, 13 \leq C_{15} \leq 15, C_{16} = 16$.

The approach in the proof of the main result leads to the following. A near-2-factor M on K_n is a 2-factor on $K_n - v$ for some vertex $v \in K_n$. A collection of near-2-factors M_i on $K_n - i$ for $1 \leq i \leq n$ is a decomposition if the pairs M_i, M_j for $i \neq j$ share exactly one edge $e \in K_n$, and every such e is the shared edge for exactly one such pair M_i, M_j. The near-2-factors are said near-hamiltonian if M is a hamiltonian cycle on $K_n - v$.

We ask whether decompositions into near-2-factors or near-hamiltonians exist for all $n \geq 4$. We prove near-hamiltonian decompositions for $4 \leq n \leq 31$ and for any prime $n = p \geq 5$. We say that a decomposition is oriented if we can orient the cycles in the near-2-factors, so that each shared edge e is oriented different for each pair of M_i, M_j. The near-hamiltonian decompositions described above are oriented except for $n = 6$ which does not have an oriented near-2-factor decomposition.

This question is motivated by the existence of a decomposition into edge-disjoint hamiltonian cycles for K_n, n odd, as by the conjecture of Seymour that any bridgeless graph can be decomposed into cycles, so that each edge is visited by exactly two cycles chosen. The work of Keevash settled the existence of large designs, and the existence of pairwise almost disjoint $STS(n)$s was obtained [5, 4].

2 The case $r = 3$ and $s = r - 1 = 2$

We settle the case $n = 7$.

Theorem 1 *We have $C_7 = 6$ (and thus $C_6 = 6$).*

PROOF: We first prove $C_7 \leq 6$ by giving six colors S_i for $0 \leq i \leq 5$ to all 35 triples on seven elements $v, 1, 2, 3, 1', 2', 3',$, where each of $1, 2, 3$ and $1', 2', 3'$ are treated modulo 3. Color S_0 contains five triples $121', 233', 12'3', v13, v1'2'$. Colors S_1, S_2 are obtained by adding 1 and 2 respectively to the elements other than v in S_0, containing five triples each. Color S_3 consists of seven triples of the forms $(1'23) + i, (v11') + i, 1'2'3'$, color S_4 consists of seven triples of the forms $(1'2'1) + i, (v13') + i, 123$, and color S_5 consists of six triples of the forms $(2'3'3) + i, (v31') + i$.

Suppose now $C_7 = 5$. Then the five colors S_i are five Steiner triple systems, so may choose one element u and each color S_i gives a matching M_i on the elements other than u which when combined with u yields the three triples in S_i containing u. We claim that the choice of the five matchings M_i is unique up to isomorphism. Indeed, M_1, M_2 form a 6-cycle R, and M_3 cannot join three opposite pairs of C or we would be left with two triangles of M_4. So M_3 must join one opposite pair and two distance two along R, and similarly for M_4, M_5.

This proves the uniqueness of the decomposition into M_i, so we may assume these M_i form the Walecki decomposition from the introduction. This decomposition has vertices -2,-1,0,1,2, modulo 5 and an extra 6th vertex v. Then M_0 is $(-2)2, (-1)1, 0v$, and four extra triples for S_0 that are either $(-2)(-1)0, 012, (-2)1v, (-1)2v$ or $(-2)(-1)v, v12, (-2)10, (-1)20$. The remaining S_i are obtained by adding i modulo 5, but adding or subtracting 1 or 2 from either choice creates a repeated triple, i.e., $(-2)(-1)0 + 2 = 012, (-2)1v + 1 = (-1)2v, (-2)(-1)v - 2 = v12, (-2)10 - 1 = (-1)20$. Therefore it is not possible to give disjoint such Steiner triple systems, proving $C_7 = 6$. ■

We show $C_n = n - 2$ for a large number of n.

Theorem 2 *We have $C_n = n - 2$ if 2 is of odd multiplicative order modulo $n - 2$, that is for*

$$\ell = n - 2 = 7, 23, 31, 47, 49, 71, 73, 79, 89, 103, 127, 151, 161, 167, 191, 199, \ldots$$

which lists all numbers $2^{2m+1} - 1$ and their divisors, $m \geq 0$.

PROOF: Since $\ell = n - 2$ divides $2^{2m+1} - 1$, it follows that $n - 2$ is not divisible by 2, 3. Consider the proof that $C_{n-2} \leq n - 2$ from the introduction. It has elements $1, 2, \ldots, n - 2$ as integers modulo $n - 2$, and color S_i has triples abc with $a + b + c = i$ (modulo $n - 2$). If we add two extra elements u, v to reach n elements, we must handle triples uva, uab, vab. We put uva in color S_i if $3a = i$, and put uab, vab in S_i if $2a + b = i, a + 2b = i$. The only problem occurs if $2a+b=2b+c=i$, in which case if uab is put in color S_i we must put vbc in color S_i (instead of ubc, which would conflict with uab).

Consider first the color S_0. Then $2a+b=0$ means $b=-2a$, $2b+c=0$ means $c=-2b=4a$, and so on. We thus consider the cyclic sequence modulo ℓ $a,-2a,4a,\ldots,a(-2)^{2m'+1}=-a,2a,-4a,\ldots,a(-2)^{2(2m'+1)}=a$ for a divisor $2m'+1$ of $2m+1$. Since this cycle is of even length, we can assign to S_0 alternatingly $ua(-2a),v(-2a)(4a),u(4a)(-8a),\ldots$.

Similarly, for color S_{2i}, we have the sequence $a+i,-2a+i,4a+i,\ldots$. We will have $(-a,2a)=(-2a+i,a+i)$ for $i=a$, but we choose independently of i the triple $u(a+i)(-2a+i)$ to start the alternation for i at $a+i$ in the same way we started the alternation for 0 at a, while the alternation for 0 after an odd number of steps gives $v(-a)(2a)$. This avoids the conflict between different i as well. Note that the consecutive differences in the above sequences is $3a,-6a,12a,\ldots,3a(-2)^i,\ldots$ which are all $2(2m'+1)$ different, and opposites $2m'+1$ apart, as required. ■

The above proof involves an oriented near-2–factor decomposition, namely $a+2b=i$ puts the edge ab in near-2-factor M_i, with the omitted a having $3a=i$.

We follow Hardy and Wright [3]. For a,p relatively prime, with p prime, recall $(\frac{a}{p})$ is 1 if a is a quadratic residue of p, and is -1 otherwise. Then $(\frac{a}{p})\equiv a^{\frac{1}{2}(p-1)}$. Furthermore $(\frac{2}{p})$ is 1 if $p=8k\pm 1$ and is -1 if $p=8k\pm 3$.

Theorem 3 *The set S of numbers ℓ such that 2 has odd multiplicative order modulo ℓ is generated by taking all the primes of the form $8k-1$, and additionally only some primes of the form $8k+1$, then taking closure under multiplication.*

Proof: If $\ell=p$ prime, then $2^{2r+1}\equiv 1 \pmod{p}$ for some r (since $\ell\in S$), so $b^2\equiv 2 \pmod{p}$ for $b=2^{r+1}$, therefore Furthermore $(\frac{2}{p})$ is 1 and $p=8k\pm 1$.

Of the form $p=8k+1$ we have e.g. $17\notin S$ and $73\in S$. For the form $p=8k-1$ we have in general $1=(\frac{2}{p})\equiv 2^{\frac{1}{2}(p-1)}\equiv 2^{4k-1} \pmod{p}$ so 2 is of odd order and $\ell=p\in S$.

If 2 has odd orders r,r' respectively modulo s,s' relatively prime, then 2 has odd order dividing rr' modulo ss'. It remains to prove the case $s=p^k$. Assume inductively that 2 has odd order r modulo $s'=p^{k-1}$. Then $2^{rp}\equiv (bp^{k-1}+1)^p\equiv 1$ (modulo p^k). ■

3 Near-2-Factor Decompositions

We examine near-2-factor decompositions.

Lemma 1 *A near-2-factor decomposition of K_n consists of n near-2-factors M_i with vertex i omitted in M_i.*

Proof: Since each edge is visited twice in the decomposition, we have $n(n-1)$ edge traversals in the decomposition, and each near-2-factor has $n-1$ edges, so the decomposition must consist of n near-2-factors.

The number of edges traversals incident to a vertex v is $2(n-1)$ since each edge is visited twice, and a near-2-factor makes two traversals of such edges if it visits v, so $n-1$ of the n near-2-factors visit v, and exacly one M_v does not visit v. ■

If K_n has a near-2-factor, then $n \geq 4$. If $n \neq 6$, then for $4 \leq n \leq 31$ there is an oriented near-hamiltonian decomposition, and for $n = p \geq 5$ prime there is an oriented near-hamiltonian decomposition.

Lemma 2 *For $n = 6$, there is a near-hamiltonian decomposition, but no oriented near-2-factor decomposition.*

PROOF: An oriented near-2-factor decomposition for $n = 6$ must be a near-hamiltonian decomposition (else we would have a cycle of length 2, which is not allowed in a near-2-factor of K_n). One of the near-hamiltonians must avoid v and traverse the cycle 12345. The near-hamiltonian M_i must traverse $(i+1)i$ for $1 \leq i \leq 5$ (modulo 5). The near-hamiltonian M_2 has possible cycles (1) $1324v$, (2) $3241v$, (3) $4132v$, (4) $5324v$, (5) $1325v$. In the edges with gap 2 along the perimeter in the 5-cycle of types (1),(2),(3), the direction is always $i(i+2)$, while for (4),(5) we have one $i(i+2)$ and one $i(i-2)$. Since all edges must be traversed in both directions, only (4),(5) can be used. Going along the perimeter, we must find two consecutive (4) of (5). For (4) we would have say $4213v$ in M_1, which would coincide in two edges $24, 4v$ with case (4) of M_2 obove. For (5) we would have say $5214v$ in M_1, which would coincide in two edges $25, 5v$ with case (5) of M_2 obove. Therefore none of the five possible cycles can be used, a contradiction.

If we do not care about orientation, we can use case (1) for each of the five added M_i, giving a near-hamiltonian decomposition. ■

We now describe three kinds of oriented near-2-factor decompositions. on K_n. If ρ is a permutation on $[n]$, and D is a near-2-factor decomposition, we say that D is ρ-invariant if for each near-2-factor M in D, the near-2-factor $\rho(M)$ obtained by applying ρ to each vertex in K_n is also in D.

In particular, (1) if K_n has vertices $0, 1, 2, \ldots, n-1$ and $\rho(i) = i+1$ modulo n, then a ρ-invariant D is said cyclic; (2) if K_n has vertices $v, 0, 1, 2, \ldots, n-2$ and $\rho(v) = v$, $\rho(i) = i+1$ modulo $n-1$, then a ρ-invariant D is said rotational; (3) If K_9 has vertices ij with $0 \leq i, j \leq 2$ and $\rho_1(ij) = (i+1)j$ modulo 3, $\rho_2(ij) = i(j+1)$ modulo 3, then a ρ_1, ρ_2-invariant D is said $Z_3 \times Z_3$ invariant.

Conjecture 1 *If $n \neq 6, n \geq 4$, then K_n has a near-hamiltioan decomposition. This decomposition is (1) cyclic if $n \neq 9$ is odd, (2) rotational is n is even, and (3) $Z_3 \times Z_3$ invariant if $n = 9$.*

We prove this conjecture for $n \leq 31$ or $n \geq 5$ prime. We verified by computer that $n = 9$ does not have a cyclic decomposition. For case (1), the subcases $n \geq 5$ prime are proven below in Theorem 4, so we are left with the cases $n = 15, 21, 25, 27$. For these cases, we give only one near-hamiltonian, and the remaining ones can be obtained by cyclic permutation. For $n = 15$, the near-hamiltion with differences $d = b - a$ between consecutive a, b is given by (1 (1) 2 (2) 4 (3) 7 (-1) 6 (6) 12 (-2) 10 (4) 14 (-5) 9 (-6) 3 (-7) 11 (-3) 8 (5) 13 (7) 5 (-4)). Here the differences must be all distinct or else two near-hamiltonians would share the same edge in the same direction. Furthermore, for differences $d, -d$ giving $a(d)a+d, a'+d(-d)a'$, the distances $a - a'$ must be all distinct, else two near-hamiltonians would share two edges. Here the distances are noted as $a - a'(d)$ given by 5 (1), 7 (2), 4 (3), 6 (4), 1 (5), 3 (6), 2

(7). The remaining cases of (1) are given in the appendix. The case (3) for $n = 9$ is similar with addition and subtraction in $Z_3 \times Z_3$ and the near-hamiltonian with differences and distances is given in the appendix. The case (2) always has near-hamiltonian (0 1 2 $\cdots$ n-2) not using v, and then a single additional near-hamiltonian up to rotations gives the result with corresponding differences and distances, with a v $a+1$ sequence in the near-hamiltonian giving distance 1 (v). This is given in the appendix.

We prove the following result.

Theorem 4 *Cyclic orientable near-hamiltonian decompositions exist for $n = p \geq 5$ prime, and cyclic orientable near-2-factor decompositions exist for n that has prime factors at least 5.*

PROOF: For $p \geq 5$ prime, take r generating the multiplicative group modulo p, and consider the near-hamiltonian cycles

$$C_i = (1 + i, r + i, r^2 + i, \ldots, r^{p-2} + i).$$

If we consider two cycles C_i, C_j, they will satisfy $ra - b = (r-1)i$, $rb - a = (r-1)j$ traversing an edge in opposite directions and in a unique pair.

If n has prime factors at least 5, then the near-2-factor M_i for $0 \leq i < n$ contains edges ab such that $2a - b = i$ (modulo n). Two near-2-factors M_i share by the equations $2a - b = i, 2b - a = j$ a unique edge in opposite directions, as the determinant 3 is coprime to n. ■

Theorem 5 *If K_n, K_m have orientable near-2-factor decompositions, then K_{nm} has an orientable near-2-factor decomposition as well.*

PROOF: A near 2-factor M_i over $K_m - i$ in an oriented decomposition can be viewed as a permutation ρ_i on $[m]$ with $\rho_i(i) = i$. Suppose similarly we have 2-factors N_j over $K_n - j$ in an oriented decomposition can beviewed as a permutation σ_j on $[n]$ with $\sigma_j(j) = j$. Then $\tau_{ij}(ab) = \rho_i(a)\sigma_j(b)$ gives near 2-factors on $K_{mn} - (ij)$. Given $\tau_{ij}(ab) = cd$, i, j can be recovered uniquely giving $\rho_i(a) = c$, $\sigma_j(b)$. If in addition $\rho_{i'}(c) = a$, $\sigma_{j'}(d) = b$, we have $\tau_{i'j'}(cd) = ab$ giving the reverse edge in the product. ■

Theorem 6 *Every number that is not of one of the forms $2m, 3m, 6m, 54m$ with m having prime factors at least 5, has an oriented near-2-factor decomposition.*

PROOF: It suffices that if $k \neq 2, 3, 6, 54$ has prime factors among 2,3, then K_k has an oriented near-2-factor decomposition, as the result then folllows form Theorems 4 and 5.

If $k = 2^i$, $i \geq 2$, then $k = 4^j$ or $k = 8 \times 4^j$. If $k = 3^i$, $i \geq 2$, then $k = 9^j$ or $k = 27 \times 9^j$. Then the result follows for Theorem 5 from the cases below 31. Otherwise we can handle the parts with factors 2,3 separately and then multiply, unless $k = 3 \times 2^i$, which is handled as $k = 12 \times 4^j$ or $k = 24 \times 4^j$; or $k = 2 \times 3^i$, which is hadled as $k = 18 \times 4^j$ or $k = 18 \times 27 \times 4^j$ except for $k = 54$. ■

Appendix

The $Z_3 \times Z_3$ oriented near-hamiltonian decomposition for 9 is (00 (1-1) 12 (10) 22 (0-1) 21 (11) 02 (-11) 20 (-10) 10 (01) 11 (-1-1)) with distances 10 (1-1), 01 (10), 11 (01), 1-1 (11).

For $5 \leq n \leq 31$, $n \neq 9$ odd we obtain a cyclic oriented near-hamiltonian decomposition given for n prime by Theorem 4, and for the remaining odd numbers as follows.

For 15, (1 (1) 2 (2) 4 (3) 7 (-1) 6 (6) 12 (-2) 10 (4) 14 (-5) 9 (-6) 3 (-7) 11 (-3) 8 (5) 13 (7) 5 (-4)) with distances 5 (1), 7 (2), 4 (3), 6 (4), 1 (5), 3 (6), 2 (7).

For 21, (1 (1) 2 (2) 4 (-1) 3 (3) 6 (4) 10 (-2) 8 (10) 18 (-3) 15 (-8) 7 (5) 12 (8) 20 (-6) 14 (-9) 5 (-7) 19 (-10) 9 (7) 16 (-5) 11 (6) 17 (-4) 13 (9)) with distances 2 (1), 6 (2), 9 (3), 7 (4), 4 (5), 3 (6), 10 (7), 5 (8), 8 (9), 1 (10).

For 25, (1 (1) 2 (2) 4 (-1) 3 (3) 6 (4) 10 (-5) 5 (6) 11 (11) 22 (-4) 18 (-2) 16 (8) 24 (-9) 15 (-6) 9 (12) 21 (-8) 13 (10) 23 (-3) 20 (-12) 8 (9) 17 (-10) 7 (7) 14 (5) 19 (-7) 12 (-11)) with distances 2 (1), 11 (2), 8 (3), 12 (4), 9 (5), 4 (6), 5 (7), 3 (8), 7 (9), 6 (10), 10 (11), 1 (12).

For 27, (1 (1) 2 (2) 4 (-1) 3 (3) 6 (4) 10 (-5) 5 (6) 11 (-3) 8 (13) 21 (-2) 19 (-10) 9 (9) 18 (5) 23 (-8) 15 (11) 26 (-9) 17 (7) 24 (10) 7 (-12) 22 (-6) 16 (-4) 12 (8) 20 (-7) 13 (12) 25 (-11) 14 (-13)) with distances 2 (1), 10 (2), 5 (3), 6 (4), 13 (5), 11 (6), 4 (7), 3 (8), 8 (9), 12 (10), 1 (11), 9 (12), 7 (13).

For $5 \leq n \leq 31$, $n \neq 6$ even we obtain a rotational oriented near-hamiltonian decomposition as follows.

For 4, (1 v 2 (-1)) with distances 1 (v).

For 8, (2 v 3 (2) 5 (-1) 4 (-3) 1 (-2) 6 (3)) with distances 1 (v), 3 (2), 2 (3).

For 10, (2 v 3 (4) 7 (3) 1 (-4) 6 (2) 8 (-3) 5 (-1) 4 (-2)) with distances 1 (v), 4 (2), 2 (3), 3 (4).

For 12, (1 v 2 (2) 4 (4) 8 (-1) 7 (-2) 5 (5) 10 (-4) 6 (-3) 3 (-5) 9 (3)) with distances 1 (v), 3 (2), 5 (3), 2 (4), 4 (5).

For 14, (1 v 2 (2) 4 (-5) 12 (-2) 10 (-3) 7 (-4) 3 (3) 6 (5) 11 (-6) 5 (4) 9 (-1) 8 (6)) with distances 1 (v), 5 (2), 4 (3), 2 (4), 6 (5), 3 (6).

For 16, (1 v 2 (2) 4 (3) 7 (6) 13 (-2) 11 (-1) 10 (-7) 3 (5) 8 (4) 12 (-3) 9 (-4) 5 (-6) 14 (7) 6 (-5)) with distances 1 (v), 6 (2), 5 (3), 3 (4), 2 (5), 7 (6), 4 (7).

For 18, (1 v 2 (2) 4 (-1) 3 (4) 7 (6) 13 (-4) 9 (-3) 6 (5) 11 (3) 14 (-2) 12 (-7) 5 (-6) 16 (-8) 8 (7) 15 (-5) 10 (8)) with distances 1 (v), 7 (2), 5 (3), 6 (4), 4 (5), 8 (6), 3 (7), 2 (8).

For 20, (1 v 2 (2) 4 (-1) 3 (3) 6 (-9) 16 (-2) 14 (-3) 11 (4) 15 (-5) 10 (7) 17 (-8) 9 (-4) 5 (8) 13 (-6) 7 (5) 12 (6) 18 (9) 8 (-7)) with distances 1 (v), 7 (2), 8 (3), 6 (4), 3 (5), 5 (6), 9 (7), 4 (8), 2 (9).

For 22, (1 v 2 (2) 4 (-1) 3 (3) 6 (4) 10 (6) 16 (-4) 12 (8) 20 (-6) 14 (-5) 9 (10) 19 (9) 7 (-10) 18 (-7) 11 (-3) 8 (7) 15 (-2) 13 (-8) 5 (-9) 17 (5)) with distances 1 (v), 10 (2), 5 (3), 6 (4), 8 (5), 4 (6), 3 (7), 7 (8), 2 (9), 9 (10).

For 24, (1 v 2 (2) 4 (-1) 3 (3) 6 (4) 10 (-5) 5 (9) 14 (6) 20 (11) 8 (7) 15 (-6) 9 (10) 19 (-2) 17 (-10) 7 (-8) 22 (-4) 18 (-7) 11 (5) 16 (-3) 13 (8) 21 (-9) 12 (-11)) with distances 1 (v), 8 (2), 10 (3), 11 (4), 6 (5), 5 (6), 3 (7), 9 (8), 7 (9), 2 (10), 4 (11).

For 26, (1 v 2 (2) 4 (-1) 3 (3) 6 (4) 10 (-5) 5 (6) 11 (11) 22 (-10) 12 (8) 20 (-2) 18 (-3) 15 (-7) 8 (9) 17 (-4) 13 (-6) 7 (-11) 21 (-12) 9 (10) 19 (5) 24 (-8) 16 (7) 23 (-9) 14 (12)) with distances 1 (v), 9 (2), 12 (3), 7 (4), 11 (5), 2 (6), 8 (7), 4 (8), 6 (9), 3 (10), 10 (11), 5 (12).

For 28, (1 v 2 (2) 4 (-1) 3 (3) 6 (4) 10 (-5) 5 (6) 11 (-3) 8 (-13) 22 (-7) 15 (-2) 13 (10) 23 (-6) 17 (9) 26 (-10) 16 (8) 24 (-4) 20 (-8) 12 (13) 25 (-11) 14 (5) 19 (-12) 7 (11) 18 (-9) 9 (12) 21 (7)) with distances 1 (v), 11 (2), 5 (3), 13 (4), 9 (5), 12 (6), 6 (7), 4 (8), 8 (9), 3 (10), 7 (11), 2 (12), 10 (13).

For 30, (1 v 2 (2) 4 (-1) 3 (3) 6 (4) 10 (-5) 5 (6) 11 (-3) 8 (8) 16 (10) 26 (12) 9 (-11) 27 (-9) 18 (-6) 12 (13) 25 (-8) 17 (5) 22 (-2) 20 (-7) 13 (11) 24 (-10) 14 (9) 23 (-4) 19 (-12) 7 (14) 21 (7) 28 (-13) 15 (-14)) with distances 1 (v), 11 (2), 5 (3), 13 (4), 12 (5), 7 (6), 8 (7), 9 (8), 4 (9), 2 (10), 14 (11), 10 (12), 3 (13), 6 (14).

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