Obstructions to Partitions of Chordal Graphs

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Abstract

Matrix partition problems generalize graph colouring and homomorphism problems, and occur frequently in the study of perfect graphs. It is difficult to decide, even for a small matrix M, whether the M-partition problem is polynomial time solvable or NP-complete (or possibly neither), and whether M-partitionable graphs can be characterized by a finite set of forbidden induced subgraphs (or perhaps by some other first order condition). We discuss these problems for the class of chordal graphs. In particular, we classify all small matrices M according to whether M-partitionable graphs have finitely or infinitely many minimal chordal obstructions (for all matrices of size less than four), and whether they admit a polynomial time recognition algorithm or are NP-complete (for all matrices of size less than five). We also suggest questions about larger matrices.

1 Background

Let M be an m by m symmetric matrix over $0, 1, *$. An *M-partition* of a graph G is a partition $V_1, V_2, \ldots, V_m$ of $V(G)$ such that two distinct vertices in (possibly equal) parts V_i and V_j are adjacent if $M(i, j) = 1$, and nonadjacent if $M(i, j) = 0$; the entry $M(i, j) = *$ signifies no restriction. Since we admit $i = j$, a set V_i is independent if $M(i, i) = 0$, and a clique if $M(i, i) = 1$; as above, $M(i, i) = *$ means there is no internal restriction on V_i. (Below we sometimes refer to V_i as *the i-th part.*) The *M-partition problem* asks whether or not an input graph G admits an M-partition [12, 24, 25]. We will also discuss variants of this basic M-partition problem. In the *list variant*, the vertices of the input graph G have lists (of allowed parts), and an M-partition must place each vertex of G in a part that is allowed for it [2, 12].

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In the *surjective variant*, each M-partition must have all parts $V_i \neq \emptyset$ [6, 31]. In the *digraph variant*, we partition digraphs instead of graphs, and the matrix M is not required to be symmetric [15]. In the *edge-coloured variant*, we partition edge-coloured complete graphs G and the matrix M specifies which vertex colours are allowed, within, and between, the parts [4, 8]. (A two-edge-coloured complete graph may be viewed as just a graph, formed by the edges of one of the colours; in this sense, the edge-coloured case generalizes the basic M-partition problem.) There are also variants for other relational structures related to general constraint satisfaction problems [8, 7].

Here we shall for the most part discuss the basic problem (except for pointing out various applications and connections), and hence we assume that M has no diagonal $*$ – since if $M(i,i) = *$ then every graph G admits the trivial M-partition with $V_i = V(G)$. (This assumption does not apply when we discuss the list variant or the surjective variant, where diagonal $*$ are explicitly allowed.) Thus, for the basic problem we shall always assume that all diagonal entries of M are 0 or 1. In fact, by reordering the parts $V_1, \ldots, V_m$, we may assume that $V_1, \ldots, V_k$ are independent sets, and $V_{k+1}, \ldots, V_m$ cliques, i.e., that the matrix M has a block structure, consisting of a symmetric matrix A with rows and columns $1, 2, \ldots, k$ and all diagonal entries 0, a symmetric matrix B with rows and columns $k+1, k+2, \ldots, m$ and all diagonal entries 1, and a matrix C with rows $1, 2, \ldots, k$ and columns $k+1, k+2, \ldots, m$, and its transpose with rows $k+1, k+2, \ldots, m$ and columns $1, 2, \ldots, k$. (Below we shall refer to the size of B as $\ell = m - k$.)

We note that M-partition problems include all homomorphism problems. Indeed, if H is a graph, we let M be the adjacency matrix of H with 1's replaced by $*$: then an M-partition of a graph G corresponds exactly to a homomorphism of G to H. In particular, if C_m is the matrix of size m with diagonal 0's and off-diagonal $*$'s, then a C_m-partition of G is exactly an m-colouring of G.

We also note that even small matrices M yield important and nontrivial problems, cf. [12]. For matrices M of size 2, in addition to the polynomial problem of 2-colouring (C_2-partition), we meet the polynomial problem of recognizing split graphs [21]. For matrices of size 3, we encounter the polynomial problems of the existence of a clique cutset [30], or the existence of a homogeneous set [21] (both in the surjective variant), as well as the NP-complete problems of 3-colouring and the of the existence of a stable cutset [18] (the latter in the surjective variant). For matrices of size 4, we obtain several problems that have resisted solution for many years, including the problem of existence of a skew cutset, conjectured to be polynomial by Chvátal [3], proved quasi-polynomial in [12] (in the list variant), and then polynomial in [19] (again, in the list variant); an improved algorithm in [26] applies only in the non-list version. The complexity of a certain other problem with a matrix of size 4 (and nonempty parts), was posed by Peter Winkler in the 1970's, cf. e.g. [12], and proved NP-complete by Narayan Vikas [31]. (For recent progress on a related problem see [32] and [27].) The list variant of the M-partition problem for another matrix M of size 4 has been dubbed the "stubborn problem" [2] because its complexity was difficult to determine. This problem was also recently solved, and shown to be polynomial in [4]. For a further discussion of interesting related problems see [12, 6, 22].

We say that G is a *minimal obstruction* for M if G does not admit an M-partition, but each proper induced subgraph of G does admit an M-partition [10]. If M has finitely many minimal obstructions, then there is a characterization of M-partitionable graphs by a finite set of forbidden induced subgraphs, and hence a polynomial algorithm for M-partition. Of course, there are polynomial M-partition problems that have infinitely many minimal obstructions, such as, say, 2-colouring.

It is known that if M has no $*$ then it has finitely many minimal obstructions [10]. In fact, denoting the sizes of the blocks A, B by k, ℓ, respectively, as described above, it is shown in [10] that each minimal obstruction for M has at most $(k+1)(\ell+1)$ vertices, and there are at most two minimal obstructions with $(k+1)(\ell+1)$ vertices.

On the other hand, if M has an $*$ in its block A or B, then it is called *unfriendly*, cf. [16], where it is proved that an unfriendly matrix always has infinitely many minimal obstructions. It is also shown in [16] that each *friendly* matrix (i.e., one that is not unfriendly) of size $m < 6$ has finitely many minimal obstructions, while there exists a friendly matrix of size 6 with infinitely many minimal obstructions; there is even a friendly matrix M with NP-complete M-partition problem [16].

It is not known which matrices M have polynomial M-partition problems. In fact, it is not known whether all M-partition problems are polynomial or NP-complete; for the digraph variant (and just restricted to matrices without 1's) this would imply the Dichotomy Conjecture of Feder and Vardi [17].

Thus the classification of matrices M, both with respect to the complexity of the M-partition problem, and with respect to the finiteness of the number of minimal obstructions, remain open. Nevertheless, considerable effort has gone into classification of small matrices, for all the variants discussed: see [12] for the complexity of the basic problem and matrices of size $m < 5$, see [2, 4, 12] for the complexity of the list version and $m < 5$, see

[6, 32, 27] for the complexity of the surjective variant and $m < 5$, see [15] for the complexity of the digraph variant and $m < 4$, and see [16] for the finiteness of the number of minimal obstructions in the basic variant and $m < 4$.

From now on, we will only focus on the basic M-partition problem. Some of these results were first obtained in [28], and were also summarized without proofs in [14].

2 Chordal Graphs

Consider now the M-partition problem for restricted input graphs. A graph is *perfect* if it and all its induced subgraphs have the chromatic number equal to the maximum clique size. A graph is *chordal* if it does not have an induced cycle of length greater than four. We shall start by restricting input graphs to be perfect. This may seem like a good idea, since their definition implies that an m-colouring of a perfect graph exists if and only if it does not contain a complete graph with $m+1$ vertices; thus the matrix for m-colouring has just one minimal perfect obstruction. In fact, it is easy to see that if M has no 1's (and so it corresponds to a graph H as explained above), then a perfect graph G has an M-partition if and only if G does not contain a clique of size greater than the maximum clique of H. However, it was shown in [9] that if each M-partition problem for perfect graphs is polynomial or NP-complete, then the Dichotomy Conjecture mentioned earlier holds.

Thus arbitrary matrix partitions still seem badly behaved for perfect graphs. Looking at subclasses of perfect graphs, we mention in passing that for the class of cographs it is known that all matrices yield only finitely many minimal cograph obstructions [5, 11].

In this paper we focus on the class of chordal graphs. Generalizations of colouring are often well behaved on chordal graphs; here is a typical example.

Theorem 2.1 *[23] A chordal graph can be partitioned into k independent sets and ℓ cliques if and only if it does not contain an induced $(\ell+1)K_{k+1}$.*

This partition problem corresponds to the matrix M with all off-diagonal entries equal to $*$, and with blocks A of size k and B of size ℓ. There is a more general class of matrices for which the number of chordal (and even perfect) minimal obstructions is known to be finite. A matrix M is *normal* if M does not have two off-diagonal entries $e = *$ and $e' \neq *$ in the same block, A, B, or C. In [9, 13], it is shown that normal matrices result in finitely many chordal (respectively perfect) minimal obstructions, and (different) bounds are given on the size of chordal (respectively perfect) minimal obstructions of normal matrices. It will follow from Theorem 2.3 that being normal is not a necessary condition for having a finite number of chordal minimal obstructions.

A more general class of matrices M ensures that the M-partition problem is polynomial for chordal graphs [13]. A matrix M is *crossed* if its block C contains a set of rows and columns without $*$ that together cover all the entries of C different from $*$. The class of crossed matrices contains all normal matrices, all matrices without $*$, and other classes of matrices we discussed. Nevertheless, Theorem 2.2 yields many non-crossed matrices with polynomial M-partition problem on chordal graphs, such as the matrix

$$M = \begin{pmatrix} 0 & * & * & 0 \\ * & 0 & 0 & * \\ * & 0 & 1 & * \\ 0 & * & * & 1 \end{pmatrix}.$$

M is not crossed because C has neither a row nor a column without $*$, and so its two zero entries cannot be covered.

Thus the classification of matrices M with respect to complexity, as well as with respect to the finiteness of the number of minimal obstructions, also remains open for the class of chordal graphs. However, there are in this case no results classifying small matrices, for either of the problems. This is our goal here, see Theorems 2.2 and 2.3.

For the complexity of the M-partition problem for chordal graphs, we derive the following result.

Theorem 2.2 *If M is a matrix of size $m < 5$, then the M-partition problem for chordal graphs is polynomial.*

Proof: According to Theorems 6.1 and 6.2 from [12], all matrices M which do not contain C_3 or its complement have polynomial M-partition problems. This proves the claim for any matrix without three diagonal 0's or three diagonal 1's (such as the matrix mentioned above). For matrices with three diagonal 0's or three diagonal 1's and size $m < 5$, the matrix C has only one row or one column and so it is automatically crossed, thus the M-partition problem is polynomial by [13]. $\diamond$

As mentioned earlier, in [13] there are examples of matrices M with NP-complete M-partition problems for chordal graphs. The matrices constructed in [13] have a fairly large size, say, in the neighbourhood of thirty rows and columns; we do not know what is the smallest size of a matrix M with an NP-complete M-partition problem.

For the number of minimal obstructions, the situation is more difficult, even for small matrices. The theorem below summarizes our findings; small matrices with constant diagonal were first handled in the third author's master's thesis [28].

Theorem 2.3 *If M is a matrix of size $m < 4$, then M has finitely many chordal minimal obstructions, except for the following two matrices, which have infinitely many chordal minimal obstructions.*

$$M_1 = \begin{pmatrix} 0 & * & * \\ * & 0 & 1 \\ * & 1 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & * & * \\ * & 0 & 1 \\ * & 1 & 1 \end{pmatrix}$$

We first handle the two exceptional cases.

Lemma 2.4 *The matrices M_1, M_2 have infinitely many chordal minimal obstructions.*

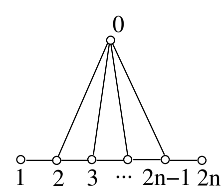

Figure 1: An infinite family of chordal minimal obstructions for M_1 (or M_2)

Proof: It turns out that the same infinite family, depicted in Figure 1, applies to both matrices, and the proofs are similar. We focus on M_1; the proof for M_2 is similar, with easy modifications. (This also follows from the general result proved in Theorem 3.1.)

The graphs are obviously chordal. Next we show that these graphs do not admit an M_1-partition (for $n > 2$). Indeed, suppose that there was such an M-partition. The vertex 0 could not be placed in the first part (corresponding to the first diagonal entry of M_1), since this would require all vertices from 2 to $2n-1$ to go to the other two parts, which are connected by all possible edges, so this would result in a four-cycle (as long as $n > 2$). Without loss of generality suppose that 0 is placed in the second part. Then 2 and $2n-1$ are placed in the first and third part, in some order; they cannot be placed in the same part because of parity. Suppose 2 is in the first part and $2n-1$ in the third part (the other case is similar). Now 1 cannot go to any of the parts, a contradiction.

It remains to verify that if any vertex of the depicted graph is deleted, then an M-partition exists. This is clear if the deleted vertex is 0; in all other cases it follows from the fact that with a vertex removed, parity is no longer a constraint and if 0 is placed in the second part, both 1 and $2n$ (if not removed) can be placed in the first part. $\diamond$

We now show that all other symmetric matrices of size less than four have only finitely many chordal minimal obstructions.

Lemma 2.5 *If M has size $m \leq 2$, then M has finitely many chordal minimal obstructions.*

Proof: All such matrices are normal, and we conclude by the corresponding result from [9]. $\diamond$

For matrices of size $m = 3$, we first focus on those having constant diagonal. The following three matrices are of interest. (We note that chordal graphs are not closed under complementation, so exchanging 0's and 1's in a matrix leads in general to a different problem; the reader may notice that M_5 is obtained this way from M_1, after a suitable permutation.)

$$M_3 = \begin{pmatrix} 0 & 1 & * \\ 1 & 0 & 1 \\ * & 1 & 0 \end{pmatrix} \; M_4 = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ * & 0 & 1 \end{pmatrix} \; M_5 = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ * & * & 1 \end{pmatrix}$$

Lemma 2.6 *If M has size $m = 3$ and a constant diagonal, and if $M \neq M_i$ for $i = 1, 3, 4, 5$, then M has finitely many chordal minimal obstructions.*

Proof: Consider first a general symmetric matrix of size $m = 3$ with zero diagonal.

$$M = \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix}.$$

If M is normal, we conclude by [9], so we may assume that one of a, b, c is $*$ and one is not $*$. Without loss of generality, let $b = *$ and $c \neq *$. If $c = 0$, then the M-partition problem is equivalent to the C_2-partition problem, since any M-partition of a graph G can be modified to avoid the second part, by moving all vertices from the second part to the third part. For chordal graphs, C_2 has a single minimal obstruction, namely K_3. Thus we may assume $c = 1$. If $a = *$, we have $M = M_1$, otherwise if $a = 0$ we have again a problem equivalent to C_2-partition, and if $a = 1$ then $M = M_3$.

Similarly, consider a matrix

$$M = \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix}$$

and suppose without loss of generality that $b = *$, $a \neq *$. By a similar argument we must have $a = 0$ and then $c = *$ or $c = 0$, and we obtain $M = M_4$ or $M = M_5$. $\diamond$

Lemma 2.7 *The matrix* $M_3 = \begin{pmatrix} 0 & 1 & * \\ 1 & 0 & 1 \\ * & 1 & 0 \end{pmatrix}$ *has three chordal minimal obstructions, depicted in Figure 2.*

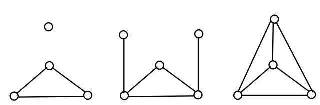

Figure 2: The chordal minimal obstructions for M_3

Proof: It is easy to check that each of the graphs in Figure 2 is not M_3-partitionable, but with any vertex removed an M_3-partition is possible.

Now we show that there are no other chordal minimal obstructions. Let G be a chordal graph that does not contain any of the three graphs as an induced subgraph. We will show that it is M_3-partitionable. If G is bipartite, then it can be partitioned using the first and third part. Otherwise, G contains a triangle (since it is chordal). The absence of the forbidden three induced subgraphs now implies that the following three statements hold for any triangle abc: (i) every vertex of $G - \{a, b, c\}$ is adjacent to at least one vertex from a, b, c; (ii) two distinct vertices of $G - \{a, b, c\}$ that are only adjacent to one vertex from a, b, c must be non-adjacent to each other and adjacent to the same vertex from a, b, c; and (iii) no vertex of $G - \{a, b, c\}$ is adjacent to all three vertices a, b, c.

We first note that for any triangle abc in G, two vertices of $G - \{a, b, c\}$ must have a common neighbour on abc. Otherwise, say, for some triangle abc, a vertex u is adjacent to a but not b, and v is adjacent to b but not a. Since G is chordal, u and v are not adjacent. By (ii), we conclude that u or v is adjacent to c; suppose u is adjacent to c. Then a, u, c forms a triangle, and v is not adjacent to a, u; thus by (i) applied to the triangle auc we must have v also adjacent to c, and u and v have a common neighbour c.

Next we suppose that G contains a vertex a adjacent to all other vertices. Since G is 3-colourable, $G - a$ must be bipartite and hence admits an M_3-partition where a is the only vertex in the second part.

Finally, we show that if G has no vertex adjacent to all other vertices then it is complete 3-partite. Consider a triangle abc in G, and vertices a', b', c' non-adjacent to a, b, c respectively. It is easy to see that a', b', c' must be distinct, by (iii) and the fact that two vertices of $G - \{a, b, c\}$ must have a common neighbour on abc. The latter fact now also implies that every vertex of $G - \{a, b, c\}$ has exactly two neighbours on abc, for any triangle abc. If G was not complete 3-partite then some a'' adjacent to b, c and some b'' adjacent to a, c would not be adjacent to each other. However, this contradicts the fact that b' has two neighbours on abc'. Thus G is a complete 3-partite graph, and the 3-partition is also an M_3-partition. $\diamond$

For the matrices M_4, M_5, it is harder to explicitly describe all the chordal minimal obstructions, and we merely prove that there are only finitely many.

Lemma 2.8 *The matrix* $M_4 = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ * & 0 & 1 \end{pmatrix}$ *has finitely many chordal minimal obstructions.*

Proof: We proceed as follows. Let G be a chordal graph with independence number α; thus G can be partitioned into α cliques. If $\alpha \leq 2$, then G is M_4-partitionable, by placing the vertices in the first and third parts. Hence there are no minimal obstructions with $\alpha \leq 2$. When $\alpha \geq 4$, then G is not M_4-partitionable, and G contains $\overline{K_4}$. Thus $\overline{K_4}$ is the only minimal obstruction with $\alpha \geq 4$. It remains to prove that there are only finitely many chordal minimal obstructions with $\alpha = 3$. Suppose G is such a chordal minimal obstruction, and let v_1, v_2, v_3 be three fixed independent vertices. We will give an upper bound on the number of vertices of G.

We first give an upper bound on the number of vertices of a chordal graph H with three specified independent vertices v_1, v_2, v_3 which is minimal in the following sense: H does not admit an M_4-partition in which each v_i is placed in the i-th part, but any induced subgraph of H admits an M_4-partition in which each v_i that is in the subgraph is in the i-th part of the partition. Such a graph H is called a *minimal labeled obstruction* to M_4-partition. If all chordal minimal labeled obstructions H have at most N vertices, then all minimal obstructions G have at most $6N$ vertices. Indeed, G must contain, for each bijective assignment of v_1, v_2, v_3 to the three parts of M_4, some (at most) N vertices that prevent v_1, v_2, v_3 from being placed in the corresponding parts. But the set of these (at most) $6N$ vertices already induces a subgraph of G that is not M_4-partitionable (since v_1, v_2, v_3 must be placed bijectively in the three parts in any M_4-partition). Since G is a minimal obstruction, it must not have other vertices, i.e., G has at most $6N$ vertices. (In other words, we obtain an upper bound on the number of vertices of a minimal obstruction G by adding together the upper bounds on each minimal labeled obstruction, over all six assignments of v_1, v_2, v_3 to the three parts of M_4.)

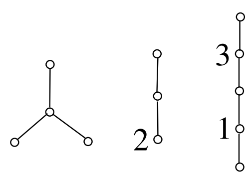

Figure 3: The chordal minimal labeled obstructions for M_4

It remains to give an upper bound on the number of vertices in a chordal minimal labeled obstruction. In fact, this bound is $N = 5$, as we shall show that all the chordal minimal labeled obstructions are given in Figure 3. (Each vertex v_i is labeled by i.) Note that a minimal labeled obstruction need not contain all the labeled vertices – the first obstruction in fact contains none of them. (Unlabeled vertices in the obstruction may correspond to any vertex of G, labeled or not.) It is easy to check that each of the depicted graphs is a chordal minimal labeled obstruction.

Let G be a chordal graph with $\alpha = 3$ and three independent vertices v_1, v_2, v_3 labeled $1, 2, 3$, that contains none of the obstructions in Figure 3. We shall show that G admits an M_4-partition with the labeled vertices placed in the corresponding parts. The vertices of G are partitioned into the following sets. The set $S(v_1, v_2, v_3)$ consists of the vertices adjacent to all of v_1, v_2, v_3. The set $S(v_1, v_2)$ consisting of the vertices adjacent to v_1, v_2 but not v_3, and the corresponding sets $S(v_1, v_3), S(v_2, v_3)$ are defined analogously. Finally, the set $S(v_1)$ consists of v_1 and the vertices adjacent to v_1 but not to v_2, v_3, and the corresponding sets $S(v_2), S(v_3)$ are defined analogously. As $\alpha = 3$, the union of all these sets is $V(G)$.

We first note that G is claw-free (it does not contain the first obstruction from Figure 3), and hence we must have $S(v_1, v_2, v_3) = \emptyset$. The absence of the second obstruction from Figure 3 implies that $S(v_1, v_2) = S(v_2, v_3) = \emptyset$. Next we note that $S(v_1, v_3)$ is a clique – if $a, b \in S(v_1, v_3)$ were non-adjacent, then a, b, v_1, v_3 would induce a four-cycle without chords, contradicting the fact that G is chordal. Moreover, our assumption that $\alpha(G) = 3$ implies that $S(v_1)$, $S(v_2)$ and $S(v_3)$ are also cliques.

Now we focus on the set $S(v_1, v_3)$. We claim that each vertex of $S(v_1, v_3)$ is adjacent to all vertices of $S(v_1)$ or to all vertices of $S(v_3)$. Otherwise, there is vertex $v \in S(v_1, v_3)$ non-adjacent to both a vertex $u \in S(v_1)$ and a vertex $w \in S(v_3)$ – and the induced path u, v_1, v, v_3, w is the last obstruction from Figure 3. Thus we can partition $S(v_1, v_3)$ into a set X of vertices adjacent to all vertices of $S(v_1)$ and a set Y of vertices adjacent to all vertices of $S(v_3)$. Since $S(v_1, v_3)$ and $S(v_1)$ are cliques, so is $S(v_1) \cup X$, and similarly for $S(v_3) \cup Y$. It is now easy to check that placing $S(v_1) \cup X$ in the first part, $S(v_2)$ in the second part, and $S(v_3) \cup Y$ in the third part, is an M_4-partition placing the labeled vertices in their corresponding parts. (For instance, using the absence of the second obstruction, we see that $S(v_2)$ has no edges to $S(v_1) \cup X \cup S(v_3) \cup Y$.) $\diamond$

For the most involved case, of the matrix M_5, we proceed similarly, but the proof is more technical.

Lemma 2.9 *The matrix* $M_5 = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ * & * & 1 \end{pmatrix}$ *has finitely many chordal minimal obstructions.*

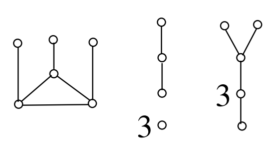

Figure 4: The chordal minimal labeled obstructions for M_5

Proof: It will again suffice to prove that there are only finitely many chordal minimal obstructions with $\alpha = 3$. An upper bound on the number of vertices of such an obstruction will again follow from an upper bound on the number of vertices in a chordal minimal labeled obstruction, with some three independent vertices v_1, v_2, v_3, labelled by $1, 2, 3$ respectively. The chordal minimal labeled obstructions are given in Figure 4. It can again be checked that each depicted labeled graph is in fact a chordal minimal labelled obstruction. Thus assume that G is a chordal graph, with $\alpha = 3$, and three independent vertices v_1, v_2, v_3 labeled $1, 2, 3$, that contains none of the obstructions in Figure 4. We shall again show that G admits an M_5-partition with the labeled vertices placed in the corresponding parts.

The vertices of G are again partitioned into the same sets $S(v_1, v_2, v_3)$, $S(v_1, v_2)$, $S(v_1, v_3), S(v_2, v_3), S(v_1), S(v_2)$, and $S(v_3)$. As before, using the chordality of G, we conclude that each of these sets is a clique. Moreover, the chordality of G also implies that each vertex of $S(v_1, v_2, v_3)$ is adjacent to all vertices of $S(v_1, v_3) \cup S(v_2, v_3)$. The absence of the last obstruction from Figure 4 implies that each vertex of $S(v_1, v_2, v_3)$ is also adjacent to all vertices of $S(v_3)$. The absence of the second obstruction from Figure 4 implies that $S(v_1, v_2) = \emptyset$, and no vertex of $S(v_1)$ is adjacent to a vertex of $S(v_2)$.

Since a vertex of $S(v_1)$ is non-adjacent to v_2, v_3 it can only be placed in the first part, in any M_5-partition. Similarly, vertices of $S(v_2)$ must be placed in the second part, and vertices of $S(v_3)$ in the third part; furthermore, all vertices of $S(v_1, v_2, v_3)$ must go to the third part. By a similar consideration, we see that vertices of $S(v_1, v_3)$ must go to the first and third parts, and vertices of $S(v_2, v_3)$ to the second and third parts.

Let A denote the set of all vertices of $S(v_1, v_3)$ that have a non-neighbour in $S(v_3)$; these must be placed in the first part. Let B the set of all vertices of $S(v_2, v_3)$ that have a non-neighbour in $S(v_3)$; these must be placed in the second part. Let C consist of all those remaining vertices of $S(v_1, v_3)$ that have a neighbour in $B \cup S(v_2)$ or have a non-neighbour in $S(v_1)$; these must be placed in the third part. Let D consist of all those remaining vertices of $S(v_2, v_3)$ that have a neighbour in $A \cup S(v_1)$ or have a non-neighbour in $S(v_2)$; these must also be placed in the third part.

Let $E = S(v_1, v_3) - A - C$ and $F = S(v_2, v_3) - B - D$. To decide how to place vertices of E and F, we shall further partition these sets. Let E_1 denote the set of all vertices of E that have a non-neighbour in D. Vertices of E_1 must be placed in the first part. Let F_1 be the set of all vertices of F that have a non-neighbour in C. Vertices of F_1 must be placed in the second part. Let E_2 be the set of all vertices of $E - E_1$ with a neighbour in F_1. These must be placed in the third part. Up to this point there was symmetry between the first and second part; however now we define F_2 to consist of all vertices of $F - F_1$ that have a non-neighbour in E_2. These must be placed in the second part. Finally, let $E_3 = E - E_1 - E_2$ and $F_3 = F - F_1 - F_2$. We will place all vertices of E_3 in the first part and all the vertices of F_3 in the third part. (Note the asymmetry.)

In summary, the first part consists of $v_1, S(v_1), A, E_1$, and E_3; the second part consists of $v_2, S(v_2), B, F_1$, and F_2; and the third part consists of v_3, $S(v_3), S(v_1, v_2, v_3), C, D, E_2$, and F_3. We shall show that each part is a clique and that there are no edges between the first and second parts.

To see that the first part is a clique, we first recall that $S(v_1)$, and $S(v_1, v_3)$ are cliques. Then we observe that every vertex u of A is adjacent to every vertex of $S(v_1)$. Indeed, u has a non-neighbour w in $S(v_3)$, and if u has also a non-neighbour v in $S(v_1)$, then the vertices u, v, w, v_1, v_3 form a chordless cycle (if vw is an edge) contradicting the chordality of G, or the four vertices u, v, w, v_2 are independent (if vw is not an edge) contradicting $\alpha(G) = 3$. Since every vertex of E is adjacent to every vertex of $S(v_1)$, according to the definition of C, we conclude that the vertices placed in the first part form a clique.

A symmetric argument shows that the vertices taken to the second part form a clique. Thus consider the vertices placed in the third part. We have already observed that v_3 together with $S(v_3) \cup S(v_1, v_2, v_3)$ form a clique, and every vertex of $S(v_1, v_2, v_3)$ is adjacent to all vertices of $S(v_1, v_3) \cup S(v_2, v_3)$. Also, $C \cup E_2$ is a part of the clique $S(v_1, v_3)$, and $D \cup F_3$ a part of the clique $S(v_2, v_3)$. The definition of A ensures that every vertex of $C \cup E_2$ is adjacent to all vertices of $S(v_3)$, and the definition of B ensures that every vertex of $D \cup F_3$ is adjacent to all vertices of $S(v_3)$. The definition of E_1 ensures that ever vertex of E_2 is adjacent to all vertices of D, and the definition of F_1 ensures that every vertex of F_3 is adjacent to all vertices of C.

It remains to show that every vertex of C is adjacent to every vertex of D. Recall that there are three possible reasons for a vertex v to belong to C (and similarly for D) – it can have a neighbour in B, a neighbour in $S(v_2)$, or a non-neighbour in $S(v_1)$. If $v \in C$ has a neighbour t in $S(v_2)$ and is non-adjacent to some vertex $u \in D$, then the cycle u, z, v, t, y, u contains a chordless cycle of length at least four, contradicting the chordality of G. By symmetry, any $u \in D$ with a neighbour in $S(v_1)$ is adjacent to all $v \in C$. Similarly, if $v \in C$ has a neighbour s in in B and is non-adjacent to some vertex $u \in D$, then consider a vertex $t \in S(v_3)$ nonadjacent to s: the four vertices s, u, t, v form a chordless four-cycle (t is adjacent to u as $u \notin B$, and similarly for v). By symmetry, any $u \in D$ with a neighbour in A is adjacent to all $v \in C$. Finally, consider non-adjacent $v \in C, u \in D$ where v has non-neighbour $w \in S(v_1)$ and u has a non-neighbour $t \in S(v_2)$. Recall that we have proved that such w, t must be non-adjacent. Moreover, u cannot be adjacent to w otherwise we would have the last obstruction from Figure 4; and v cannot be adjacent to t, for the same reason. The four vertices u, v, w, t now contradict $\alpha(G) = 3$. This completes the proof that the third part is a clique.

We now show that there are no edges between the first and second part. So suppose $v \in S(v_1) \cup A \cup E_1 \cup E_3$ is adjacent to $u \in S(v_2) \cup B \cup F_1 \cup F_2$. We have already observed that we cannot have $v \in S(v_1)$ and $u \in S(v_2)$. If $v \in A$ and $u \in S(v_2)$, then consider a $t \in S(v_3)$ non-adjacent to v. If t is adjacent to u, we have a chordless four-cycle v, u, t, z, and if t is non-adjacent to u, then v, x, t, z, u induce the last obstruction from Figure 4. For symmetric reasons, we cannot have $v \in S(v_1)$ and $u \in B$. If $v \in A$ and $u \in B$, and if v, u have a common non-neighbour w in $S(v_3)$, then u, v, z, x, y, w induce the first obstruction from Figure 4. Otherwise there are vertices $w, w' \in S(v_3)$ such that w is adjacent to u but not v and w' is adjacent to v but not u: then u, v, w, w' form a chordless four-cycle.

The definition of E ensure that it contains no neighbours of $S(v_2)$, and similarly F contains no neighbours of $S(v_1)$. If $v \in E_1$ and $u \in F_1$, we obtain a chordless four-cycle with a non-neighbour of v in D and a non-neighbour of u in C. If $v \in E_1, u \in F_2$, then there exists a $w \in E_2$ non-adjacent to v, and a $t \in F_1$ adjacent to w. As we have just shown, v, t are non-adjacent, so u, w, t, v is a chordless four-cycle. Finally suppose $v \in E_3$ and $u \in F_2$. (Note that $u \notin F_1$ by the definition of E_2.) Then u has a non-neighbour $w \in E_2$, and w a neighbour $t \in F_1$; as before we obtain a chordless four-cycle v, u, w, t. Thus there are no edges joining the first and second parts, and G is M_5-partitioned. $\diamond$

For matrices with mixed diagonal, we single out three additional interesting cases:

$$M_6 = \begin{pmatrix} 0 & * & * \\ * & 0 & 0 \\ * & 0 & 1 \end{pmatrix}$$

$$M_7 = \begin{pmatrix} 0 & * & 0 \\ * & 1 & * \\ 0 & * & 1 \end{pmatrix}$$

$$M_8 = \begin{pmatrix} 0 & * & 1 \\ * & 1 & * \\ 1 & * & 1 \end{pmatrix}$$

Lemma 2.10 *If M has size $m = 3$ and a mixed diagonal, and if $M \neq M_i$ for $i = 2, 6, 7, 8$, then M has finitely many chordal minimal obstructions.*

Proof: We proceed as in the proof of Lemma 2.6. For matrices

$$M = \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 1 \end{pmatrix},$$

we may assume that $a = *$, else M is friendly, and as $m < 6$ the conclusion follows from [16]. We may again assume that one of b, c is $*$ and the other is not; by symmetry assume $b = *$. Now we obtain the matrices M_2 and M_6 as the only choices.

For

$$M = \begin{pmatrix} 0 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix},$$

the arguments are similar, yielding M_7 and M_8.

$\diamond$

Lemma 2.11 *The matrix* $M_6 = \begin{pmatrix} 0 & * & * \\ * & 0 & 0 \\ * & 0 & 1 \end{pmatrix}$ *has the three chordal minimal obstructions given in Figure 5.*

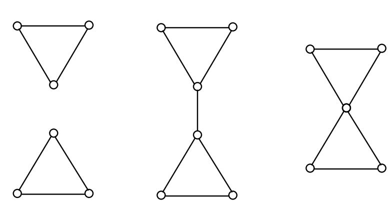

Figure 5: The chordal minimal obstructions for M_6

Proof: We first note that it is easy to see that each of the three chordal graphs in Figure 5 is a chordal minimal obstructions to M_6-partitionability. Now suppose G is a chordal minimal obstruction. Then G must have an induced $2K_2$; otherwise it is a split graph, and it can be partitioned using the first and third parts.

Furthermore, G cannot have a vertex of degree smaller than two. Indeed, if u were adjacent to at most one other vertex, say v, then $G - u$ admits a partition, and depending on where v was placed, u can always be placed in the first or the second part.

A chordal graph that contains an induced $2K_2$ and all vertices have degree at least two must contain an induced copy of one of the graphs from Figure 5. If each of the K_2 belongs to a cycle, then by chordality it belongs to a triangle, and the triangles are edge-disjoint, yielding the three graphs depicted in Figure 5. Otherwise, at least one of the copies of K_2 lies on a unique path joining two cycles, whence G must contain an induced copy of the first graph in Figure 5. $\diamond$

Lemma 2.12 *The matrix* $M_7 = \begin{pmatrix} 0 & * & 0 \\ * & 1 & * \\ 0 & * & 1 \end{pmatrix}$ *has finitely many chordal minimal obstructions.*

Proof: Let H be a chordal graph. As before, if H has no induced $2K_2$, it is a split graph and hence M_7-partitionable. So suppose edges ab, cd induce a $2K_2$. It is easy to see that in any M_7-partition of H one of a, c must be in the third part. We will consider the size of a chordal minimal labelled obstruction placing a in the third part, and then similarly for c. Specifically, we will bound the maximum size s of a chordal minimal labeled obstruction G, with a fixed vertex x in the third part. Then the maximum size of a chordal minimal obstruction will be at most $2s$, by adding (as before) the two bounds of s vertices each.

We will trace all vertices that are needed to ensure that G does not admit an M_7-partition with x in the third part. Let C denote all neighbours of x and N the set of all non-neighbours of x. Placing x in the third part forces all vertices of C into the second and third parts, and all vertices of N into the first and second parts.

Consider first the case when C is a clique. Since in this case any partition of C into the second and third parts is consistent with an M_7-partition of G, there must be a problem in partitioning N within the first and second parts. (Either N is not a split graph, or no split partition of N respects the constraints on non-adjacency between the first and the third part.) If N is not a split graph, it contains four vertices that induce $2K_2$, and we have a labeled obstruction with five vertices (x and the four vertices of the induced $2K_2$). Otherwise, there is a partition of N into a clique K and an independent set S. We choose one such partition and call it the *reference* *partition*; we also note that any other partition of N into a clique and an independent set is obtained from K, S by at most two moves, one from K to S, and one from S to K. (This is so because an independent set and a clique have at most one vertex in common.) Now we tentatively assign all vertices of K to the second part and all vertices of S to the first part. Consider a vertex v of C and its adjacencies to $N = K \cup S$. If v is not adjacent to any vertices of S, it can be placed in the third part. If v is adjacent to all vertices of K, it can be placed in the second part. Thus we obtain an M_7 partition of G, unless some vertex v is adjacent to at least one vertex u of S and non-adjacent to at least one vertex w of K. In such a case we will consider separately what happens when we move u to K and what happens when we move w to S. In either case, we either succeed in forming an M_7 partition, or we have another vertex v' adjacent to at least one vertex u' of the new set S and non-adjacent to at least one vertex w' of the new set K. If each of these cases also fails, it results in a vertex v" adjacent to at least one vertex u" of S and non-adjacent to at least one vertex w" of K. In this case, we know that no M_7 partition with x in the third part is possible, as the non-neighbours of x would move more than two vertices from the reference partition. Thus the minimal labeled obstruction G only needs to contain the vertex x, the three vertices v, u, w, the six vertices v', u', w' (three for moving u to K and three for moving w to S), and the twelve vertices v", u", w" (three for each of the four possibilities considered). Even if all these considered vertices are distinct, we only have 22 vertices in G, i.e., $s \leq 22$.

(As an aid to the reader's intuition, we offer the following example. Suppose that we have three distinct problem vertices, v_1, v_2, v_3, each with distinct pairs $u_1 \in S, w_1 \in K, u_2 \in S, w_2 \in K$, and $u_3 \in S, w_3 \in K$. Then whichever way we try to resolve these conflicts (by moving u_i into K or w_i into S), we will fail, as we would be changing the reference partition by more than one move from S to K and one move from K to S. Therefore these nine vertices ($v_i, u_i, w_i, i = 1, 2, 3$), together with x, are already an obstruction.)

In the second case, when C is not a clique, the neighbours of x include two non-adjacent vertices, say $y, z \in C$. If there are three independent vertices in C, then they, together with x already comprise a minimal obstruction, as C should be partitionable into two cliques. We define, for any subset X of $\{x, y, z\}$, the set $S(X)$ to consist of all vertices of $G - \{x, y, z\}$ adjacent to every vertex of X and no vertex of $\{x, y, z\} - X$. We again write $S(x, y, z), S(x, y), \ldots, S(x), S(\emptyset)$. Since G is chordal, we must have $S(y, z) = \emptyset$. Since C does not contain three independent vertices, we must have $S(x) = \emptyset$. We assume that y is in the second part and z in the third part. (The opposite assumption, that z is in the second part and y in the third part results in the same number of vertices, and the total bound is the sum of these two bounds.) Now all vertices of $S(x,y)$ must be placed in the second part, and if two of them are not adjacent, then they, together with x, y, z form already an obstruction (to placing x and z in the third part and y in the second part). Thus $S(x,y)$ is a clique, and by a similar argument, we may assume that $S(x,z)$ is also a clique, and its vertices are placed in the third part. Similarly, $S(z) = \emptyset$, else any vertex in $S(z)$, together with x, y, z, would be an obstruction. (Recall that y is in the second part and z in the third part.) Thus $C = \{x, y, z\} \cup S(x,y) \cup S(x,z) \cup S(x,y,z)$ and $N = S(y) \cup S(\emptyset)$. Note that all vertices of $S(\emptyset)$ must be placed in the first part (being non-adjacent to y), and if one of them is adjacent to a vertex in $S(x,z)$, or if two of them are adjacent to each other, we have a small obstruction. Let A denote the set of all vertices in $S(y)$ that have a non-neighbour in $S(x,y)$; these must be placed in the first part, and if this is not possible, there is a small obstruction. Let B denote the set of all vertices in $S(y)$ that have a neighbour in $S(x,z)$; these must be placed in the second part, and if this is not possible, there is a small obstruction. We are left with the problem of partitioning $S(x,y,z)$ into the second a third parts and $S(y)$ into the first and second parts, which is identical with the problem discussed above; thus it also yields only small minimal obstructions. $\diamond$

Lemma 2.13 *The matrix* $M_8 = \begin{pmatrix} 0 & * & 1 \\ * & 1 & * \\ 1 & * & 1 \end{pmatrix}$ *has finitely many chordal minimal obstructions.*

Proof: Consider a chordal minimal obstruction G. Since G cannot be covered by two cliques (that could go to the second and third parts), it must contain three independent vertices. If there were M_8-partitions of G, one (in fact, two) of these three vertices would have to go to the first part, so G only needs enough vertices to prevent each of them going to the first part. In other words, if t is the maximum size of a chordal minimal labelled obstruction H that takes a fixed vertex x to the first part, then G has at most $3t$ vertices. Let again C denotes the set of neighbours of x in H, and N the set of non-neighbours of x in H. The vertices of C must go to the second and third parts, the vertices of N must go to the first and second parts. From this point on, the proof is very similar to the proof of Lemma 2.12. The differences are due to the requirement that the first and third part are completely adjacent rather than completely non-adjacent as in M_7-partitions. Thus, for instance, when C is a clique, and we have chosen a reference partition $N = K \cup S$, tentatively placing all vertices of S into the first part and all vertices of K into the second part, a vertex v in C that is adjacent to all vertices of S can go to the third part, and a vertex v that is adjacent to all vertices of K can go to the second part. This leaves us with considering a vertex v non-adjacent to at least one vertex u of S and at least one vertex w of K; we again consider separately what happens when we move u to K and what happens when we move w to S, and conclude exactly as in the proof of Lemma 2.12. When C is not a clique, the modifications are similar. $\diamond$

In Lemmas 2.12, 2.13, it may be possible to work out concrete chordal minimal labelled obstructions as in Lemmas 2.8, 2.9. However, we feel the additional technique pursued here may be useful for attacking the general case

3 Large Matrices

Let M be any m by m symmetric matrix over $0, 1, *$. If all off-diagonal entries are $*$, then Theorem 2.1 shows that there is a unique chordal minimal obstruction. If no entry is $*$, then according to [10], there are only finitely many minimal obstructions, and hence chordal minimal obstructions. If no entry is 1, then M corresponds to an undirected graph H as described in the first section, where G admits an M-partition if and only if it admits a homomorphism to H. A chordal graph G is perfect, and hence admits a homomorphism to H if and only if it has no subgraph $K_{\chi(H)+1}$ [24]. Thus a matrix M without 1's has only one chordal minimal obstruction. The same is true if M has no 0's, as can be seen by considering the complementary input graphs, and noting that complements of chordal graphs are also perfect.

We now introduce a class of large matrices with infinitely many chordal minimal obstructions. We first focus on generalizing the matrix M_1. Let M be a block matrix consisting of diagonal blocks X and Y, both having all diagonal entries 0 and all off-diagonal entries 1, and the off-diagonal blocks Z and its transpose. Assume $m \geq 3$, both X and Y are non-empty, and Z contains a row with two $*$. (The last assumption implies that M_1 is a principal submatrix of M.) Then we claim that M has infinitely many chordal minimal obstructions. We will prove a more general version of this fact; the more general class of matrices, introduced in the next theorem generalizes both the matrices M_1 and M_2.

Theorem 3.1 *Suppose M is a block matrix*

$$M = \begin{pmatrix} X & Z & P & R \\ Z^t & Y & S & T \\ P^t & S^t & U & V \\ R^t & T^t & V^t & W \end{pmatrix},$$

where X and Y have zero diagonal and 1's off-diagonal, U, W have 1's on the diagonal, and the matrices P and T (and hence also their transposes P^t, T^t) consist of 1's.

Suppose that $m \geq 3$, that X and Y are non-empty, and that M_1 or M_2 is a principal submatrix of M.

Then M has infinitely many chordal minimal obstructions.

Below we offer an example of a matrix from the theorem. Each question mark can be 0 or 1 or $*$ independently. Here X is a two by two matrix, Y is a two-by-two matrix, U is a one-by-one matrix, and W is a two-by-two matrix.

$$M = \left(\begin{array}{cc|cc|c|cc} 0 & 1 & ? & ? & 1 & ? & ? \\ 1 & 0 & ? & ? & 1 & ? & ? \\ - & - & - & - & - & - & - \\ ? & ? & 0 & 1 & ? & 1 & 1 \\ ? & ? & 1 & 0 & ? & 1 & 1 \\ - & - & - & - & - & - & - \\ 1 & 1 & ? & ? & 1 & ? & ? \\ - & - & - & - & - & - & - \\ ? & ? & 1 & 1 & ? & 1 & ? \\ ? & ? & 1 & 1 & ? & ? & 1 \end{array}\right),$$

We note that if the parts are permuted so that M becomes the matrix

$$M = \begin{pmatrix} X & P & Z & R \\ P^t & U & S & V \\ Z^t & S^t & Y & T \\ R^t & V^t & T^t & W \end{pmatrix},$$

then the off-diagonal matrices Z, R, S, V are completely arbitrary. Thus an M-partition of G consists of two groups of parts (corresponding to X, U and Y, V), with any kind of connections $(0, 1$ or $*)$ between the groups (corresponding to Z, R, S, V). In each group, there are some independent sets (X in the first group, Y in the second), completely interconnected to each other, and to the cliques in their group (if any) (corresponding to the matrices U, W). Note that M must have at least two diagonal zeros.

Proof: We shall again show that each large member G of the family of chordal graphs in Figure 1 is a minimal obstruction to M-partition. Our first observation is that the path $\Pi = 1, 2, \ldots, 2n$ must be placed in a very particular way. Each of the parts that is a clique (the parts corresponding to the diagonal elements of U and of W), can contain at most two vertices of Π. Since M is fixed, for large n, there will be vertices of Π that are placed into parts that are independent sets (corresponding to the diagonal elements of X and of Y). However, at most one independent set in each group (corresponding to X and to Y) can contain more than one vertex of Π (since Π has no C_4). Thus if n is sufficiently large there will be an independent set in each group that contains many vertices of Π. Because of the 1's in matrices P, T, it now follows that all other parts that are cliques contain no vertices of Π. In other words, the entire path Π is placed alternatingly into one part from X and one part from Y. The vertices 1 and $2n$ are by parity placed in different parts, which now means that it is impossible to place the vertex 0. Thus G is not M-partitionable. Since M contains M_1 or M_2 by assumption, each proper subgraph of G is M-partitionable. $\diamond$

This observation allows us to point out that the case of infinitely many chordal minimal obstructions occurs with at least some frequency.

Let $T_{k,\ell}$ denote the number of symmetric m by m matrices over $0, 1, *$, with k diagonal 0's followed by ℓ diagonal 1's (where $m = k + \ell$). Then $T_{k,\ell} = 3^{\binom{m}{2}}$. Consider now the number of such matrices that satisfy the requirements of Theorem 3.1. These are matrices with arbitrary entries in U, W (except the diagonal), as well as Z, R, S, V. Their number is the greatest when X and Y have the same size, $k/2$, and U and W have the same size, $\ell/2$. In this case there are $\binom{\ell}{2}$ undecided entries in U, V, W, there are $(k/2)^2 = k^2/4$ undecided entries in Z, and there are $(k/2)(\ell/2) = k\ell/4$ undecided entries in each of R, S, except for the two entries that need to be $*$ to satisfy the requirement that the matrix contains M_1 or M_2. Thus the number of matrices with infinitely many chordal minimal obstructions is at least $3^{\frac{k^2}{4}+2\frac{k\ell}{4}+\binom{\ell}{2}-2} = 3^{\frac{m^2}{4}+\frac{\ell^2-2\ell}{4}-2}$. In particular, when $\ell = 0$, the number of matrices with infinitely many chordal minimal obstructions is of the order of $\sqrt{T_{k,\ell}}$. On the other hand, when ℓ is near m (recall that k has to be at least two for the matrix to contain M_1 or M_2), the number of matrices with infinitely many chordal minimal obstructions is of the order of $T_{k,\ell}^{1-\epsilon}$.

For comparison, matrices with finitely many chordal minimal obstructions include all matrices without $*$ [10] or without 1 [14] or without 0 [14], so there are at least $3 \cdot 2^{\binom{m}{2}}$ such matrices for any k, ℓ. We do not know whether there are more matrices with finitely or with infinitely many chordal minimal obstructions.

Note that we have no examples of matrices with a diagonal consisting of 1's (or possibly with one 0) which admits infinitely many chordal minimal obstructions, and possibly there are none. In particular, when the diagonal consists of 1's and there are no 1's off the diagonal, we seem to be encountering some of the most interesting cases, such as the matrices M_4 and M_5 above.

4 Conclusions

We have provided some information about the complexity and number of minimal obstructions of chordal partitions with few parts. Perhaps this evidence could be useful for finding possible classifications for all partitions of chordal graphs; in any event it suggests new techniques that may be useful.

We close with the following note. If M has finitely many minimal obstructions, then the positive instances of the M-partition problem can be described by a first order formula. When M has no 1's, it is known that no other matrices have this property, i.e., that if the positive instances of the M-partition problem can be described by a first order formula, then M has finitely many minimal obstructions [1, 29]. This is open for general matrices M.

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