15-780: Grad AI
Lecture 22: MDPs

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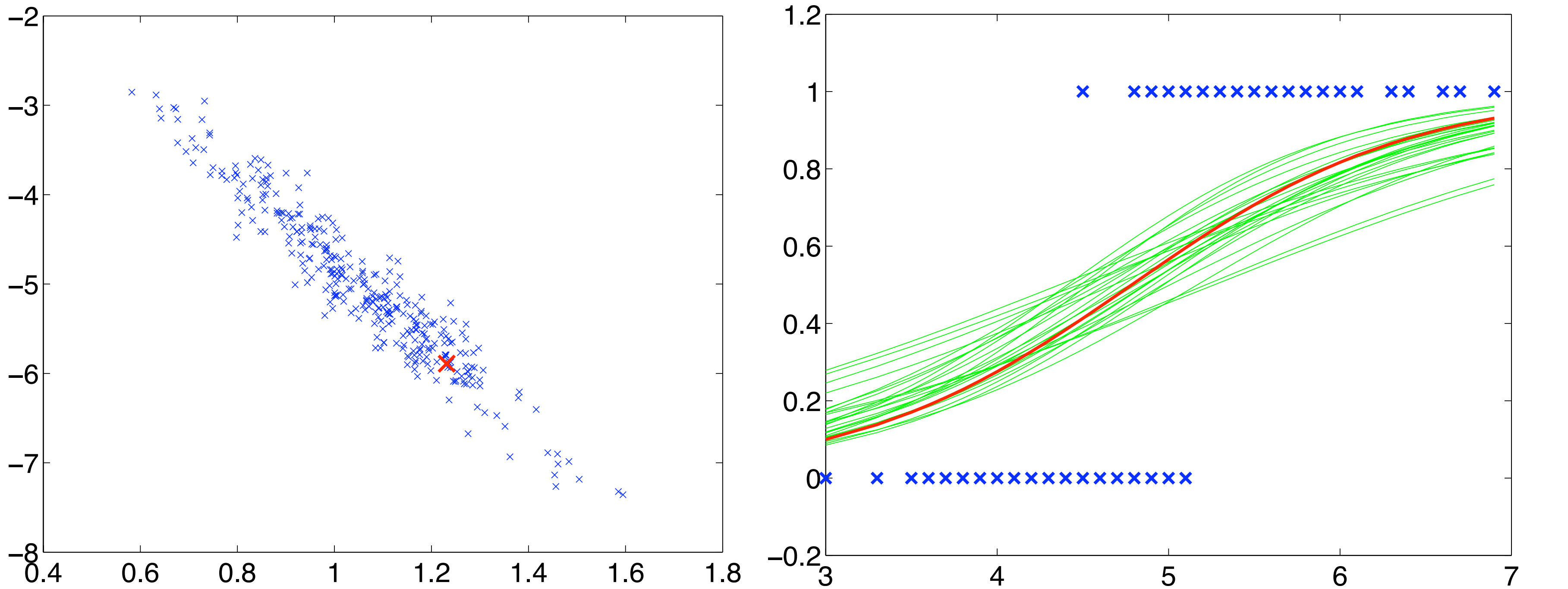
Review: Bayesian learning

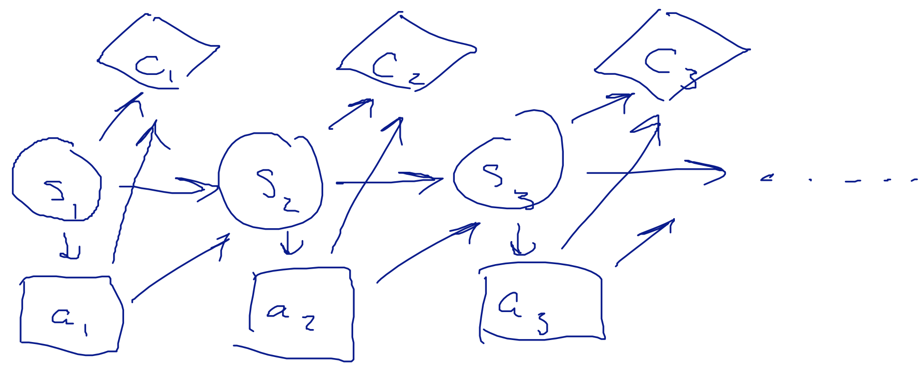
- Bayesian learning: $P(\theta \mid \mathbf{D}) = P(\mathbf{D} \mid \theta)\, P(\theta) / Z$
  - $P(\theta)$: prior over parametric model class
  - $P(\mathbf{D} \mid \theta)$: likelihood
  - or, $P(\theta \mid \mathbf{X}, \mathbf{Y}) = P(\mathbf{Y} \mid \theta, \mathbf{X})\, P(\theta) / Z$ as long as $\mathbf{X} \perp \theta$
- Predictive distribution

Review: Bayesian learning



- Exact Bayes w/ conjugate prior, or numerical integration—this example: logistic regression
- Or, MLE/MAP

Review: MDPs



- Sequential decision problem under uncertainty
- States, actions, costs, transitions, discounting
- Policy, execution trace
- State-value (J) and action-value (Q) function
  - $(1-\gamma) \times$ immediate cost $+ \gamma \times$ future cost

Review: MDPs

- Tree search
- Receding horizon tree search w/ heuristic
- Dynamic programming (value iteration)
- Pruning (once we realize a branch is bad, or subsampling scenarios)
- Curse of dimensionality

Alternate algorithms for "small" systems—policy evaluation

$$
\begin{aligned}
Q^\pi(s,a) &= (1-\gamma)C(s,a) + \gamma\mathbb{E}[J^\pi(s') \mid s' \sim T(\cdot \mid s,a)] \\
J^\pi(s) &= \mathbb{E}[Q^\pi(s,a) \mid a \sim \pi(\cdot \mid s)]
\end{aligned}
$$

- Linear equations: so, Gaussian elimination, biconjugate gradient, Gauss-Seidel iteration, …
  - VI is essentially the Jacobi iterative method for matrix inversion
- Stochastic-gradient-descent-like
  - TD(λ), Q-learning

Alternate algorithms for "small" systems—policy optimization

- Policy iteration: alternately
  - use any above method to evaluate current π
  - replace π with ***greedy*** policy: at each state s, $\pi(s) := \arg\max_a Q(s,a)$
- Actor-critic: like policy iteration, but ***interleave*** solving for J^π and updating π
  - e.g., run biconjugate gradient for a few steps
  - warm start: each J^π probably similar to next
- SARSA = AC w/ TD(λ) critic, ϵ-greedy policy

Alternate algorithms for "small" systems—policy optimization

- (Stochastic) policy gradient
  - pick a parameterized policy class $\pi_\theta(a \mid s)$
  - compute or estimate $g = \nabla_\theta J^\pi(s_1)$
  - $\theta \leftarrow \theta - \eta g$, repeat
- More detail:
  - can estimate g quickly by simulating a few trajectories
  - can also use ***natural*** gradient to get faster convergence

Alternate algorithms for “small” systems—policy optimization

- Linear programming
  - analogy: use an LP to compute min(3, 6, 5)
  - note min v. max

$$\max\ J \quad \text{s.t.}$$
$$J \leq 3$$
$$J \leq 6$$
$$J \leq 5$$

Linear programming

$$\max \ J(s_1) \ \text{ s.t.}$$

$$Q(s,a) = (1-\gamma)C(s,a) + \gamma\mathbb{E}[J(s') \mid s' \sim T(\cdot \mid s,a)]$$

$$J(s) \leq Q(s,a) \qquad \forall s, a$$

- Variables J(s) and Q(s,a) for all s, a
- Note: dual of this LP is interesting
  - generalizes single-source shortest paths

Model requirements

- What we have to know about the MDP in order to plan?
  - full model
  - simulation model
  - no model: only the real world

Model requirements

- VI and LP require full model
- PI and actor-critic inherit requirements of policy-evaluation subroutine
- TD(λ), SARSA, policy gradient: OK with simulation model or no model
  - horribly data-inefficient if used directly on real world with no model—don't do this!
  - note: model can be just { all of my data }

A word on performance measurements

- Multiple criteria we might care about:
  - data (from real world)
  - runtime
  - calls to model (under some API)
- Measure convergence rate of:
  - $J(s)$ or $Q(s, a)$
  - $\pi(s)$
  - actual (expected total discounted) cost

Building a model

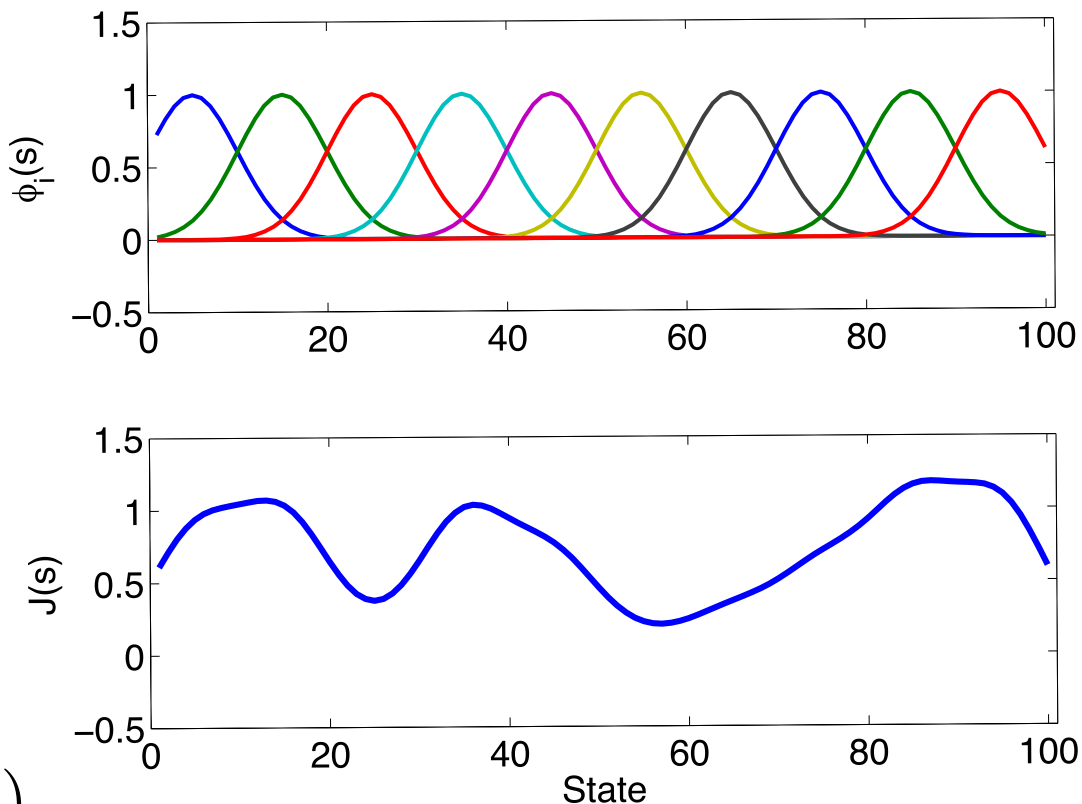
- How to handle lack of model without horrible data inefficiency? Build one!
  - hard inference problem; getting it wrong is bad
  - this is why { all of my data } is a popular model
- What do we do with posterior over models?
  - just use MAP model ("certainty equivalent")
  - compute posterior over π^*: slow, still wrong
  - even slower: $\max_{\pi} \mathbb{E}(J^{\pi}(s) \mid \text{data, model class})$
    - except policy gradient (Ng's helicopter)

Algorithms for large systems

- Policy gradient: no change
- Any value-based method: can't even write down J(s) or Q(s,a)
- So,

$$J(s) = \sum_i w_i \phi_i(s)$$

$$Q(s,a) = \sum_i w_i \phi_i(s,a)$$



Algorithms for large systems

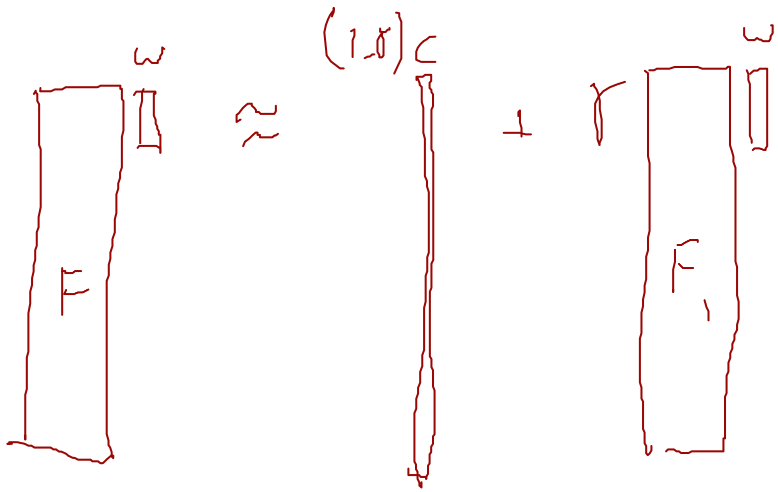
- Evaluation: TD(λ), LSTD
- Optimization:
  - policy iteration or actor-critic
    - e.g., LSTD $\rightarrow$ LSPI
  - approximate LP
  - value iteration: only special cases, e.g., finite-element grid

Least-squares temporal differences (LSTD)

$$
\begin{aligned}
Q^\pi(s,a) &= (1-\gamma)C(s,a) + \gamma\mathbb{E}[J^\pi(s') \mid s' \sim T(\cdot \mid s,a)] \\
J^\pi(s) &= \mathbb{E}[Q^\pi(s,a) \mid a \sim \pi(\cdot \mid s)]
\end{aligned}
$$

- Data: $\tau = (s_1, a_1, c_1, s_2, a_2, c_2, \ldots) \sim \pi$
- Want $Q(s_t, a_t) \approx (1-\gamma)c_t + \gamma Q(s_{t+1}, a_{t+1})$
  - $w^T\Phi(s_t, a_t) \approx (1-\gamma)c_t + \gamma w^T\Phi(s_{t+1}, a_{t+1})$
  - Φ = vector of k features, w = weight vector

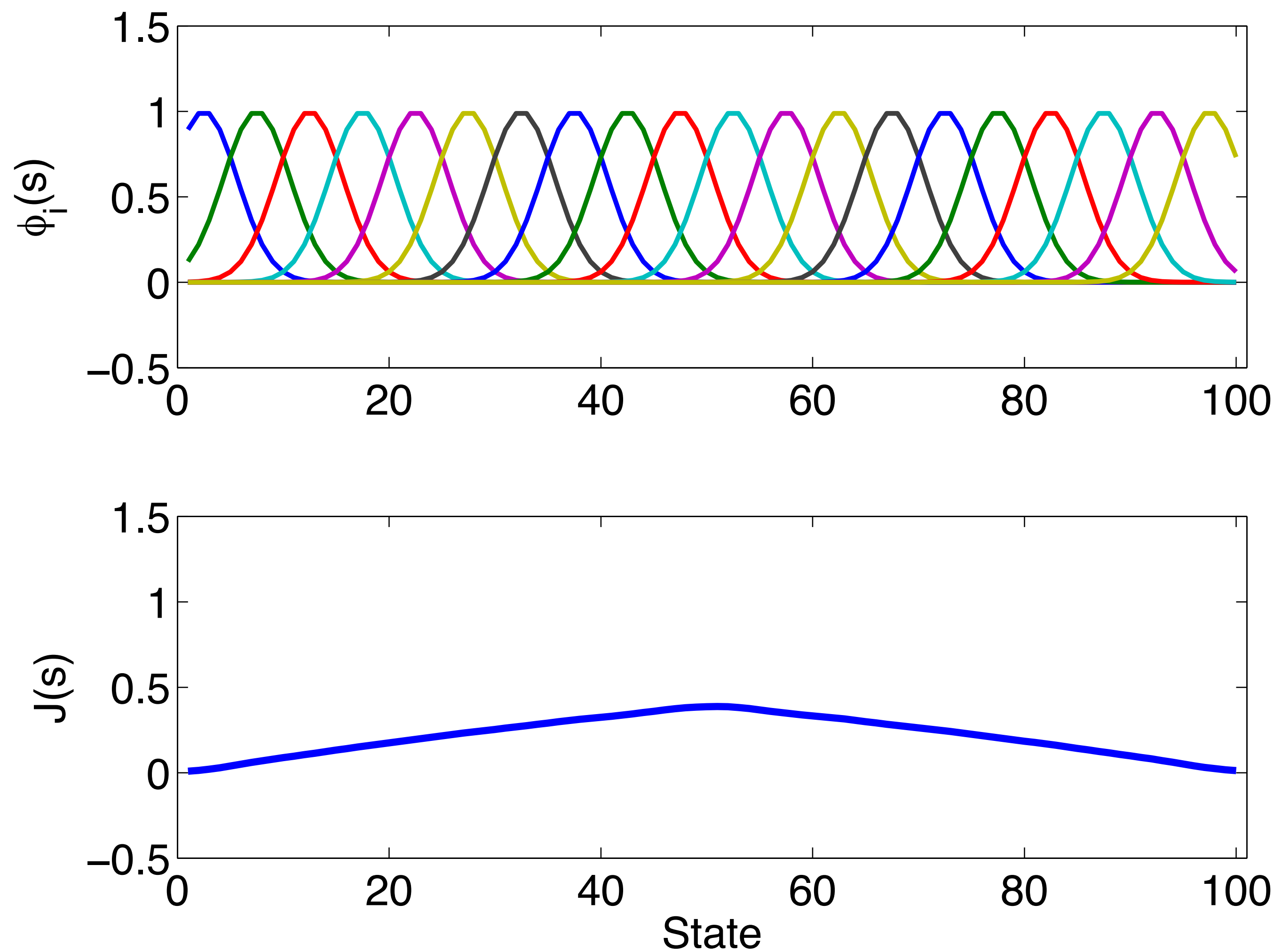
LSTD



- $w^T\Phi(s_t, a_t) \approx (1-\gamma)c_t + \gamma w^T\Phi(s_{t+1}, a_{t+1})$
- Vector notation:
  - $Fw \approx (1-\gamma)c_{\cancel{t}} + \gamma F_1 w$
- Overconstrained: multiply both sides by F
  - $F^T F w = (1-\gamma)F^T c_{\cancel{t}} + \gamma F^T F_1 w$

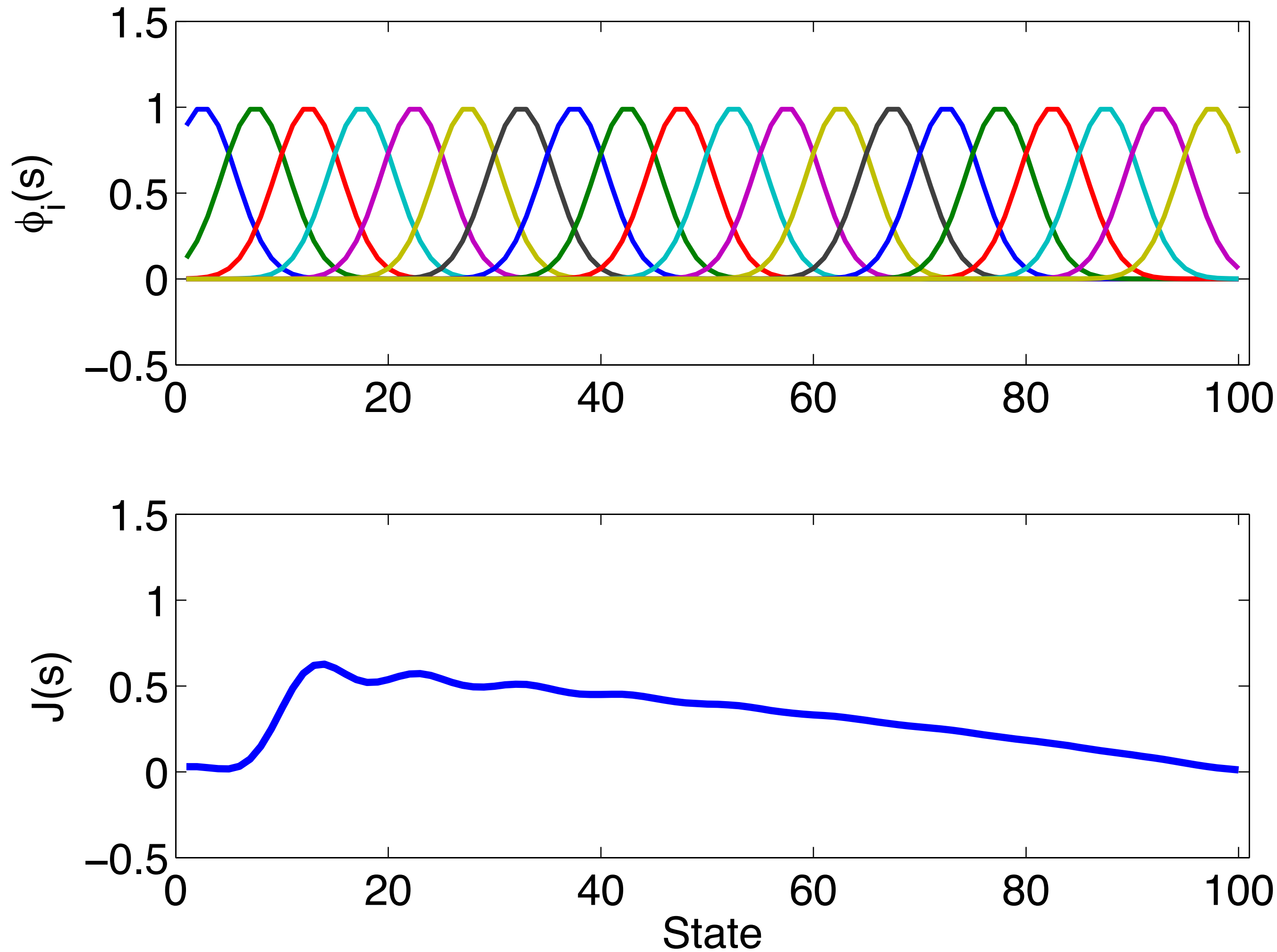
LSTD: example

- 100 states in a line; move left or right at cost 1 per state; goals at both ends; discount 0.99
- optimal policy

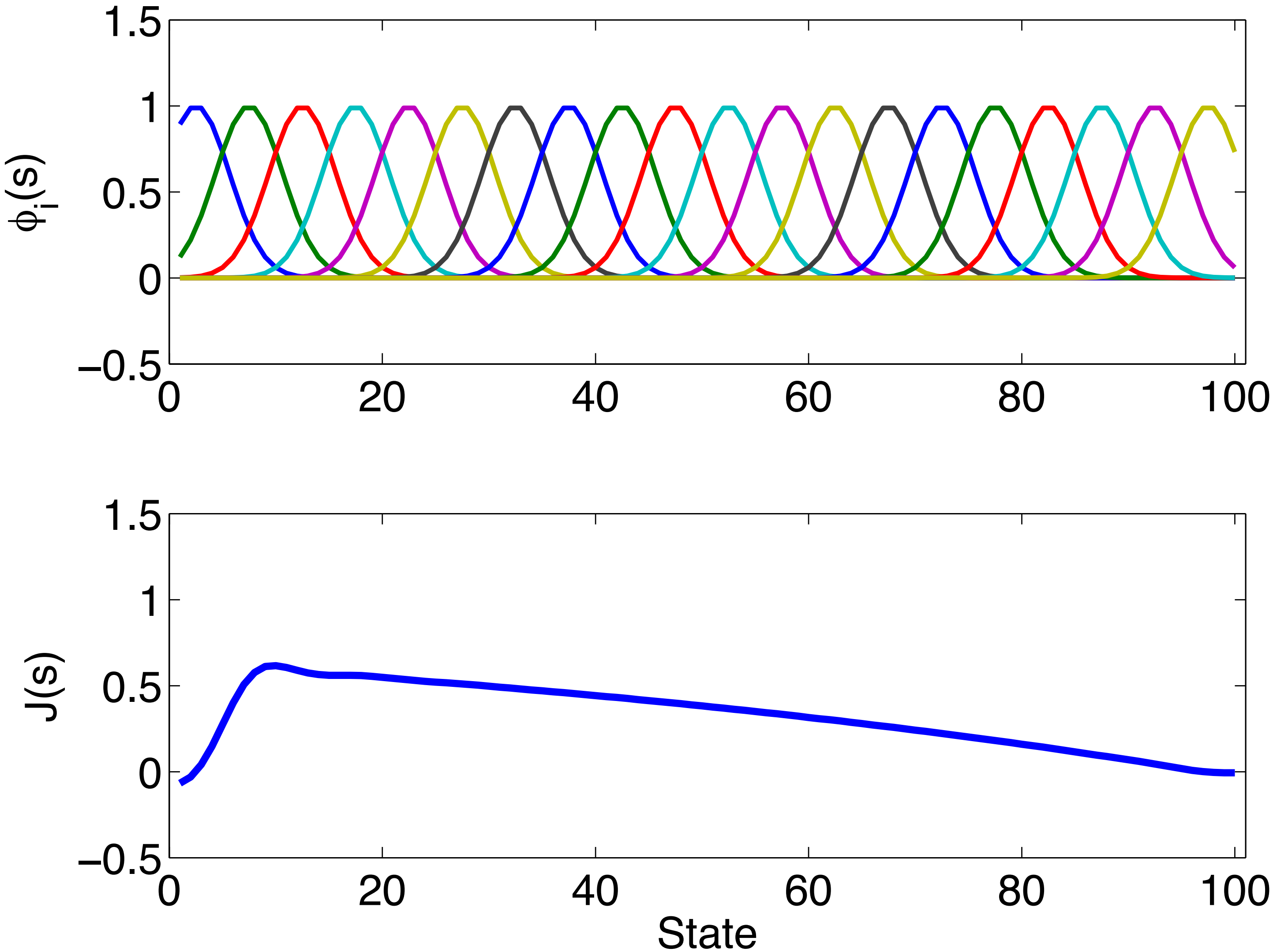


LSTD: example

- Compare: suboptimal policy
- True J(x) has discontinuity at x=10



LSPI



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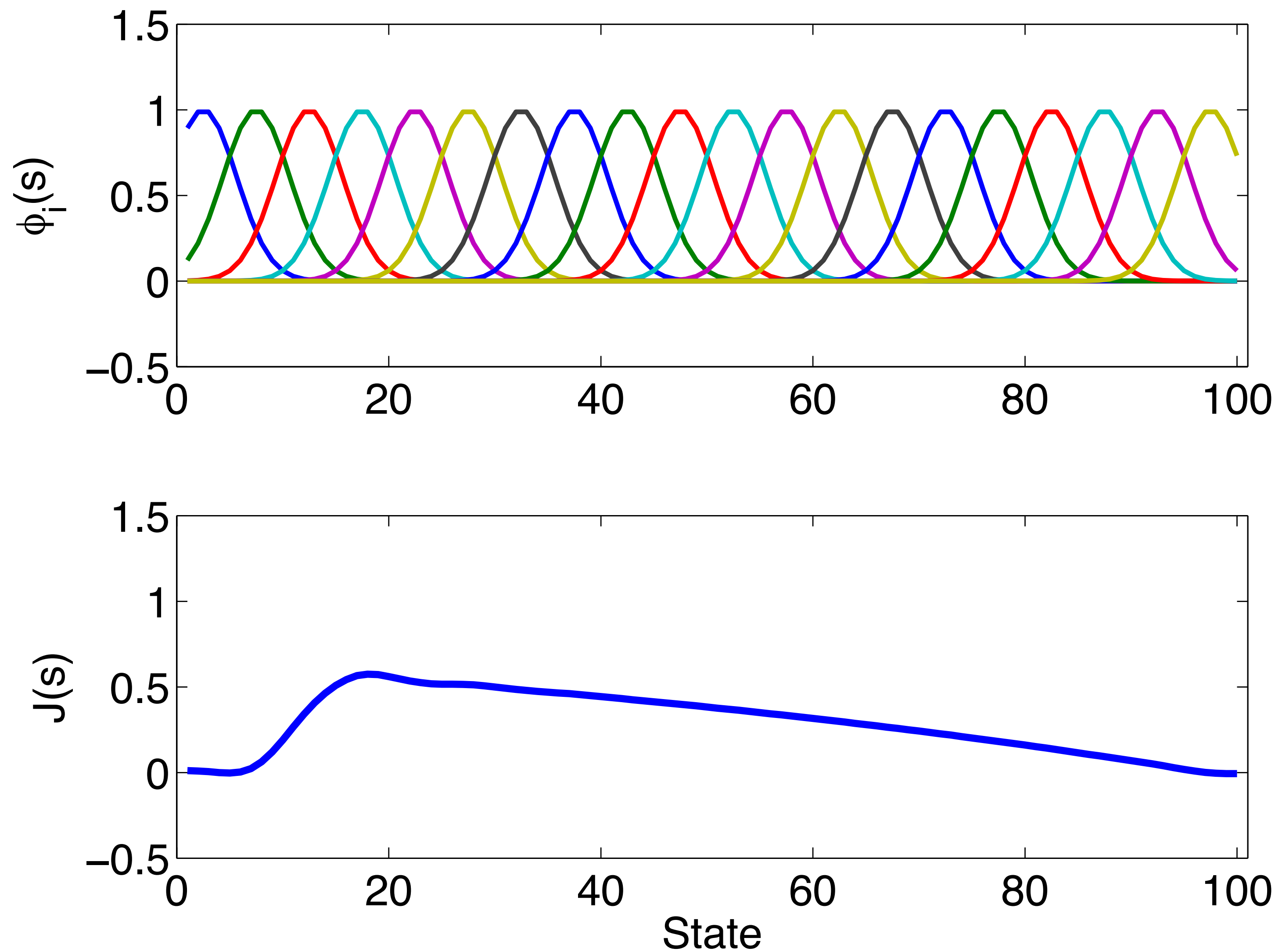
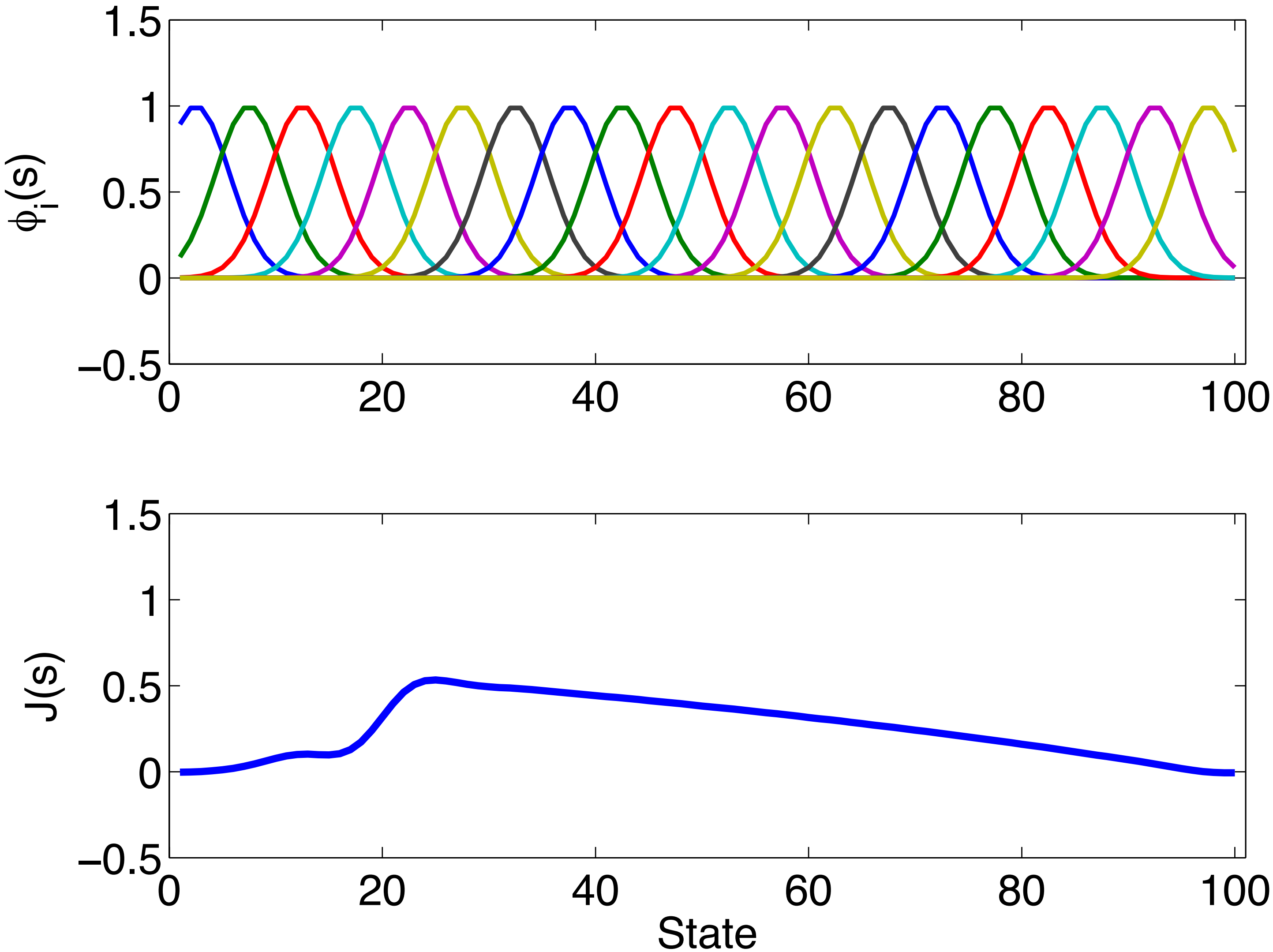
J(s)
State

LSPI



LSPI



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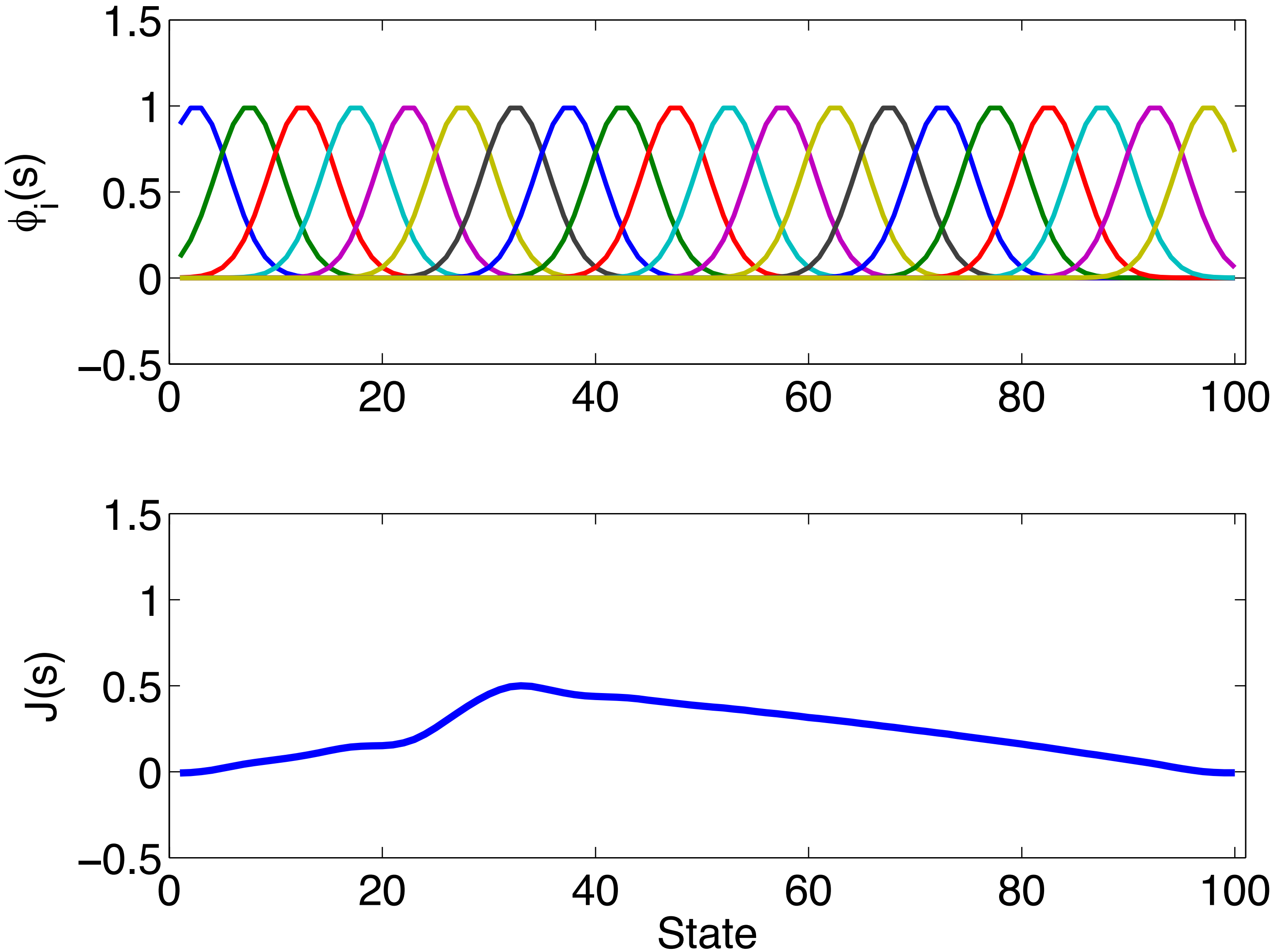

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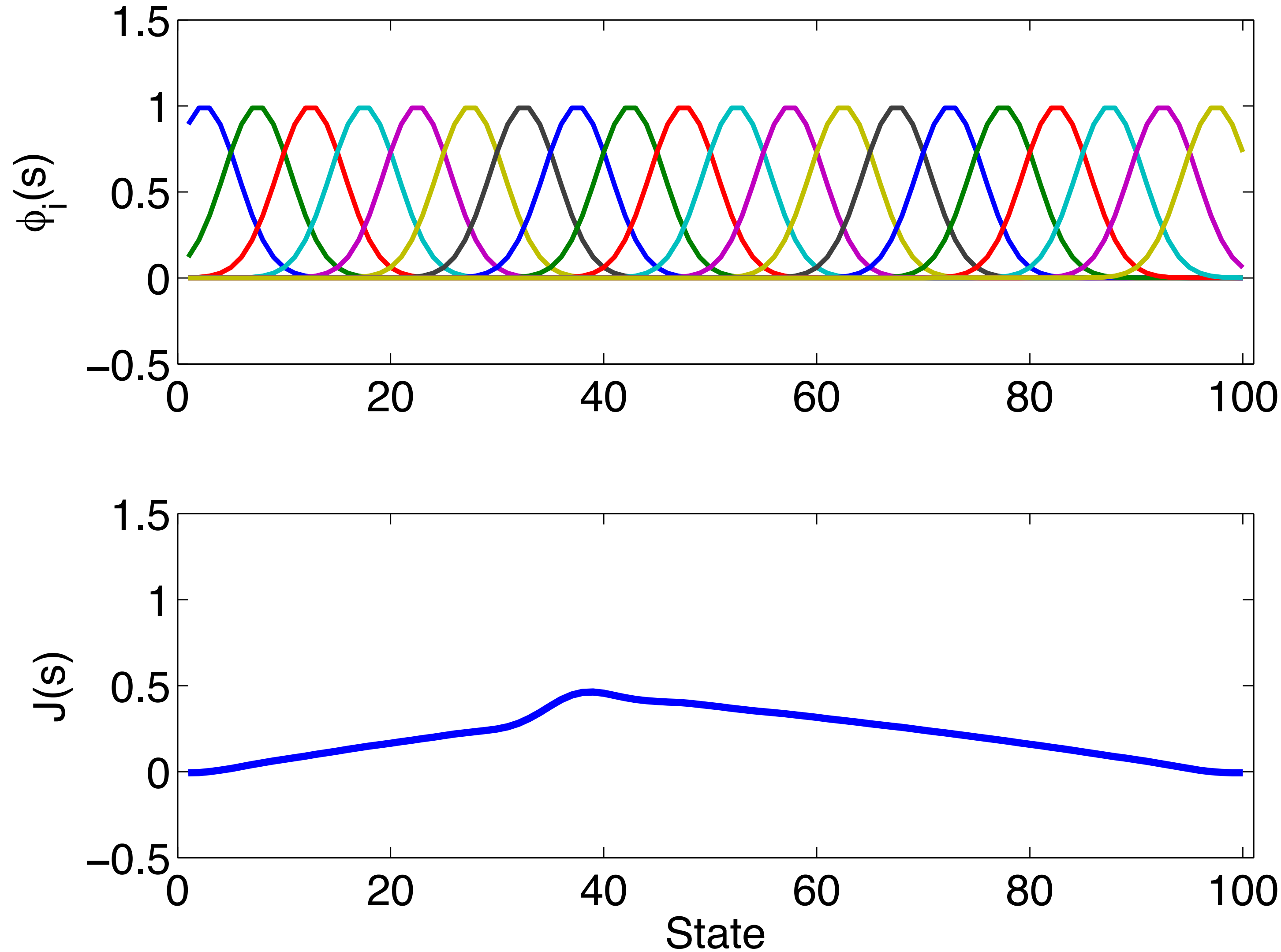

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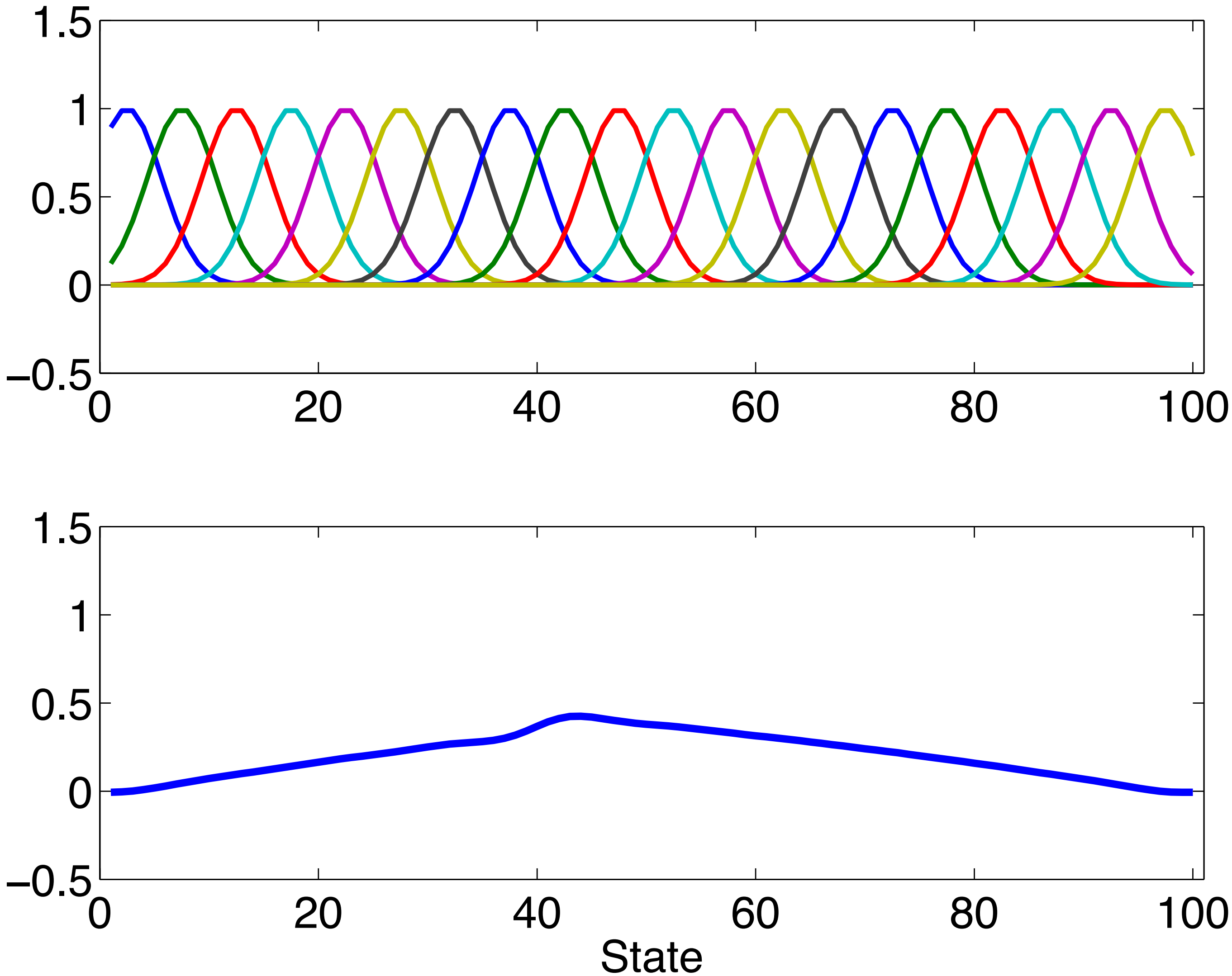

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