Statistical Entropy in General Equilibrium Theory

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This essay seeks to develop an integrated account of the workings of statistical mechanics and thermodynamics as a theory of economic equilibrium. It begins with a probabilistic description of general systems (made out of numerous elements), based on the practice of statistical physics and the work of E. T. Jaynes, and a self-contained overview of the arguments that lead to the concept of statistical entropy as a measure of uncertainty or disorder and the maximum statistical entropy principle. This provides the conceptual setting for developing a statistical mechanical model of general equilibrium in pure exchange economies, inspired by the statistical theory of markets of Duncan K. Foley. Emphasis is placed in the derivation of the properties of the entropy function of an economy—the maximized statistical entropy as a function of the amounts of resources in that economy. We then show that the statistical equilibrium theory of pure exchange economies gives rise to a phenomenological or 'macro' theory of resource allocation in the image of classical thermodynamics (and the generalized thermodynamics of L. I. Rozonoer). We thus establish the fundamental principle of the phenomenological theory—the maximum entropy principle—and illustrate its use for the study of isolated and small open economies.

0. Introduction

This essay seeks to develop an integrated account of the workings of statistical mechanics and thermodynamics as a theory of economic equilibrium. It is worth recalling[1] that *statistical mechanics* (or *statistical physics*) is concerned with the study of macroscopic physical systems from the microscopic point of view (i.e. on the basis of the atomic or molecular structure of matter). The part of statistical mechanics that is concerned with systems in equilibrium is usually referred to as *statistical thermodynamics*; it forms a bridge between classical thermodynamics and molecular physics. *Classical (or equilibrium) thermodynamics* or simply *thermodynamics*[2], on the other hand, is a macroscopic or phenomenological physical theory, that is, a theory in which the atomic or molecular structure of matter is ignored. It is grounded on generalizations of empirical regularities in the form of axioms, known as the laws of thermodynamics, from which one derives mathematical relations between observable properties of macroscopic physical systems in equilibrium. These laws

[1] See, for example, Balian (1991) and McQuarrie (2000).

[2] There are a few alternative formulations of thermodynamics. Unless otherwise stated, thermodynamics in this essay is identified with its neo-Gibbsian formulation, exemplified by Callen (1985), Tisza (1977), and Wightman (1979). Since it is only at the end of the essay that nonequilibrium thermodynamics will be mentioned, we will use the term thermodynamics as an abbreviation for equilibrium thermodynamics.

are obtained as theorems in statistical thermodynamics[3], which, in addition, provides the means to calculate physical properties (as opposed to thermodynamics which provides only relations between many properties).

In the course of the past ten years, Foley (1994, 19996a, b, 2003) has elaborated a statistical equilibrium theory of markets along the lines of statistical thermodynamics. The intention of that work is to transcend the confines of the Walrasian paradigm and thereby the Arrow-Debreu model[4] of general equilibrium [Arrow and Hahn (1971), Debreu (1959)]. Earlier, Wilson (1970) worked out a systematic statistical mechanical approach to urban and regional modeling[5]. That statistical mechanics should be relevant for the study of non-physical systems is not paradoxical. After the pioneering work of E. T. Jaynes[6], statistical physics may be seen as an instance of a formalism or *generalized statistical mechanics*, which is independent of physical properties and therefore applicable to a variety of disciplines[7]. To put it otherwise, statistical mechanics may be viewed as applied probability theory, more specifically as probability theory with constraints [Sornette (2004)].

To avoid confusion with the physical entropy of thermodynamics, we shall follow the Balian (1991) vocabulary regarding key statistical mechanical and thermodynamic terms. Thus when entropy signifies a measure of uncertainty or disorder associated with a probability distribution, it will be called *statistical entropy* rather than 'entropy'. Accordingly, the 'principle of maximum entropy' of Jaynes will be referred to as the *maximum statistical entropy principle.* The term *entropy function*, which corresponds to the physical entropy of thermodynamics, will be reserved for the maximum value function associated with certain constrained statistical entropy maximization problems, i.e. for the *maximized statistical entropy* as a function of the constraint constants.

While the subjects of statistical mechanics and thermodynamics are firmly interconnected in the realm of physics, this has not been the case with their counterparts in other disciplines. One of the objectives of this essay is to show that such a deep interconnection does obtain in economics as well. We intend to construct a coherent narrative that begins with generalized statistical mechanics as general systems theory, continues with the combination of statistical mechanics and microeconomics that results in the statistical theory of markets, and concludes with the derivation of a phenomenological theory of resource allocation in the image of thermodynamics. For if such a theoretical narrative cannot be told, the separate applicability of statistical mechanics and thermodynamics to economics, however insightful and important, is nothing but the transfer of useful techniques from one discipline to another; it does not constitute an alternative, unified theoretical perspective.

[3] This is a simplified description of the relation between statistical physics and classical thermodynamics, but adequate for the purposes of this essay—see, for instance, Yi (2003) for the subtleties of this relationship.
[4] This is the prevalent name of the theory, even though there other important contributors, most notably McKenzie (2003).
[5] The recent outburst of writings on the use of statistical mechanical methods in finance, also known as econophysics, is beyond the scope of this essay—it will be taken up in future papers.
[6] See Rosenkrantz (1989) for a collection of important papers by Jaynes on probability, statistics, and statistical physics, including his first two seminal papers published in *Physical Review* in 1957. See also Jaynes (2003) for amore detailed account of his views on probability theory.
[7] See, for example, Kapur and Kesavan (1992).

The influence of thermodynamics to the formation of mathematical economics is well-known. The methods of classical thermodynamics, as perfected by the great American physicist J. Willard Gibbs (1839-1903), inspired Samuelson's path-breaking 1947 book on constrained optimization and comparative statics in economic analysis [Samuelson (1983)]. Samuelson (1989) himself gives a fascinating account of the influence of Gibbs in modern economics. He makes it crystal clear, though, that the benefit to theoretical economics comes from the 'mathematical isomorphisms' between the maximum-minimum structures of thermodynamics and the cost-profit-utility systems of economics.

Yet numerous attempts have been made to show that the affinity of thermodynamics to economics is much closer[8]. The generalized thermodynamics of Rozonoer (1973) is particularly interesting for our purposes: it is a thorough and systematic attempt to obtain a phenomenological theory of resource allocation from a synthesis of thermodynamics and economics, including the general equilibrium theory of pure exchange economies. More recently, Smith and Foley (2002) have also developed a systematic thermodynamic reading of neoclassical utility and general equilibrium theory. The great virtue of the paper by Smith and Foley is their demonstration that "the correspondence of utility theory to thermodynamics defines a whole consistent methodology, and not just a set of analogies" [Smith and Foley (2002, p. 22)]. Indeed, in a remarkable paper Candeal et al. (2001) have shown that the mathematical structure of the utility representation problem is identical to that of the entropy representation problem in the Caratheodory version of classical thermodynamics. The preference ordering of an individual agent corresponds to the ordering of the states of a macroscopic system (dictated by the Second Law of thermodynamics). So this mathematical association relates individual agents to the macroscopic systems of classical thermodynamics, and not to the particles of Newtonian mechanics.

While immensely insightful, the formal equivalence between utility and thermodynamic entropy does not address the main concern of this essay, namely, the interconnection between the statistical mechanical and thermodynamic accounts. We will follow a different route—the route opened up by Foley's statistical theory of markets. We will be able to show (by means of the maximum statistical entropy principle) that every trader in a pure exchange economy is fully characterized by her entropy function, which looks like a neoclassical utility function, but is a fundamentally different concept. It describes our best prediction of the agent's behavior on the basis of knowing just her preferred action set (i.e. the consumption bundles weakly preferred to her endowment) and the expected value of her action.

In **Section 1**, we present a probabilistic description of general systems (made out of numerous elements), based on the practice of statistical physics and the work of Jaynes, and a self-contained overview of the arguments that lead to the concept of statistical entropy as a measure of uncertainty or disorder and the maximum statistical entropy principle. This provides the conceptual setting for developing a statistical mechanical model of general equilibrium in pure exchange economies in **Sections 2 and 3**, inspired by the statistical theory of markets of Foley. Emphasis is placed in the derivation of the properties of the entropy function of an economy—the maximized statistical

[8] See, for example, Burley and Foster (1994).

entropy as a function of the amounts of resources in that economy. We then show in **Section 4** that the statistical equilibrium theory of pure exchange gives rise to a phenomenological or 'macro' theory of resource allocation in the image of classical thermodynamics (and the generalized thermodynamics of Rozonoer). We thus establish the fundamental principle of the phenomenological theory—the maximum entropy principle. In **Section 5**, we develop a generalized version of the maximum entropy principle and illustrate its workings in the cases of fully heterogeneous agents and small open economies. Finally, **Section 6** is devoted to some concluding remarks.

1. Probabilistic Description of Systems and Statistical Entropy

Following the practice of statistical mechanics [Balian (1991)] we use the term *micro-state* to signify a complete description of a system under investigation—all properties of the system are fully specified once its micro-state is known. Any given system is associated with a space Γ of all possible micro-states; to avoid issues of convergence of infinite sums or measure theory we will presume that Γ is a finite set. We will be concerned with situations where, as in statistical thermodynamics, the number of macroscopic data and associated constraints is far from sufficient to specify the system's micro-state. That is, there are many micro-states in Γ that are compatible with the available data. With this sort of insufficient information about the state of the system, one typically resorts to a statistical treatment.

This amounts to giving Γ the structure of a probability space. Single-point subsets of Γ are to be viewed as elementary events; we only need to assign probabilities to the latter in order to specify a probability measure on the assembly of all events or subsets of Γ. Such a probability assignment is usually referred to as a probability distribution or probability law on Γ in the physics or other applied probability theory literature. The microscopic description of the system's *macro-state* is just a probability distribution on Γ.

Suppose next that Γ consists of n distinct micro-states, where n is a positive integer[9], and let $p_1, \cdots, p_n$ be the respective probabilities, which should be nonnegative and add up to one. Thus a probability distribution is a vector in the n-1 dimensional unit simplex

$$\Delta := \left\{ p \in \Re_+^n \middle| p_1 + \cdots + p_n = 1 \right\} \tag{1.1}$$

with p_i being the probability of the *ith* micro-state (i=1,…,n) and $p := (p_1, \cdots, p_n)$. Here $\Re_+^n$ is the non-negative orthant of the n-dimensional Euclidian space $\Re^n$; the symbol := means 'by definition equal'. Very schematically, micro-level theory deals with the elements that constitute the macroscopic system (along with their interactions) and determines the set Γ of possible micro-states. All properties of the system are functions of the micro-state and hence random variables on Γ. Their moments can in principle be specified once p is known. Thus the selection of p is a crucial step for linking the micro and macro descriptions of the system under study.

[9] The nature of the micro-states or elementary events is immaterial at this point—what counts is that there are a finite number of possible micro-states.

Suppose we do have an exhaustive list of mutually exclusive outcomes of a conceptual experiment—the set Γ of all possible micro-states of a system under study—but we know nothing else. There is no basis—say, on account of symmetry considerations—to single out any particular outcome to be more likely than another. Under these circumstances, the *principle of insufficient reason* (PIR) dictates that we should assign all the micro-states equal probabilities[10]. That is, the PIR would prescribe a uniform probability distribution, $p_i = 1/n$ for $i = 1, \cdots, n$.

The case where the n micro-states are equiprobable represents a situation of the greatest uncertainty or disorder regarding the predictability of the micro-state. In contrast, there is no uncertainty or disorder at all if p happens to be a *degenerate probability distribution*: one of the probabilities $p_1, \cdots, p_n$ is equal to one and all others are equal to zero. The degree of uncertainty or disorder is in between these two extremes for all $p \in \Delta$ that are neither degenerate nor uniform probability distributions. The lack of information or measure of uncertainty or degree of disorder associated with each probability distribution in Δ is quantified by the concept of *statistical entropy*: a function $H : \Delta \to \Re$ defined by

$$H(p_1, \cdots, p_n) := -k \sum_{i=1}^{n} p_i \ln p_i, \tag{1.2}$$

with k being an arbitrary positive constant, which we may set equal to one without loss of generality[11]. It is understood in (1.2) that $0 \ln 0 := 0$.

It is useful to review some of the properties of statistical entropy in order to confirm that it is a reasonable measure of uncertainty or disorder. We note first that H is positive semi-definite and that $H(p) = 0$ if and only if p is a degenerate probability distribution, signifying (as we have just seen) complete certainty. The statistical entropy is positive, on the other hand, for any non-degenerate distribution in Δ, that is, when the outcome of a conceptual experiment is uncertain. The uniform probability distribution $(1/n, \cdots, 1/n)$ maximizes H over Δ, consistently with the notion that the greatest uncertainty or *complete disorder* obtains when all outcomes are equally likely, namely, $p_i = 1/n$ for $i = 1, \cdots, n$. The associated degree of disorder

$$H(1/n, \cdots, 1/n) = k \ln n \tag{1.3}$$

is strictly increasing with the number n of micro-states, as expected. Further, it is easy to see that H is a continuous and strictly concave function[12].

[10] See, for example, Grandy (1987) and references therein.
[11] The term 'statistical entropy' originated in statistical physics and is still maintained in some theoretical texts such as Balian (1991) to whom we refer for more discussion and an excellent account of the history of the entropy concept. Usually, statistical entropy is referred to simply as entropy; the term Shannon's entropy is also used.
[12] For a more detailed account of the properties of statistical property see, among others, Balian (1991), Khinchin (1957), or Kapur and Kesavan (1992).

Turn next to the additivity properties of statistical entropy [Balian (1991), Kapur and Kesavan (1992), Khinchin (1957), Wannier (1987)]. To articulate these properties, consider two systems *A* and *B* with respective spaces of micro-states Γ^a and Γ^b; and let *L* and *M* be the respective numbers of micro-states, where *L* and *M* are positive integers. The corresponding probability assignments are designated by $p^a := (p_1^a, \cdots, p_L^a)$ and $p^b := (p_1^b, \cdots, p_M^b)$, where $p^i \in \Delta^i$ for $i = a, b$. Δ^a designates the *L*-1 dimensional unit simplex, while Δ^b stands for the *M*-1 dimensional unit simplex. Instead of separate experiments involving the individual systems, we may think of a joint experiment for the ensemble of A and B whose possible outcomes are pairs of micro-states of A and B. This composite system is denoted by AB; its space of micro-states is the Cartesian product $\Gamma^a \times \Gamma^b$ of Γ^a and Γ^b. The associated joint probability distribution is denoted by

$$p^a * p^b := (p_{11}, \cdots, p_{LM}) \tag{1.4}$$

with p_{lm} being the probability that A is in micro-state *l* *and* B in micro-state *m* $(l = 1, \cdots, L; m = 1, \cdots, M)$. $p^a * p^b$ is a vector in the *LM*-1 dimensional unit simplex. The statistical entropy $H(p^a * p^b)$ of the joint probability distribution $p^a * p^b$ is given by

$$H(p^a * p^b) = -k \sum_{l=1}^{L} \sum_{m=1}^{M} p_{lm} \ln p_{lm}. \tag{1.5}$$

Begin with a situation where A and B are statistically independent in the sense that

$$p_{lm} = p_l^a \cdot p_m^b \tag{1.6}$$

for $l = 1, \cdots, L$ and $m = 1, \cdots, M$. In turn, insert (1.6) into (1.5) and recall the definitions of $H(p^a)$ and $H(p^b)$ along with the normalization conditions for p^a and p^b to obtain

$$H(p^a * p^b) = H(p^a) + H(p^b), \tag{1.7}$$

which is referred to as the *additivity property* of statistical entropy. Thus for two independent probability distributions, the statistical entropy of the joint probability distribution equals the sum of the statistical entropies of the individual distributions. It is easy to see that the additivity property holds for any finite number of independent probability distributions.

If A and B are not statistically independent and hence (1.6) does not hold, the joint distribution $p^a * p^b$ is the primary concept with the marginal distributions p^a and p^b defined by

$$p_l^a := \sum_{m=1}^{M} p_{lm} \qquad (l = 1, \cdots, L) \tag{1.8}$$

and

$$p_m^b := \sum_{l=1}^{L} p_{lm} \qquad (m = 1, \cdots, M), \tag{1.9}$$

respectively. To see how (1.7) gets modified under conditions of non-independence, observe that instead of (1.6) we have

$$p_{lm} = p_l^a \cdot q_{lm}^b, \tag{1.10}$$

where q_{lm}^b is the conditional probability that the system B is in micro-state *m* given that A is in the *lth* micro-state[13]. Substituting (1.10) for p_{lm} in (1.5) we obtain

$$H(p^a * p^b) = -k \sum_{l=1}^{L} p_l^a \ln p_l^a \sum_{m=1}^{M} q_{lm}^b - k \sum_{l=1}^{L} p_l^a \sum_{m=1}^{M} q_{lm}^b \ln q_{lm}^b. \tag{1.11}$$

From (1.8) and (1.10) we infer that $\sum_{m=1}^{M} q_{lm}^b = 1$ for any $l = 1, \cdots, L;$ further, we may regard

$$H(q_{l\cdot}^b) := -k \sum_{m=1}^{M} q_{lm}^b \ln q_{lm}^b \tag{1.12}$$

as the statistical entropy of the conditional probability distribution

$$q_{l\cdot}^b := (q_{l1}^b, \cdots, q_{lM}^b) \tag{1.13}$$

of the micro-states of B when A is known to be in micro-state *l*. Hence (1.11) becomes

$$H(p^a * p^b) = H(p^a) + \sum_{l=1}^{L} p_l^a H(q_{l\cdot}^b), \tag{1.14}$$

which is known as the *strong additivity property* of statistical entropy.

Using the convexity of the function $x \mapsto x \ln x,$ one can show[14] that the second term in the right side of (1.14) cannot exceed $H(p^b)$ and hence (1.14) yields

$$H(p^a * p^b) \leq H(p^a) + H(p^b), \tag{1.15}$$

the *subadditivity property* of statistical entropy. It states that when p^a and p^b are not necessarily independent, the statistical entropy of the joint probability distribution is less than or equal to the

[13] The conditional probability in (1.10) is defined only for $p_l^a > 0.$ The possibility that $p_l^a = 0$ for some *l* does not cause any problem because in such a case we would have $p_{lm} = 0$ as well (for all *m*).

[14] See, for example, Kapur and Kesavan (1992).

sum of the statistical entropies of the (marginal) probability distributions associated with the individual systems. To put it in another way, inequality (1.14) states that we lack less information about the composite system AB when we know the correlations between the micro-states of A and B than when we know the separate marginal probabilities of the micro-states of each system [Balian (1991, p. 106)]. The generalization of (1.15) to any finite number of systems is straightforward.

The properties derived from the expression (1.2) for statistical entropy are all desired features for a plausible measure of uncertainty or disorder[15]. But we only need to posit some of these properties for a measure of uncertainty in order to show that it must of the form (1.2). Shannon (1948) was the first to derive such a result. We summarize here Khinchin's (1957) version of this result, his uniqueness theorem. Return to our original system with a set Γ of n micro-states and the associated space of probability distributions: the n-1 unit simplex Δ as defined in (1.1). Let $H:\Delta\to\Re$ be a function which is defined and continuous for any positive integer n and, in addition, has the following properties: a) for given n, the uniform probability distribution $(1/n,\cdots,1/n)$ maximizes H over Δ; b) H satisfies the strong additivity property as in (1.14); and c) Adding an impossible micro-state or any number of impossible micro-states does not change the value of H, that is

$$H(p_1,\cdots,p_n,0)=H(p_1,\cdots,p_n). \tag{1.16}$$

Then, Khinchin has shown, H must be of the form (1.2), which is the expression for statistical entropy.

The method of deriving the probability distribution $p:=(p_1,\cdots,p_n)$, i.e. the system's macro-state, on the basis of macroscopic data was principally established[16] by Jaynes (1957), which he termed 'the principle of maximum entropy'. Following Balian (1991), we shall call it the *maximum statistical entropy principle*. It stipulates that amongst all probability distributions that are compatible with the available data we must choose the most disordered macro-state, namely the probability distribution that has the greatest statistical entropy. Any other choice of macro-state, the argument goes, would entail an arbitrary attribution of order for which there is no foundation.

It is important to distinguish two different types of available information. One type consists of data given with certainty. This kind of information is taken into account by appropriately restricting the space Γ of allowable micro-states. If this is the only information we have, the maximum statistical entropy principle coincides with the PIR: it entails that the system's macro-state should be a uniform probability distribution. Things change, though, if there is information in the form of data of a statistical nature: if the expected values of some macroscopic variables are predetermined. Suppose there are m macro-variables, real-valued functions of the micro-state $i\mapsto g_{ri}, i=1,\cdots,n, r=1,\cdots,m,$ where m is a positive integer less than n-1, whose expectation values are given. Then we have m constraints of the form $\sum_{i=1}^{n}p_i g_{ri}=x_r, r=1,\cdots,m,$ where $x_1,\cdots,x_m$ are known real constants. The maximum statistical entropy principle amounts to the following constrained maximization problem:

[15] To avoid confusion with the notion of uncertainty in economics, we will predominantly refer to statistical entropy as a measure of disorder from now on. Besides the term disorder is overwhelmingly used in statistical thermodynamics, and is more intuitively appealing for our purposes.
[16] For other scholars who independently contributed to the method see Kapur and Kesavan (1992, p. 36).

Choose p to

$$\max[-\sum_{i=1}^{n} p_i \ln p_i] \tag{1.17}$$

subject to

$$\sum_{i=1}^{n} p_i = 1, \tag{1.18}$$

$$\sum_{i=1}^{n} p_i g_{ri} = x_r, \qquad r = 1, \cdots, m, \tag{1.19}$$

$$p_i \geq 0, \qquad i = 1, \cdots, n. \tag{1.20}$$

Note that the constant k in the expression (1.2) for statistical entropy has been normalized to one—a practice to be followed hereafter. We will also find it convenient to write separately the normalization requirement (1.18) and the non-negativity condition (1.20) for p instead of the compact statement $p \in \Delta$; it turns out [Kapur and Kesavan (1992)] that inequalities (1.20) will be automatically satisfied.

Rather than examining the constrained maximization problem (1.17)—(1.20) in general terms, we turn to the study of problems of this sort in the context of a statistical mechanical model of pure exchange, inspired by the statistical equilibrium theory of markets of Foley (1994, 1996a,b, 2003).

2. A Statistical Mechanical Model of Pure Exchange

In this and the following section we develop a statistical mechanical model of pure exchange that draws more specifically upon Foley (1994, 2003). We begin with the specification of a pure exchange economy in an Arrow-Debreu fashion [Debreu (1998)]. Thus we posit a world of L distinct goods or useful (desirable) entities referred to as resources, inhabited by N individual *agents* or *traders*. Note that we prefer to use the neutral (and broader) term 'resource' a la Rozonoer (1973) instead of the standard Arrow-Debreu 'commodity'. All activities or endowments are ordered L-lists of quantities of resources or points in the *resource space* $\Re^L$. Resources are typically indexed by $l = 1, \cdots, L$, while agents are indexed by $n = 1, \cdots, N$; L and N are positive integers. The consumption or use of the lth resource $(l = 1, \cdots, L)$ by the n th agent $(n = 1, \cdots, N)$ is a nonnegative real number $s_l^n \geq 0$; and the agent's consumption activity or *action* is the vector $s^n := (s_1^n, \cdots, s_L^n) \in \Re_+^L$, with $\Re_+^L$ being her *consumption set*. It is assumed that all agents have the same consumption set $\Re_+^L$.

Every agent $n \in \{1, \cdots, N\}$ is characterized by the pair $(e^n, \succeq_n)$ of the endowment $e^n \in \Re_+^L$ and of the preference relation $\succeq_n$ on $\Re_+^L$, where $\succeq$ stands for 'at least as good as' or weak preference; the agent's willingness to trade as she enters the market may be represented by her *preferred action set* Γ^n : the set of actions or resource bundles in $\Re_+^L$ that are at least as good as her endowment[17]. That is,

$$\Gamma^n := \left\{ s^n \in \Re_+^L \middle| s^n \succeq e^n \right\}. \tag{2.1}$$

Any point in the preferred action set represents a potential trading activity. The endowment of each agent is by definition an element of her preferred action set—inaction or no trade is always a possible action.

The data of a pure exchange economy E with L resources and N agents are summarized by the array

$$\mathrm{E} := (e^n, \succeq_n)_{n=1,\cdots,N}. \tag{2.2}$$

It is assumed that the *total endowment* of the economy E, defined by

$$e := \sum_{n=1}^{N} e^n , \tag{2.3}$$

is strictly positive in all its components, that is, $e \in \Re_{++}^L$. Even though we could allow for individual endowment vectors with some or all components equal to zero, we will assume that $e^n \in \Re_{++}^L$ for all individuals $n \in \{1, \cdots, N\}$ for the sake of simplicity. The total endowment coincides with the total supply of E if the latter is an isolated economy. Aside from the standard properties of reflexivity, totality (or completeness), and transitivity, we will make no other assumption regarding the preference relation. In particular, we do **not** require that the preferred action sets as in (2.1) be convex. In fact, pretty much as in Foley (1994), we will make the simplifying assumption that $\forall n \in \{1, \cdots, N\}$, Γ^n is a large but finite subset of $\Re_{++}^L$.

To motivate the probabilistic considerations of the model, we will make the additional assumption [after Foley (1994)] that the set of agents is partitioned into I equivalence classes or *types* according to a predetermined rule $n \mapsto i(n) \in \{1, \cdots, I\}$, where I is a known positive integer[18]. Agents of the same type have the same preferred action set. Thus in effect there are only I distinct preferred action sets, indexed by $i = 1, \cdots, I$. The preferred action set of agent n is $\Gamma^{i(n)}$, while Γ^i designates the common preferred action set of all agents of type i, that is, of all agents n for whom $i(n) = i$. There

[17] The preferred action set corresponds to Foley's concept of a transaction or offer set. But where Foley (1994, 2003) defines the latter in terms of net demands or transactions, our definition is in terms of agent actions or gross demands. Further, unlike Foley, our preferred action set is a derived concept rather than a primitive of the theory.
[18] As shown in Section 5, this assumption is not essential for the validity of the statistical mechanical model.

are N_i agents of type i, $i = 1, \cdots, I$, where $N_1, \cdots, N_I$ are pre-specified large positive integers, which sum up to the total number of agents, i.e.

$$N_1 + \cdots + N_I = N. \tag{2.4}$$

For concreteness, we may presume that agents of type 1 are labeled by $1, \cdots, N_1$, agents of type 2 are labeled by $N_1 + 1, \cdots, N_1 + N_2$, and so forth. A complete specification of the state of the economy or a *micro-state* $\vec{s}$ is an ordered list of agent actions:

$$\vec{s} := (s^1, \cdots, s^{N_1}, s^{N_1+1}, \cdots, s^{N_1+N_2}, \cdots, s^N). \tag{2.5}$$

This concept of a micro-state corresponds to Foley's definition of a "market allocation." Not every vector $\vec{s} \in \Re_+^{LN}$ is a possible micro-state: no individual agent will engage in activities outside her preferred action set. Accordingly, a micro-state as in (2.5) is possible if $s^n \in \Gamma^{i(n)} \forall n \in \{1, \cdots, N\}$, namely if

$$\vec{s} \in \Gamma := (\Gamma^1)^{N_1} \times \cdots \times (\Gamma^I)^{N_I}. \tag{2.6}$$

Here $(\Gamma^i)^{N_i}$ is the $N_i - fold$ Cartesian product of Γ^i $(i = 1, \cdots, I)$. Thus Γ as given in (2.6) is the space of all possible micro-states for our exchange economy E. A microscopically possible micro-state $\vec{s} := (s^1, \cdots, s^{N_1}, s^{N_1+1}, \cdots, s^{N_1+N_2}, \cdots, s^N) \in \Gamma$ is *globally feasible* if it satisfies the *market-clearing condition*

$$\sum_{n=1}^{N} s^n = e. \tag{2.7}$$

Clearly, there are a very large number of globally feasible micro-states. In the Arrow-Debreu model, agents are faced with a vector of resource (commodity) prices (cried out by the Walrasian auctioneer); they choose the action that is highest in their preference scale, subject to the budget constraint that the value of the selected action cannot exceed the value of their endowment. Under appropriate conditions, individual actions or gross demands are determined as functions of the price vector. So does the total demand on the left-hand side of (2.7). The price vector is determined by the latter (up to a multiplicative constant). This entails in turn the determination of all individual actions (given that they are homogeneous of degree zero in prices).

The Arrow-Debreu method of removing indeterminacy leaves completely open the question of how a price system comes to form in the first place. The virtue of the statistical mechanical approach is that the *emergence* of prices is the hallmark of equilibrium [Foley (2003)]. In this approach, the indeterminacy of the economy's micro-state is dealt with in a fundamentally different fashion. In the absence of information about trading practices at the micro level, it is posited that all globally feasible micro-states are equally probable. The market is considered in the words of Foley (2003, p.102) as "a chaotic process that tends to explore all feasible patterns of market transactions." The objective of the theory is no longer the prediction of the action of each individual agent; rather, we want to find out how the agents of every type are distributed over their preferred action sets. The equal-probability premise entails that the agent distribution that is most likely to obtain is the one that gets realized in the greatest number of microscopic ways. This is tantamount—as we shall see shortly—to positing a probability distribution on Γ whose statistical entropy gets maximized, subject to the market-clearing condition (2.7)[19]. The latter will be interpreted as a statistical datum—namely, as knowledge of the mean total action or demand (with respect to the probability distribution under determination).

Assume that individual actions are *statistically independent*: the probability $p(\vec{s})$ of a market allocation $\vec{s} \in \Gamma$ equals the product of the probabilities of the individual actions that constitute that allocation. Thus we only need to know the probability $p^i(s)$ that an agent of type i takes action s for all $s \in \Gamma^i$ and for all $i \in \{1, \cdots, I\}$ in order to ascertain the probability of any micro-state. That is, the probability of the micro-state $\vec{s}$ as in (2.5)—which is the economy's macro-state—is given by

$$p(\vec{s}) = p^1(s^1) \cdots p^1(s^{N_1}) \cdot p^2(s^{N_1+1}) \cdots p^2(s^{N_1+N_2}) \cdots p^I(s^N). \tag{2.8}$$

The macro-state $[p(\vec{s})]_{\vec{s}\in\Gamma}$ is in effect specified by a list $\{[p^i(w)]_{w\in\Gamma^i} \,|\, i = 1, \cdots, I\}$ of I probability vectors; we shall often summarize the latter by $p := (p^1, \cdots, p^I)$, where $p^i := [p^i(w)]_{w\in\Gamma^i}, i \in \{1, \cdots, I\}$. Given that once p is known, the macro-state $\vec{s} \mapsto p(\vec{s})$ is then determined via (2.8), we shall often use the same symbol to summarize the latter as well. The probabilities of individual actions should be nonnegative and satisfy the normalization conditions

$$\sum_{w\in\Gamma^i} p^i(w) = 1, \qquad i = 1, \cdots, I. \tag{2.9}$$

For large N_i, the number of agents of type i that take action $w \in \Gamma^i$ is approximately equal to $N_i p^i(w)$. That is, $\sum_{i(n)=i} s^n \cong N_i \sum_{w\in\Gamma^i} p^i(w)w$; since $\sum_{n=1}^{N} s^n = \sum_{i=1}^{I} \sum_{i(n)=i} s^n$, the market-clearing condition (2.7) becomes

$$\sum_{i=1}^{I} N_i \sum_{w\in\Gamma^i} p^i(w)w = e. \tag{2.10}$$

The left side of (2.10) is the mean or expected value of aggregate action or total demand $s^1 + \cdots + s^{N_1} + s^{N_1+1} + \cdots + s^{N_1+N_2} + \cdots + s^N$ with respect to the probability distribution $(p(\vec{s}))_{\vec{s}\in\Gamma}$ as in (2.8); thus the market-clearing condition in the form of (2.10) is tantamount to pre-specifying the mean aggregate action in the spirit of (1.19). The only data we have about this unknown probability distribution (besides the requirement of non-negativity) are the constraints (2.9)

[19] We are in effect referring here to the 'method of the most probable distribution' of statistical physics, originally developed by L. Boltzman in 1871 and articulated in the classic little book of Schrodinger (1952). See also McQuarrie (2000, Ch. 2) for a brief but very clear account of this method.

and (2.10). Given the statistical independence of individual actions, the additivity property of statistical entropy allows us to write the statistical entropy $H(p)$ of p as follows:

$$H(p) := -\sum_{i=1}^{I} N_i \sum_{w\in\Gamma^i} p^i(w) \ln p^i(w). \tag{2.11}$$

Before taking up the constraint maximization of (2.11), we briefly illustrate the connection between statistical entropy and degree of disorder[20]. Since the number of agents of type i, N_i, is assumed to be large (for any $i = 1,\cdots,I$), the probabilities of individual actions may be interpreted (as already alluded to) as relative frequencies: $p^i(w)$ is approximately equal to the *proportion* of agents of type i that take action $w \in \Gamma^i$; and $N_i p^i(w)$ is then the *occupation number* of w, namely the *number* of those agents that take action w. The number $W_i(p^i)$ of different ways (i.e. agent assignments to actions) that may result in the occupation numbers $[N_i p^i(w)]_{w\in\Gamma^i}$ is given by

$$W_i(p^i) := N_i! / \prod_{w\in\Gamma^i} (N_i p^i(w))!, \qquad i = 1,\cdots,I. \tag{2.12}$$

Accordingly, a certain probability distribution or macro-state p can materialize in

$$W(p) := \prod_{i=1}^{I} W^i(p^i) \tag{2.13}$$

distinct ways. Hence $\ln W(p) = \sum_{i=1}^{I} \ln W^i(p^i)$; by (2.12) and Sterling's approximation for large N_i we have $\ln W^i(p^i) \cong -N_i \sum_{w\in\Gamma^i} p^i(w) \ln p^i(w)$; accordingly, (2.11) yields

$$H(p) = \ln W(p). \tag{2.14}$$

Thus the statistical entropy $H(p)$ of a probability distribution p is a strictly increasing function of the number of ways $W(p)$ in which the occupation numbers $[N_i p^i(w)]_{w\in\Gamma^i}, i = 1,\cdots,I,$ can be realized.

The concept of statistical entropy and the principle of its maximization do not require 'repeated experiments' and the prevalent frequency interpretation of probability [Jaynes (2003)]. But in the statistical equilibrium setting of a pure exchange economy (as in basic statistical thermodynamics) a macro-state or probability distribution p is often naturally associated with relative frequencies and occupation numbers. As a consequence, statistical entropy—which in the Shannon tradition is a measure of uncertainty or missing information—can also be interpreted by means of (2.14) as an objective measure of disorder. The constrained maximization of statistical entropy amounts to

[20] See Foley (1994) and Footnote 19.

selecting the most disordered of the feasible macro-states—the macro-state that can be realized in the greatest number of ways and hence the most likely to materialize.

3. Statistical Equilibrium Prices

The *statistical equilibrium* for the economy E is the probability distribution with the highest statistical entropy under the constraint of the market-clearing condition. That is, a statistical equilibrium is a collection $p := (p^1, \cdots, p^I)$ of nonnegative vectors that maximizes (2.11) subject to (2.9) and (2.10). The objective of this section is to address the questions of existence, uniqueness, and characterization of such equilibrium[21]. As with any constrained-optimization problem, we begin with the properties of its objective function and constraint set.

Regarding the objective function, we need to recall the discussion of statistical entropy in Section 1: letting the positive integer g_i be the cardinality of the preferred action set Γ^i and Δ^i be the associated $g_i - 1$ dimensional unit simplex, we infer from that discussion that $p^i \mapsto -\sum_{w \in \Gamma^i} p^i(w) \ln p^i(w)$ is continuous and strictly concave on Δ^i $(i = 1, \cdots, I)$. Given that $N_1, \cdots, N_I$ are all positive integers, it follows that the statistical entropy (2.11) is **continuous** and **strictly concave** on $\Delta := \Delta^1 \times \cdots \times \Delta^I$.

The *constraint set* is defined as the collection of nonnegative vectors $p := (p^1, \cdots, p^I)$, $p^i \in \Re_+^{g_i}, i \in \{1, \cdots, I\}$, which satisfy (2.9) and (2.10). It is useful for our purposes to embed this set into a family of constraint sets by replacing the total endowment e in (2.10) by an *arbitrary total supply* $x := (x_1, \cdots, x_L) \in \Re_+^L$ to yield

$$\sum_{i=1}^{I} N_i \sum_{w \in \Gamma^i} p^i(w) w = x. \tag{3.1}$$

Associated with (3.1) is the constraint set $C[x]$ defined by

$$C[x] := \left\{ p \in \Delta \,\middle|\, \sum_{i=1}^{I} N_i \sum_{w \in \Gamma^i} p^i(w) w = x \right\}. \tag{3.2}$$

We are thus led to the study of a parametric family of maximization problems

[21]The term ‘statistical equilibrium’ is due to Foley (1994), who has fully worked out the existence, uniqueness and characterization of statistical equilibrium in his model. His analysis certainly carries over to the version of the statistical mechanical model advanced here. The development of our arguments regarding these issues, though, takes a somewhat different, constructivist route, given our emphasis in establishing the properties of the maximized statistical entropy as a function of the total amounts of resources.

$$\max\{H(p) \mid p \in C[x]\}, \tag{3.3}$$

indexed by the points x in some subset of $\Re_+^L$.

The notation in (3.1)-(3.3) intends to signify that *one may vary the total supply of resources while holding agent endowments constant*. Aside from being analytically useful, the discrepancy between total supply and total endowment will acquire operational significance when our economy E is opened up to trade with another economy. But as long as we consider an isolated economy with total endowment *e*, the pertinent statistical equilibrium is the solution to (3.3) when $x = e$.

This is the case we take up first. Clearly, $C[e] := \left\{ p \in \Delta \middle| \sum_{i=1}^{I} N_i \sum_{w \in \Gamma^i} p^i(w) w = e \right\}$ is the constraint set of the isolated economy E, namely the collection of the nonnegative probability vectors that satisfy (2.9) and (2.10). The set $C[e]$ is **non-empty**. This follows from the fact that a micro-state $\vec{s}$ as in (2.6) with $s^n = e^n \, \forall n \in \{1, \cdots, N\}$ (i.e. a micro-state in which the action of every agent is to stick with her initial endowment) is microscopically possible [that is, condition (2.6) is fulfilled] and globally feasible [on account of (2.3) and (2.7)]. Further, it is easy to show that $C[e]$ is **compact** and **convex**. . From the continuity of statistical entropy and the fact that the constrained set $C[e]$ is non-empty and compact we infer by means of the Weierstrass theorem[22] that the problem (3.3) when $x = e$ has a solution; in addition, since the constraint set $C[e]$ is convex and H is strictly concave, the solution is unique[23]; the unique solution is designated

$$p(e) := \arg\max\{H(p) \mid p \in C[e]\} \tag{3.4}$$

We will soon establish that under plausible restrictions there is an open set $U \subseteq \Re_{++}^L$ containing the total endowment *e* such that for every $x \in U$ the set $C[x]$ is non-empty. It is easy to show that $C[x]$ is, in addition, compact and convex $\forall x \in \Re_+^L$. Hence the constraint set (3.2) is non-empty, compact, and convex for every $x \in U$, for some open set $U \subseteq \Re_{++}^L$ containing the economy's total endowment *e*. The arguments that led to (3.4) apply to the problem (3.3) for every $x \in U$. Accordingly, the solution to (3.3)

$$p(x) := \arg\max\{H(p) \mid p \in C[x]\} \tag{3.5}$$

and the maximized statistical entropy

$$S(x) := \max\{H(p) \mid p \in C[x]\}, \tag{3.6}$$

[22] See, for example, Moore (1999, Theorem 3.26).
[23] See, for example, Sundaram (1996, Theorem 7.13).

are well defined for every total supply $x \in U$. The maximum value function $S : U \to \Re$ defined by (3.6) corresponds to the entropy concept of classical thermodynamics; it will be referred to as the *entropy function* of the economy and will play a central role in the sequel.

Regarding the characterization of (3.5), recall from our general discussion Section 1 that (by the nature of the functional form (2.11) for statistical entropy) the non-negativity constraint on the unknown probability distribution is not binding at the maximum. The maximization problem (3.3) boils down to finding a positive-valued probability distribution that maximizes (2.11) subject to the linear constraints (2.9) and (3.1). The Lagrange multiplier theorem (for maximization problems with equality constraints) becomes relevant[24]. To ensure its applicability to the problem at hand, we only need to *assume* a) that the sum $g := \sum_{i=1}^{I} g_i$ of the cardinalities of the preferred action sets is greater than the total number $I+L$ of constraints in (2.9) and (3.1), $g > I + L$—an essential presupposition for the applicability of the maximum statistical entropy principle; and b) that the $I+L$ of constraints in (2.9) and (3.1) satisfy the *non-degenerate constraint qualification* (NDCQ), namely that the coefficient matrix associated with the left-hand sides of the linear equations (2.9) and (3.1) has rank $I+L$.

Let

$$\Lambda(p) := -\sum_{i=1}^{I} N_i \sum_{w \in \Gamma^i} p^i(w) \ln p^i(w) - \sum_{i=1}^{I} N_i (m_i - 1)\Big(\sum_{w \in \Gamma^i} p^i(w) - 1\Big) - p \cdot \Big(\sum_{i=1}^{I} N_i \sum_{w \in \Gamma^i} p^i(w) w - x\Big) \quad (3.7)$$

be Lagrangean function for the problem of maximizing (2.11) subject to (2.9) and (3.1), with $m := (m_1, \cdots, m_I) \in \Re^I$ and $p := (p_1, \cdots, p_L) \in \Re^L$ being vectors of Lagrange multipliers[25]. The Lagrange multiplier theorem entails that if $p = p(x)$ is the solution (3.5) with $p \in \Re^g_{++}$ [which is tantamount to saying that $p \in \Re^g_{++}$ is a maximizer of (2.11) subject to (2.9) and (3.1)], then there exist unique vectors of Lagrange multipliers, $m \in \Re^I$ and $p \in \Re^L$, such that the Lagrangean function (3.7) has a critical point at $p = p(x)$, i.e.

$$\partial \Lambda / \partial p^i(w) = 0, \quad w \in \Gamma^i, i \in \{1, \cdots, I\}. \quad (3.8)$$

Since the first term of the Lagrangean function (3.7) is strictly concave while the other two terms are linear in the unknown probability distribution, the Lagrangean function is concave irrespective of the signs of the multipliers. Accordingly, the first-order (necessary) conditions (3.8) along with (2.9) and (3.1) are also sufficient[26] for a positive-valued solution to the parametric maximization problem (3.3). To put it more succinctly, the triple (p, m, p) just defined is characterized by equations (2.9),

[24] See, for example, Simon and Blume (1994, Theorem 18.2) or Sydsaeter (1981, Theorem 5.20).

[25] The writing of the multiplier for the ith constraint in (2.9) as $N_i(m_i - 1)$ is just a matter of convenience.

[26] See, for example, Carter (2001, Corollary 5.2.4) or Sydsaeter (1981, Theorem 5.21).

(3.1), and (3.8). The properties of statistical equilibrium distributions and associated multipliers are embodied in the latter system of equations. We, therefore, now turn to the study of that system.

In view of (3.7), equation (3.8) yields $\ln p^i(w) = -m_i - p \cdot w$ and hence

$$p^i(w) = \exp(-m_i)\exp(-p \cdot w), \quad w \in \Gamma^i, i \in \{1, \cdots, I\}. \tag{3.9}$$

Insert (3.9) into (2.9) to obtain

$$p^i(w) = \exp[-p \cdot w] / Z^i(p), \quad w \in \Gamma^i, i = 1, \cdots, I, \tag{3.10}$$

and $\exp m_i = Z^i(p)$ or

$$m_i = \ln Z^i(p), \quad i = 1, \cdots, I, \tag{3.11}$$

where

$$Z^i(p) := \sum_{w \in \Gamma^i} \exp[-p \cdot w] \tag{3.12}$$

is a normalization factor, known as the *partition function* for agents of type i $(i = 1, \cdots, I)$. The statistical equilibrium distribution (3.10) is known in statistical thermodynamics as the *canonical* or *Boltzmann-Gibbs distribution*. The dependence of $p^i(w)$ [in (3.10)] on x is suppressed; it should be clear from the context from this point on that $p = p(x)$ whenever p refers to the equilibrium probability distribution. The vector $p := (p_1, \cdots, p_L)$ is determined by the requirement that the market-clearing condition (3.1) be satisfied; insert (3.10) into (3.1) to obtain the determining system of equations in p :

$$\sum_{i=1}^{I} N_i \sum_{w \in \Gamma^i} [\exp(-p \cdot w) / Z^i(p)] w = x. \tag{3.13}$$

Following Foley (1994), we call p_l the *entropy price* of the *lth* resource $(l = 1, \cdots, L)$, and $p := (p_1, \cdots, p_L)$ the *entropy price vector*; we will also refer to p as the *statistical equilibrium price vector*. The left-hand side of (3.11) is the mean total action or gross demand of the economy expressed as a function of entropy prices, which may be seen as adjusting to balance total demand to the total supply. But the entropy prices here clear the market by appropriately distributing agents over their preferred action sets, rather than by coordinating their utility-maximizing choices (as in the Arrow-Debreu model). We need to recall that when our economy E is isolated, the total supply x is equal to the total endowment *e*, the actually given total supply of E. Accordingly, the statistical equilibrium price vector $p^* := (p_1^*, \cdots, p_L^*)$ for E obtains when $x = e$, and the associated canonical distribution is specified by setting $p = p^*$ in (3.10) and (3.12). The statistical equilibrium with $x \neq e$ becomes relevant when the given exchange economy E interacts with other economies.

Whether the economy E is isolated or open, the emergence of entropy prices as determined by (3.13) and the corresponding canonical distribution in (3.10) signal the attainment of statistical equilibrium in E—that the economy has reached its most disordered macro-state, given the total supply $x \in U \subseteq \Re^L_{++}$. The significance of such an equilibrium concept stems from the *economic assumption* that underlies the statistical mechanical perspective—that the spontaneous trading practices among individual agents drive the economy to its most disordered macro-state, compatible with the available resources. It is important to reiterate that in the statistical equilibrium approach, unlike the Arrow-Debreu model, we do not begin with a price vector to which individual agents react; rather agents come to the market with their preferred action sets and try to strike deals to their best advantage. As a consequence of repeated (possibly multilateral) exchanges the most disordered macro-state eventually sets in, with entropy prices emerging as the social outcome of these interactions at the micro-level. The statistical equilibrium price vector p serves to define the *entropy cost* $p \cdot w$ of any individual action w and a scalar probability field, $w \mapsto \exp(-p \cdot w)$, applicable to all agents and thus signifying that all participate in the same market[27]. Actions of higher entropy cost are less likely to materialize, though the likelihood varies with the type of agents—different preferred action sets entail different partition functions.

We now turn to the question of existence of an open set [containing the total endowment e] on which the constraint set (3.2) is non-empty and to the study of the properties of the entropy function. It turns out that the system (3.13) plays a pivotal role in deciding the answers to these questions. To facilitate the understanding of (3.13), focus on the transformation $T : \Re^L \to \Re^L$ defined by the left-hand side of (3.13), namely

$$T(p) := \sum_{i=1}^{I} N_i \Big(\sum_{w \in \Gamma^i} [\exp(-p \cdot w) / Z^i(p)] w \Big). \tag{3.14}$$

We may view T as a vector-valued function or an ordered L-list $T := (T_1, \cdots, T_L)$ of real-valued functions, with the lth component function $T_l : \Re^L \to \Re$ being defined by

$$T_l(p) := \sum_{i=1}^{I} N_i \Big(\sum_{w \in \Gamma^i} [\exp(-p \cdot w) / Z^i(p)] w_l \Big), \forall l \in \{1, \cdots, L\}. \tag{3.15}$$

We first note that the transformation T is infinitely continuously differentiable or of class C^∞ everywhere in its domain; to put it otherwise, each of the component functions $T_1, \cdots, T_L$ of the transformation is of class C^k or k times continuously differentiable to any desired order k. That T is of class C^1 is all that we will need in the sequel. By differentiating every component function (3.15) with respect to each entropy price, we infer that the Jacobian matrix (or Jacobian derivative) $DT(p) := [(\partial T_l / \partial p_m)(p)]^{L \times L}$ of T at $p \in \Re^L$ is given by

[27] Our interpretation of statistical equilibrium and entropy prices draws on the work of Foley, especially on Foley (2003).

$$DT(p) = -\sum_{i=1}^{I} N_i \Sigma^i(p), \tag{3.16}$$

where $\Sigma^i(p)$ is the variance-covariance matrix of the random vector $w \in \Gamma^i, i \in \{1, \cdots, I\}$, relative to the normal distribution (3.10).

For any given entropy price vector $p \in \Re^L$, we know that the variance-covariance matrix $\Sigma^i(p)$ has to be *non-negative definite*[28] $\forall i \in \{1, \cdots, I\}$; barring degenerate situations, namely supposing that each of these variance-covariance matrices is positive-definite, it follows from (3.16) that the Jacobian matrix $DT(p)$ of T is *negative-definite* and hence *non-singular* $\forall p$. As a consequence, the image $T(\Re^L)$ of $\Re^L$ is an *open set*[29]; by inspection of (3.12) and (3.14), $T(\Re^L) \subseteq \Re^L_{++}$. It also follows that T has C^1 local inverse everywhere in its domain[30], although it need not have a single inverse defined on the entire image set $T(\Re^L)$. Thus for any given entropy price vector p there is an open neighborhood $B(p) \subseteq T(\Re^L)$ of p on which T is invertible; letting $T_p := T|B(p)$ be the restriction of T on $B(p)$, the Jacobian matrix $DT_p^{-1}(T(p))$ of its inverse T_p^{-1} is given by

$$DT_p^{-1}(T(p)) = (DT_p(p))^{-1}; \tag{3.17}$$

Actually, we can go beyond this local-inverse result by integrating our existence/uniqueness and characterization analysis. We have already established that the maximization problem (3.3) has a unique solution

$$p(e) := \arg\max \{H(p) | p \in C[e]\} \tag{3.18}$$

when $x = e$. By the Lagrange multiplier theorem, there is a unique vector $p^* \in \Re^L$ of multipliers [associated with the market-clearing condition (2.10) or (3.13) with $x = e$] such that the pertinent first-order conditions are satisfied; thus (2.10) or (3.13) with $x = e$ is satisfied and hence, by (3.14), $T(p^*) = e$, entailing that $e \in T(\Re^L) \subseteq \Re^L_{++}$. In sum, we have shown that the image set $T(\Re^L)$ is an *open set* in $\Re^L_{++}$ and *contains the total endowment vector* of the exchange economy E.

When x is not restricted to be equal to the total endowment e, we first note that for any given total supply $x \in T(\Re^L)$, there is $p \in \Re^L$ such that $T(p) = x$ [by the very definition of the image set $T(\Re^L)$]. This means that the associated positive vector of probabilities (3.10) satisfies (2.9) and (3.1) and hence it is an element of the constraint set (3.2). Accordingly, *there is indeed an open set*

$$U := T(\Re^L) \subseteq \Re^L_{++} \tag{3.19}$$

[28] See, for example, Searle and Willett (2001, p. 79).
[29] See, for example, Buck (1978, p. 356: Theorem 15).
[30] Here we invoke the inverse function theorem—see, for example, Carter (2001, Theorem 4.4) or Sydsaeter (1981, Theorem 3.8).

containing the total endowment such that the constraint set (3.2) is non-empty $\forall x \in U$; the unique solution (3.5) and the corresponding entropy function (3.6) are well-defined on (3.19) as anticipated.

Since for any given total supply x in (3.19), the maximization problem (3.3) has a unique solution $p(x) \in \Re^{g}_{++}$, the Lagrange multiplier theorem once again entails that there is a *unique* $p \in \Re^{L}$ such that $T(p) = x$. Accordingly, the transformation $T : \Re^{L} \to \Re^{L}$ is globally 1-to-1 in $\Re^{L}$ and hence there is an inverse transformation T^{-1} defined on the open set (3.19). On account of the fact that T is of class C^1 with non-singular Jacobian matrix everywhere in its domain, it follows[31] that T^{-1} is of class C^1 on $U := T(\Re^{L}) \subseteq \Re^{L}_{++}$, and its Jacodian matrix is given by

$$DT^{-1}(x) = (DT(p))^{-1}, \tag{3.20}$$

where $x = T(p)$ or, equivalently,

$$p = T^{-1}(x), \ \ x \in U. \tag{3.21}$$

It immediately follows that the entropy function S is of class C^1 on the open set $U \subseteq \Re^{L}_{++}$ given by (3.19); indeed, insert $p^i(w)$ from (3.10) into (2.11) and take into account (3.13) to derive

$$S(x) = p \cdot x + \sum_{i=1}^{I} N_i \ln Z^i(p), \tag{3.22}$$

which is a C^1 function on account of the fact that p is C^1 vector-valued function of x, given by (3.21). Hence the familiar envelope result

$$p = \nabla S(x), \ \ \ x \in U, \tag{3.23}$$

holds true. But (3.21) is of class C^1 and hence, by (3.23), *the entropy function S is of class C^2*. Moreover, from (3.20), (3.21), and (3.23) we have

$$D^2 S(x) = DT^{-1}(x), \ \ \ x \in U. \tag{3.24}$$

Here $D^2 S(x) := [S''_{lm}(x)]^{L \times L}$ is the Hessian matrix of S at x, summarizing the second-order partial derivatives of S at x. Since the Jacobian matrix $DT(p)$ of T is negative-definite $\forall p$, the Jacobian matrix $DT^{-1}(x)$ of the inverse transformation T^{-1}, given by (3.20)-(3.21), is also negative definite $\forall x \in U$. It follows from (3.24) that the Hessian matrix of the second-order partial derivatives of S is

[31] See Buck (1978, p. 358: Theorem 16). This result is also known as the inverse function rule—for a weaker version of this rule, see, for example, Carter (1999, Exercise 4.27).

negative-definite $(\forall x \in U)$, entailing that S is a *regular strictly concave function*. The designation of S as a *regular* strictly concave function (not a standard term) intends to signify (and underscore) that the Hessian matrix of the entropy function is negative-definite *everywhere* in its domain.

With finite preferred action sets, it is expected that the domain of S is bounded above—no sufficiently large individual actions are possible so as to fulfill the market-clearing condition (3.13) for positive total supply vectors of arbitrarily large norm. Only when the preferred action sets are (denumerable or non-denumerable) unbounded sets (so that arbitrarily 'large' actions are individually possible as in the Arrow-Debreu world), is the constraint set (3.2) expected to remain non-empty for arbitrarily large total supplies of resources. But even under those conditions there may be restrictions on U 'from below', stemming from the fact that every preferred action set is typically restricted from below by the indifference set associated with the agent's endowment. We surmise—in anticipation of the extendibility of the statistical mechanical model to the case where the preferred action sets are infinite and unbounded—that the domain of S is of the form

$$U := \left\{ x \in \Re^L_{++} \middle| a_l < x_l < +\infty, l = 1, \cdots, L \right\} \tag{3.25}$$

for some $a := (a_1, \cdots, a_L) \in \Re^L_+$. Further, it is rather innocuous to presume that there is a continuous extension of S defined on the closure $\overline{U} := \left\{ x \in \Re^L_{++} \middle| a_l \leq x_l < +\infty, l = 1, \cdots, L \right\}$ of (3.25), even though this not a an inference of the statistical mechanical model. We shall henceforth pretend that $\overline{U} = \Re^L_+$ without loss of generality. For if the vector of constants in (3.25) has some positive components, we can always introduce new variables z by the transformation $x = a + z$ so that $x \in \overline{U}$ if and only if $z \in \Re^L_+$. This change of variables would be tantamount to measuring the total supply of resources by its excess $x - a$ over the constant vector a.

In sum, we may take for granted from now on that the entropy function of the economy, defined by (3.6), is continuous on $\Re^L_+$ and, in addition, of class C^2 and regular strictly concave on $\Re^L_{++}$. Strictly speaking, only the last two properties (of class C^2 and regular strictly concave on $\Re^L_{++}$) have been rigorously deduced from the statistical mechanical model; the existence of a continuous extension of the entropy function with domain $\Re^L_+$ has been added as a harmless premise.

4. Connection with Classical Thermodynamics

As already indicated in Section 0, the Laws of classical thermodynamics are obtained as theorems in statistical thermodynamics, which, in addition, provides the means to calculate physical properties (as opposed to thermodynamics which provides only relations between many properties). In a parallel fashion, we seek to derive a phenomenological theory[32] of pure exchange from the statistical mechanical theory of markets. Such a derivation is essential for ensuring the internal coherence of the statistical mechanical model of pure exchange; at the same time it will deepen our understanding

[32] We prefer to use the term 'phenomenological' [Rozonoer (1973)] rather than 'macro' to designate a theory of exchange of resources between macroscopic subsystems without a detailed microscopic description of the interactions among agents.

of the statistical mechanical model itself and of the analytical significance of the entropy function. Moreover, it will become clear that Rozonoer's analysis of 'systems with additive effects' is naturally linked to the statistical mechanical theory of pure exchange.

The concept of a 'composite system' made out, actually or notionally, of macroscopic 'equilibrium subsystems' plays a key role in thermodynamics [Callen (1985), Tisza (1977)] and the generalized thermodynamics of Rozonoer (1973). Accordingly, our starting point is to view the pure-exchange economy E as a *composite system* made out of I *subsystems*. Subsystem i consists of all agents of type $i, i \in \{1, \cdots, I\}$.

Begin with the situation where the subsystems are *isolated*: a pre-trade condition. Each subsystem i is in possession of a vector $\sum_{i(n)=i} e^n \in \Re^L_{++}$ of resources, where, recall, $e^n \in \Re^L_{++}$ is the endowment of agent n in that subsystem. In the language of thermodynamics, the no-trade situation is tantamount to having each subsystem enclosed by a wall that is impermeable to resource flows; constraints that prevent the flow of one or more resources are also known as *internal* constraints in thermodynamics [Callen (1985, p. 26)]. In an L-resource world there are L different kinds of walls or internal constraints; the lth wall or internal constraint is blocking the flow of the lth resource, but permits the flow of all other resources $m \neq l \ \ (l, m = 1, \cdots, I)$.

If some or all internal constraints are removed, i.e. if agents of different types are free to exchange resources, then resources may be redistributed among the subsystems. The objective of the phenomenological theory is to predict the 'equilibrium allocation' that would emerge as a result of this redistribution. An *allocation* among the I subsystems is by definition a vector

$$x := (x^1, \cdots, x^I) \in \Re^{LI}_{+}, \tag{4.1}$$

where $x^i := (x^i_1, \cdots, x^i_L) \in \Re^L_+$ is the vector of resources allocated to the subsystem $i \in \{1, \cdots, I\}$, with $x^i_l \geq 0$ being the amount of the lth resource acquired by subsystem i, namely by the group of agents of type i as a whole. Before taking up the question of equilibrium, we need to identify which allocations are feasible. When the economy is an *isolated system*, the total amount of each resource is predetermined:

$$\sum_{i=1}^{I} x^i = e := \sum_{n=1}^{N} e^n = \sum_{i=1}^{I} \sum_{i(n)=i} e^n . \tag{4.2}$$

In order to directly link the analysis to the statistical mechanical model of the preceding section, we will study the allocation problem for an arbitrary given total supply $x \in \Re^L_+$, and get the solution corresponding to (4.2) by setting $x = e$. Thus (4.2) is replaced by the more general condition

$$\sum_{i=1}^{I} x^i = x, \quad x \in \Re^L_+. \tag{4.3}$$

To put it otherwise, resources are not wasted and hence are conserved in the process of exchange. Thus (4.2) or (4.3) is a summary of the *conservation laws* of our pure-exchange economy. The set of *feasible allocations* $F[x]$ consists of the allocations that satisfy the conservation laws:

$$F[x] := \left\{ x \in \Re_+^{LI} \middle| \sum_{i=1}^{I} x^i = x \right\}. \tag{4.4}$$

By inspection of the objective function (2.11) and the linear constraints (2.9) and (3.1), it is rather natural to envision a two-stage optimization procedure for solving problem (3.3), i.e. for finding the most disordered macro-state of the economy. This observation is at the hart of the argument linking the statistical mechanical model to a phenomenological theory of resource allocation—whose key proposition is an equilibrium principle that selects the allocation in (4.4) that will eventually prevail when all internal constraints are removed.

In Stage I, we find the most disordered macro-state $p^i := [p^i(w)]_{w \in \Gamma^i}$ in each subsystem $i \in \{1, \cdots, I\}$ associated with an arbitrary allocation (4.1); in other words, we choose $p^i \in \Re_+^{g_i}$ to

$$\max[-N_i \sum_{w \in \Gamma^i} p^i(w) \ln p^i(w)] \tag{4.5}$$

subject to

$$\sum_{w \in \Gamma^i} p^i(w) = 1, \tag{4.6}$$

$$N_i \sum_{w \in \Gamma^i} p^i(w) w = x^i. \tag{4.7}$$

It is clear from constraint (4.7) that the assignment of an arbitrary allocation $x := (x^1, \cdots, x^I)$ is tantamount to specifying the levels of mean action,

$$x^i / N_i := (1/N_i) x^i = (x_1^i / N_i, \cdots, x_L^i / N_i), \tag{4.8}$$

in the ith subsystem, $i = 1, \cdots, I$. The maximization problem (4.5) for each type of agents (or subsystem) involves $L+1$ constraints [equations (4.6) and (4.7)] for the unknown probability distribution that is associated with that type of agents. The two key assumptions made earlier need to be strengthened in order to handle Stage I, i.e. (4.5)-(4.7). First, since our finite preferred sets are simplifying approximations of infinite sets, we may comfortably presume that $L+1$ is less than the number g_i of the unknown probabilities $\forall i \in \{1, \cdots, I\}$ and hence the maximum statistical entropy principle becomes applicable to each subsystem of the economy[33]. Second, we assume that the

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[33] The assumption $L + I < g$ is now a consequence of positing $L + 1 < g_i$ for all $i = 1, \cdots, I$ and is no longer needed as a separate presupposition.

constraints (4.6) and (4.7) satisfy the NDCQ[34]. On account of these two assumptions and the observation that the non-negativity constraint is not binding at the maximum, it follows that the Lagrange multiplier theorem is once again applicable; hence the problem of maximizing (4.5), a strictly concave objective function, subject to the linear constraints(4.6) and (4.7), has at most one solution for each x^i, which is characterized by the associated first-order conditions. By solving the latter with respect to the unknown probability vector p^i, it is easy to establish that the solution must be of the canonical form

$$p^i(w) = \exp(-p^i \cdot w)/Z^i(p^i), \;\; w \in \Gamma^i, i = 1, \cdots, I; \tag{4.9}$$

Here $p^i := (p_1^i, \cdots, p_L^i) \in \Re^L$ is the vector of Lagrange multipliers associated with the constraint (4.7) and are interpreted as the entropy prices in the ith subsystem; and $Z^i(p^i)$ is the partition function of the ith subsystem as given by (3.11) with p being replaced by p^i:

$$Z^i(p^i) := \sum_{w \in \Gamma^i} \exp[-p^i \cdot w]. \tag{4.10}$$

Insert (4.10) into (4.7) to obtain the system of equations—the counterpart of (3.13)—that determine the entropy price vector p^i:

$$N_i \sum_{w \in \Gamma^i} [\exp(-p^i \cdot w)/Z^i(p^i)]w = x^i, \;\; i = 1, \cdots, I. \tag{4.11}$$

In analogy to the role of entropy prices in the economy as a whole, the emergence of entropy prices as determined by (4.11) and the corresponding canonical distribution (4.9) in each subsystem signal that the subsystem has reached statistical equilibrium, i.e. its most disordered macro-state, compatible with the assigned vector of resources.

The concept that parallels the transformation T [given by (3.14)] is the transformation $T^i : \Re^L \to \Re$ defined by the right-hand side of (4.11), namely

$$T^i(p^i) := N_i \sum_{w \in \Gamma^i} [\exp(-p^i \cdot w)/Z^i(p^i)]w = x^i, \;\;\; i = 1, \cdots, I. \tag{4.12}$$

Our discussion surrounding (3.16) implies that T^i is of class C^1 (actually, it is of class C^∞) and that the Jacobian matrix $DT^i(p^i) := [(\partial T_l^i / \partial p_m^i)(p^i)]^{L \times L}$ of T^i at p^i is negative-definite and hence non-singular $\forall p^i \in \Re^L, \forall i \in \{1, \cdots, I\}$. As a consequence, each image $T^i(\Re^L)$ of $\Re^L$ under T^i is an open set[35], it obviously contains the total endowment $\sum_{i(n)=i} e^n$ of the ith subsystem and contains only positive vectors, i.e. $T^i(\Re^L) \subseteq \Re^L_{++}$.

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[34]. It is not hard to show that as a consequence of this assumption, the constraints (2.9) and (3.1) are linearly independent and thus satisfy the NDCQ as well—we no longer need to assume this.

[35] See footnote 20.

To demonstrate that a solution to (4.5)-(4.7) in fact exists, it useful to encapsulate the constraints $p^i \in \Re_+^L$, (4.6), and (4.7) by means of the *ith constraint set*

$$C^i\left[x^i / N_i\right] := \left\{ p^i \in \Delta^i \middle| N_i \sum_{w \in \Gamma^i} p^i(w) w = x^i \right\}, \tag{4.13}$$

and to restate the *ith* maximization problem in the form

$$\max\left\{ -N_i \sum_{w \in \Gamma^i} p^i(w) \ln p^i(w) \middle| p^i \in C^i\left[x^i / N_i\right] \right\}. \tag{4.14}$$

For every assignment of resources $x^i \in T^i(\Re^L)$ to the *ith* subsystem, there is an entropy vector $p^i \in \Re^L$ such that (4.11) is satisfied; thus the associated canonical distribution (4.9) satisfies the constraint (4.7) and belongs to the unit simplex Δ^i [because it is a positive vector and satisfies the normalization condition (4.6)]. Accordingly, the constraint set (4.13) is *non-empty* $\forall x^i \in T^i(\Re^L)$; since it is obviously compact as well and the statistical entropy—the objective function in the maximization problem (4.14)—is continuous, the Weierstrass theorem ensures that (4.14) has a solution. We may directly re-confirm uniqueness by noticing that the constraint set (4.13) is convex and the objective function in (4.14) is strictly concave. It follows that *the entropy function* $S^i : T^i(\Re^L) \to \Re$ *of the ith subsystem*, with

$$S^i(x^i) := \max\left\{ -N_i \sum_{w \in \Gamma^i} p^i(w) \ln p^i(w) \middle| p^i \in C^i\left[x^i / N_i\right] \right\}, \tag{4.15}$$

is well-defined $(i = 1, \cdots, I)$.

Since for any given $x^i \in T^i(\Re^L)$, the problem (4.14) has a unique solution $p^i \in \Re_{++}^{g_i}$, the Lagrange multiplier theorem implies that there is a *unique* $p^i \in \Re^L$ such that (4.11) is satisfied, i.e. $T^i(p^i) = x^i$. This entails that the transformation $T^i : \Re^L \to \Re$ is globally 1-1 in $\Re^L$ and hence that there is an inverse transformation $(T^i)^{-1}$ defined on the open set $T^i(\Re^L)$. In view of the earlier established fact that T^i is of class C^1 with negative-definite and hence non-singular Jacobian matrix everywhere in its domain, it follows[36] that the inverse transformation $(T^i)^{-1}$ is of class C^1 on its domain $T^i(\Re^L)$, and its Jacobian matrix is given by

$$D(T^i)^{-1}(x^i) = (DT^i(p^i))^{-1}, \tag{4.16}$$

where $x^i = T^i(p^i)$ or, equivalently,

$$p^i = (T^i)^{-1}(x^i), \quad x^i \in T^i(\Re^L), i \in \{1, \cdots, I\}. \tag{4.17}$$

[36] See footnote 21.

Clearly, the Jacobian matrix of the inverse transformation $(T^i)^{-1}$ is negative definite and non-singular everywhere in its domain $T^i(\Re^L), i = 1, \cdots, I$.

The properties of the transformations $T^1, \cdots, T^I$ determine, respectively, the properties of the entropy functions $S^1, \cdots, S^I$. To see this, insert $p^i(w)$ from (4.9) into the expression for statistical entropy in (4.14) and take into account (4.10) and (4.11) to obtain

$$S^i(x^i) = p^i \cdot x^i + N_i \ln Z^i(p^i), \tag{4.18}$$

where p^i stands for the vector-valued function (4.17). Since the latter is of class C^1, we infer from (4.18) that the entropy function $S^i : T^i(\Re^L) \to \Re$ of the subsystem $i \in \{1, \cdots, I\}$ is of class C^1 as well. Accordingly, we have the counterpart of the envelope result (3.23)

$$p^i = \nabla S^i(x^i), \quad x^i \in T^i(\Re^L), i \in \{1, \cdots, I\}. \tag{4.19}$$

By (4.16), (4.17), and (4.19), the entropy function is of class C^2 and regular strictly concave.

In sum, the entropy function $S^i : T^i(\Re^L) \to \Re$ of the subsystem $i \in \{1, \cdots, I\}$ in statistical equilibrium is *of class C^2* and *regular strictly concave*. Once again, we may presume that there is a continuous extension of each S^i on the closure of the image set $T^i(\Re^L)$, which closure we take to be $\Re^L_+$ for all $i \in \{1, \cdots, I\}$ for the sake of simplicity. In this fashion, the entropy function of each subsystem is defined and continuous on $\Re^L_+$ and, in addition, of class C^2 and regular strictly concave on $\Re^L_{++}$.

That every subsystem separately has reached statistical equilibrium is not sufficient in itself to ensure that the economy has attained its most disordered macro-state (in the absence of internal constraints or barriers to trade)—the entropy price vector may very well vary across subsystems. The equilibrium *among* subsystems is attained **in Stage II:** the allocation (4.1) is feasibly adjusted to maximize the sum of the entropy functions of the subsystems; thus an *equilibrium allocation* associated with total supply $x \in \Re^L_+$ is a maximizer in the problem

$$\max\left\{\sum_{i=1}^{I} S^i(x^i) \middle| x \in F[x]\right\}. \tag{4.20}$$

There is no standard term for the function $x = (x^1, \cdots, x^I) \mapsto \sum_{i=1}^{I} S^i(x^i)$; we shall call it the *global entropy expression* of the economy. We restrict our attention to potential solutions of (4.20) in the interior of $F[x]$—a presumption to be justified *ex post*. It is not hard to see that the rank of the $L \times LI$ coefficient matrix associated with the linear system (4.3) is equal to L and hence the constraint (4.3), which defines $F[x]$ in (4.4), satisfies the NDCQ. Further, $F[x]$ is obviously a convex set $\forall x \in \Re^L_+$ and the global entropy expression of the economy is strictly concave (because it is the sum of strictly concave functions). Accordingly, the Lagrange multiplier theorem entails that an allocation $x \in \Re^{LI}_{++}$ solves the maximization problem (4.20) if and only if there is a vector $l \in \Re^L$ of Lagrange multipliers [associated with the constraints (4.3)] such that

$$\nabla S^i(x^i) = l \quad \forall i \in \{1, \cdots, I\}. \tag{4.21}$$

We can now show how the two-stage process for solving (3.5) works out. We first note with the aid of (4.19) and (4.21) that in an equilibrium allocation of resources the vectors p^i of multipliers get equalized across subsystems:

$$p^i = l, \; i = 1, \cdots, I. \tag{4.22}$$

In turn, we insert (4.22) into (4.9)-(4.11); sum (4.11) after the preceding substitution over $i \in \{1, \cdots, I\}$ and take into account (4.3) to derive

$$\sum_{i=1}^{I} N_i \sum_{w \in \Gamma^i} [\exp(-l \cdot w) / Z^i(l)] w = x. \tag{4.23}$$

By inspection of (3.13) and (4.23), we see that p and l satisfy the same system of equations (which has been shown to have a unique solution (3.21) for any $x \in T(\Re^L)$, with the image set being presumed to coincide with $\Re^L_{++}$ without loss of generality); hence $l = p$ for all $x \in \Re^L_{++}$: the vector l of Lagrange multipliers that was introduced in Stage II in connection with the conservation laws [summarized by (4.3)] coincides with the economy's statistical equilibrium price vector p given by (3.21) [presume $U = \Re^L_{++}$]. It follows that

$$p^i(x^i(x)) = p = T^{-1}(x), \forall i \in \{1, \cdots, I\}, \tag{4.24}$$

where, recall, T^{-1} is the inverse of the transformation T defined by (3.14), and where

$$x(x) := \arg\max \left\{ \sum_{i=1}^{I} S^i(x^i) \middle| x \in F[x] \right\}; \tag{4.25}$$

of course, for the statistical equilibrium associated with total endowment e we have

$$p^i(x^i(e)) = p^* = T^{-1}(e), \forall i \in \{1, \cdots, I\}. \tag{4.26}$$

Further, from (3.22), (4.3), (4.18), (4.10), and (4.25) we obtain

$$\max \left\{ \sum_{i=1}^{I} S^i(x^i) \middle| x \in F[x] \right\} = \sum_{i=1}^{I} S^i(x^i(x)) = S(x), \tag{4.27}$$

as expected.

We may now summarize the two-step procedure for obtaining the most disordered macro-state of the economy as follows. The maximum statistical entropy principle is applied to each subsystem for an arbitrary allocation of resources x [as in (4.1)] among subsystems, fixing the mean action of each agent type. As a result we obtain an entropy function and entropy price vector for every subsystem. Every subsystem is in statistical equilibrium—that is, in the most disordered macro-state consistent with the pre-assigned allocation of resources. This is also the most disordered macro-state of the economy as a whole provided that the subsystems remain isolated from one another. The global entropy expression $\sum_{i=1}^{I} S^i(x^i)$ is the greatest attainable disorder under these circumstances. When all internal constraints are removed—so that agents of different types may engage in exchange—the economy's degree of maximum disorder is the maximum of the global entropy expression over the set (4.4) of feasible allocations, which is determined by the total supply vector $x \in \Re^L_{++}$.This maximization is attained via the equalization of the entropy price of every resource across subsystems.

Clearly, in order for the economy to be in statistical equilibrium, it is necessary for each subsystem to be in equilibrium. Hence we may always presume the latter and represent each group of identical agents by the associated entropy function, which supplants the traditional utility function of the 'representative agent' in that group. With the subsystem entropy functions known, the attainment of the most disordered macro-state for the economy is accomplished by finding the equilibrium allocation $x(x)$ as given by (4.25), which corresponds to the concept of *thermodynamic equilibrium* in classical thermodynamics. Statements (4.25) and (4.27) correspond to the *maximum entropy principle* of thermodynamics[37]—the principle that encompasses its standard Laws [Balian (1991, Ch. 6)]—and Rozonoer's equilibrium principle for 'systems with additive effects'. The existence and properties of the subsystem and economy-wide entropy functions along with the maximum entropy principle have been derived from the statistical mechanical theory of markets (which we view as the economic counterpart of statistical thermodynamics).Thus the statistical mechanical theory of markets provides the foundation for a phenomenological theory of resource allocation in the image of classical thermodynamics, along the lines of Rozonoer (1973).

5. The Maximum Entropy Principle Generalized: Heterogeneous Agents and Open Economies

Thus far our statistical mechanical model has presumed the existence of groups of identical agents. This premise serves the useful function of enabling one to interpret the probability of an individual action as the proportion of agents (with the same preferred action set) who take that action. The theoretical discourse in the preceding sections should have made it apparent that such a presupposition is not required for the development of the statistical mechanical formalism and the subsequent derivation of the maximum entropy principle. So we want to begin by showing how things work out when such an assumption is abandoned.

[37] This is a theorem rather than principle, derived from first principles, most notably the maximum statistical entropy principle.

By inspection of (4.5)-(4.8) or (4.13)-(4.15), we see that the entropy function of the *ith* subsystem can be written as

$$S^i(x^i) = N_i s^i\left(x^i / N_i\right), \tag{5.1}$$

where

$$s^i\left(x^i / N_i\right) := \max\left\{-\sum_{w\in\Gamma^i} p^i(w)\ln p^i(w) \,\middle|\, p^i \in C^i\left[x^i / N_i\right]\right\}. \tag{5.2}$$

Accordingly, every agent $n \in \{1,\cdots,N_1,N_1+1,\cdots,N_1+N_2,\cdots,N\}$ may be considered as an *elementary subsystem* whose entropy function is given by (5.2), where $i = i(n)$. Since the agents of any given type $i \in \{1,\cdots,I\}$ are by definition identical, they are characterized by the same entropy function. To put it otherwise, the set of traders of the *ith* type

$$E_i := \{N_{i-1}+1,\cdots,N_{i-1}+N_i\},\ i \in \{1,\cdots,I\}, \tag{5.3}$$

constitute a composite subsystem $\mathrm{E}_i := (e^n, \succeq_n)_{n\in E_i}$, made out of N_i identical elementary subsystems[38]; hence the entropy function of the ensemble of agents of type i takes the simple form (5.1). It should be noted that an individual agent corresponds to a macroscopic thermodynamic system and not to the particle of Newtonian mechanics[39].

If the agents in a certain subsystem E_i are not assumed to be identical, then the entropy function of that subsystem would be given by

$$S^i(x^i) = \max\left\{\sum_{n\in E_i} s^n(x^n) \,\middle|\, x^n \in \Re_+^L \forall n \in E_i \, and \sum_{n\in E_i} x^n = x^i\right\} \tag{5.4}$$

instead of (5.1). Here the entropy function s^n of the elementary subsystem $n \in E_i$ is defined by

$$s^n\left(x^n\right) := \max\left\{-\sum_{w\in\Gamma^n} p^n(w)\ln p^n(w) \,\middle|\, p^n \in C^n\left[x^n\right]\right\}, \tag{5.5}$$

where

[38] We understand that $N_0 := 0$ in (5.3).

[39] See also Smith and Foley (2002) to this effect.

$$C^n\left[x^n\right] := \left\{ p^n \in \Delta^n \,\middle|\, \sum_{w\in\Gamma^n} p^n(w)w = x^n \right\}. \tag{5.6}$$

As the notation in (5.5) and (5.6) indicates, the probability distribution $p^n := [p^n(w)]_{w\in\Gamma^n}$ may vary among individual traders in E_i; it takes the canonical form (4.9), with the index i in (4.9)-(4.11) being replaced by n and the range E_i of the latter taking the place of the range of the former. Δ^n is the $g_n - 1$ dimensional unit simplex, where g_n is the cardinality of the preferred action set Γ^n of agent n. As indicated in (5.6), the assignment of the vector of resources $x^n \in \Re^L_+$ to agent n amounts to specifying the expected value of her action. Recalling that the cardinality g_n of Γ^n has been presumed to be a large positive integer, we may comfortably posit that $L+1 < g_n$; accordingly, we may invoke the maximum statistical entropy principle to determine the most disordered macro-state [as described by the maximization problem (5.5)] of the elementary subsystem n, which turns out to be the canonical distribution. When the latter prevails, trader n is said to be an elementary *equilibrium subsystem.*

The above generalization regarding heterogeneous agents may be extended to any segment of a pure exchange economy E, including E itself. The determination and interpretation of statistical equilibrium prices along with the maximum entropy principle of the phenomenological theory hold true when all N agents in E are heterogeneous. Of course, the pertinent formulas and equations would have to be modified accordingly. A macro-state of the economy E is now specified by the N-list $p := (p^1, \cdots, p^N)$ of probability vectors $p^n := [p^n(w)]_{n\in\Gamma^n}$, $n \in \{1, \cdots, N\}$. The normalization conditions (2.9), the form (2.11) of the statistical entropy of p, the market-clearing condition (3.1), and the constraint set (3.2) are, respectively, modified as follows:

$$\sum_{w\in\Gamma^n} p^n(w) = 1, \;\; n = 1, \cdots, N, \tag{5.7}$$

$$H(p) := -\sum_{n=1}^{N} \sum_{w\in\Gamma^n} p^n(w) \ln\, p^n(w), \tag{5.8}$$

$$\sum_{n=1}^{N} \sum_{w\in\Gamma^n} p^n(w)w = x, \tag{5.9}$$

$$\tilde{C}[x] := \left\{ p \in \Delta \,\middle|\, \sum_{n=1}^{N} \sum_{w\in\Gamma^n} p^n(w)w = x \right\}, \tag{5.10}$$

where $\Delta := \Delta^1 \times \cdots \times \Delta^N$. The statistical equilibrium (3.5) is now given by the appropriate modifications of (3.10) and (3.12):

$$p^n(w) = \exp[-p \cdot w] / Z^n(p), \quad w \in \Gamma^n, n = 1, \cdots, N, \tag{5.11}$$

$$Z^n(p) := \sum_{w \in \Gamma^n} \exp[-p \cdot w], \; n = 1, \cdots, N. \tag{5.12}$$

The system of equations (3.13) that determines the entropy price vector p now reads

$$\sum_{n=1}^{N} \sum_{w \in \Gamma^n} [\exp(-p \cdot w) / Z^n(p)] w = x. \tag{5.13}$$

We may next recast the two-stage procedure [as developed in Section 4] for arriving at the most disordered macro-state (5.11) of the pure exchange economy E as follows. **In Stage I**, we obtain the most disordered macro-state of every elementary subsystem $n \in \{1, \cdots, N\}$ associated with an arbitrary allocation $x := (x^1, \cdots, x^N) \in \Re_+^{LN}$, where $x^n := (x_1^n, \cdots, x_L^n) \in \Re_+^L$ is the vector of resources assigned to trader n. That is, we solve the maximization problem (5.5), and determine an entropy function s^n for each trader or elementary subsystem $n \in \{1, \cdots, N\}$. **In Stage II**, the allocation x is feasibly adjusted to maximize the global entropy expression of the economy, which in this instance is equal to the sum of the entropy functions of the economy's elementary subsystems. Thus the *equilibrium allocation* $x(x)$ associated with total supply $x \in \Re_+^L$ is given by the counterpart of (4.25)

$$x(x) := \arg\max \left\{ \sum_{n=1}^{N} s^n(x^n) \middle| x \in \tilde{F}[x] \right\}, \tag{5.14}$$

with the set of feasible allocations (4.4) being appropriately modified to read

$$\tilde{F}[x] := \left\{ x \in \Re_+^{LI} \middle| \sum_{n=1}^{N} x^i = x \right\}. \tag{5.15}$$

The entropy function of the economy S, defined by

$$S(x) := \max \left\{ -\sum_{n=1}^{N} \sum_{w \in \Gamma^n} p^n(w) \ln p^n(w) \middle| p \in \tilde{C}[x] \right\}, \tag{5.16}$$

is equal to the maximized global entropy expression and hence the sum of the entropies of the elementary subsystems, evaluated at the equilibrium allocation (5.14):

$$S(x) = \max \left\{ \sum_{n=1}^{N} s^n(x^n) \middle| x \in \tilde{F}[x] \right\} = \sum_{n=1}^{N} s^n(x^n(x)). \tag{5.17}$$

Statements (5.14) and (5.17) constitute a more general version of the maximum entropy principle when an exchange economy is considered as a composite of its elementary subsystems with no assumption of classes of homogeneous agents. The derivation of the principle from (5.16) would proceed pretty much along the lines of the argument developed in Section 4 so it will be omitted.

Likewise, the properties of the entropy functions are identical with those established in Sections 3 and 4 and so are the methods of proof. In a nutshell, the entropy function of an elementary equilibrium subsystem or any equilibrium finite collection of such subsystems (with no internal constraints) is of class C^2 and regular strictly concave on $\Re^L_{++}$; it may be assumed to be defined and continuous on the entire resource space $\Re^L_{+}$.

In most general terms, we may consider an isolated exchange economy which is a composite system made out of an arbitrary collection of J subsystems, indexed by $j \in \{1, \cdots, J\}$, where J is a positive integer. Each subsystem is an exchange economy itself of the form (2.2) for an appropriate choice of N, possibly an elementary one consisting of a single agent. *An equilibrium subsystem j* (that is, a subsystem j in the most disordered macro-state, given the vector of resources allocated to it) is associated with a continuous entropy function $x^j \mapsto S^j(x^j), x^j \in \Re^L_{+}$, which is of class C^2 and regular strictly concave on $\Re^L_{++}$; an *allocation* among the J subsystems is an ordered list $x := (x^1, \cdots, x^J) \in \Re^{LJ}_{+}$; for any predetermined total supply $x \in \Re^L_{+}$, the conservation laws are summarized by

$$\sum_{j=1}^{J} x^j = x \tag{5.18}$$

rather than by (4.3). Correspondingly, the set of *feasible allocations* $\Phi[x]$ is now given by

$$\Phi[x] := \left\{ x \in \Re^{LI}_{+} \middle| \sum_{j=1}^{J} x^j = x \right\}, \ x \in \Re^L_{+}, \tag{5.19}$$

rather than by (4.4) or (5.15); and an *equilibrium allocation* $x(x)$ [associated with total supply x] is defined via an appropriate modification of (4.25) or (5.14):

$$x(x) := \arg\max \left\{ \sum_{j=1}^{J} S^j(x^j) \middle| x \in \Phi[x] \right\}. \tag{5.20}$$

That the maximization problem

$$\max \left\{ \sum_{j=1}^{J} S^j(x^j) \middle| x \in \Phi[x] \right\}, \ x \in \Re^L_{+}, \tag{5.21}$$

has a solution follows from the fact that the global entropy expression (which is the objective function) is continuous and the constraint set (5.19) is non-empty and compact. From the functional form of the statistical entropy and the maximum-statistical-entropy underpinnings of the phenomenological theory we may infer that a solution to (5.21) should have no zero components; accordingly, the solution to (5.21) is also unique [as anticipated by (5.20)]. This is so because the global entropy expression is strictly concave on $\Re^{LJ}_{++}$ and $\Phi[x] \cap \Re^{LJ}_{++}$ is non-empty and convex for $x \in \Re^L_{++}$. Once again, the entropy function $S : \Re^L_{+} \to \Re$ of the composite system in statistical equilibrium is equal to the sum of the entropies of the equilibrium subsystems—the counterpart of (4.27) or (5.17) reads

$$S(x) = \max\left\{\sum_{j=1}^{J} S^j(x^j) \middle| x \in \Phi[x]\right\} = \sum_{j=1}^{J} S^j(x^j(x)). \tag{5.22}$$

In the context of pure exchange economies, we may view statements (5.20) and (5.22) as the most general formulation of the maximum entropy principle—which is the fundamental principle of the phenomenological theory of resource allocation. This is in effect the theory of 'systems with additive effects' of Rozonoer (1973, Paper I). But here the existence and properties of the entropy functions (twice continuously differentiable, regular strictly concave) as well as the maximum entropy principle are all derived from the statistical mechanical theory of markets (rather than being postulated).

It is worth emphasizing that the equilibrium behavior of a composite system as a whole is fully determined by its entropy function—we do not need to know the make up of the system that gives rise to the latter. Once the entropy function of the composite system is known, the equilibrium entropy price vector is determined by the envelope result (3.23) [with $U = \Re^L_{++}$]. Further, from (5.19) and (5.20) it follows that the equilibrium allocation (5.20) is fully determined by the entropy functions of the equilibrium subsystems.

Even though the foundations of the phenomenological theory are provided by the statistical equilibrium model of pure exchange, we should view the former theory as an autonomous level of analysis. Accordingly, it is not surprising that the needs of the phenomenological theory itself dictate some additional restrictions on entropy functions. Some of them are merely simplifying assumptions (e.g. assumptions that ensure that all components of the equilibrium allocation vector are positive), while others are more substantive desiderata (e.g. that entropy prices should be positive), suggesting the need for adding more structure at the level of statistical equilibrium theory. Thus in reference to the maximization problem (5.6) we **posit** that for every $j \in \{1, \cdots, J\}$, the entropy function $S^j : \Re^L_+ \to \Re$ (aside from being continuous and, by statistical equilibrium theory, of class C^2 and regular strictly concave on $\Re^L_{++}$) has the following **additional properties**[40]: (a) S^j has positive partial derivatives on $\Re^L_{++}$, namely

$$\nabla S^j(x^j) \in \Re^L_{++}, \forall x^j \in \Re^L_{++}; \tag{5.23}$$

(b) S^j satisfies the Inada conditions

$$\partial S^j / \partial x^j_l \to +\infty \ as \ x^j_l \to 0+ \ \ (l = 1, \cdots, L), \tag{5.24}$$

$$\partial S^j / \partial x^j_l \to 0 \ \ as \ \ x^j_l \to +\infty \ \ (l = 1, \cdots, L). \tag{5.25}$$

[40] The adoption of properties (a) and (b) is after Rozonoer (1973).

The Inada condition (5.24) ensures that the non-negativity constraint on x is not binding at a solution of the maximization problem (5.21), without having to invoke its statistical-equilibrium underpinning.

With the above additional properties, entropy functions may take the Cobb-Douglas form:

$$S^j(x^j) := \prod_{l=1}^{L} (x_l^j)^{a_l}, \quad a_1, \cdots, a_L > 0, \sum_{l=1}^{L} a_l < 1. \tag{5.26}$$

This is an illustration of what should be apparent on general grounds: the entropy function of an equilibrium system, whether elementary or not, looks like a neoclassical utility function. But even if an equilibrium subsystem consists of a single trader, the associated entropy function is **not** a representation of the trader's preference ordering of alternative resource (consumption) bundles. Preferences or wants do affect the preferred action sets and, as a consequence, the entropy functions, but the latter do not represent those wants. There is no implication that a situation like $S^j(x^j) > S^j(y^j),\ x^j, y^j \in \Re_+^L,$ signifies any sort of superiority of x^j over y^j (in terms of preference or welfare) in the eyes of the agents that constitute the jth subsystem.

Rozonoer (1973) treats the entropy function—his structure function—as analogous to utility in his systems with non-additive effects or as analogous to profit or revenue in his systems with additive effects. There is indeed, as we have already mentioned a deep correspondence of thermodynamic entropy (the analogue of our entropy function) to neoclassical utility [Candeal et al. (2001), Smith and Foley (2002)]. That such a connection is absent in our approach reflects our premise that statistical thermodynamics is the fundamental theory; classical or macroscopic thermodynamics is basically derived from the latter. In more discipline-neutral terms, our point of entry to the study of pure-exchange economies is their probabilistic description and the maximum statistical entropy principle (consistently with the statistical theory of markets of Foley). The entropy function of the phenomenological theory is just the maximum of the statistical entropy over the set of probability distributions compatible with the resource constraints on mean actions. This is precisely the perspective that emanates from statistical physics [Balian (1991, p. 247)].

The significance of the requirement (5.23) becomes apparent once we write the conditions that characterize (5.20). The conservation laws (5.18) do satisfy the NDCQ—it is easy to see that the $L \times LJ$ matrix associated with the linear system (5.18) has rank L. Hence the Lagrange multiplier theorem is applicable to the maximization problem (5.21)—that the other assumptions of the theorem besides the NDCQ are satisfied should be obvious. Accordingly, for every total supply vector $x \in \Re_{++}^L$, the solution $x := x(x) \in \Re_{++}^{LJ}$ to (5.21) and the vector $p := p(x) \in \Re^L$ of Lagrange multipliers associated with the conservation laws (5.18) satisfy the first-order conditions

$$\nabla S^j(x^j) = p, \quad j = 1, \cdots, J. \tag{5.27}$$

In view of the statistical mechanical underpinning of the phenomenological model, the multipliers are the equilibrium entropy prices, which now, in view of (5.23) and (5.27) and the fact that $x \in \Re_{++}^{LJ}$, *are all positive.*

We may (locally) invert (5.27) to obtain $x^1, \cdots, x^J$ as vector-valued functions of the entropy price vector p and subsequently insert them into (5.18)—which summarizes the economy's conservation laws—to yield the system of equations that implicitly determines p as a vector-valued function of the total supply $x \in \Re^L_{++}$. The solution of (5.27) is in turn determined as a *J*-list of vector-valued functions of x and hence so is the economy-wide entropy function (5.22). The relationship between total supply and entropy prices is also captured by the envelope result

$$p = \nabla S(x). \tag{5.28}$$

Since the entropy function S is of class C^2 and regular strictly concave on $\Re^L_{++}$, the transformation (5.28) has a local C^1 inverse, denoted by $x = x(p)$; this local inverse would become useful in situations in which entropy prices rather the amounts of resources in the economy play the role of independent variables. Such situations may indeed materialize when an exchange economy E is *open*; the maximum entropy principle is then applicable not to the economy E per se but to the isolated composite system made out of E and its complement or *environment* E^c. The latter consists of all agents outside the economy E with whom its members engage in trade—so E^c is the 'rest of the world' outside E, viewed as another pure exchange economy with the same resource space $\Re^L$. The pertinent total endowment of the composite system would be the vector sum of the total endowments of E and E^c. Letting $e^c \in \Re^L_{++}$ and S^c be the total endowment and the entropy function of E^c, respectively, the equilibrium allocation of resources $(\hat{x}, \hat{x}^c)$ between the economy E and its environment E^c is determined by the maximum entropy principle:

$$(\hat{x}, \hat{x}^c) := \arg\max \left\{ S(x) + S^c(x^c) \middle| x, x^c \in \Re^L_+; x + x^c = e + e^c \right\} \tag{5.29}$$

Typically, the entropy price vector of an exchange economy varies with the amounts of resources that are available to the economy—see, for example, equation (5.28) and recall the general properties of entropy functions or think of the Cobb-Douglass form (5.26). But it is of special interest to consider systems with *linear entropy functions*, i.e. economies whose entropy price vector is not affected by the amounts of their resources. These are Rozonoer's 'unsaturatable systems', which are generalized versions of the 'thermal reservoirs' of thermodynamics[41]. Accordingly, we propose to call the system with a linear entropy function a *resource reservoir*[42]. Suppose, in particular, that the environment E^c of the economy E is a resource reservoir with the entropy function

$$S^c(x^c) := \bar{p} \cdot x^c, \tag{5.30}$$

[41] See, for example, Callen (1985, p.106).

[42] The reservoir here encompasses all resources, although one could introduce a separate reservoir for each resource. If a "small" subsystem is in equilibrium with a "large" subsystem, then under appropriate conditions the large subsystem functions as a resource reservoir for the small subsystem [Rozonoer (1973), Paper I].

where $\bar{p}$ is a predetermined vector in $\Re^L_{++}$. In view of (5.29) and (5.30), the equilibrium allocation $\hat{x}$ for the economy E satisfies (5.28) at $p = \bar{p}$ and hence it is given by the value $\hat{x} = x(\bar{p})$ of the local inverse of (5.28) at $\bar{p}$.

When an exchange economy E is open and in equilibrium with a resource reservoir, it is more usefully described by the Legendre transform of its entropy function:

$$\Pi(p) := \max\left\{S(x) - p \cdot x \,\middle|\, x \in \Re^L_+\right\} = S(x(p)) - p \cdot x(p), \tag{5.31}$$

where $x = x(p)$ is the local inverse of (5.28). The validity of the last expression in (5.31) derives from the observation that the maximizer of $x \mapsto S(x) - p \cdot x$ is characterized by (5.28). Assumption (5.25) on S (in addition to its other properties) is crucial[43] for ensuring that (5.28) has a local C^1 inverse *for every* $x \in \Re^L_{++}$. The function (5.31) is a 'thermodynamic potential' in the vocabulary of thermodynamics; we call Π the *characteristic function* of the economy E [after Rosonoer (1973)].

By (5.31), the characteristic function is of class C^1 and hence the envelope theorem applies to yield

$$x(p) = -\nabla\Pi(p), \tag{5.32}$$

which in turn establishes that in effect Π is *of class* C^2 on $\Re^L_{++}$. Further, from (5.28) and (5.32) it follows that the characteristic function Π is *regular strictly convex* (on account of the entropy function being regular strictly concave). Equation (5.32) makes it clear that the characteristic function fully describes the equilibrium properties of a small open economy—an economy in equilibrium with the 'rest of the world', functioning as a resource reservoir for the economy in discussion. The equilibrium allocation $\hat{x}$ pertaining to (5.29)-(5.30) is obtained by simply calculating the gradient of the characteristic function at 'world prices' $\bar{p}$:

$$\hat{x} = -\nabla\Pi(\bar{p}). \tag{5.33}$$

The existence and properties of the characteristic function have a transparent statistical mechanical foundation. In the model of Sections 3 and 4 with groups of homogeneous agents, a mere comparison of equations (3.22)-(3.23) with (5.31) yields

$$\Pi(p) = \sum_{i=1}^{N} N_i \ln Z^i(p). \tag{5.34}$$

When all agents are heterogeneous, it is not hard to see from (5.8), (5.11), (5.12), and (5.16) that the counterpart of equation (3.22) is

$$S(x) = p \cdot x + \sum_{n=1}^{N} \ln Z^n(p) \tag{5.35}$$

[43] This is our reading of Theorem B in Rozonoer (1973, Paper I).

and hence (5.34) generalizes to

$$\Pi(p) = \sum_{n=1}^{N} \ln Z^n(p). \tag{5.36}$$

6. Concluding Remarks

Theory construction in economics along the lines of statistical and classical thermodynamics is both possible and fruitful. The cornerstone for this endeavor is the generalized statistical mechanics pioneered by E. T. Jaynes, which is independent of any physical properties. As a showcase of this approach, we have presented a comprehensive theory of pure exchange economies, inspired by the statistical equilibrium theory of markets of Foley and the generalized thermodynamics of resource allocation of Rozonoer. Our theoretical discourse is grounded on a probabilistic description of general systems and a self-contained account of the arguments that lead to the concept of statistical entropy as a measure of uncertainty or disorder and the maximum statistical entropy principle.

Naturally, the statistical mechanical mode of theorizing has to rely on microeconomics for specifying the spectrum of all possible micro-states of the system under investigation—which in this essay is a pure exchange economy. A key contribution of the Foley model pertains exactly to this point—the description of agents by their 'transaction set', namely the set of trades that they are willing and able to carry out. We have adopted a more or less neoclassical version of this concept, the preferred action set—the set of actions or resource bundles which in the trader's preference scale are at least as good as her initial endowment. While the description of the agents is largely neoclassical (but with less restrictive assumptions about preferences), the concept of statistical equilibrium is distinctly non-Walrasian—equilibrium entropy prices clear the markets by appropriately distributing traders over their preferred action sets, rather than by coordinating their utility-maximizing choices (as in the Arrow-Debreu model). Aside from requiring that the relevant constraints should be independent, the main assumption needed for the propositions of the statistical mechanical model is shown to be disarmingly simple—we only need to posit that the cardinality of y preferred action set of every trader is sufficiently high.

One of the main results of this essay is the proposition that the statistical mechanical model of markets gives rise to a phenomenological theory of resource allocation in the image of classical thermodynamics. Crucial in the demonstration of this proposition is the concept of the entropy function and its properties—defined as the maximum value function associated with the constrained maximization of statistical entropy. Thus the entropy function of the pure exchange economy under discussion is just the maximized statistical entropy, viewed as a function of the vector of resources in the economy. The entropy function of any segment of the economy (or subsystem) is defined in a similar fashion. The equilibrium behavior of an exchange economy or any of its subsystems is fully characterized by its entropy function. With this concept in place, we have been able to portray the establishment of statistical equilibrium as a two-stage optimization process; the second stage of the process yields the fundamental principle of the phenomenological theory.

More specifically, in the first stage of the process, we partition the economy into a collection of notionally isolated subsystems and assign an arbitrary allocation of resources among them. We let each separate subsystem attain statistical equilibrium and thus determine its entropy function. In the second stage, we let the subsystems interact through the unrestrained exchange of resources; it is shown that the attainment of maximum disorder in the economy is tantamount to finding a feasible allocation among the subsystems (i.e. an allocation obeying the resource conservation laws) that maximizes the global entropy expression of the economy, namely the sum of the subsystem entropy functions. The entropy function corresponds to thermodynamic entropy, and the just described constrained maximization of the global entropy expression corresponds to the maximum entropy principle of classical thermodynamics; it is the fundamental principle of the phenomenological theory of resource allocation. Thus the principle of maximum entropy here emerges not by analogy to its thermodynamic counterpart (as, for instance, in the Rosonoer system) but as a proposition derived from the maximum statistical entropy principle in the context of a pure exchange economy.

When an exchange economy is open, the maximum entropy principle is then applicable not to the economy per se but to the composite system made out of the economy and its environment. In the case of a small open economy, its environment functions like a resource reservoir, fixing the entropy price vector for the economy. The vector of resources in the economy is now determined endogenously (as a result of unrestricted trade with the rest of the world) by means of the economy's characteristic function, which a Legendre transform of its entropy function. The characteristic function of the economy has a transparent micro foundation: it is shown to be equal to the sum of the natural logarithms of the partition functions of the individual traders that constitute the economy in discussion.

The derivation of the principle of maximum entropy from first economic principles opens the way for the systematic use of the vast analytical apparatus of equilibrium and non-equilibrium thermodynamics in economic theory. While there is still a lot of ground to be covered at the equilibrium level, it is worth noticing that the exploration of the workings of 'process thermodynamics' in an economic context is now within reach as well. Indeed, our two-stage optimization procedure points out to the possibility of local equilibrium (i.e. the economy is made out of parts which are separately in equilibrium) without global or economy-wide equilibrium. Loosely speaking, this observation corresponds to the starting point of nonequilibrium thermodynamics [Kondepudi and Prigogine (1998, Ch. 15)][44].

If one is willing to accept the interpretation of probability as state of knowledge—which is the interpretation that underlies the generalized statistical mechanics of Jaynes—then we can dispense with the assumption of the Foley model that the ensemble of an economy's agents is partitioned into large equivalence classes of agent types. The statistical mechanical theory of pure exchange economies holds true even in situations of completely heterogeneous traders. Every agent is an elementary subsystem, with a determinate entropy function. The latter looks like a neoclassical utility function, but it is a fundamentally different concept. It allows us to state our best prediction regarding the agent's behavior when the only information about the probability distribution over her preferred action set is the expected value of her action.

[44] See Isard and Liossatos (1979) for some earlier attempts to use the Prigogine version of nonequilibrium thermodynamics in spatial economics.

This brings us to a host of methodological issues surrounding the use of the probability formalism, which warrant further investigation. The general question of how probability is to be understood even in the context of statistical physics is still a subject of debate[45]. We need more clarity on this matter in economics as well, especially when we envision probability at the foundation of economic theory. We will take up the theme of the concept of probability in generalized statistical mechanics in future papers.

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[45] See, for example, Jaynes (2003), D' Agostini (2003), and Guttman (1999).

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