Serre Duality

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1 Introduction

Serre duality is a fundamental result in algebraic geometry. The goal of this note is to provide a complete proof of Serre duality, with some comments and explanations.

Here's the Serre duality theorem that we will prove:

Theorem (Serre duality)**.** *Let X be a projective Cohen–Macaulay scheme of pure dimension $d \geq 0$ over a field k. Then, for any coherent sheaf $\mathcal{F} \in \mathrm{Coh}(X)$, we have a natural isomorphism*

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \cong H^{d-i}(X, \mathcal{F})^*$$

in $\mathcal{F}$ for all $0 \leq i \leq d$, where ω_X is a dualizing sheaf of X.

2 Ingredients

Here, we list results that we are going to use without proofs in this note.

Lemma 1. *Let k be a field and V and W be finite-dimensional k-vector spaces. Let $f : V \times W \to k$ be a bilinear map. Then, the following is equivalent:*

- *f is a perfect pairing:* i.e., *the induced maps $V \to W^*$ and $W \to V^*$ are isomorphisms.*
- *f is nondegenerate in both arguments:* i.e., *for any nonzero $v \in V$, there exists $w \in W$ such that $f(v, w) \neq 0$, and for any nonzero $w \in W$, there exists $v \in V$ such that $f(v, w) \neq 0$.*

Comments. The proof is straightforward linear algebra. When I first encountered the term "perfect pairing," I couldn't find its definition in standard textbooks (including Axler, Lang, Eisenbud, Hartshorne, Vakil, etc.), where they either assume that the reader already knows it or do not mention this term. Nonetheless, the definition is quite natural, and can be roughly understood as the nondegeneracy of a bilinear map (at least in the finite-dimensional case). We will use this lemma to establish $V \cong W^*$ by constructing a bilinear map $f : V \times W \to k$ and showing that it is nondegenerate.

Lemma 2. *Let X be a locally noetherian scheme, $\mathcal{L} \in \mathrm{Pic}(X)$ be an ample line bundle on X, and $\mathcal{F} \in \mathrm{Coh}(X)$ be a coherent sheaf on X. Then, there exists integers $n, d \geq 1$ and a surjective morphism*

$$\bigoplus_{i=1}^{n} \mathcal{L}^{-d} \twoheadrightarrow \mathcal{F},$$

where d can be chosen to be arbitrarily large. In particular, for k a field, $X = \mathbb{P}^n_k$, and $\mathcal{L} = \mathcal{O}(1)$, any coherent sheaf $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}^n_k)$ admits a surjective morphism

$$\mathcal{O}(-d)^{\oplus n} \twoheadrightarrow \mathcal{F}$$

for some integers $n, d \geq 1$, where d can be chosen to be arbitrarily large.

Comments. This is a useful lemma about ample line bundles (in fact, it is more or less equivalent to a line bundle being ample). We will use this lemma several times to compute a resolution of $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}^n_k)$ by direct sums of $\mathcal{O}(-d)$'s.

Lemma 3 (Projective Noether normalization)**.** *Let X be a projective scheme of dimension $d \geq 0$ over a field k. Then, there exists a finite morphism $f : X \to \mathbb{P}^d_k$. Moreover, for any closed immersion $\iota : X \hookrightarrow \mathbb{P}^n_k$ with $n > 0$, we can find $N > 0$ such that $f^*\mathcal{O}_{\mathbb{P}^d_k}(1) = \mathcal{O}_X(N)$, where $\mathcal{O}_X(N) := \iota^*\mathcal{O}_{\mathbb{P}^n_k}(N)$.*

Comments. This is a useful lemma about projective schemes. It is analogous to the affine Noether normalization lemma. The second half of the statement sounds a bit technical, but it naturally follows from the construction of the finite morphism in the proof of the first half. We will use this lemma to prove the existence of dualizing sheaves on projective schemes.

Lemma 4. *For abelian categories $\mathcal{A}$ and $\mathcal{B}$, let (T^i, δ^i) be a contravariant δ-functor from $\mathcal{A}$ to $\mathcal{B}$. If for any object $A \in \mathcal{A}$, there exists an epimorphism $P \to A$ such that $T^i(P) = 0$ for all $i > 0$, then T^i is a universal δ-functor (with respect to T^0). In particular, it is unique up to unique isomorphism.*

Comments. It is a standard result from homological algebra. Basically, this lemma tells us that if two δ-functors share "enough" acyclic objects, then they are the same. It is useful when we want to show that two homological functors are isomorphic.

Lemma 5. *Let $f : X \to Y$ be an affine morphism of locally noetherian schemes. Then, the category of quasicoherent (resp. coherent) sheaves on X is equivalent to the category of quasicoherent (resp. coherent) sheaves of $f_*\mathcal{O}_X$-modules on Y (*i.e.*, $f_*\mathcal{O}_X$-modules in $\mathrm{QCoh}(Y)$ (resp. $\mathrm{Coh}(Y)$)). The equivalence is given by the pushforward functor f_* and its inverse.*

Comments. This seemingly technical lemma can be proven using the relative Spec construction. We will use this lemma to prove the existence of the upper shriek functor $f^!$, which plays an important role in defining dualizing sheaves.

Lemma 6 (Miracle flatness)**.** *Let $f : (R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be a local homomorphism of noetherian local rings. If R is regular, $\dim(R) = \dim(S)$, and $\sqrt{\mathfrak{m}_R S} = \mathfrak{m}_S$, then f is flat if and only if S is Cohen–Macaulay.*

Comments. This is a criterion that connects flatness and Cohen–Macaulayness. Note that the condition $\sqrt{\mathfrak{m}_R S} = \mathfrak{m}_S$ means that the fiber over $\mathfrak{m}_R$ is precisely $\{\mathfrak{m}_S\}$ (f being a local homomorphism means $\mathrm{Spec}(f)(\mathfrak{m}_S) = \mathfrak{m}_R$, so the fiber over $\mathfrak{m}_R$ can generally be larger than just $\{\mathfrak{m}_S\}$). Honestly, I don't fully understand what's so "miraculous" about this lemma (at least not yet), but after all, it enables us to convert flatness to Cohen–Macaulayness and vice versa, which is useful. The proof involves quite a bit of commutative algebra.

Lemma 7 (Serre vanishing)**.** *Let X be a projective scheme over a noetherian ring R and $\iota : X \hookrightarrow \mathbb{P}^n_R$ be a closed immersion. Define $\mathcal{O}_X(N) := \iota^*\mathcal{O}_{\mathbb{P}^n_R}(N)$ for $N \in \mathbb{Z}$. Then, for any coherent sheaf $\mathcal{F} \in \mathrm{Coh}(X)$, we have*

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(N)) = 0$$

for all $i > 0$ and sufficiently large N.

Comments. This is one of the fundamental results about sheaf cohomology on projective schemes. It is so fundamental that there's not much to say about it.

3 Proof of Serre duality

We begin with the following special case of Serre duality, which will serve as the foundation for the general case.

Proposition 8. *Let k be a field and $n \geq 1$ be an integer. For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$, there is a natural isomorphism*

$$\mathrm{Hom}(\mathcal{F}, \mathcal{O}(-n-1)) \cong H^n(\mathbb{P}^n_k, \mathcal{F})^*.$$

in $\mathcal{F}$.

Proof. This can be directly shown using the explicit computation of cohomology on $\mathbb{P}^n_k$.

First, from $H^n(\mathbb{P}^n_k, \mathcal{O}(-n-1)) \cong k$, we can find an isomorphism $\lambda : H^n(\mathbb{P}^n_k, \mathcal{O}(-n-1)) \xrightarrow{\sim} k$. We construct a natural map

$$\alpha_{\mathcal{F}} : \operatorname{Hom}(\mathcal{F}, \mathcal{O}(-n-1)) \to H^n(\mathbb{P}^n_k, \mathcal{F})^*$$

as follows: given a morphism $f : \mathcal{F} \to \mathcal{O}(-n-1)$, we define $\alpha_{\mathcal{F}}(f)$ to be the composition

$$H^n(\mathbb{P}^n_k, \mathcal{F}) \xrightarrow{H^n(f)} H^n(\mathbb{P}^n_k, \mathcal{O}(-n-1)) \xrightarrow{\lambda} k.$$

The naturality follows from the functoriality of the cohomology functor $H^n(\mathbb{P}^n_k, -)$ (specifically, see the commutative diagram below for a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$).

$$\begin{array}{ccc} \operatorname{Hom}(\mathcal{G}, \mathcal{O}(-n-1)) & \longrightarrow & \operatorname{Hom}(\mathcal{F}, \mathcal{O}(-n-1)) \\ \downarrow & & \downarrow \\ \operatorname{Hom}(H^n(\mathbb{P}^n_k, \mathcal{G}), k) & \longrightarrow & \operatorname{Hom}(H^n(\mathbb{P}^n_k, \mathcal{F}), k) \end{array} \qquad \begin{array}{ccc} g & \longmapsto & g \circ \varphi \\ \downarrow & & \downarrow \\ \lambda \circ H^n(g) & \longmapsto & \begin{array}{c} \lambda \circ H^n(g \circ \varphi) \\ = \lambda \circ H^n(g) \circ H^n(\varphi) \end{array} \end{array}$$

Next, we show that $\alpha_{\mathcal{F}}$ is an isomorphism when $\mathcal{F} = \mathcal{O}(-d)$ for any integer d. To do this, first define a bilinear map

$$\beta : \operatorname{Hom}(\mathcal{O}(-d), \mathcal{O}(-n-1)) \times H^n(\mathbb{P}^n_k, \mathcal{O}(-d)) \to H^n(\mathbb{P}^n_k, \mathcal{O}(-n-1)) \cong k$$

given by $(\varphi, \xi) \mapsto H^n(\varphi)(\xi)$. This is clearly bilinear, so it suffices to check the equivalent condition in Lemma 1. To see this, we need the explicit description of the cohomology on $\mathbb{P}^n_k$. First, under the identification $\operatorname{Hom}(\mathcal{O}(-d), \mathcal{O}(-n-1)) \cong H^0(\mathbb{P}^n_k, \mathcal{O}(d-n-1))$, note that the bilinear map

$$\beta : H^0(\mathbb{P}^n_k, \mathcal{O}(d-n-1)) \times H^n(\mathbb{P}^n_k, \mathcal{O}(-d)) \to H^n(\mathbb{P}^n_k, \mathcal{O}(-n-1))$$

is simply given by $(a, [b]) \mapsto [ab]$, where $[\cdot]$ denotes the corresponding cohomology class.

Next, recall that from the Čech cohomology computation, $H^n(\mathbb{P}^n_k, \mathcal{O}(-d))$ is isomorphic to the degree $-d$ part of $k[T_0^{\pm 1}, \dots, T_n^{\pm 1}] / \operatorname{im}(\oplus_{i=0}^n k[T_0^{\pm 1}, \dots, T_i, \dots, T_n^{\pm 1}] \to k[T_0^{\pm 1}, \dots, T_n^{\pm 1}])$, where the map in the denominator is the alternating sum of the natural inclusions. Hence, the only "surviving" monomials in $H^n(\mathbb{P}^n_k, \mathcal{O}(-d))$ are of the form $\frac{1}{T_0^{e_0} T_1^{e_1} \cdots T_n^{e_n}}$ with $e_i \geq 1$ and $\sum_{i=0}^n e_i = d$. This gives us the following identifications:

$$\begin{aligned} H^0(\mathbb{P}^n_k, \mathcal{O}(d-n-1)) &\cong k[T_0, \dots, T_n]_{d-n-1}, \\ H^n(\mathbb{P}^n_k, \mathcal{O}(-d)) &\cong \left(\frac{1}{T_0 \cdots T_n} k[T_0^{-1}, \dots, T_n^{-1}] \right)_{-d}, \\ H^n(\mathbb{P}^n_k, \mathcal{O}(-n-1)) &\cong \frac{1}{T_0 \cdots T_n} k. \end{aligned}$$

Now, for $a = T_0^{e_0} T_1^{e_1} \cdots T_n^{e_n} \in H^0(\mathbb{P}^n_k, \mathcal{O}(d-n-1))$ and $b = \frac{1}{T_0 T_1 \cdots T_n} T_0^{f_0} T_1^{f_1} \cdots T_n^{f_n} \in H^n(\mathbb{P}^n_k, \mathcal{O}(-d))$, $[ab]$ is given by $[\frac{1}{T_0 \cdots T_n} T_0^{e_0+f_0} T_1^{e_1+f_1} \cdots T_n^{e_n+f_n}]$, which is nonzero in $H^n(\mathbb{P}^n_k, \mathcal{O}(-n-1))$ if and only if $e_i = -f_i$ for all $0 \leq i \leq n$. This shows that β is nondegenerate in both arguments. Hence, by Lemma 1, β is a perfect pairing and thus $\alpha_{\mathcal{O}(-d)}$ is an isomorphism.

Finally, for a general coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$, we conclude by using Lemma 2 and the five lemma. Specifically, by Lemma 2, we can find integers $a, b, d \geq 1$ and an exact sequence

$$\mathcal{O}(-d)^{\oplus b} \to \mathcal{O}(-d)^{\oplus a} \to \mathcal{F} \to 0.$$

Then, we have the following commutative diagram with exact rows, where we use the facts that Hom is left exact, $H^{n+1}(\mathbb{P}^n_k, -) = 0$, both Hom and H^n commute with finite direct sums, and α is natural.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}(\mathcal{F}, \mathcal{O}(-n-1)) & \longrightarrow & \operatorname{Hom}(\mathcal{O}(-d), \mathcal{O}(-n-1))^{\oplus a} & \longrightarrow & \operatorname{Hom}(\mathcal{O}(-d), \mathcal{O}(-n-1))^{\oplus b} \\ & & \downarrow{\scriptstyle \alpha_{\mathcal{F}}} & & \sim \downarrow{\scriptstyle \alpha_{\mathcal{O}(-d)}^{\oplus a}} & & \sim \downarrow{\scriptstyle \alpha_{\mathcal{O}(-d)}^{\oplus b}} \\ 0 & \longrightarrow & H^n(\mathbb{P}^n_k, \mathcal{F})^* & \longrightarrow & H^n(\mathbb{P}^n_k, \mathcal{O}(-d))^{* \oplus a} & \longrightarrow & H^n(\mathbb{P}^n_k, \mathcal{O}(-d))^{* \oplus b} \end{array}$$

Since the middle and right vertical maps are isomorphisms, by the five lemma, $\alpha_{\mathcal{F}}$ is also an isomorphism. This completes the proof. □

Comments. This is almost the only theorem that has "substantive" content in the proof of Serre duality. The other parts are essentially (non-trivial) extensions of this case with lots of abstract nonsense.

We now extend Proposition 8 to all Ext^i groups beyond $i = 0$ on $\mathbb{P}^n_k$.

Proposition 9. *Let k be a field and $n \geq 1$ be an integer. For $i \geq 0$ and a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_k$, there is a natural isomorphism*

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{O}(-n-1)) \cong H^{n-i}(\mathbb{P}^n_k, \mathcal{F})^*.$$

in $\mathcal{F}$.

Proof. First, by Lemma 2, we can find a surjective morphism $\mathcal{O}(-d)^{\oplus k} \twoheadrightarrow \mathcal{F}$ for some integers $k, d \geq 1$. Since both $\mathrm{Ext}^i(-, \mathcal{O}(-n-1))$ and $H^{n-i}(\mathbb{P}^n_k, -)^*$ are contravariant δ-functors that agree when $i = 0$ by Proposition 8 (here, importantly, $H^{n-i}(\mathbb{P}^n_k, -)^*$ is the "reverse" of the usual (covariant) sheaf cohomology functor; this makes sense because $H^{n+1}(\mathbb{P}^n_k, -) = 0$), by Lemma 4, it suffices to show that $\mathcal{O}(-d)^{\oplus k}$ is acyclic for both Ext^i and H^{n-i}.

First, $\mathrm{Ext}^i(\mathcal{O}(-d), \mathcal{O}(-n-1)) \cong H^i(\mathbb{P}^n_k, \mathcal{O}(d-n-1))$ because Ext is the right derived functor of $\mathrm{Hom}(\mathcal{O}(-d), -) \cong \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n_k}, \mathcal{O}(d) \otimes -) \cong \Gamma(\mathbb{P}^n_k, \mathcal{O}(d) \otimes -)$, from which we get the right-hand side as the right derived functor. Then, by the cohomology computation on $\mathbb{P}^n_k$, we have $H^i(\mathbb{P}^n_k, \mathcal{O}(d-n-1)) = 0$ for all $i \geq 1$ (since $d - n - 1 > -n - 1$).

Next, by the cohomology computation on $\mathbb{P}^n_k$ again, we have $H^{n-i}(\mathbb{P}^n_k, \mathcal{O}(-d)) = 0$ for all $i \geq 1$ (since $-d < 0$).

Hence, $\mathcal{O}(-d)$ is indeed acyclic for both δ-functors, and so is $\mathcal{O}(-d)^{\oplus k}$. By Lemma 4, we conclude that $\mathrm{Ext}^i(\mathcal{F}, \mathcal{O}(-n-1)) \cong H^{n-i}(\mathbb{P}^n_k, \mathcal{F})^*$. □

Comments. The key idea here is to note that any coherent sheaf on $\mathbb{P}^n_k$ admits an acyclic resolution by (direct sums of) $\mathcal{O}(-d)$'s. The rest follows directly from standard homological algebra.

Now, we move on to the general case of Serre duality on projective schemes. To state Serre duality, we need to define dualizing sheaves.

Definition. *Let X be a projective scheme of dimension $d \geq 0$ over a field k. A* dualizing sheaf *on X is a coherent sheaf $\omega_X \in \mathrm{Coh}(X)$ such that for any coherent sheaf $\mathcal{F} \in \mathrm{Coh}(X)$, there is a natural isomorphism*

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \cong H^d(X, \mathcal{F})^*$$

in $\mathcal{F}$.

By the Yoneda lemma, it is clear that if a dualizing sheaf exists, then it is unique up to isomorphism. To show the existence of dualizing sheaves, we need the following lemma.

Lemma 10. *Let $f : X \to Y$ be a finite morphism of locally noetherian schemes. Then, there exists a right adjoint $f^! : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$ to the pushforward functor $f_* : \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$.*

Proof. First, for $\mathcal{F} \in \mathrm{Coh}(Y)$, we claim that $\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{F})$ is a coherent sheaf of $f_*\mathcal{O}_X$-modules on Y. This is because $f_*\mathcal{O}_X$ is finitely presented as a sheaf of $\mathcal{O}_Y$-modules (since f is finite; check locally) and thus admits an exact sequence

$$\mathcal{O}_Y^{\oplus a} \to \mathcal{O}_Y^{\oplus b} \to f_*\mathcal{O}_X \to 0$$

for some integers $a, b \geq 1$. Applying the left exact functor $\mathcal{H}om_{\mathcal{O}_Y}(-, \mathcal{F})$, we get an exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{F}) \to \mathcal{F}^{\oplus b} \to \mathcal{F}^{\oplus a},$$

which shows that $\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{F})$ is coherent (because $\mathrm{Coh}(Y)$ is an abelian category). Moreover, it is naturally a sheaf of $f_*\mathcal{O}_X$-modules. This proves the claim.

Hence, by Lemma 5, there exists a coherent sheaf on X, which we call $f^!\mathcal{F}$, such that $f_*f^!\mathcal{F} = \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{F})$ as sheaves of $f_*\mathcal{O}_X$-modules on Y. Next, we show that $f^!$ is indeed a right adjoint to f_*. For $\mathcal{G} \in \mathrm{Coh}(X)$ and $\mathcal{F} \in \mathrm{Coh}(Y)$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, f^!\mathcal{F}) &\cong \mathrm{Hom}_{f_*\mathcal{O}_X}(f_*\mathcal{G}, f_*f^!\mathcal{F}) \quad \text{(by Lemma 5)} \\
&\cong \mathrm{Hom}_{f_*\mathcal{O}_X}(f_*\mathcal{G}, \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{F})) \\
&\cong \mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{G}, \mathcal{F}),
\end{aligned}$$

where the last natural isomorphism is basically a "sheaf" version of the adjunction $\mathrm{Hom}_B(M, N) = \mathrm{Hom}_A(M, \mathrm{Hom}_B(A, N))$ for a ring homomorphism $B \to A$, an A-module M, and a B-module N. This can be proven by constructing natural maps in both directions and checking that they are inverses (we omit this because it is straightforward yet somewhat tedious). □

Comments. In general, f_* has a *left* adjoint functor f^* (the pullback functor). However, when the underlying schemes are locally noetherian, the morphism is finite, and we restrict to coherent sheaves, Lemma 10 shows that f_* also admits a *right* adjoint $f^!$. The notation $f^!$ clashes with the true upper shriek functor defined in derived categories, but it seems they agree when f is finite and flat (according to Vakil), which is the case we will assume.

With this lemma, we can now prove the existence of dualizing sheaves for projective schemes.

Proposition 11. *Let X be a non-empty projective scheme over a field k. Then, X admits a dualizing sheaf $\omega_X \in \mathrm{Coh}(X)$.*

Proof. Let $d = \dim X \geq 0$. By Lemma 3, there exists a finite morphism $f : X \to \mathbb{P}^d_k$. Also, we know that $\omega_{\mathbb{P}^d_k} \cong \mathcal{O}(-d-1)$ is a dualizing sheaf on $\mathbb{P}^d_k$ by Proposition 8. We claim that $\omega_X := f^! \omega_{\mathbb{P}^d_k}$ is a dualizing sheaf on X. To see that, for any $\mathcal{F} \in \mathrm{Coh}(X)$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) &= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, f^! \omega_{\mathbb{P}^d_k}) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^d_k}}(f_*\mathcal{F}, \omega_{\mathbb{P}^d_k}) \quad \text{(by Lemma 10)} \\
&\cong H^d(\mathbb{P}^d_k, f_*\mathcal{F})^* \quad \text{(by Proposition 8)} \\
&\cong H^d(X, \mathcal{F})^* \quad \text{(by the affineness of } f).
\end{aligned}$$

Hence, ω_X is indeed a dualizing sheaf on X. □

Comments. The proof is very simple and satisfying given the adjunction established in Lemma 10.

Next, we need the following lemma to deal with the Cohen–Macaulay condition that appears in the statement of Serre duality.

Lemma 12. *Let X be a projective Cohen–Macaulay scheme of pure dimension $d \geq 0$ over a field k and $f : X \to \mathbb{P}^d_k$ be a finite morphism. Then, $f_*\mathcal{O}_X$ is finite locally free over $\mathcal{O}_{\mathbb{P}^d_k}$.*

Proof. First, let $x \in X$ and $p = f(x) \in \mathbb{P}^d_k$. Then, $\varphi : \mathcal{O}_{\mathbb{P}^d_k,p} \to \mathcal{O}_{X,x}$ is a local homomorphism of noetherian local rings. Since f is finite, for $x_0, x_1 \in X$ such that $x_0 \rightsquigarrow x_1$ and $x_0 \neq x_1$, we have $f(x_0) \rightsquigarrow f(x_1)$ with $f(x_0) \neq f(x_1)$ in $\mathbb{P}^d_k$. Hence, the fiber over p (of the local homomorphism φ) is precisely the single-point set of the maximal ideal of $\mathcal{O}_{X,x}$. Also, since both $\mathbb{P}^d_k$ and X are of pure dimension d and both $\mathcal{O}_{\mathbb{P}^d_k,p}$ and $\mathcal{O}_{X,x}$ are Cohen–Macaulay and thus catenary, we also get $\dim(\mathcal{O}_{\mathbb{P}^d_k,p}) = \dim(\mathcal{O}_{X,x})$. Moreover, $\mathcal{O}_{\mathbb{P}^d_k,p}$ is regular and $\mathcal{O}_{X,x}$ is Cohen–Macaulay. Hence, we can apply Lemma 6, which tells us that $\varphi : \mathcal{O}_{\mathbb{P}^d_k,p} \to \mathcal{O}_{X,x}$ is flat.

Next, since f is finite, we have $(f_*\mathcal{O}_X)_p = \oplus_{x \in f^{-1}(p)} \mathcal{O}_{X,x}$. Since $f_*\mathcal{O}_X$ is coherent over $\mathcal{O}_{\mathbb{P}^d_k}$ (by Lemma 5) and $\mathcal{O}_{\mathbb{P}^d_k,p} \to \mathcal{O}_{X,x}$ is flat for all $x \in f^{-1}(p)$, $(f_*\mathcal{O}_X)_p$ is a finitely generated flat module over $\mathcal{O}_{\mathbb{P}^d_k,p}$. But for finitely generated modules over noetherian local rings, flatness is equivalent to being finite free. Hence, $(f_*\mathcal{O}_X)_p$ is finite free over $\mathcal{O}_{\mathbb{P}^d_k,p}$ for all $p \in \mathbb{P}^d_k$, and thus $f_*\mathcal{O}_X$ is finite locally free over $\mathcal{O}_{\mathbb{P}^d_k}$. □

Comments. This is a nice lemma that translates the mysterious Cohen–Macaulay condition into a more intuitive property involving finite locally free sheaves. In fact, X being Cohen–Macaulay is *equivalent* to $f_*\mathcal{O}_X$ being finite locally free over $\mathcal{O}_{\mathbb{P}^d_k}$ (which follows from the equivalence established in the miracle flatness lemma), although we won't need the converse direction in the proof of Serre duality.

Here's the last lemma we need before proving Serre duality.

Lemma 13. *Let X be a locally noetherian scheme and $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module. Then, $\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is also finite locally free, and for any $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$, there is a natural isomorphism*

$$\mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{E}, \mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{F}, \mathcal{E}^\vee \otimes \mathcal{G})$$

in $\mathcal{F}$ and $\mathcal{G}$ for all $i \geq 0$.

Proof. First, for any $x \in X$, find an affine open neighborhood $U = \operatorname{Spec} A$ of x such that $\mathcal{E}|_U$ corresponds to a finite free A-module. Then, on U, $\mathcal{E}^\vee|_U$ corresponds to the dual module of that finite free A-module, which is also finite free. This shows that $\mathcal{E}^\vee$ is also finite locally free.

Next, note that we have the following natural isomorphism:

$$\begin{aligned}\operatorname{Hom}(\mathcal{F} \otimes \mathcal{E}, \mathcal{G}) &\cong \operatorname{Hom}(\mathcal{F}, \mathcal{H}om(\mathcal{E}, \mathcal{G})) \\ &\cong \operatorname{Hom}(\mathcal{F}, \mathcal{E}^\vee \otimes \mathcal{G}).\end{aligned}$$

This shows the case when $i = 0$. To see the general case, first note that the functor $\mathcal{E}^\vee \otimes -$ is exact because $\mathcal{E}^\vee$ is finite locally free. Then, $\operatorname{Ext}^i(\mathcal{F} \otimes \mathcal{E}, -)$ and $\operatorname{Ext}^i(\mathcal{F}, \mathcal{E}^\vee \otimes -)$ are the right derived functors of the same left exact functor $\operatorname{Hom}(\mathcal{F} \otimes \mathcal{E}, -) \cong \operatorname{Hom}(\mathcal{F}, \mathcal{E}^\vee \otimes -)$ (which is natural in both $\mathcal{F}$ and $-$). Hence, by the uniqueness of derived functors, we get the desired natural isomorphisms. □

Comments. This is a nice technical lemma that allows us to "move" finite locally free sheaves between the two arguments of Ext groups.

We are finally ready to prove Serre duality.

Theorem 14 (Serre duality)**.** *Let X be a projective Cohen–Macaulay scheme of pure dimension $d \geq 0$ over a field k. Then, for any coherent sheaf $\mathcal{F} \in \operatorname{Coh}(X)$, we have a natural isomorphism*

$$\operatorname{Ext}^i(\mathcal{F}, \omega_X) \cong H^{d-i}(X, \mathcal{F})^*$$

in $\mathcal{F}$ for all $0 \leq i \leq d$, where ω_X is a dualizing sheaf of X.

Proof. By Lemma 3, there exists a finite morphism $f : X \to \mathbb{P}^d_k$ with $f^*\mathcal{O}_{\mathbb{P}^d_k}(1) \cong \mathcal{O}_X(N)$ for some $N > 0$ (where $\mathcal{O}_X(N) \cong \iota^*\mathcal{O}_{\mathbb{P}^n_k}(N)$ for any given embedding $\iota : X \hookrightarrow \mathbb{P}^n_k$; note that $\mathbb{P}^d_k$ and $\mathbb{P}^n_k$ are not necessarily the same). Then, by Lemma 12, $f_*\mathcal{O}_X$ is finite locally free over $\mathcal{O}_{\mathbb{P}^d_k}$.

We claim that for any $\mathcal{F} \in \operatorname{Coh}(X)$, $\mathcal{G} \in \operatorname{Coh}(\mathbb{P}^d_k)$, and $i \geq 0$, we have a natural isomorphism $\operatorname{Ext}^i(f_*\mathcal{F}, \mathcal{G}) \cong \operatorname{Ext}^i(\mathcal{F}, f^!\mathcal{G})$ in $\mathcal{F}$. First, note that both $\operatorname{Ext}^i(f_*-, \mathcal{G})$ and $\operatorname{Ext}^i(-, f^!\mathcal{G})$ are δ-functors in $\mathcal{F}$ (by the exactness of f_* when f is affine) that agree when $i = 0$ (by Lemma 10). Next, note that $\mathcal{O}_X(-Nr)$ is an acyclic object for both δ-functors for a sufficiently large integer r. This is because

$$\begin{aligned}\operatorname{Ext}^i(f_*\mathcal{O}_X(-Nr), \mathcal{G}) &\cong \operatorname{Ext}^i(f_*f^*\mathcal{O}_{\mathbb{P}^d_k}(-r), \mathcal{G}) \\ &\cong \operatorname{Ext}^i(f_*\mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^d_k}(-r), \mathcal{G}) \quad \text{(by the projection formula)} \\ &\cong \operatorname{Ext}^i(\mathcal{O}_{\mathbb{P}^d_k}(-r), (f_*\mathcal{O}_X)^\vee \otimes \mathcal{G}) \quad \text{(by Lemma 13)} \\ &\cong H^i(\mathbb{P}^d_k, (f_*\mathcal{O}_X)^\vee \otimes \mathcal{G} \otimes \mathcal{O}_{\mathbb{P}^d_k}(r)) \quad \text{(as in the proof of Proposition 9)},\end{aligned}$$

and

$$\operatorname{Ext}^i(\mathcal{O}_X(-Nr), f^!\mathcal{G}) \cong H^i(X, f^!\mathcal{G} \otimes \mathcal{O}_X(Nr)) \quad \text{(as in the proof of Proposition 9)},$$

where both vanish for all $i \geq 1$ when r is sufficiently large by Lemma 7. Since any coherent sheaf on X admits a surjection from some $\mathcal{O}_X(-Nr)$ for sufficiently large r (by Lemma 2), the claim follows from Lemma 4.

Now, we have the following natural isomorphisms in $\mathcal{F}$:

$$\begin{aligned}\operatorname{Ext}^i(\mathcal{F}, \omega_X) &\cong \operatorname{Ext}^i(\mathcal{F}, f^!\omega_{\mathbb{P}^d_k}) \\ &\cong \operatorname{Ext}^i(f_*\mathcal{F}, \omega_{\mathbb{P}^d_k}) \quad \text{(by the claim above)} \\ &\cong H^{d-i}(\mathbb{P}^d_k, f_*\mathcal{F})^* \quad \text{(by Proposition 9)} \\ &\cong H^{d-i}(X, \mathcal{F})^* \quad \text{(by the affineness of } f\text{)}.\end{aligned}$$

This completes the proof. □

Comments. The proof is very clean and satisfying given all the lemmas we have set up.

Acknowledgments

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