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Linear Algebra - Groups, Vector Spaces, Matrix Transformations

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Overview

- 6 properties in number theory
- Groups, Rings, Fields
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Set: a set of elements
Binary operator: an operator than works on two elements and produces one element

6 properties in number theory:

One binary operator (eg.: +):

- Closure
- Associative
- Identity
- Inverse
- Commutative

Two binary operators (eg. +, .):

- Distributive

6 properties in number theory:

One binary operator (eg.: +):

- **Closure** — $\forall a, b \in S \Rightarrow a \star b \in S$
- **Associative** — $a \star (b \star c) = (a \star b) \star c$
- **Identity** — $\exists \mathbf{0} \in S \mid a \star \mathbf{0} = a$
- **Inverse** — $\exists b \in S \mid a \star b = \mathbf{0}$
- **Commutative** — $a \star b = b \star a$

Two binary operators (eg. +, .):

- **Distributive** — $a \triangle (b \star c) = (a \triangle b) \star (a \triangle c)$

Example:

$$S = \mathbb{N}, \mathbb{W}, \mathbb{Z}$$

With **addition** operation, check closure, associative, identity, inverse, commutative.
With **addition** and **multiplication**, check distributive.

Group:

A group consists of a non-empty set G and a binary operator $\star$ s.t. (assume $a, b, c \in G$):

- $\star$ is **closed** under G, i.e. $\forall a, b \in G, (a \star b) \in G$
- $\star$ is **associative**, i.e. $\forall a, b, c \in G, a \star (b \star c) = (a \star b) \star c$
- G contains the **identity** element e of $\star$, defined as: $\exists e \in G \mid \forall a \in G, a \star e = e \star a = a$
- G contains **inverse** elements, i.e. $\forall a \in G, \exists z \in G \mid (a \star z) = e$

In addition, if $\star$ is **commutative** in G, i.e. $\forall a, b \in G, a \star b = b \star a$, G is called an **abelian group**.

Example:
Check if $(\mathbb{N}, +)$, $(\mathbb{Z}, +)$, $(\mathbb{R}, .)$ are groups, and/or abelian groups.

Ring:

A structure $(R, +, .)$ is a **ring** if R is a non-empty set, $+$ and . are binary operations s.t.:

- $(R, +)$ is an abelian group, i.e. **Closure, Associative, Identity, Inverse, Commutative**
- $(R, .)$ satisfies **Closure, Associative**
- . **distributes** over $+$, i.e. $\forall a, b, c \in R, a.(b + c) = a.b + a.c$ and $(a + b).c = a.c + b.c$

Example:
Check if $(\mathbb{Z}, +, .)$, $(\mathbb{Z}_n, +, .)$, $(\mathbb{R}, +, .)$ are rings.

Field:

A structure $(R, +, .)$ is a **field** if R is a non-empty set, $+$ and $.$ are binary operations s.t.:

- $(R, +)$ is an abelian group, i.e. **Closure, Associative, Identity, Inverse, Commutative**
- $(R\backslash\{0\}, .)$ is an abelian group, i.e. **Closure, Associative, Identity, Inverse, Commutative**
- . **distributes** over $+$, i.e. $\forall a, b, c \in R, a.(b + c) = a.b + a.c$ and $(a + b).c = a.c + b.c$

Example:
Check if $(\mathbb{Z}, +, .)$, $(\mathbb{Q}, +, .)$, $(\mathbb{R}, +, .)$ are fields.

Vector Space:

V is a **vector space** or **linear space** over the field R if $(a, b \in R, u, v \in V)$:

- Addition $(V, +)$ is an abelian group, i.e. **Closure, Associative, Identity, Inverse, Commutative**
- Scalar Multiplication is **Associative**, i.e. $a.(b.\mathbf{v}) = (a.b).\mathbf{v}$
- Scalar Multiplicative **Identity**, i.e. $\exists 1 \in R \mid 1.\mathbf{v} = \mathbf{v}$
- Addition and Scalar Multiplication are **Distributive**, i.e. $a.(\mathbf{u} + \mathbf{v}) = a.\mathbf{u} + a.\mathbf{v}$ and $a.(\mathbf{u} + \mathbf{v}) = a.\mathbf{u} + a.\mathbf{v}$

Example:
Check if $\mathbb{R}^n$ is a vector space.

Transformation of vector spaces

Linear Transformation:

$$L : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

such that

- $L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$
- $L(a.\mathbf{v}) = a.L(\mathbf{v})$

L can be represented as a matrix $A \in \mathbb{R}^{m \times n}$ s.t.

$$L(\mathbf{v}) = \mathbf{Av}$$

The set of all real (non-singular) $n \times n$ matrices with matrix multiplication forms a **group**.

Affine Transformation:

Link


Thank You