Arabic arithmetic in context: al-Hawārī's Essential Commentary

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In 1999, eight years after finishing my Ph.D. in mathematics, I was looking to change the direction of my research. Having the freedom afforded by a teaching institution to make that change truly drastic, I returned to an interest in medieval Arabic mathematics that I had developed during an undergraduate course in History of Science. Given that I nearly majored in history in college and that I had recently finished writing a 560-page book on the history of railroad tie preservation (no room to explain that here!), history of mathematics was a natural choice. My first step in this direction was to build a website that grew to list over 1,300 books and articles on Arabic mathematics, arranged by topic, that had been published since about 1950 [5]. Then, with the secondary literature under control and without really knowing where it would lead, I turned my attention to algebra. I began with al-Khwārazmī's early-ninth-century *Book of algebra* (*Kitāb al-jabr wa l-muqābala*), the earliest extant Arabic book on the topic, but because I could not read Arabic I had to work from Latin and English translations [2; 3]. I soon discovered a couple of interesting features that no one had written about, so to really understand the matter I enlisted my Palestinian colleague Haitham Alkhateeb to teach me Arabic. With his help, and with what seemed to be unwarranted determination, I was soon able to read the texts myself.

At first, I naively approached the algebra as if it were modern algebra written in Arabic prose. But observing the ways that certain operations are worded and taking into account the overall procedures of many solutions, I discovered almost immediately that such a reading is untenable. Those curious yet consistent differences between Arabic algebra and the algebra we practice today turned out to be signals of a radically different way of understanding monomials, polynomials, and equations. And as I learned only later, the algebraic expressions are themselves grounded in the practical Arabic understanding of 'number'.

With this approach of paying careful attention to the wording of the texts, and with sensitivity to the Arabic authors' potentially different ways of conceiving of mathematical objects, I became a specialist in history of mathematics. I have since published over twenty research articles, not just on Arabic mathematics but also on Greek mathematics in one direction and Medieval, Renaissance, and early modern European mathematics in the other. And just recently, Mahdi Abdeljaouad and I have published the book *Al-Hawārī's Essential Commentary: Arabic Arithmetic in the Fourteenth Century*, in which we present an edition, translation, and commentary of a fourteenth-century Arabic arithmetic textbook. This is the latest installment of Edition Open Sources, a collaborative venture between the Max Planck Institute for the History of Science in Berlin and the University of Oklahoma. As with all EOS books, ours is free to read online or download [1]. It is this book that I now describe.

Abd al- 'Azī z al-Hawārī was a student of the polymath Ibn al-Bannāʿ in Marrakesh in the early years of the fourteenth century CE. Ibn al-Bannāʿ was lecturing on his *Condensed [Book] on the Operations of Arithmetic* (*Talkhī s a 'māl al-hisāb*), a book *so* condensed that it included not one numerical example to illustrate the rules. While still attending lectures, al-Hawārī began writing a commentary on his teacher's book with the main goal of providing those numerical examples. He completed his *Essential Commentary on the Condensed [Book] on the Operations of Arithmetic* (*al-Lubāb fī sharh Talkhī s a 'māl al-hisāb*) in 1305 CE.

Al-Hawārī followed Ibn al-Bannā "s book chapter by chapter, beginning with a description of different kinds of numbers and how to write them with Indian numerals. (We call them "Arabic numerals" because Europeans learned them from Arabic sources.) He continued with operations on whole numbers, operations on fractions, square root calculations, problem-solving via proportion, double false position, and algebra, and finally a short section on finding secret numbers. Like many medieval Arabic textbooks, this one is a splendid hybrid, combining the rules of Indian calculation with techniques of Middle Eastern finger-reckoning, and mixed with Greek number theory. To add to this complexity, Ibn al-Bannāʿ had copied many passages word-for-word from earlier textbooks, and al-Hawārī included many passages from his teacher's own commentary, titled *Lifting the Veil from the Faces of the Operations of Arithmetic* (*Rafʿ al-hijāb 'an wujūuh a 'māl al-hisāb*) [4].

Our book begins with a 28-page general introduction to Arabic arithmetic, something that has been lacking in the literature. There we explain the collisions between the arithmetics of earlier cultures brought together in Arabic texts as well as the roles played by algebra and other arithmetical problem-solving methods. A literal English translation of al-Hawārī's book comes next, followed by a comprehensive mathematical, conceptual, linguistic, and historical commentary that is longer than the translation. Because our book is meant to be read on a screen, we include links throughout, so that, for instance, one can click between the translation and the commentary. Following the commentary we give appendices, including a conspectus of problems, translations of some worked-out problems from other books, a chronological list of mathematicians and other scholars, and a glossary of Arabic terms with links back to the translation and the commentary. This is followed by a bibliography and an index of people that is also linked back to the text. The Arabic edition of al-Hawārī's book comes at the end, with its own introduction. If the book were printed in physical form, the Arabic portion would come first for Arabic readers, since Arabic is written right-to-left.

Al-Hawārī was not a brilliant mathematician with innovative ideas and theoretical insights. His aim was simply to provide students with a clear and comprehensive guide to the practical arithmetic of his time. He offers no proofs or philosophical discussions, just explanations and examples of the rules. The book must have enjoyed a modest success, since at least fourteen manuscripts are extant.

There is much to be gained from reading textbooks like this one, beginning with the window they provide into contemporary forms of mathematical practice. Not only are we treated to different techniques for performing calculations and solving problems, but the way the procedures are presented testifies to the different role that books and writing played in a predominantly oral culture. Ibn al-Bannāʿ was following standard practice by reciting his book aloud to his students, and those students would have been expected to memorize its contents. Calculations were worked out in notation on a dust-board or other ephemeral surface, and if one wished to include a calculation in a book, then a rhetorical version was composed. The notation used in working out the problems is shown in books only as figures illustrating what should be written on the board. It is because some modern writers have been unaware of dust-board calculations that they believe that al-Khwārazmī and other algebraists worked out their problems verbally!

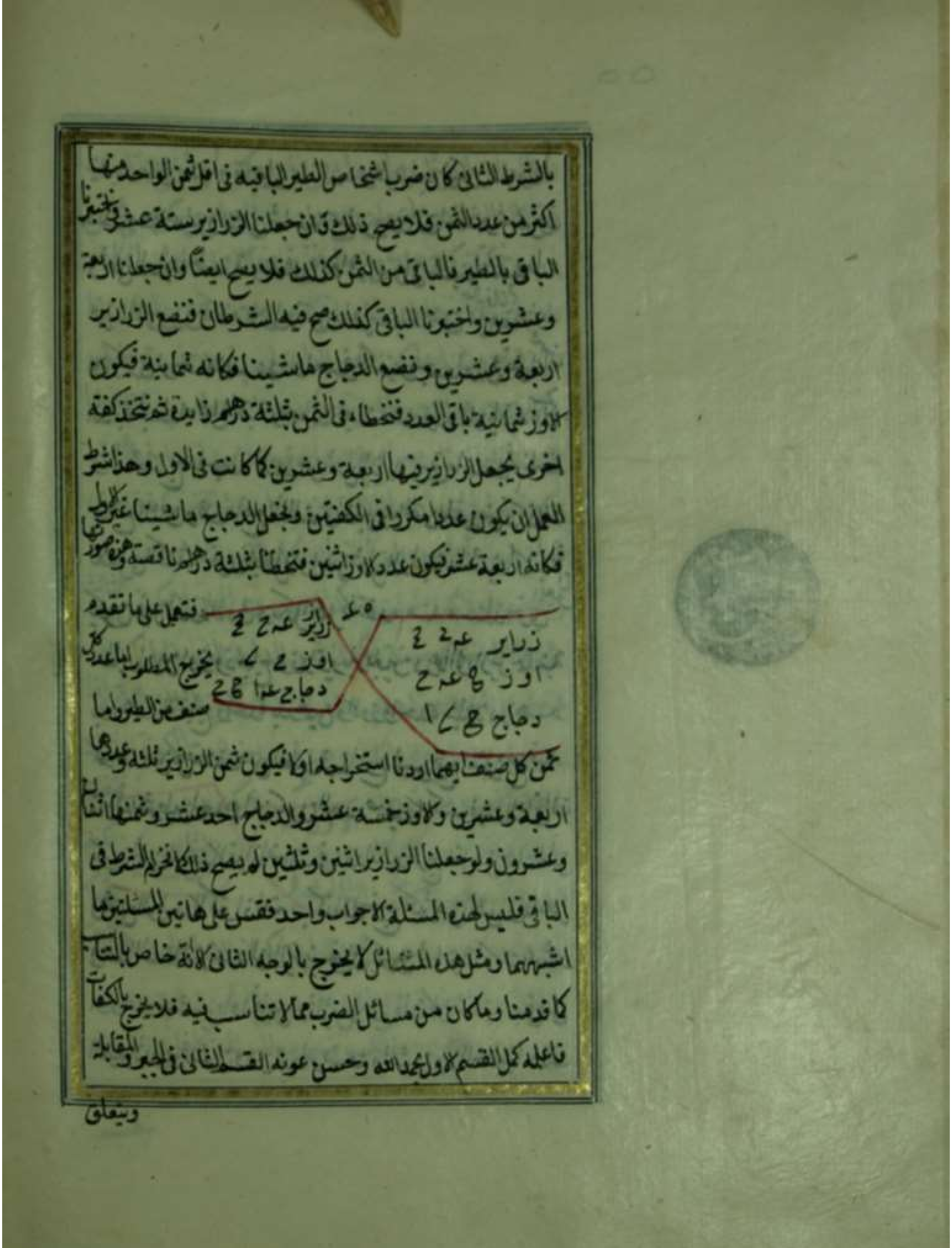
بالشرط الثاني كان ضرب اشخاص الطير الباقيه في اقل ثمن الواحد منها
اكثر من عدد الثمن فلا يصح ذلك فان جعلنا الزرازير ستة عشر واختبرنا
الباقي بالطير فالباقي من الثمن كذلك فلا يصح ايضا وان جعلنا الاربعة
وعشرين واختبرنا الباقي كذلك صح فيه الشرطان فنضع الزرازير
اربعة وعشرين ونضع الدجاج ما شئنا فكانه ثمانية فيكون
الاوز ثمانية باقي العدد فنحطا في الثمن بثلثة دراهم زايدة ثم نتخذ كفة
اخرى نجعل الزرازير فيها اربعة وعشرين كما كانت في الاول وهذا شرط
العمل ان يكون عددا مكررا في الكفتين ونجعل الدجاج ما شئنا غير الاول
فكانه اربعة عشر فيكون عدد الاوز اثنين فنحطنا بثلثة دراهم ناقصة وهذه صورتها
فنعمل على ما تقدم
فنخرج المطلوب لما عدد كل
صنف من الطيور واما
ثمن كل صنف منهما اردنا استخراجه اولا فيكون ثمن الزرازير ثلثة وعددها
اربعة وعشرين وللاوز خمسة عشر والدجاج احد عشر وثمنها اثنان
وعشرون ولو جعلنا الزرازير اثنين وثلثين لم يصح ذلك لانخرام الشرط في
الباقي فليس لهذه المسئلة الا جواب واحد فقس على هاتين المسئلتين ما
اشبههما ومثل هذه المسائل لا تخرج بالوجه الثاني لانه خاص بالتناسب
كما قدمنا وما كان من مسائل الضرب مما لا تناسب فيه فلا يخرج بالكفات
تم عليه كمل القسم الاول بحمد الله وحسن عونه القسم الثاني في الجبر والمقابلة
ويتعلق

Figure 1. A sample page from the Medina manuscript of al-Hawārī's book, copied in 1345 CE.

But there is more to learn from elementary textbooks than merely "how they did things then". For the case of medieval Arabic arithmetic, these books can be particularly valuable for what their explanations, wording, and methods reveal about how differently people conceived of numbers and algebraic expressions, which in turn explains their seemingly curious procedures. Numbers today are regarded as existing independently of whatever units they may count or measure, but this was not the case in premodern arithmetic. Numbers in al-Hawārī and other Arabic authors on practical arithmetic are always numbers of some divisible unit, whether material or intelligible. Examples of material units include horses, bricks, parasangs (a unit of length), dirhams (a denomination of coin), hours, and mithqals (a unit of weight), while intelligible units were measured in generic "units", which were often labeled "in number", or again, as "dirhams". Numbers could be any positive amount, including fractions and irrational roots, which makes them incompatible with the multitudes composed of indivisible, intelligible units, as in Book VII of Euclid's *Elements*. Even the foundation of the numbers of Arabic practitioners is different from both Euclid's and our own: they are validated through practice, not via philosophical definitions or axioms. It is on this practical foundation that Arabic algebraic expressions were conceived, which are again radically different from their modern counterparts. Very briefly, Arabic polynomials are not built from arithmetical operations like ours, but were simply aggregations of the powers of the unknown.

This background on the nature of premodern numbers and expressions should then be taken into account in evaluating the contributions of more original authors like al-Karajī, al-Samaw ʻal, al-Khayyam, al-Fārisī, and al-Kāshī, who all developed their ideas on arithmetic and algebra from the practical tradition. Medieval Europeans, too, learned their computational methods directly from this tradition, so that the foundation one gains from studying the Arabic books applies equally to the underlying concepts at play in earlier figures such as Fibonacci, Jean de Murs, and Luca Pacioli, as well as later authors writing throughout the sixteenth century. To understand why Michael Stifel (1544) avoided irrational coefficients in algebra, such as multiplying-in our notation- $\sqrt{6}$ by x to get $\sqrt{6x^2}$, or why even François Viète (d. 1603) would not write multiple roots, insisting on forms like $\sqrt{3888}$ and $\sqrt{\frac{45}{64}}$ instead of $36\sqrt{3}$ and $\frac{3}{8}\sqrt{5}$, one should return to the Arabic sources.

Al-Hawārī reproduces the whole of Ibn al-Bannā "s book. He cites a passage, then gives his examples and comments, and then repeats the pattern with the next passage. We put Ibn al-Bannā "s passages in bold font in both the edition and translation. Al-Hawārī also quoted from Ibn al-Bannā "s own commentary, and we distinguish these passages by rendering them in an all-caps font in the translation. We also indicate in footnotes which passages Ibn al-Bannā' borrowed from other authors.

Our commentary follows the medieval paradigm. We remark on al-Hawārī's book passage-by-passage, with occasional digressions to expand on broader aspects of the mathematics. In addition to explaining the conceptual issues mentioned above, we illustrate in detail how the different rules of calculation function, sometimes bringing in examples from other books. We also identify the origins of various definitions, techniques, and ideas, whether from Euclid, Nicomachus, the finger-reckoning tradition, another Arabic author, or some other source. With all that is going on in al-Hawārī's seemingly mundane textbook, we hope that others will find it as fascinating as we have.

References

[1] Abdeljaouad, Mahdi, and Jeffrey Oaks (2021) *Al-Hawārī's Essential Commentary: Arabic Arithmetic in the Fourteenth Century.* Berlin: Max Planck Institute for the History of Science. **https://edition-open-sources.org/sources/14/**.

[2] Al-Khwārazmī. (1831) *The Algebra of Mohammed ben Musa*. Translated and edited by Frederic Rosen. London: Nachdruck der Ausgabe.

[3] Hughes, Barnabas, ed. (1986) Gerard of Cremona's Translation of al-Khwārizmī's *Al-Jabr*: A Critical Edition. *Mediaeval Studies* 48, 211-263.

[4] Ibn al-Bannā'55. (1994) *Raf' al-hijāb'an wujūh a'māl al-hisāb*. Edited by Muhammad Aballagh. Fās: Jāmi'at Sīdī Muhammad ibn 'Abd Allāh.

[5] Oaks, Jeffrey (2006) Bibliography of the Mathematical Sciences in the Medieval Islamic World (by Topic). University of Indianapolis. **https://uindy.edu/cas/mathematics/oaks/biblio/**. By now this bibliography is very out of date.