

# ABCExceptions

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# Chapter 1

## Introduction

In this paper, we use tools from analytic number theory to estimate the number of triples of a given height satisfying the *abc* conjecture. Associated to any non-zero integer  $n$  is its radical

$$\text{rad}(n) = \prod_{p|n} p.$$

We say that a triple  $(a, b, c) \in \mathbb{N}^3$  with  $\gcd(a, b, c) = 1$  is an *abc triple of exponent  $\lambda$*  if

$$a + b = c, \quad \text{rad}(abc) < c^\lambda.$$

The well-known *abc* conjecture of Masser and Oesterlé asserts that, for any  $\lambda < 1$ , there are only finitely many *abc* triples of exponent  $\lambda$ . The best unconditional result is due to Stewart and Yu [12], who have shown that finitely many *abc* triples satisfy  $\text{rad}(abc) < (\log c)^{3-\varepsilon}$ . Recently, Pasten [11] has proved a new subexponential bound, assuming that  $a < c^{1-\varepsilon}$ , via a connection to Shimura curves. In this paper we shall focus on counting the number  $N_\lambda(X)$  of *abc* triples of exponent  $\lambda$  in a box  $[1, X]^3$ , as  $X \rightarrow \infty$ .

**Definition 1.1.** For  $\lambda > 0$  define  $N_\lambda(X)$  as the number of triples  $(a, b, c) \in \mathbb{N}^3$  with  $a + b = c$ ,  $\gcd(a, b, c) = 1$  and  $\text{rad}(abc) < c^\lambda$ .

Given  $\lambda > 0$ , an old result of de Bruijn [4] implies that

**Lemma 1.2.** For any  $\varepsilon > 0$ , we have

$$\#\{n \leq x : \text{rad}(n) \leq x^\lambda\} \ll_\varepsilon x^{\lambda+\varepsilon}. \quad (1.0.1)$$

*Proof.*

It suffices to show that for any integer  $k \geq 2$  we have

$$|\{n \leq X : \text{rad}(n) = k\}| \ll X^{O(1/\log \log X)}. \quad (1.0.2)$$

To prove (1.0.2), write  $k = p_1 \cdots p_r$  as distinct primes  $p_1 < \cdots < p_r$ . Then  $\text{rad}(n) = k$  implies that  $n = p_1^{m_1} \cdots p_r^{m_r}$  for some integers  $m_1, \dots, m_r \geq 1$ . Therefore, the number of  $n \leq X$  in question is

$$\begin{aligned} |\{n \leq X : \text{rad}(n) = k\}| &\leq \sum_{m_1, \dots, m_r \geq 1} \mathbf{1}_{\sum_{j \leq r} m_j (\log p_j) \leq \log X} \\ &\leq \sum_{m_1, \dots, m_r \geq 1} \int_{\mathbb{R}^r} \mathbf{1}_{\sum_{j \leq r} t_j (\log p_j) \leq \log X} \mathbf{1}_{t_j \in (m_j-1, m_j] \forall j \leq r} dt_1 \cdots dt_r \\ &= \text{vol}\left(\{(t_1, \dots, t_r) \in \mathbb{R}_{\geq 0}^r : t_1 (\log p_1) + \cdots + t_r (\log p_r) \leq \log X\}\right) \\ &= (\log X)^r e^{O(r)} r^{-r} \prod_{j=1}^r \frac{1}{\log p_j} \end{aligned}$$

by calculating the volume of a simplex. Moreover, we have

$$\prod_{j=1}^r \log p_j \geq \prod_{j=2}^{r+1} \log j = \exp\left(\sum_{j=2}^{r+1} \log \log j\right) \gg (\log r)^r e^{-O(r)}.$$

Hence for some  $C_0 > 1$

$$|\{n \leq X : \text{rad}(n) = k\}| \ll \left( \frac{C_0 \log X}{r \log r} \right)^r \ll \exp \left( O \left( \frac{\log X}{\log \log X} \right) \right),$$

since  $r = \omega(k) \leq (1 + o(1))(\log X)/(\log \log X)$  by the prime number theorem.  $\square$

Any triple  $(a, b, c)$  counted by  $N_\lambda(X)$  must satisfy  $\text{rad}(abc) < X^\lambda$ , and so we must have  $\min\{\text{rad}(a)\text{rad}(b), \text{rad}(b)\text{rad}(c), \text{rad}(a)\text{rad}(c)\} < X^{2\lambda/3}$ , since  $a, b, c$  are pairwise coprime. An application of (1.0.1) now leads to the following “trivial bound”.

**Proposition 1.3.** *Let  $\lambda > 0$ . Then  $N_\lambda(X) = O_\varepsilon(X^{2\lambda/3+\varepsilon})$ , for any  $\varepsilon > 0$ .*

*Proof.* Any triple  $(a, b, c)$  counted by  $N_\lambda(X)$  must satisfy  $\text{rad}(abc) < X^\lambda$ , and so we must have  $\min\{\text{rad}(a)\text{rad}(b), \text{rad}(b)\text{rad}(c), \text{rad}(a)\text{rad}(c)\} < X^{2\lambda/3}$ , since  $a, b, c$  are pairwise coprime. An application of 1.2 now leads to the following “trivial bound”.  $\square$

The primary goal of this paper is to give the first power-saving improvement over this simple bound for values of  $\lambda$  close to 1.

**Theorem 1.4.** *Let  $\lambda \in (0, 1.001)$  be fixed. Then  $N_\lambda(X) = O(X^{33/50})$ .*

Here we note that  $33/50 = 0.66$ . By comparison, the trivial bound in Proposition 1.3 would give  $N_1(X) = O(X^{0.666+\varepsilon})$  and  $N_{1.001}(X) = O(X^{0.6674})$ . Moreover, we see that Theorem 1.4 gives a power-saving when  $\lambda \in (0.99, 1.001)$ . We emphasise that this power-saving represents a proof of concept of the methods; we expect that the exponent can be reduced with substantial computer assistance.

Theorem 1.4 also applies for  $\lambda$  slightly greater than 1, which places it in the realm of a question by Mazur [10]. Given a fixed  $\lambda > 1$ , he asked whether or not  $N_\lambda(X)$  has exact order  $X^{\lambda-1}$ . In fact, Mazur studies the refined counting function

**Definition 1.5.** For  $\alpha, \beta, \gamma > 0$ , define  $S_{\alpha, \beta, \gamma}(X)$  as the number of  $(a, b, c) \in \mathbb{N}^3$  with  $\gcd(a, b, c) = 1$  such that

$$a, b, c \in [1, X], \quad a + b = c, \quad \text{rad}(a) \leq X^\alpha, \quad \text{rad}(b) \leq X^\beta, \quad \text{rad}(c) \leq X^\gamma.$$

The argument used to prove Proposition 1.3 readily yields

**Lemma 1.6.**

$$S_{\alpha, \beta, \gamma}(X) \ll_\varepsilon X^{\min\{\alpha+\beta, \alpha+\gamma, \beta+\gamma\}+\varepsilon}, \tag{1.0.3}$$

for any  $\varepsilon > 0$ .

*Proof.*  $\square$

Mazur then asks whether  $S_{\alpha, \beta, \gamma}(X)$  has order  $X^{\alpha+\beta+\gamma-1}$  if  $\alpha + \beta + \gamma > 1$ . Evidence towards this has been provided by Kane [9, Theorems 1 and 2], who proves that

$$X^{\alpha+\beta+\gamma-1-\varepsilon} \ll_\varepsilon S_{\alpha, \beta, \gamma}(X) \ll_\varepsilon X^{\alpha+\beta+\gamma-1+\varepsilon} + X^{1+\varepsilon},$$

for any  $\varepsilon > 0$ , provided that  $\alpha, \beta, \gamma \in (0, 1]$  are fixed and satisfy  $\alpha + \beta + \gamma > 1$ . This result gives strong evidence towards Mazur’s question when  $\alpha + \beta + \gamma \geq 2$ , but falls short of the trivial bound (1.0.3) when  $\alpha + \beta + \gamma < 3/2$ .

When considering  $abc$  triples of exponent  $\lambda < 1$ , we always have  $\alpha + \beta + \gamma \leq \lambda < 1$ , and the methods of Kane give no information in this regime. Indeed, we are not aware of any general estimates when  $\lambda < 1$ , beyond Proposition 1.3. Nonetheless, there do exist specific Diophantine equations which are covered by the  $abc$  conjecture and where bounds have been given for the number of solutions. For example, it follows from work of Darmon and Granville [5] that there are only finitely many coprime integer solutions to the Diophantine equation  $x^p + y^q = z^r$ , when  $p, q, r \in \mathbb{N}$  are given and satisfy  $1/p + 1/q + 1/r < 1$ .

## Proof outline

We now describe the main ideas behind the proof of Theorem 1.4. In terms of the counting function  $S_{\alpha,\beta,\gamma}(X)$ , our task is to show that whenever  $\alpha, \beta, \gamma \in (0, 1]$  satisfy  $\alpha + \beta + \gamma \leq \lambda$ , we have  $S_{\alpha,\beta,\gamma}(X) \ll X^{2\lambda/3-\eta}$ , for some  $\eta > 0$ . A simple factorisation lemma (Proposition 2.6) will reduce the problem of bounding  $S_{\alpha,\beta,\gamma}(X)$  to the problem of bounding the number of solutions to various Diophantine equations of the shape

$$\prod_{j \leq d} x_j^j + \prod_{j \leq d} y_j^j = \prod_{j \leq d} z_j^j,$$

with specific constraints  $x_i \sim X^{a_i}$ ,  $y_i \sim X^{b_i}$ ,  $z_i \sim X^{c_i}$  on the size of the variables, for admissible values of  $a_i, b_i, c_i$  (depending on  $\alpha, \beta, \gamma$ ). We then bound the number of solutions to these Diophantine equations using four different methods. The first of these (Proposition 3.1) uses Fourier analysis and Cauchy-Schwarz to estimate the number of solutions, leading to a bound that works well if two of the exponent vectors  $(a_i)_i, (b_i)_i, (c_i)_i$  are somewhat “correlated”. The second method (Proposition 3.2) uses the geometry of numbers and gives good bounds when one of  $a_1, b_1, c_1$  is large. The remaining tools come from the determinant method of Heath-Brown (Proposition 3.14) and uniform upper bounds for the number of solutions to Thue equations (Proposition 3.15). For every choice of the exponents  $a_i, b_i, c_i$  we shall need to take the minimum of these bounds, which leads to a rather intricate combinatorial optimisation problem. This is solved by showing that at least one of the four methods always gives a power-saving over Proposition 1.3 when  $\lambda$  is close to 1.

## Notation

We shall use  $x \sim X$  to denote  $x \in [X, 2X]$  and we put  $[d] = \{1, \dots, d\}$ . We denote by  $\tau(n) = \sum_{d|n} 1$  the divisor function.

## Chapter 2

# Reduction to Diophantine equations

We will work with a variant of  $S_{\alpha,\beta,\gamma}(X)$

**Definition 2.1.** Let  $S_{\alpha,\beta,\gamma}^*(X)$  to be the number of  $(a, b, c) \in \mathbb{N}^3$  with  $\gcd(a, b, c) = 1$  and

$$c \in [X/2, X], \quad a + b = c, \quad \text{rad}(a) \sim X^\alpha, \quad \text{rad}(b) \sim X^\beta, \quad \text{rad}(c) \sim X^\gamma.$$

We begin by noting that by the pigeonhole principle,

**Lemma 2.2.** *We have*

$$N_\lambda(X) \ll (\log X)^4 \max_{\substack{\alpha,\beta,\gamma>0 \\ \alpha+\beta+\gamma \leq \lambda}} S_{\alpha,\beta,\gamma}^*(X). \quad (2.0.1)$$

*Proof.*

□

**Theorem 2.3.** *There exists  $\varepsilon > 0$  such that for all  $\mathbf{c} \in \mathbb{Z}^3$  and  $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{R}_{>0}^d$ . we have*

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll X^{0.66-\varepsilon}. \quad (2.0.2)$$

*Proof.*

□

*Proof of Theorem 1.4.*

□

The following result allows us to bound  $S_{\alpha,\beta,\gamma}^*(X)$  in terms of the number of solutions to certain monomial Diophantine equations. In order to state it, we need to introduce the quantity  $B_d$

**Definition 2.4.** For  $\mathbf{c} \in \mathbb{Z}^3$  and  $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{R}_{>0}^d$ . we have

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) := \# \left\{ (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbb{N}^{3d} : \begin{array}{l} x_i \sim X_i, y_i \sim Y_i, z_i \sim Z_i \\ c_1 \prod_{j \leq d} x_j^j + c_2 \prod_{j \leq d} y_j^j = c_3 \prod_{j \leq d} z_j^j \\ \gcd(c_1 \prod_{j \leq d} x_j, c_2 \prod_{j \leq d} y_j, c_3 \prod_{j \leq d} z_j) = 1 \end{array} \right\}. \quad (2.0.3)$$

**Lemma 2.5.** *Let  $\varepsilon \in (0, 1/2)$ , and let  $2 \leq n \leq X$  be an integer. Then there exists a factorisation*

$$n = c \prod_{j \leq \frac{5}{2}\varepsilon^{-2}} x_j^j,$$

*for positive integers  $x_j, c$  such that  $c \leq X^{\varepsilon/2}$ , the  $x_j$  are pairwise coprime, and*

$$X^{-\varepsilon} \prod_{j \leq \frac{5}{2}\varepsilon^{-2}} x_j \leq \text{rad}(n) \leq X^\varepsilon \prod_{j \leq \frac{5}{2}\varepsilon^{-2}} x_j.$$

*Proof.* Fix  $2 \leq n \leq X$  and let  $K = 2\lceil \varepsilon^{-1} \rceil$ ,  $M = \lfloor \frac{5}{2}\varepsilon^{-2} \rfloor$ . Define

$$y_j := \prod_{p^j \parallel n} p.$$

For  $j \leq M$ , we set

$$x_j := \begin{cases} y_j & \text{for } j \neq K, \\ y_j \prod_{m>M} y_m^{\lfloor m/K \rfloor} & \text{for } j = K, \end{cases} \quad \text{and} \quad c := \prod_{m>M} y_m^{m-K\lfloor m/K \rfloor}.$$

All the  $x_j$  are pairwise coprime, since the  $y_j$  are pairwise coprime.

Note that by definition  $c \prod_{j \leq M} x_j^j = \prod_{m \geq 1} y_m^m = n \leq X$ . In particular,

$$\prod_{m \geq M} y_m \leq \left( \prod_{m \geq M} y_m^m \right)^{1/M} \leq X^{1/M}.$$

Then, since  $m - K\lfloor m/K \rfloor \leq K$ , it follows from the definition of  $c$  that

$$c \leq \prod_{m \geq M} y_m^K \leq X^{K/M} \leq X^{\varepsilon/2}.$$

Thus

$$\text{rad}(n) \leq \text{rad}(c) \prod_{j \leq M} \text{rad}(x_j) \leq X^{\varepsilon/2} \prod_{j \leq M} x_j.$$

On the other hand, we have

$$x_K = y_K \prod_{m>M} y_m^{\lfloor m/K \rfloor} \leq \left( y_K^K \cdot \prod_{m>M} y_m^m \right)^{1/K} \leq n^{1/K} \leq X^{\varepsilon/2}.$$

Recalling that the  $y_j$  are squarefree and pairwise coprime for  $j \neq K$ , gives the lower bound

$$\text{rad}(n) = \prod_{m \geq 1} y_m \geq \prod_{\substack{j \leq M \\ j \neq K}} x_j \geq X^{-\varepsilon/2} \prod_{j \leq M} x_j,$$

as claimed.  $\square$

**Proposition 2.6.** *Let  $\alpha, \beta, \gamma \in (0, 1]$  be fixed and let  $X \geq 2$ . For any  $\varepsilon > 0$  there exists an integer  $d = d(\varepsilon) \geq 1$  such that the following holds. There exist  $X_1, \dots, X_d, Y_1, \dots, Y_d, Z_1, \dots, Z_d \geq 1$  satisfying*

$$X^{\alpha-\varepsilon} \ll_{\varepsilon} \prod_{j=1}^d X_j \leq 2X^{\alpha+\varepsilon}, \quad X^{\beta-\varepsilon} \ll_{\varepsilon} \prod_{j=1}^d Y_j \leq 2X^{\beta+\varepsilon}, \quad X^{\gamma-\varepsilon} \ll_{\varepsilon} \prod_{j=1}^d Z_j \leq 2X^{\gamma+\varepsilon} \quad (2.0.4)$$

and

$$\prod_{j=1}^d X_j^j \leq X, \quad \prod_{j=1}^d Y_j^j \leq X, \quad X^{1-\varepsilon^2} \ll_{\varepsilon} \prod_{j=1}^d Z_j^j \leq X \quad (2.0.5)$$

and pairwise coprime integers  $1 \leq c_1, c_2, c_3 \leq X^{\varepsilon}$ , such that

$$S_{\alpha, \beta, \gamma}^*(X) \ll_{\varepsilon} X^{\varepsilon} B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}).$$

*Proof of Proposition 2.6.* We may assume that  $X$  is large enough in terms of  $\varepsilon$ , since otherwise the claim is trivial. Let  $(a, b, c)$  be a triple counted by  $S_{\alpha, \beta, \gamma}^*(X)$ . Apply Lemma 2.5 (with  $\varepsilon^2/2$  in place of  $\varepsilon$ ) to each of  $a, b, c$  to obtain factorisations of the form

$$a = c_1 \prod_{j \leq d} x_j^j, \quad b = c_2 \prod_{j \leq d} y_j^j, \quad c = c_3 \prod_{j \leq d} z_j^j,$$

where  $d = \lfloor 10\varepsilon^{-4} \rfloor$  and  $1 \leq c_1, c_2, c_3 \leq X^{\varepsilon^2/4}$ . Since  $(a, b, c)$  is counted by  $S_{\alpha, \beta, \gamma}^*(X)$ , we have  $\gcd(a, b, c) = 1$  and  $a+b=c$ , so  $a, b, c$  are pairwise coprime. Hence, all the  $3d+3$  numbers  $x_i, y_i, z_i, c_1, c_2, c_3$  are pairwise coprime. Note also that by the properties of the factorisation given by Lemma 2.5, we have

$$\begin{aligned} X^{-\varepsilon/2} \prod_{j \leq d} x_j &\leq \text{rad}(a) \leq X^{\varepsilon/2} \prod_{j \leq d} x_j, & X^{-\varepsilon/2} \prod_{j \leq d} y_j &\leq \text{rad}(b) \leq X^{\varepsilon/2} \prod_{j \leq d} y_j, \\ X^{-\varepsilon/2} \prod_{j \leq d} z_j &\leq \text{rad}(c) \leq X^{\varepsilon/2} \prod_{j \leq d} z_j. \end{aligned}$$

Since  $\text{rad}(a) \sim X^\alpha, \text{rad}(b) \sim X^\beta, \text{rad}(c) \sim X^\gamma$  for all triples under consideration, this implies

$$X^{\alpha-\varepsilon} \leq \prod_{j \leq d} x_j \leq X^{\alpha+\varepsilon}, \quad X^{\beta-\varepsilon} \leq \prod_{j \leq d} y_j \leq X^{\beta+\varepsilon}, \quad X^{\gamma-\varepsilon} \leq \prod_{j \leq d} z_j \leq X^{\gamma+\varepsilon}.$$

By dyadic decomposition, we can now find some  $X_i, Y_i, Z_i$  such that (2.0.4) and (2.0.5) hold, and such that

$$S_{\alpha, \beta, \gamma}^*(X) \ll_\varepsilon (\log X)^{3d} \sum_{\substack{\mathbf{c} \in \mathbb{N}^3 \\ c_1, c_2, c_3 \leq X^{\varepsilon/4}}} B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}).$$

Now the claim follows from the pigeonhole principle. □

## Chapter 3

# Upper bounds for integer points

### 3.1 Fourier analysis

The following result uses basic Fourier analysis to bound the quantity defined in (2.0.3).

**Proposition 3.1** (Fourier analysis bound). *Let  $d \geq 1$ ,  $\varepsilon > 0$  and  $A \geq 1$  be fixed. Let*

$$X_1, \dots, X_d, Y_1, \dots, Y_d, Z_1, \dots, Z_d \geq 1$$

and put

$$\Delta = \max_{1 \leq i \leq d} (X_i Y_i Z_i). \quad (3.1.1)$$

Let  $\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{Z}^3$  satisfy  $0 < |c_1|, |c_2|, |c_3| \leq \Delta^A$ . Then

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll \Delta^\varepsilon \frac{\prod_{j \leq d} (X_j Y_j Z_j (Y_j + Z_j))^{\frac{1}{2}}}{\max_{i > 1} \prod_{j \equiv 0 \pmod i} Z_j^{\frac{1}{2}}}.$$

*Proof.* By the orthogonality of characters, we have

$$\begin{aligned} B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) &\leq \int_0^1 \sum_{x_j \sim X_j} \sum_{y_j \sim Y_j} \sum_{z_j \sim Z_j} e\left(\alpha \left(c_1 \prod_{j \leq d} x_j^j + c_2 \prod_{j \leq d} y_j^j - c_3 \prod_{j \leq d} z_j^j\right)\right) d\alpha \\ &= \int_0^1 S_1(\alpha) S_2(\alpha) S_3(-\alpha) d\alpha, \end{aligned}$$

where

$$\begin{aligned} S_1(\alpha) &= \sum_{x_1 \sim X_1, \dots, x_d \sim X_d} e(\alpha c_1 x_1 x_2^2 \cdots x_d^d), & S_2(\alpha) &= \sum_{y_1 \sim Y_1, \dots, y_d \sim Y_d} e(\alpha c_2 y_1 y_2^2 \cdots y_d^d), \\ S_3(\alpha) &= \sum_{z_1 \sim Z_1, \dots, z_d \sim Z_d} e(\alpha c_3 z_1 z_2^2 \cdots z_d^d). \end{aligned}$$

Then Cauchy-Schwarz gives

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \leq \left( \int_0^1 |S_1(\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \left( \int_0^1 |S_2(\alpha)|^2 |S_3(\alpha)|^2 d\alpha \right)^{\frac{1}{2}} =: \sqrt{I_1 I_2}. \quad (3.1.2)$$

By Parseval's identity and the divisor bound, we have

$$\begin{aligned} I_1 &= \int_0^1 |S_1(\alpha)|^2 d\alpha = \sum_{x_j \sim X_j \forall j} \#\{(x'_1, \dots, x'_d) : x'_j \sim X_j \forall j, x_1 x_2^2 \cdots x_d^d = x'_1 x'_2{}^2 \cdots x'_d{}^d\} \\ &\ll \prod_{j \leq d} X_j^{1+\varepsilon}, \end{aligned} \quad (3.1.3)$$



for any  $\varepsilon > 0$ . Using Cauchy-Schwarz again, for any  $i \leq d$  we have

$$|S_3(\alpha)|^2 = \left| \sum_{z_j \sim Z_j, \forall j \leq d} e(\alpha c_3 z_1 \cdots z_d^d) \right|^2 \leq T(\alpha) \prod_{j \not\equiv 0 \pmod i} Z_j,$$

where

$$T(\alpha) = \sum_{z_j \sim Z_j, \forall j \not\equiv 0 \pmod i} \left| \sum_{z_j \sim Z_j, \forall j \equiv 0 \pmod i} e(\alpha c_3 z_1 \cdots z_d^d) \right|^2.$$

Let  $ir$  be the largest multiple of  $i$  in  $[1, d]$ . Then

$$\begin{aligned} I_2 &= \int_0^1 |S_2(\alpha)|^2 |S_3(\alpha)|^2 d\alpha \\ &\leq \prod_{j \not\equiv 0 \pmod i} Z_j \cdot \int_0^1 |S_2(\alpha)|^2 T(\alpha) d\alpha \\ &= \prod_{j \not\equiv 0 \pmod i} Z_j \cdot \tilde{N}, \end{aligned} \tag{3.1.4}$$

where  $\tilde{N}$  is the number of

$$(z_1, \dots, z_d, z'_i, \dots, z'_{ir}) \in \mathbb{N}^{d+r}, \quad (y_1, \dots, y_d) \in \mathbb{N}^d, \quad (y'_1, \dots, y'_d) \in \mathbb{N}^d$$

such that

$$y_j, y'_j \sim Y_j, \quad z_j \sim Z_j, \quad z'_j \sim Z_j$$

for all  $j \leq d$ , and

$$c_3 \left( \prod_{j \equiv 0 \pmod i} z_j^j - \prod_{j \equiv 0 \pmod i} z_j'^j \right) \prod_{j \not\equiv 0 \pmod i} z_j^j + c_2 \prod_{j \leq d} y_j^j - c_2 \prod_{j \leq d} y_j'^j = 0.$$

Let us write

$$\tilde{N} = \tilde{N}_1 + \tilde{N}_2, \tag{3.1.5}$$

where  $\tilde{N}_1$  is the contribution to  $\tilde{N}$  from tuples with  $\prod_{j \equiv 0 \pmod i} z_j^j = \prod_{j \equiv 0 \pmod i} z_j'^j$ , and  $\tilde{N}_2$  is the contribution of the complementary tuples.

Then by the divisor bound we have

$$\tilde{N}_1 = \#\left\{ y_j, y'_j \sim Y_j, z_j \sim Z_j \forall j \leq d : \prod_j y_j^j = \prod_j y_j'^j \right\} \ll \prod_j Z_j Y_j^{1+\varepsilon}, \tag{3.1.6}$$

for any  $\varepsilon > 0$ . In order to bound  $\tilde{N}_2$ , we first note that  $a - b \mid a^i - b^i$  for any integers  $a \neq b$  and  $i \geq 1$ . Thus for any integers  $n \neq 0$  and  $i \geq 2$ ,

$$\#\{(a, b) \in \mathbb{Z}^2 : a^i - b^i = n\} \leq \tau(|n|) \max_{d \mid n} \#\{b \in \mathbb{Z} : (b+d)^i - b^i = n\} \ll_\varepsilon |n|^\varepsilon.$$

This follows from the divisor bound and the fact that  $(x+d)^i - x^i - n$  is a polynomial of degree  $i-1$ . (Importantly, this argument fails when  $i=1$ , since then the polynomial  $(x+n)^i - x^i - n$  is identically 0.) Hence, on appealing to the divisor bound, we obtain

$$\begin{aligned} \tilde{N}_2 &\ll \prod_{j \leq d} Y_j^2 \cdot \max_{0 < |n| \leq \Delta^{A+k_2}} \#\left\{ (z_1, \dots, z_d, z'_i, \dots, z'_{ir}) \in \mathbb{N}^{d+r} : z_j \sim Z_j, z'_j \sim Z'_j \forall j \right. \\ &\quad \left. c_3 (\prod_{j \equiv 0 \pmod i} z_j^j - \prod_{j \equiv 0 \pmod i} z_j'^j) \prod_{j \not\equiv 0 \pmod i} z_j^j = n \right\} \\ &\ll \prod_{j \leq d} Y_j^2 Z_j^{\varepsilon/2} \cdot \max_{0 < |n| \leq \Delta^{A+k_2}} \#\left\{ (a, b, c) \in \mathbb{N}^3 : c(a^i - b^i) = n \right\} \\ &\ll \Delta^\varepsilon \prod_{j \leq d} Y_j^2. \end{aligned} \tag{3.1.7}$$

Combining (3.1.4), (3.1.5), (3.1.6) and (3.1.7), we deduce that

$$\begin{aligned} I_2 &\ll \Delta^\varepsilon \prod_{j \not\equiv 0 \pmod i} Z_j \cdot \left( \prod_j Y_j Z_j + \prod_j Y_j^2 \right) \\ &\ll \Delta^\varepsilon \prod_{j \equiv 0 \pmod i} Z_j^{-1} \cdot \prod_j (Y_j Z_j^2 + Y_j^2 Z_j). \end{aligned} \quad (3.1.8)$$

Plugging (3.1.3) and (3.1.8) back into (3.1.2), we conclude that

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \leq \sqrt{I_1 I_2} \ll \Delta^\varepsilon \prod_{j \equiv 0 \pmod i} Z_j^{-\frac{1}{2}} \cdot \prod_j (X_j Y_j Z_j (Y_j + Z_j))^{\frac{1}{2}},$$

which is the desired bound.  $\square$

## 3.2 Geometry of numbers

We can supplement Proposition 3.1 with the following bound, where  $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$  is defined in (2.0.3).

**Proposition 3.2** (Geometry of numbers bound). *Let  $d \geq 1$  and  $\varepsilon > 0$  be fixed, and let*

$$X_1, \dots, X_d, Y_1, \dots, Y_d, Z_1, \dots, Z_d \geq 1.$$

*Let  $\mathbf{c} \in (c_1, c_2, c_3) \in \mathbb{Z}^3$  have non-zero and pairwise coprime coordinates. Then for  $\Delta$  as in (3.1.1),*

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll \Delta^\varepsilon \min_{I, I', I'' \subset [d]} \left( \prod_{i \in I} X_i \prod_{i \in I'} Y_i \prod_{i \in I''} Z_i \right) \left( 1 + \frac{\prod_{i \notin I} X_i \prod_{i \notin I'} Y_i \prod_{i \notin I''} Z_i}{\max\{|c_1| \prod_i X_i, |c_2| \prod_i Y_i, |c_3| \prod_i Z_i\}} \right).$$

*Proof.* Take any sets  $I, I', I'' \subset [d]$ . Let  $(x_1, \dots, x_d, y_1, \dots, y_d, z_1, \dots, z_d)$  be a tuple counted by  $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$ . We fix a choice of  $x_i, y_i, z_i \in \mathbb{Z}$  for all indices  $i$  in  $I, I', I''$ , respectively, and define

$$\begin{aligned} a_1 &= c_1 \prod_{i \in I} x_i^i, & a_2 &= c_2 \prod_{i \in I'} y_i^i, & a_3 &= c_3 \prod_{i \in I''} z_i^i, \\ x &= \prod_{i \notin I} x_i^i, & y &= \prod_{i \notin I'} y_i^i, & z &= \prod_{i \notin I''} z_i^i, \\ X &= \prod_{i \notin I} X_i^i, & Y &= \prod_{i \notin I'} Y_i^i, & Z &= \prod_{i \notin I''} Z_i^i. \end{aligned}$$

Then  $\gcd(a_1, a_2, a_3) = 1$  and  $\gcd(x, y, z) = 1$ . According to Heath-Brown [6, Lemma 3], the number of triples  $(x, y, z)$  that contribute to  $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$  is

$$\ll 1 + \frac{XYZ}{\max\{|a_1|X, |a_2|Y, |a_3|Z\}}.$$

Moreover, by the divisor bound, any triple  $(x, y, z)$  corresponds to  $O_\varepsilon(\Delta^\varepsilon)$  choices of  $x_i, y_i, z_i$  with  $i \notin I, j \notin I', r \notin I''$ . We arrive at the desired upper bound by summing over the choices  $x_i, y_i, z_i \in \mathbb{Z}$ , with  $i$  in  $I, I', I''$ .  $\square$

**Theorem 3.3.** *Let  $\gcd(a_1, a_2, a_3) = 1$ . Then for  $X_1, X_2, X_3 > 1$ , we have*

$$\#((x_1, x_2, x_3) \in \mathbb{Z}^3 : \gcd(x_1, x_2, x_3) = 1, |x_i| \leq X_i, a_1 x_1 + a_2 x_2 + a_3 x_3 = 0) \ll 1 + \frac{X_1 X_2 X_3}{\max_i \{|a_i| X_i\}}.$$

*Proof.*  $\square$

## 3.3 Determinant method

In this section we will record a bound for  $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$  in (2.0.3) that proceeds via the determinant method of Bombieri-Pila [2] and Heath-Brown [8]. See [1] for a gentle introduction to the determinant method. We first record a basic fact about the irreducibility of certain polynomials.

**Lemma 3.4.** *Let  $r \geq 1$  and let  $g \in \mathbb{C}[x]$  be a polynomial which has at least one root of multiplicity 1. Then the polynomial  $g(x) - y^r$  is absolutely irreducible.*

*Proof.* We may assume a factorisation  $g(x) = l_1(x)^{e_1} \dots l_t(x)^{e_t}$ , with pairwise non-proportional linear polynomials  $l_1, \dots, l_t \in \mathbb{C}[x]$  and exponents  $e_1, \dots, e_t \in \mathbb{N}$  such that  $e_1 = 1$ . But  $\mathbb{C}[x]$  is a unique factorisation domain and so we can apply Eisenstein's criterion with the prime  $l_1$  in order to deduce that  $g(x) - y^r$  is irreducible over  $\mathbb{C}[y]$ . It then follows that  $g(x) - y^r$  is irreducible over  $\mathbb{C}$ , as claimed in the lemma.  $\square$

Let  $p, q, r$  be positive integers and let  $a_1, a_2, a_3 \in \mathbb{Z}_{\neq 0}$ . We shall require a good upper bound for the counting function

$$N(X, Y, Z) = \# \left\{ (x, y, z) \in \mathbb{Z}_{\neq 0}^3 : \begin{array}{l} |x| \leq X, |y| \leq Y, |z| \leq Z \\ \gcd(x, y) = \gcd(x, z) = \gcd(y, z) = 1 \\ a_1 x^p + a_2 y^q + a_3 z^r = 0 \end{array} \right\},$$

for given  $X, Y, Z \geq 1$ . This is achieved in the following result.

Bombieri-Pila [2, Theorem 4]

**Theorem 3.5.** *Let  $f(x)$  be a  $C^\infty$  function on a closed subinterval of  $[0, N]$ , and suppose that  $F(x, f) = 0$ , where  $F(x, y) \in \mathbb{R}[x, y]$  is absolutely irreducible of degree  $d \geq 2$ . Suppose that  $|f'(x)| \leq 1$ . Then*

$$\#\{(x, f(x)) \in \{1, \dots, N\}^2\} \ll_d (\log N)^{O(N)} N^{1/d}$$

*Proof.*  $\square$

Heath-Brown Theorem 15

**Theorem 3.6.** *Let  $F \in \mathbb{Z}[x_1, \dots, x_n]$  be an absolutely irreducible polynomial of degree  $d$ , and let  $\varepsilon > 0$  and  $B_1, \dots, B_n \geq 1$  be given. Define*

$$B = \max_{(e_1, \dots, e_n)} \left( \prod_{i \leq n} B_i^{e_i} \right)$$

where the maximum is taken over all integer  $n$ -tuples  $(e_1, \dots, e_n)$  for which the corresponding monomial  $x_1^{e_1} \dots x_n^{e_n}$  occurs in  $F$  with non-zero coefficient.

Then there exists  $D = D(n, d, \varepsilon)$  and an integer  $k$  with

$$k \ll_{n, d, \varepsilon} T^\varepsilon (\log \|F\|)^{2n-3} \exp \left( (n-1) \left( \frac{\prod_{i \leq n} \log B_i}{\log B} \right)^{1/(n-1)} \right)$$

satisfying the following: There are  $k$  polynomials  $F_1, \dots, F_k \in \mathbb{Z}[x_1, \dots, x_n]$  coprime to  $F$ , with  $\deg F_i \leq D$ , such that every root of  $F(x_1, \dots, x_n) = 0$  with  $x_i \leq B_i$  is also a root of  $F_j(x_1, \dots, x_n) = 0$ , for some  $j \leq k$ .

*Proof.*  $\square$

**Theorem 3.7.** *Given integers  $n, a_1, a_2 \neq 0$ , there are at most  $O_{\varepsilon, D}(|na_1 a_2 X_1 X_2|^\varepsilon)$  many solutions  $(x_1, x_2) \in \mathbb{Z}^2$  such that  $|x_i| \leq X_i$  and*

$$n = a_1 x_1^2 + a_2 x_2^2.$$

*Proof.* Suppose  $n = a_1 x_1^2 + a_2 x_2^2$ . Then  $a_1 n = a_1^2 x_1^2 + a_1 a_2 x_2^2$ . Let  $D$  be the squarefree part of  $a_1 a_2$ . Then the number of solutions  $(x_1, x_2)$  with  $|x_i| \leq X_i$  to this equations is at most the number of solutions  $(m_1, m_2)$  to  $a_1 n = m_1^2 + D m_2^2$  with  $|m_i| \leq X_i a_1 a_2$ . For any such solution, we have  $m_1 + \sqrt{-D} m_2 \mid a_1 n$  in  $\mathbb{Q}(\sqrt{-D})$ . The claim follows from the divisor bound in quadratic fields, in Lemma 3.8.  $\square$

**Lemma 3.8.** *Let  $\varepsilon > 0$ . Let  $D \geq 1$  be a squarefree integer, and set  $K = \mathbb{Q}(\sqrt{-D})$ . Then for all  $\alpha \in K$ , the number of ideals dividing  $(\alpha)$  in  $K$  is  $O(N_\alpha^\varepsilon)$ .*

*Proof.* Mimics the proof over the integers using the fundamental theorem of arithmetic, but with ideals [See link in comment]  $\square$

**Theorem 3.9.** *Given integers  $n, a_1, a_2 \neq 0$  and  $p \geq 3$ , there are at most  $O(p^{1+\omega(|n|)})$  many solutions  $(x_1, x_2) \in \mathbb{Z}^2$  such that  $|x_i| \leq X_i$  and*

$$n = a_1 x_1^p + a_2 x_2^p.$$

*Proof.* □

**Definition 3.10.** Let  $\omega(n)$  denote the number of distinct prime factors of an integer  $n$ .

**Lemma 3.11.** For any  $n \geq 2$ , we have  $\omega(n) \ll \log(3n)/\log \log(3n)$ .

*Proof.* □

**Lemma 3.12.** Let  $\varepsilon > 0$  and  $D \geq 1$  and assume that  $p, q, r \in [1, D]$  are integers. Then

$$N(X, Y, Z) \ll_{\varepsilon, D} Z \min \left( X^{\frac{1}{q}}, Y^{\frac{1}{p}} \right) (XY)^{\varepsilon},$$

where the implied constant only depends on  $\varepsilon$  and  $D$ . Furthermore, if  $p = q \geq 2$ , then we have

$$N(X, Y, Z) \ll_{\varepsilon, D} Z (|a_1 a_2 a_3| XYZ)^{\varepsilon}.$$

*Proof.* We fix a choice of non-zero integer  $z \in [-Z, Z]$ , of which there are  $O(Z)$ . When  $z$  is fixed, the resulting equation defines a curve in  $\mathbb{A}^2$  and we can hope to apply work of Bombieri-Pila [2, Theorem 4], which would show that the equation has  $O_{\varepsilon, D}(\max(X, Y)^{\frac{1}{\max(p, q)} + \varepsilon})$  integer solutions in the region  $|x| \leq X$  and  $|y| \leq Y$ , where the implied constant only depends on  $\varepsilon$  and  $D$ . This is valid only when the curve is absolutely irreducible, which we claim is true when  $z \neq 0$ . But, for fixed  $z \in \mathbb{Z}_{\neq 0}$  the polynomial  $a_2 y^q + a_3 z^r$  has non-zero discriminant as a polynomial in  $y$ . Hence the claim follows from Lemma ???. Rather than appealing to Bombieri-Pila, however, we can get a sharper bound by using work of Heath-Brown [8, Theorem 15]. For fixed  $z \in \mathbb{Z}_{\neq 0}$  this gives the bound  $O_{\varepsilon, D}(\min(X^{\frac{1}{q}}, Y^{\frac{1}{p}})(XY)^{\varepsilon})$  for the number of available  $x, y$ . □

**Lemma 3.13.** Let  $\varepsilon > 0$  and  $D \geq 1$  and assume that  $p = q, r \in [1, D]$  are integers. Then

$$N(X, Y, Z) \ll_{\varepsilon, D} Z (|a_1 a_2 a_3| XYZ)^{\varepsilon}.$$

where the implied constant only depends on  $\varepsilon$  and  $D$ .

*Proof.* Suppose now that  $p = q \geq 2$ . Then, for given  $z \in \mathbb{Z}_{\neq 0}$ , we are left with counting the number of integer solutions to the equation  $N = a_1 x^p + a_2 y^p$ , with  $|x|, |y| \leq \max(X, Y)$ , and where  $N = -a_3 z^r$ . For  $p = 2$  this is a classical problem in quadratic forms. The bound  $O_{\varepsilon, D}((|a_1 a_2 N| XYZ)^{\varepsilon})$  follows from Heath-Brown [7, Theorem 3], for example. For  $p \geq 3$  we obtain a Thue equation. According to work of Bombieri and Schmidt [3], there are at most  $O(p^{1+\omega(|N|)})$  solutions, for an absolute implied constant. Using the bound  $\omega(|N|) \ll \log(3|N|)/(\log \log(3|N|))$ , this is  $O_{\varepsilon, D}((|a_3| Z)^{\varepsilon})$ , which thereby completes the proof of the lemma. □

Using these lemmas we can now supplement Propositions 3.1 and 3.2 with further bounds for  $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$ , as defined in (2.0.3).

**Proposition 3.14.** Let  $d \geq 1$ , and let

$$X_1, \dots, X_d, Y_1, \dots, Y_d, Z_1, \dots, Z_d \geq 1.$$

Let  $\mathbf{c} \in (c_1, c_2, c_3) \in \mathbb{Z}_{\neq 0}^3$ . Then for  $\Delta$  as in (3.1.1), we have

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll \Delta^{\varepsilon} \prod_{j \leq d} X_j Y_j Z_j \cdot \min_{p, q \geq 1} \left( (X_p Y_q)^{-1} \min \left( X_p^{\frac{1}{q}}, Y_q^{\frac{1}{p}} \right) \right).$$

*Proof.* Let  $(x_1, \dots, x_d, y_1, \dots, y_d, z_1, \dots, z_d)$  be a tuple counted by  $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$ . For any integers  $p, q \geq 1$ , we fix all but  $x_p$  and  $y_q$  and apply the first part of Lemma 3.12. This gives

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll \Delta^{\varepsilon} \prod_{\substack{j \leq d \\ j \neq p}} X_j \prod_{\substack{j \leq d \\ j \neq q}} Y_j \prod_{j \leq d} Z_j \cdot \min \left( X_p^{\frac{1}{q}}, Y_q^{\frac{1}{p}} \right),$$

from which the statement of the lemma easily follows. □

**Proposition 3.15.** *Let  $d \geq 1$ , and let*

$$X_1, \dots, X_d, Y_1, \dots, Y_d, Z_1, \dots, Z_d \geq 1.$$

*Let  $\mathbf{c} \in (c_1, c_2, c_3) \in \mathbb{Z}_{\neq 0}^3$ . Then*

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll \Delta^\varepsilon \prod_{j \leq d} X_j Y_j Z_j \cdot \min_{p \geq 2} \prod_{\substack{j \leq d \\ p|j}} (X_j Y_j)^{-1},$$

*where  $\Delta$  is given by (3.1.1).*

*Proof.* Let  $(x_1, \dots, x_d, y_1, \dots, y_d, z_1, \dots, z_d)$  be a tuple counted by  $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$ . We collect together all indices which are multiples of  $p$  for any integer  $p \geq 2$ . Applying the second part of Lemma 3.13 now gives

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll \Delta^\varepsilon \prod_{\substack{j \leq d \\ j \not\equiv 0 \pmod p}} X_j Y_j Z_j \prod_{\substack{j \leq d \\ j \equiv 0 \pmod p}} Z_j,$$

from which the statement easily follows. □

## Chapter 4

# Combining the upper bounds

### 4.1 Preliminaries

Throughout this section, let  $\varepsilon > 0$  be small but fixed. We now have everything in place to prove Theorem 1.4 for any

$$\lambda \in (0, 1 + \delta - \varepsilon).$$

and

$$\delta \leq 0.001 - \varepsilon.$$

We shall write  $\delta$  as a symbol rather than its numerical value in order to clarify the argument. As will be clear from the proof, a somewhat larger value of  $\delta$  would also work.

Our goal is to prove that the upper bound in (??) holds for  $S_{\alpha, \beta, \gamma}^*(X)$ , for any  $\alpha, \beta, \gamma$  such that

$$\alpha + \beta + \gamma \leq \lambda < 1 + \delta - \varepsilon. \quad (4.1.1)$$

The following result allows us to limit the range of  $\alpha, \beta, \gamma$  under consideration.

**Proposition 4.1.** *Let  $\alpha, \beta, \gamma > 0$ , and let  $\varepsilon > 0$  be fixed. Then*

$$S_{\alpha, \beta, \gamma}^*(X) \ll X^{0.66 - \varepsilon^2/2},$$

*unless  $\min\{\alpha + \beta, \beta + \gamma, \gamma + \alpha\} \geq 0.66 - \varepsilon^2$ .*

*Proof.* This is an immediate consequence of (1.0.3) (with  $\varepsilon^2/2$  in place of  $\varepsilon$ ).  $\square$

In the light of Proposition 2.6 (with  $\varepsilon^2$  in place of  $\varepsilon$ ), we wish to bound  $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$  for any pairwise coprime integers  $1 \leq |c_1|, |c_2|, |c_3| \leq X^{\varepsilon^2}$ , any fixed  $d \geq 1$ , and any choice of  $X_i, Y_i, Z_i \geq 1$ , for  $1 \leq i \leq d$  that satisfies (2.0.4) and (2.0.5). Moreover,  $\alpha, \beta, \gamma$  satisfy (4.1.1).

It will be convenient to define

**Definition 4.2.** Define  $a_i, b_i, c_i \in \mathbb{R}_{\geq 0}$  via

$$X_i = X^{a_i}, \quad Y_i = X^{b_i}, \quad Z_i = X^{c_i},$$

for  $1 \leq i \leq d$ , and  $a_i = b_i = c_i = 0$  for  $i > d$ .

**Lemma 4.3.**

$$\sum_{i \leq d} ia_i \leq 1, \quad \sum_{i \leq d} ib_i \leq 1, \quad 1 - \varepsilon^2 \leq \sum_{i \leq d} ic_i \leq 1. \quad (4.1.2)$$

*Proof.* Follows from 2.6  $\square$

In particular, in the light of (4.1.1) and Proposition 4.1, we may henceforth assume that

**Lemma 4.4.** *We have*

$$\sum_{i \leq d} (a_i + b_i) \geq 0.66 - \varepsilon^2, \quad \sum_{i \leq d} (a_i + c_i) \geq 0.66 - \varepsilon^2, \quad \sum_{i \leq d} (b_i + c_i) \geq 0.66 - \varepsilon^2 \quad (4.1.3)$$

and

$$\sum_{i \leq d} (a_i + b_i + c_i) \leq 1 + \delta - \varepsilon. \quad (4.1.4)$$

*Proof.* Follows from (4.1.1) and Proposition 4.1. □

**Definition 4.5.** It will be convenient to henceforth define

$$\nu = 2\varepsilon^2 + \frac{\log B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})}{\log X}.$$

Our goal is now to show that  $\nu \leq 0.66$ , since then (??) is a direct consequence of Proposition 2.6. This will then imply Theorem 1.4, via (2.0.1) and (??).

Before proceeding to the main tools that we shall use to estimate  $\nu$ , we first show that we may assume that

$$0.32 - \delta \leq \sum_{i \leq d} a_i, \sum_{i \leq d} b_i, \sum_{i \leq d} c_i \leq 0.34 + \delta - \frac{1}{2}\varepsilon,$$

using the argument of Proposition 4.1.

**Proposition 4.6.**

$$0.32 - \delta \leq \sum_{i \leq d} a_i, \sum_{i \leq d} b_i, \sum_{i \leq d} c_i \leq 0.34 + \delta - \frac{1}{2}\varepsilon, \quad (4.1.5)$$

*Proof.* Indeed, suppose that  $\sum_{i \leq d} c_i > 0.34 + \delta - \varepsilon/2$ . Then (4.1.4) implies that

$$\sum_{i \leq d} (a_i + b_i) < 0.66 - \frac{1}{2}\varepsilon,$$

whence (1.0.3) yields  $\nu < 0.66$ . This shows that we may suppose that the upper bound in (4.1.5) holds. Suppose next that  $\sum_{i \leq d} c_i < 0.32 - \delta$ . Then, by the upper bound in (4.1.5), we have

$$\sum_{i \leq d} (b_i + c_i) < 0.66 - \frac{1}{2}\varepsilon,$$

which is again found to be satisfactory, via (1.0.3). □

Thus we may proceed under the assumption that the parameters  $a_i, b_i, c_i$  satisfy (4.1.4)-(4.1.5).

## 4.2 Summary of the main bounds

We now recast our bounds in Sections 3.1–3.3 in terms of an upper bound for  $\nu$ , using the parameters  $a_i, b_i, c_i$ . In all of the following bounds, we may freely permute the exponent vectors  $(a_i), (b_i), (c_i)$ .

**Proposition 4.7** (Fourier bound).

$$\nu < \frac{1}{2} \left( 1 + \delta + \sum_{i \leq d} \max(a_i, b_i) - \max_{m \geq 1} (a_m, b_m) \right).$$

*Proof.* It follows from Proposition 3.1 that

$$\nu \leq 3\varepsilon^2 + \frac{1}{2} \sum_{i \leq d} \left( a_i + b_i + c_i + \max(a_i, b_i) - \max_{m \geq 1} b_m \right).$$

Permuting variables, the claim now follows from (4.1.4). □

**Proposition 4.8** (Geometry bound).

$$\nu < \delta + \min_{I, I', I'' \subset [d]} \left( \max \left( 1, \sum_{i \in I} i a_i + \sum_{i \in I'} i b_i + \sum_{i \in I''} i c_i \right) - \sum_{i \in I} a_i - \sum_{i \in I'} b_i - \sum_{i \in I''} c_i \right).$$

*Proof.* Applying Proposition 3.2, we obtain

$$\nu \leq 3\varepsilon^2 + \min_{I, I', I''} \left( \sum_{i \notin I} a_i + \sum_{i \notin I'} b_i + \sum_{i \notin I''} c_i + \max \left( 0, \sum_{i \in I} i a_i + \sum_{i \in I'} i b_i + \sum_{i \in I''} i c_i - \sum_{i \in [d]} i c_i \right) \right),$$

where the minimum runs over subsets  $I, I', I'' \subset [d]$ . Taking the lower bound  $\sum_{i \in [d]} ic_i \geq 1 - \varepsilon^2$ , from (4.1.2), it follows that

$$\nu \leq 4\varepsilon^2 + \min_{I, I', I''} \left( \sum_{i \notin I} a_i + \sum_{i \notin I'} b_i + \sum_{i \notin I''} c_i + \max \left( 0, \sum_{i \in I} ia_i + \sum_{i \in I'} ib_i + \sum_{i \in I''} ic_i - 1 \right) \right). \quad (4.2.1)$$

The proof now follows from (4.1.4).  $\square$

**Proposition 4.9** (Determinant Bound).

$$\nu < \min_{p, q \geq 1} \left( 1 + \delta - a_p - b_q + \min \left( \frac{a_p}{q}, \frac{b_q}{p} \right) \right).$$

*Proof.* Proposition 3.14 implies that

$$\nu \leq 3\varepsilon^2 + \sum_{i \leq d} (a_i + b_i + c_i) - \max_{p, q \geq 1} \left( \min \left( \frac{a_p}{q}, \frac{b_q}{p} \right) - a_p - b_q \right).$$

The claimed bound now follows from (4.1.4).  $\square$

**Proposition 4.10** (Thue bound).

$$\nu < 1 + \delta - \max_{p \geq 2} \sum_{p|i} (a_i + b_i).$$

*Proof.* This easily follows from Proposition 3.15 and (4.1.4).  $\square$

### 4.3 Completion of the upper bound for $\nu$

Assuming that  $\delta \leq 0.001$  and  $\varepsilon > 0$  is sufficiently small, the remainder of this paper is devoted to a proof of the upper bound

$$\nu \leq 0.66,$$

for any choice of parameters  $a_i, b_i, c_i$  satisfying the properties recorded in (4.1.2)- (4.1.5). From this point onward, we will tacitly assume those properties hold.

**Definition 4.11.** It will be convenient to define constants  $\delta_a, \delta_b, \delta_c$  via

$$\sum_{i \leq d} a_i = \frac{1}{3} - \delta_a, \quad \sum_{i \leq d} b_i = \frac{1}{3} - \delta_b, \quad \sum_{i \leq d} c_i = \frac{1}{3} - \delta_c, \quad (4.3.1)$$

together with

$$\delta_{ab} := \delta_a + \delta_b, \quad \delta_{ac} := \delta_a + \delta_c, \quad \delta_{bc} := \delta_b + \delta_c,$$

and  $\delta_s := \delta_a + \delta_b + \delta_c$ .

**Lemma 4.12.** *We have*

$$\delta_{ab}, \delta_{ac}, \delta_{bc} \leq 0.00\bar{6} + \varepsilon^2, \quad (4.3.2)$$

$$-0.00\bar{6} - \delta \leq \delta_a, \delta_b, \delta_c \leq 0.01\bar{3} + \delta + \varepsilon, \quad (4.3.3)$$

and

$$-\delta < \delta_s \leq 0.01 + \varepsilon. \quad (4.3.4)$$

*Proof.* It follows from (4.1.3) that

$$\delta_{ab}, \delta_{ac}, \delta_{bc} \leq 0.00\bar{6} + \varepsilon^2,$$

and from (4.1.5) that

$$-0.00\bar{6} - \delta \leq \delta_a, \delta_b, \delta_c \leq 0.01\bar{3} + \delta + \varepsilon,$$

and from (4.1.4) that  $1 - \delta_s \leq 1 + \delta$ . Moreover, (4.3.2) implies  $2\delta_s = \delta_{ab} + \delta_{ac} + \delta_{bc} \leq 0.02 + 3\varepsilon^2$ , so we must have

$$-\delta < \delta_s \leq 0.01 + \varepsilon.$$

$\square$



**Definition 4.13.** Define  $s_i := a_i + b_i + c_i$ .

Referring to (4.1.2), it will be convenient to record the inequalities

$$\sum_{i \geq 2} (i-1)a_i \leq \frac{2}{3} + \delta_a, \quad \sum_{i \geq 3} (i-2)a_i \leq \frac{1}{3} + a_1 + 2\delta_a, \quad \sum_{i \geq 4} (i-3)a_i \leq 2a_1 + a_2 + 3\delta_a, \quad (4.3.5)$$

that follow by subtracting. Similar relations hold for  $b_i$  and  $c_i$ , and thus for  $s_i$ .

**Proposition 4.14.** To show  $\nu \leq 0.66$ , it suffices to assume

$$a_j + b_j, a_j + c_j, b_j + c_j < 0.34 + \delta, \quad (4.3.6)$$

for each  $j \geq 2$ , and moreover,

$$a_2 + a_4 + b_2 + b_4, a_2 + a_4 + c_2 + c_4, b_2 + b_4 + c_2 + c_4 < 0.34 + \delta. \quad (4.3.7)$$

Hence

$$s_5, s_3, s_2 + s_4 < 0.51 + \frac{3}{2}\delta. \quad (4.3.8)$$

*Proof.* By the Thue bound, we have

$$\nu < 1 + \delta - \max_{p \geq 2} \sum_{p|i} (a_i + b_i),$$

and similarly for  $a_i + c_i$  and  $b_i + c_i$ . Now (4.3.6) follows by taking  $p = j$  and restricting the sum to  $i = j$ , while (4.3.7) follows by taking  $p = 2$  and restricting  $i \leq 4$ .

Finally, (4.3.6) and (4.3.7) imply

$$s_5, s_3, s_2 + s_4 < \frac{3}{2}(0.34 + \delta) \leq 0.51 + \frac{3}{2}\delta. \quad (4.3.9)$$

□

**Lemma 4.15.** We may assume

$$s_1 + s_2 \leq 0.34 + \delta. \quad (4.3.10)$$

*Proof.* If  $s_1 + s_2 > 0.34 + \delta$  then the Geometry bound and (4.3.8) imply that

$$\begin{aligned} \nu &\leq \max(1, s_1 + 2s_2) - s_1 - s_2 + \delta \\ &= \max(1 - s_1 - s_2, s_2) + \delta \\ &< \max(0.66, 0.51 + 3\delta) = 0.66. \end{aligned}$$

Thus we may proceed under the premise that

$$s_1 + s_2 \leq 0.34 + \delta.$$

□

**Lemma 4.16.** For any  $j \geq 3$ , allow  $\tau_j$  to be an element

$$\tau_j \in \{a_j, b_j, c_j, s_j, a_j + b_j, a_j + c_j, b_j + c_j\}. \quad (4.3.11)$$

Then

$$\tau_j \in \left(0.34 - s_1 - s_2 + \delta, \frac{0.66 - s_2 - \delta}{j-1}\right) \implies \nu < 0.66 \quad (4.3.12)$$

and

$$\tau_3 \in \left(0.34 - s_1 + \delta, 0.33 - \frac{1}{2}\delta\right) \implies \nu < 0.66. \quad (4.3.13)$$

*Proof.* The Geometry bound gives

$$\begin{aligned}\nu &\leq \max(1, s_1 + 2s_2 + j\tau_j) - s_1 - s_2 - \tau_j + \delta \\ &= \max(1 - s_1 - s_2 - \tau_j, s_2 + (j-1)\tau_j) + \delta.\end{aligned}$$

Thus we have  $\nu < 0.66$  if  $\tau_j \in (0.34 - s_1 - s_2 + \delta, \frac{0.66-s_2-\delta}{j-1})$ . In particular when  $j = 3$ , we have  $\nu < 0.66$  if  $\tau_3 \in (0.34 - s_1 - s_2 + \delta, 0.33 - \frac{1}{2}s_2 - \frac{\delta}{2})$ . Similarly, by the Geometry bound, we have

$$\begin{aligned}\nu &\leq \max(1, s_1 + 3\tau_3) - s_1 - \tau_3 + \delta \\ &= \max(1 - s_1 - \tau_3, 2\tau_3) + \delta.\end{aligned}$$

Thus  $\nu < 0.66$  if  $\tau_3 \in (0.34 - s_1 + \delta, 0.33 - \frac{\delta}{2})$ . □

**Lemma 4.17.** *We have*

$$\begin{aligned}a_3 &\geq \frac{1}{3} - 4\delta_a - 3a_1 - 2a_2, \\ a_3 &\geq \frac{1}{3} - \frac{5}{2}\delta_a - 2a_1 - \frac{3}{2}a_2 - \frac{1}{2}a_4.\end{aligned}\tag{4.3.14}$$

*Therefore*

$$s_3 \geq 1 - 4\delta_s - 3s_1 - 2s_2,\tag{4.3.15}$$

*and*

$$s_3 \geq 1 - \frac{5}{2}\delta_s - 2s_1 - \frac{3}{2}s_2 - \frac{1}{2}s_4.\tag{4.3.16}$$

*Proof.* Note that (4.3.5) gives  $\sum_{i \geq 4} a_i \leq \sum_{i \geq 4} (i-3)a_i \leq 2a_1 + a_2 + 3\delta_a$ . Similarly, we have  $\sum_{i \geq 5} a_i \leq \frac{1}{2} \sum_{i \geq 5} (i-3)a_i \leq \frac{1}{2}(2a_1 + a_2 - a_4 + 3\delta_a)$ . These imply that

$$a_3 = \frac{1}{3} - \delta_a - a_1 - a_2 - \sum_{i \geq 4} a_i \geq \frac{1}{3} - 4\delta_a - 3a_1 - 2a_2,$$

*and*

$$a_3 = \frac{1}{3} - \delta_a - a_1 - a_2 - a_4 - \sum_{i \geq 5} a_i \geq \frac{1}{3} - \frac{5}{2}\delta_a - 2a_1 - \frac{3}{2}a_2 - \frac{1}{2}a_4.$$

Analogous bounds hold for  $b_3$  and  $c_3$ , and so we obtain

$$\begin{aligned}s_3 &\geq 1 - 4\delta_s - 3s_1 - 2s_2, \\ s_3 &\geq 1 - \frac{5}{2}\delta_s - 2s_1 - \frac{3}{2}s_2 - \frac{1}{2}s_4.\end{aligned}$$

□

We shall need to split the argument according to whether  $s_2 < 0.3$  or  $s_2 \geq 0.3$ . Without loss of generality, we shall assume that  $a_3 \geq b_3 \geq c_3$  in all that follows.

**Case 1: Assume  $s_2 \geq 0.3$ .**

**Lemma 4.18.** *If  $s_2 \geq 0.3$ ,*

$$s_1 \leq 0.04 + \delta\tag{4.3.17}$$

*and  $s_4 < 0.21 + \frac{3}{2}\delta$ .*

*Proof.* Note that (4.3.10) gives

$$s_1 \leq 0.34 - s_2 + \delta \leq 0.04 + \delta,\tag{4.3.18}$$

*and (4.3.8) gives*

$$s_4 < 0.51 - s_2 + \frac{3}{2}\delta \leq 0.21 + \frac{3}{2}\delta.$$

□

We further split into subcases.

**Subcase 1.1: Assume  $b_3 \leq 0.34 - s_1 - s_2 + \delta$ .**

**Lemma 4.19.**  $\nu \leq 0.66$  in the case  $s_2 \geq 0.3$  and  $b_3 \leq 0.34 - s_1 - s_2 + \delta$ .

*Proof.* Then

$$\begin{aligned} b_3 + c_3 &\leq 2b_3 \leq 2(0.34 - s_1 - s_2 + \delta) \\ &\leq 0.68 - 2s_2 + 2\delta \\ &\leq 0.33 - \frac{1}{2}s_2 - \frac{1}{2}\delta, \end{aligned}$$

for  $s_2 \geq 0.3$  and  $\delta \leq 0.001$ . Hence, in view of (4.3.13), we may assume  $b_3 + c_3 \leq 0.34 - s_1 - s_2 + \delta$ . But then it follows from (4.3.16), (4.3.8) and (4.3.17) that

$$\begin{aligned} a_3 = s_3 - (b_3 + c_3) &\geq 1 - \frac{5}{2}\delta_a - 2s_1 - \frac{3}{2}s_2 - \frac{1}{2}s_4 - (0.34 - s_1 - s_2 + \delta) \\ &= 0.66 - \frac{5}{2}\delta_a - s_1 - \frac{1}{2}(s_2 + s_4) - \delta \\ &\geq 0.66 - \frac{5}{2}\delta_a - (0.04 + \delta) - \frac{1}{2}(0.51 + \frac{3}{2}\delta) - \delta \\ &\geq 0.365 - \frac{5}{2}\delta_a - 3\delta. \end{aligned}$$

But (4.3.1) implies that  $\frac{1}{3} - \delta_a \geq a_3 \geq 0.365 - \frac{5}{2}\delta_a - 3\delta$ . Thus (4.3.3) implies that

$$0.031\bar{6} < \frac{3}{2}\delta_a + 3\delta \leq \frac{3}{2}(0.01\bar{3} + \delta + \varepsilon) + 3\delta \leq 0.02 + 5\delta.$$

This contradicts our assumption  $\delta \leq 0.001$ . □

**Subcase 1.2: Assume  $b_3 > 0.34 - s_1 - s_2 + \delta$ .**

**Lemma 4.20.**  $\nu \leq 0.66$  in the case  $s_2 \geq 0.3$  and  $b_3 > 0.34 - s_1 - s_2 + \delta$ .

*Proof.* By (4.3.13) we may assume

$$b_3 \geq 0.33 - \frac{1}{2}s_2 - \frac{1}{2}\delta. \quad (4.3.19)$$

By permuting the variables in (4.3.5), we have

$$\sum_{i \geq 4} (i-2)b_i \leq \frac{1}{3} - b_3 + b_1 + 2\delta_b.$$

We also have  $b_1 \leq s_1 \leq 0.34 - s_2 + \delta$  by (4.3.10). Thus

$$\begin{aligned} \sum_{i \geq 4} b_i &\leq \frac{1}{2} \left( \frac{1}{3} + b_1 - b_3 + 2\delta_b \right) \leq \frac{1}{2} \left( \frac{1}{3} + 0.34 - s_2 + \delta - (0.33 - \frac{s_2}{2} - \frac{\delta}{2}) + 2\delta_b \right) \\ &< 0.33 - \frac{1}{2}s_2 - \frac{\delta}{2}, \end{aligned}$$

since (4.3.3) ensures that  $\delta_b \leq 0.01\bar{3} + \delta + \varepsilon$  (and we have  $\delta \leq 0.001$ ). A fortiori the same bound holds for  $\sum_{i \geq 4} a_i$ . Thus, in the light of (4.3.13), taking  $\varepsilon > 0$  small we may assume that

$$a_4, b_4, a_5, b_5, a_6, b_6 \leq 0.34 - s_1 - s_2 + \delta.$$

Now write  $M_i = \max(a_i, b_i)$  and  $m_i = \min(a_i, b_i)$ , so that  $m_i + M_i = a_i + b_i$ . By the Fourier bound we have

$$\nu < \frac{1}{2} \left( 1 + \delta + \sum_{i \leq d} \max(a_i, b_i) - \max(a_2, b_2) \right) = \frac{1}{2} \left( 1 + \delta + \sum_{i \neq 2} M_i \right).$$

On using (4.3.1), this implies that

$$\begin{aligned}
2\nu - 1 - \delta &< \sum_{i \neq 2} M_i \leq \sum_{2 \neq i \leq 6} M_i + \sum_{i \geq 7} (a_i + b_i) \\
&= \sum_{2 \neq i \leq 6} M_i + \frac{2}{3} - \delta_{ab} - \sum_{i \leq 6} (a_i + b_i) \\
&= \frac{2}{3} - \delta_{ab} - \sum_{2 \neq i \leq 6} m_i - (a_2 + b_2).
\end{aligned}$$

Next we lower bound  $a_2 + b_2$ . To do this, we observe that by (4.3.5) we have

$$4 \sum_{i \geq 7} a_i \leq \sum_{i \geq 7} (i-3)a_i = 2a_1 + a_2 + 3\delta_a - a_4 - 2a_5 - 3a_6,$$

whence

$$\frac{1}{3} - \delta_a = \sum_i a_i \leq \sum_{i \leq 6} a_i + \frac{1}{4}(2a_1 + a_2 + 3\delta_a - a_4 - 2a_5 - 3a_6) = \frac{1}{4} \sum_{i \leq 6} (7-i)a_i + \frac{3}{4}\delta_a.$$

Thus  $a_2 \geq \frac{4}{15} - \frac{1}{5} \sum_{2 \neq i \leq 6} (7-i)a_i - \frac{7}{5}\delta_a$ , and similarly  $b_2 \geq \frac{4}{15} - \frac{1}{5} \sum_{2 \neq i \leq 6} (7-i)b_i - \frac{7}{5}\delta_b$ . Since  $m_3 = b_3$ , it now follows that

$$\begin{aligned}
2\nu - 1 - \delta &< \frac{2}{3} - \delta_{ab} - \sum_{2 \neq i \leq 6} m_i - \left( \frac{8}{15} - \frac{1}{5} \sum_{2 \neq i \leq 6} (7-i)(a_i + b_i) - \frac{7}{5}\delta_{ab} \right) \\
&\leq \frac{2}{15} + \frac{2}{5}\delta_{ab} + \frac{1}{5} \left( 6M_1 + m_1 + 4a_3 - b_3 + 3M_4 + 2M_5 + M_6 \right). \tag{4.3.20}
\end{aligned}$$

Thus, using (4.3.19) and the bound  $a_3 + b_3 \leq 0.34 + \delta$  coming from (4.3.6), we have

$$\begin{aligned}
4a_3 - b_3 &\leq 4(0.34 - b_3 + \delta) - b_3 \\
&< 4(0.01 + \frac{s_2}{2} + \frac{3\delta}{2}) - (0.33 - \frac{s_2}{2} - \frac{\delta}{2}) \\
&= \frac{5}{2}s_2 - 0.29 + \frac{13\delta}{2}.
\end{aligned}$$

Also  $6M_1 + m_1 \leq 6s_1$  and recall  $M_4, M_5, M_6 \leq 0.34 - s_1 - s_2 + \delta$ . Hence plugging back into (4.3.20), we conclude

$$\begin{aligned}
2\nu - 1 - \delta &< \frac{2}{15} + \frac{2}{5}\delta_{ab} + \frac{1}{5} \left( 6s_1 + \left( \frac{5}{2}s_2 - 0.29 + \frac{13\delta}{2} \right) + 6(0.34 - s_1 - s_2 + \delta) \right) \\
&< \frac{2}{15} + \frac{2}{5}\delta_{ab} + \frac{1}{5} \left( 1.75 - \frac{7}{2}s_2 + 13\delta \right) \\
&\leq 0.48\bar{3} - \frac{7}{10}s_2 + \frac{2}{5}\delta_{ab} + \frac{13}{5}\delta \\
&\leq 0.48\bar{3} - \frac{7}{10}(0.3) + \frac{2}{5}(0.00\bar{6} + \varepsilon^2) + \frac{13}{5}\delta < 0.279,
\end{aligned}$$

since  $s_2 \geq 0.3$ ,  $\delta \leq 0.001$ , and (4.3.2) implies that  $\delta_{ab} \leq 0.00\bar{6} + \varepsilon^2$ . Hence  $\nu \leq \frac{1.3}{2} = 0.65$ , which is more than satisfactory.  $\square$

**Lemma 4.21.**  $\nu \leq 0.66$  in the case  $s_2 \geq 0.3$ .

*Proof.* Immediate from Lemmas 4.19 and 4.20.  $\square$

**Case 2: Assume  $s_2 < 0.3$ .**

**Lemma 4.22.**

$$b_3 < 0.17 + \frac{\delta}{2}. \tag{4.3.21}$$

*Proof.* It follows from (4.3.6) that

$$2b_3 \leq a_3 + b_3 < 0.34 + \delta, \tag{4.3.22}$$

whence  $b_3 < 0.17 + \frac{\delta}{2}$ .  $\square$

**Lemma 4.23.**

$$a_3 \geq 0.32 - 4\delta_s - s_1 - 2\delta. \quad (4.3.23)$$

*Proof.* We have  $0.17 + \frac{\delta}{2} \leq 0.33 - \frac{s_2}{2} - \frac{\delta}{2}$  since  $s_2 < 0.3$  and  $\delta \leq 0.001$ . Thus, in view of (4.3.13), we may assume that  $b_3, c_3 \leq 0.34 - s_1 - s_2 + \delta$ . Then (4.3.15) gives

$$\begin{aligned} a_3 = s_3 - (b_3 + c_3) &\geq 1 - 4\delta_s - 3s_1 - 2s_2 - 2(0.34 - s_1 - s_2 + \delta) \\ &= 0.32 - 4\delta_s - s_1 - 2\delta. \end{aligned}$$

□

We shall proceed by separately handling the subcases

$$\mathbf{2.1} : a_3 \geq 0.32 \quad \mathbf{2.2} : b_3 + c_3 < 0.33 - \frac{s_2}{2} - \frac{\delta}{2}.$$

These will be instrumental to proving the following subcases

$$\begin{aligned} \mathbf{2.3} : 4s_1 + 3s_2 &> 0.71, & \mathbf{2.5} : 0.066 \leq s_2 \leq 0.204, \\ \mathbf{2.4} : 4s_1 + s_2 &< 0.4, & \mathbf{2.6} : 2s_1 - s_2 > 0.025. \end{aligned}$$

Handling these subcases will complete the proof.

**Lemma 4.24.** Assuming 4.27, 4.28, 4.29, 4.30,  $\nu \leq 0.66$  in the case  $s_2 < 0.3$ .

*Proof.* Indeed **2.3, 2.4, 2.6** each define half-planes that cover  $[0, 1]^2 \setminus T$ , for the closed triangle  $T$  with vertices

$$(s_1, s_2) \in \{(0.06125, 0.155), (0.0785, 0.132), (0.0708\bar{3}, 0.11\bar{6})\}.$$

But then **2.5** covers  $T$ . Hence subcases **2.3–2.6** will complete the proof of Case 2. □

**Subcase 2.1: Assume  $a_3 \geq 0.32$**

**Lemma 4.25.**  $\nu \leq 0.66$  in the case  $s_2 < 0.3$  and  $a_3 \geq 0.32$ .

*Proof.*

By (4.3.6) we have  $b_3, c_3 \leq 0.34 + \delta - a_3 \leq 0.02 + \delta$ . Let  $m_i = \min(b_i, c_i)$ ,  $M_i = \max(b_i, c_i)$ , and  $t_i = b_i + c_i = m_i + M_i$ . If  $M := \max_{i \geq 4} M_i > \frac{3}{4}(0.09)$ , then using  $\delta \leq 0.001$  and the Determinant bound (with variables permuted) yields

$$\nu \leq 1 + \delta - a_3 - M + \min\left(\frac{M}{3}, \frac{a_3}{4}\right) \leq 1 + \delta - a_3 - \frac{2}{3}M \leq 1 + \delta - 0.32 - \frac{1}{2}(0.09) \leq 0.636.$$

This is satisfactory. We may therefore assume that  $M_i \leq \frac{3}{4}(0.09)$  for  $i \geq 4$ . Then  $t_i \leq 2M_i \leq 0.135$  for  $i \geq 4$ . Moreover,  $\sum_i (i-1)t_i \leq \frac{4}{3} + \delta_{bc}$ , by (4.1.2) and (4.3.1). Appealing to the Geometry bound in the form (4.2.1), we deduce that

$$\nu \leq \varepsilon + a_3 + b_3 + m_4 + \sum_{i \geq 5} t_i + \max\left(0, \sum_i i s_i - 3(a_3 + b_3) - 4m_4 - \sum_{i \geq 5} i t_i - 1\right).$$

Thus we have  $\nu \leq \max(\nu_1, \nu_2) + \varepsilon$ , where

$$\nu_1 := a_3 + b_3 + m_4 + \sum_{i \geq 5} t_i \quad \text{and} \quad \nu_2 := \sum_i i s_i - 2(a_3 + b_3) - 3m_4 - \sum_{i \geq 5} (i-1)t_i - 1.$$

Using (4.3.5), we see that

$$\begin{aligned} \nu_1 &\leq a_3 + b_3 + m_4 + t_5 + \frac{1}{5} \sum_{i \geq 6} (i-1)t_i \\ &\leq a_3 + b_3 + \frac{1}{2}t_4 + t_5 + \frac{1}{5} \left( \frac{4}{3} + \delta_{bc} - t_2 - 2t_3 - 3t_4 - 4t_5 \right). \end{aligned}$$

Using  $a_3 + b_3 \leq 0.34 + \delta$  (which follows from (4.3.21)),  $\delta_{bc} < 0.00\bar{6} + \varepsilon^2$ , and  $t_5 \leq 0.135$ , we conclude that

$$\nu_1 \leq a_3 + b_3 + \frac{1}{5} \left( \frac{4}{3} + \delta_{bc} + t_5 \right) \leq 0.34 + \delta + \frac{1}{5} \left( \frac{4}{3} + 0.00\bar{6} + \varepsilon^2 + 0.135 \right) < 0.637.$$

Similarly, on recalling  $\sum_i ia_i \leq 1$ , we have  $\sum_i is_i - 1 \leq \sum_i it_i$ , whence

$$\begin{aligned}\nu_2 &\leq \sum_i it_i - \sum_{i \geq 5} (i-1)t_i - 2(a_3 + b_3) - 3m_4 \\ &= \sum_i t_i + \sum_{i \leq 4} (i-1)t_i - 2(a_3 + b_3) - 3m_4 \\ &= \frac{2}{3} - \delta_{bc} + t_2 - 2(a_3 - c_3) + 3M_4,\end{aligned}$$

by (4.3.1). Using  $t_2 \leq s_2 < 0.3$ ,  $c_3 \leq 0.02 + \delta$ , and  $a_3 \geq 0.32$  by assumption, we conclude that

$$\nu_2 < \frac{2}{3} - \delta_{bc} + 0.3 - 2(0.3 - \delta) + 3 \cdot \frac{3}{4}(0.09) < 0.57 - \delta_{bc} + 2\delta.$$

Thus  $\nu_2 < 0.6$ , since (4.3.3) implies that  $\delta_{bc} \geq -0.01\bar{3} - 2\delta$ , and  $\delta \leq 0.001$ . Combining the bounds for  $\nu_1$  and  $\nu_2$ , we conclude that  $\nu \leq \max(\nu_1, \nu_2) + \varepsilon < 0.64$ , which suffices.  $\square$

**Subcase 2.2: Assume  $b_3 + c_3 < 0.33 - \frac{s_2}{2} - \frac{\delta}{2}$ .**

**Lemma 4.26.**  $\nu \leq 0.66$  in the case  $s.2 < 0.3$  and  $b_3 + c_3 < 0.33 - \frac{s_2}{2} - \frac{\delta}{2}$ .

*Proof.* Then by (4.3.13) we may assume  $\tau_3 = b_3 + c_3 < 0.34 - s_1 - s_2 + \delta$ . By (4.3.16) we have

$$\begin{aligned}a_3 = s_3 - (b_3 + c_3) &> 1 - \frac{5}{2}\delta_s - 2s_1 - \frac{3}{2}s_2 - \frac{1}{2}s_4 - (0.34 - s_1 - s_2 + \delta) \\ &= 0.66 - \frac{5}{2}\delta_s - s_1 - \frac{1}{2}(s_2 + s_4) - \delta.\end{aligned}$$

It follows from (4.3.4) that  $\delta_s \leq 0.01 + \varepsilon$  and from (4.3.8) that  $s_2 + s_4 < 0.51 + 3\delta/2$ . Hence

$$a_3 > 0.66 - \frac{5}{2}(0.01 + \varepsilon) - s_1 - \frac{1}{2}(0.51 + \frac{3\delta}{2}) - \delta \geq 0.38 - s_1 - 3\delta.$$

Since  $\delta \leq 0.001$ , we see that  $a_3 > 0.34 - s_1 + \delta$ . Thus it follows from (4.3.13) that we may assume  $\tau_3 = a_3 > 0.33 - \frac{\delta}{2} \geq 0.32$ . Hence Subcase 2.1 completes the proof.  $\square$

**Subcase 2.3: Assume  $4s_1 + 3s_2 > 0.71$ .**

**Lemma 4.27.**  $\nu \leq 0.66$  in the case  $s.2 < 0.3$  and  $4s_1 + 3s_2 > 0.71$ .

*Proof.* Then the inequalities  $b_3, c_3 \leq 0.34 - s_1 - s_2 + \delta$  give

$$b_3 + c_3 < 0.68 - 2(s_1 + s_2) + 2\delta < 0.325 - \frac{s_2}{2} + 2\delta.$$

Since  $\delta \leq 0.001$ , we see that  $b_3 + c_3 < 0.33 - \frac{s_2}{2} - \frac{\delta}{2}$ . Hence Subcase 2.2 completes the proof.  $\square$

**Subcase 2.4: Assume  $4s_1 + s_2 < 0.4$ .**

**Lemma 4.28.**  $\nu \leq 0.66$  in the case  $s.2 < 0.3$  and  $4s_1 + s_2 < 0.4$ .

*Proof.* In this case, (4.3.6) and (4.3.23) give

$$b_3, c_3 \leq 0.34 - a_3 + \delta \leq 0.34 - (0.32 - 4\delta_s - s_1 - 2\delta) + \delta = 0.02 + 4\delta_s + s_1 + 3\delta.$$

In view of (4.3.4) and our assumption  $4s_1 + s_2 < 0.4$ , we deduce that

$$b_3 + c_3 \leq 0.12 + 8\varepsilon + 2s_1 + 6\delta < 0.32 - \frac{s_2}{2} + 6\delta + 8\varepsilon.$$

Since  $\delta \leq 0.001$ , we have  $b_3 + c_3 \leq 0.33 - \frac{s_2}{2} - \frac{\delta}{2}$ . Hence Subcase 2.2 completes the proof.  $\square$

**Subcase 2.5:** Assume  $0.066 \leq s_2 \leq 0.204$ .

**Lemma 4.29.**  $\nu \leq 0.66$  in the case  $s_2 < 0.3$  and  $0.066 \leq s_2 \leq 0.204$ .

*Proof.* It follows from (4.3.4) and (4.3.23) that

$$a_3 \geq 0.32 - 4\delta_s - s_1 - 2\delta \geq 0.28 - 4\varepsilon - s_1 - 2\delta.$$

Thus  $a_3 > 0.34 - s_1 - s_2 + \delta$ , since  $s_2 \geq 0.066 \geq 0.062 + 4\varepsilon + 3\delta$  and  $\delta \leq 0.001$ . It now follows from (4.3.13) that we may assume  $\tau_3 = a_3 \geq 0.33 - \frac{s_2}{2} - \frac{\delta}{2}$ . Thus (4.3.6) gives  $b_3, c_3 \leq 0.34 - a_3 + \delta < 0.01 + \frac{s_2}{2} + \frac{3\delta}{2}$ , which in turn gives  $b_3 + c_3 < 0.02 + s_2 + 3\delta$ . Since  $s_2 \leq 0.204$  and  $\delta \leq 0.001$ , we deduce that  $b_3 + c_3 < 0.33 - \frac{s_2}{2} - \frac{\delta}{2}$ . Hence Subcase 2.2 completes the proof.  $\square$

**Subcase 2.6:** Assume  $2s_1 - s_2 > 0.025$ .

**Lemma 4.30.**  $\nu \leq 0.66$  in the case  $s_2 < 0.3$  and  $2s_1 - s_2 > 0.025$ .

*Proof.*

In this case we note that the intervals in (4.3.13) overlap, since  $\delta \leq 0.001$ . Hence for any  $\tau_3$  belonging to the set (4.3.11), we have

$$\tau_3 \in \left(0.34 - s_1 - s_2 + \delta, 0.33 - \frac{\delta}{2}\right) \implies \nu < 0.66. \quad (4.3.24)$$

Furthermore, in the light of Subcases **S<sub>3</sub>** and **S<sub>4</sub>**, we may assume that  $4s_1 + 3s_2 \leq 0.71$  and  $4s_1 + s_2 \geq 0.4$ . In particular, these imply that  $s_1 \leq \frac{0.71}{4} \leq 0.1775$  and  $s_2 \leq \frac{1}{3}(0.71 - 4s_1) \leq \frac{1}{3}(0.71 - 0.4 + s_2)$ , so that  $s_2 \leq 0.155$ . Then, on appealing to Subcase **S<sub>5</sub>**, we may assume that  $s_2 < 0.066$ . Similarly, it follows from Subcase **S<sub>1</sub>** that we may also assume  $a_3 < 0.32$ . Thus (4.3.24) and the bound  $\delta \leq 0.001$  imply that we may assume  $a_3 < 0.34 - s_1 - s_2 + \delta$ .

If we also had  $b_3 + c_3 < 0.34 - s_1 - s_2 + \delta$ , then we would have  $s_3 = a_3 + b_3 + c_3 < 0.68 - 2s_1 - 2s_2 + 2\delta$ . Combining this with (4.3.15), we would then conclude that

$$0.68 - 2s_1 - 2s_2 + 2\delta > s_3 \geq 1 - 4\delta_s - 3s_1 - 2s_2,$$

which implies that  $s_1 > 0.32 - 4\delta_s - 2\delta$ . Recalling (4.3.4) and the inequalities  $s_1 \leq 0.1775$  and  $\delta \leq 0.001$ , this is a contradiction. Hence we may assume that  $b_3 + c_3 \geq 0.34 - s_1 - s_2 + \delta$ , and by (4.3.24), we may assume  $\tau_3 = b_3 + c_3 \geq 0.33 - \frac{\delta}{2}$ , so  $b_3 > 0.165 - \frac{\delta}{4} > 0.164$ . Thus we have  $a_3, b_3 \in [0.164, 0.341 - s_1 - s_2]$ , since  $\delta \leq 0.001$ . In particular the interval is nontrivial, so  $s_1 + s_2 \leq 0.1775$ .

Letting  $M_i = \max(a_i, b_i)$  and  $m_i = \min(a_i, b_i)$ , it follows from the Fourier bound that

$$\nu < \frac{1}{2} \left(1 + \delta + \sum_i \max(a_i, b_i) - \max(a_3, b_3)\right) = \frac{1}{2} \left(1 + \delta + \sum_{i \neq 3} M_i\right).$$

It follows from (4.3.1) that  $\sum_i (M_i + m_i) = \frac{2}{3} - \delta_{ab}$ , and so

$$\begin{aligned} 2\nu - 1 - \delta &\leq \sum_{i \neq 3} M_i \leq M_1 + M_2 + \sum_{i \geq 4} (M_i + m_i) \\ &= M_1 + M_2 + \left(\frac{2}{3} - \delta_{ab} - \sum_{i \leq 3} (M_i + m_i)\right) \\ &\leq \frac{2}{3} - \delta_{ab} - m_1 - m_2 - m_3 - M_3. \end{aligned}$$

By (4.3.14) we have  $3a_1 + 2a_2 + a_3 \geq \frac{1}{3} - 4\delta_a$ , and similarly,  $3b_1 + 2b_2 + b_3 \geq \frac{1}{3} - 4\delta_b$ . Thus  $m_1 \geq \frac{1}{3}(\frac{1}{3} - 2M_2 - M_3 - 4\max(\delta_a, \delta_b))$ . This together with the bounds  $\delta \leq 0.001$  and (4.3.3) lead to the upper bound

$$\begin{aligned} 2\nu - 1 - \delta &\leq \frac{2}{3} - \delta_{ab} + \frac{1}{3}(2M_2 + M_3 - \frac{1}{3} + 4\max(\delta_a, \delta_b)) - m_2 - m_3 - M_3 \\ &\leq \frac{5}{9} + \frac{2}{3}M_2 + \frac{1}{3}\max(\delta_a, \delta_b) - \min(\delta_a, \delta_b) - m_3 - \frac{2}{3}M_3 \\ &\leq \frac{5}{9} + \frac{2}{3}M_2 + \frac{1}{3}(0.01\bar{3} + \delta + \varepsilon) + (0.00\bar{6} + \delta) - m_3 - \frac{2}{3}M_3 \\ &\leq 0.568 + \frac{2}{3}(M_2 - M_3) - m_3, \end{aligned}$$

using  $\delta_a, \delta_b \in [-0.006 - \delta, 0.013 + \delta + \varepsilon]$  and  $b_3 + c_3 \geq 0.33 - \frac{\delta}{2}$ . Thus we have

$$\begin{aligned}\nu &\leq 0.784 + \frac{1}{3}(\max(a_2, b_2) - \max(a_3, b_3)) - \frac{1}{2}\min(a_3, b_3) \\ &\leq 0.784 + \frac{1}{3}(\max(a_2, b_2) - a_3) - \frac{1}{2}b_3\end{aligned}\tag{4.3.25}$$

Next, if  $a_2 < b_2$ , let  $e_i := a_i$  for all  $i \geq 1$ , otherwise let  $e_i := b_i$  for all  $i \geq 1$ . In particular, note  $e_2 = \min(a_2, b_2)$  and  $e_3 \geq \min(a_3, b_3) \geq c_3$ . By a similar argument with  $(a_i, b_i)_i$  replaced by  $(e_i, c_i)_i$ , we have

$$\begin{aligned}\nu &\leq 0.784 + \frac{1}{3}(\max(e_2, c_2) - \max(e_3, c_3)) - \frac{1}{2}\min(e_3, c_3) \\ &\leq 0.784 + \frac{1}{3}(\max(e_2, c_2) - e_3) - \frac{1}{2}c_3.\end{aligned}$$

Averaging the bound with (4.3.25), we obtain

$$\begin{aligned}\nu &\leq 0.784 + \frac{1}{6}(\max(a_2, b_2) + \max(e_2, c_2) - a_3 - e_3) - \frac{1}{4}(b_3 + c_3) \\ &\leq 0.784 + \frac{1}{6}s_2 - \frac{5}{12}(b_3 + c_3) \\ &\leq 0.784 + \frac{1}{6}(0.066) - \frac{5}{12}(0.33 - \frac{\delta}{2}) \leq 0.658.\end{aligned}\tag{4.3.26}$$

Here we used  $\max(a_2, b_2) + \max(e_2, c_2) = \max(a_2 + b_2, \max(a_2, b_2) + c_2) \leq s_2$  and  $a_3 + e_3 \geq b_3 + c_3 \geq 0.33 - \frac{\delta}{2}$ .

This completes the proof. □

**Theorem 4.31.**  $\nu \leq 0.66$ .

*Proof.* Immediate from Lemmas 4.21 and 4.24. □

*Proof of Theorem 2.3.* □



## Chapter 5

# Bibliography

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