Overview of Riemann Surfaces

Norio Kosaka

Department of Economics, Mathematics and Statistics
Birkbeck, University of London

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Outline

- Objective of Thesis
- Riemann surfaces
- Uniformisation theorem
- Fuchsian groups
- Dessins d'Enfants

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Objective of Thesis

Properly define Riemann Surfaces to;

- (i) State the Uniformisation Theorem to classify the surfaces
- (ii) Determine / Study their automorphism groups and their importance
- (iii) Study the geometric representation of critical points

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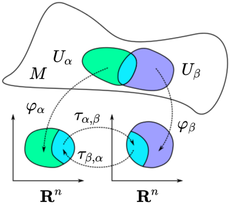
Preliminary: Manifolds

Start with reviewing a couple of concepts.

Recall: Atlas

A set of charts $\mathcal{A} := \{(U_i, \phi_i) \mid \phi_i : U_i \to D \subset \mathbb{C}\}$ is called an atlas on a surface **R** if the following hold: (i) $\bigcup_{(U_i,\phi_i)\in\mathcal{A}} U_i = R$, and (ii) transition among charts in the atlas is analytic.



When multiple atlases have analytic transition functions, then they are called **compatible**.

Formally, Riemann Surfaces are ...

The compatibility of atlases forms an equivalence relation (Exercise 4F of [JS87]) that forms the complex structure on R as follows;

Def: Conformal Structure

A **conformal structure** on R is an atlas $\mathcal{A}$ on **R** which is maximal: if (ψ, V) is a chart on R such that, for any $(\phi, U) \in \mathcal{A}$, if it is compatible with (ψ, V), then $(\psi, V) \in \mathcal{A}$.

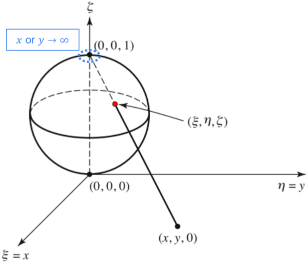
We now have all the concepts we need to define Riemann surfaces.

Def: Riemann Surfaces

A Riemann surface is a pair $(R, \mathcal{A})$, where $\mathcal{A}$ is a conformal structure on R.

Example: Riemann Sphere

A surface $S = \Sigma := \mathbb{C} \cup \{\infty\}$ is homeomorphic to the sphere S^2 with stereographic projection.



Suppose two charts: **Identity map** ($\phi_1 : z \mapsto z$ on $\mathbb{C}$) and **Reciprocal map** ($\phi_2 : z \mapsto 1/z$ with $\phi_2(\infty) = 0$ on $\Sigma \setminus \{0\}$). We have $(\phi_2 \circ \phi_1^{-1})(z) = 1/z$ which is analytic on $\phi_1(U_1 \cap U_2) = \mathbb{C} \setminus \{0\}$ and similarly for $(\phi_1 \circ \phi_2^{-1})$. Thus, this is a Riemann surface, called **Riemann sphere**.

To the next...

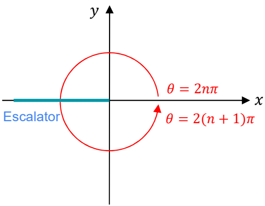
Now, we will take a look at cases where Riemann surfaces are simply connected and the classification results of those surfaces.

1. Analytic continuation
2. Covering space

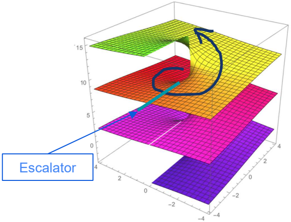
(1) Analytic continuation and Example

Analytic continuation extends the representation of a function in one region of the complex plane into another region.
Ex: For $z = re^{i\theta}$, let $f(z) = log(z) = \ln r + i(\theta + 2n\pi)$ with $n \in \mathbb{Z}$.

2D



3D



Continuation on a *simply connected region E* is **path-independent** (w/h common start/end points) by **Monodromy theorem**.
So we have a *single-valued* analytic function on E over different function elements.

(2) Covering of a Riemann surface: Example

Let $f(z) = z^n$. We have $f(0) = 0$ with multiplicity n, and as $f'(z \neq 0) \neq 0$, there are no other branch-points in $\mathbb{C}$. Similarly for $\infty \in \Sigma$, it has the multiplicity of n and thus a branch-point.

Hence, n-sheets come together at branch-points of 0 and ∞.

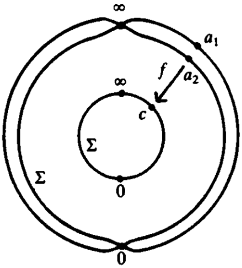


Figure: When $n = 2$.

(2) Covering of a Riemann surface: Theorem

Indeed, for connected surfaces, the covering spaces turn out to be simply connected as promised.

Thm: Regular covering space; ((10.19) of [Arm13])

Every connected surface S has a covering surface $(\tilde{S}, p)$ where p is a covering map such that $\tilde{S}$ is simply connected.

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Classify simple regions: Riemann Mapping Thm

Conformal equivalence

A **conformal equivalence** is an analytic bijection of Riemann surfaces $f : R \to S$ with any analytic inverse $f^{-1} : S \to R$.

Ex: Möbius transformations.

Riemann Mapping Theorem classifies all simply connected open subsets of $\mathbb{C}$ up to conformal equivalence.

Riemann Mapping Theorem (Corollary 16.15 of [Wil20])

If S is a simply connected open subset of $\mathbb{C}$, then either $S = \mathbb{C}$ or else $S \simeq \mathbb{D}$ (Unit disc).

Statement / Insight of Uniformisation theorem

Uniformisation Theorem (Theorem 4.17.2 of [JS87])

Every simply connected Riemann surface is conformally equivalent to one of $\Sigma = \mathbb{C} \cup \{\infty\}$ (Riemann sphere), $\mathbb{C}$ (Complex plane), or $\mathbb{D}$ (Disc).

Remark

The three Riemann surfaces are **not conformally equivalent** to each other;

- Σ is compact, so not even homeomorphic to the other two,
- $\mathbb{D}$ is bounded as a disc and by Liouville's theorem any analytic map $\mathbb{C} \to \mathbb{D}$ is constant. Thus, not conformally equivalent as not bijection.

Uniformisation and Coverings

Relationship between a Riemann surface and the covering space is indeed profound.

Corollary 16.5 of [Wil20] or Theorem 4.19.5 of [JS87].

Every Riemann surface R is conformally equivalent to a quotient $R \simeq \tilde{R}/G$ where $\tilde{R}$ is one of $\Sigma, \mathbb{C}$ or $\mathbb{D}$, and G is a (properly discontinuous) group of conformal equivalences (automorphisms) of $\tilde{R}$.

Thus, it is important to determine the automorphism groups of the Riemann surfaces to study Uniformisation theorem.

Automorphism groups

Following shows the classification of automorphism groups.

Thm 4.17.3 of [JS87]

1. $\mathrm{Aut}(\Sigma) = \mathrm{PSL}(2, \mathbb{C})$ (Möbius transformations),
2. $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b \mid a, b \in \mathbb{C}, a \neq 0\}$ (Linear transf.),
3. $\mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}(2, \mathbb{R})$ where $\mathbb{H}$ is the upper half-plane.

Remark

$\mathbb{D}$ (Disc) and $\mathbb{H}$ are conformally equivalent via a Möbius transformation $T \in \mathrm{PSL}(2, \mathbb{C}) : \mathbb{H} \to \mathbb{C}$ such that $z \mapsto \frac{z-i}{z+i}$ (Example (1) of Sec4.17 of [JS87]).

Automorphism groups

In fact, we can find a stronger result helping us to study of classifications of Riemann surfaces.

Mutual Exclusivity: Prop. 16.10 of [Wil20]

A Riemann surface R is uniformised by at most one of $\Sigma, \mathbb{C}$ and $\mathbb{D}$.

Now let us focus on the upper half-plane and the associated actions, i.e., the automorphisms group $\mathrm{PSL}(2, \mathbb{R})$, namely **Fuchsian groups**.

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What's $\mathrm{PSL}(2,\mathbb{R})$?

Intuitively, we can think of Möbius transformations in $\mathrm{PSL}(2,\mathbb{C})$ with the Euclidean distance.

Möbius transformations

It is a complex rational function of the form; $f(z) = \frac{az+b}{cz+d}$ for $z, a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

Remark

Möbius transformations can take various forms;

- *Translation*: $f(z) = b + z$ (a $=1$, c $=0$, d $=1$),
- *Rotation*: $f(z) = az$ (b $=0$, c $=0$, d $=1$),
- *Inversion*: $f(z) = 1/z$ (a $=0$, b $=1$, c $=1$, d $=0$),

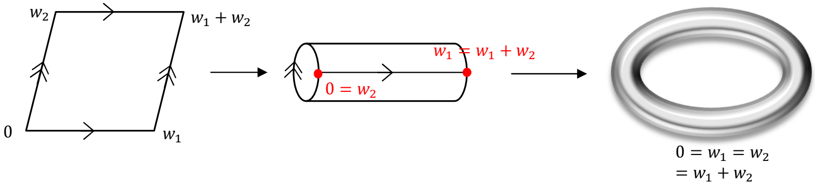
Caution: $\mathrm{PSL}(2,\mathbb{R})$ acts with the distance defined on a *Hyperbolic* space model ($\mathbb{H}$).

Preliminary to Fuchsian groups

Def: Properly discontinuous actions

Let a group G act by homomorphism on a space X. The action is said to be **properly discontinuous** if, for every compact $K \subseteq X$, the set $\{g \in G \mid g(K) \cap K \neq \emptyset\}$ is finite.

For instance, if $\Omega = \Omega(\omega_1, \omega_2)$ is a lattice in $\mathbb{C}$ then the action of Ω on $\mathbb{C}$ by **translation** is properly discontinuous. By **tessellation**, we can generate the torus (a quotient space of $\mathbb{C}/\Omega$)



Fuchsian groups

Fuchsian groups

A subgroup of $\mathrm{PSL}(2,\mathbb{R})$ that acts properly discontinuously on $\mathbb{H}$ is called a **Fuchsian group**.

Examples of Fuchsian groups

- Integer translations $T(z) = \{z + n \mid n \in \mathbb{N}\}$ form a Fuchsian group.
- More generally, as $\mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{SL}(2,\mathbb{R})$, $\mathrm{PSL}(2,\mathbb{Z})$ (**Modular group**) is a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and hence is a Fuchsian group.

Fuchsian groups and Uniformisation Theorem

Similar to the quotient space of the torus ($\mathbb{C}/\Omega$), we can define a quotient space $\mathbb{H}/\Gamma$ with $\prod : \mathbb{H} \to \mathbb{H}/\Gamma$ s.t. $z \mapsto [z]_\Gamma$ and $[z]_\Gamma$ is the Γ-orbits.
The following result shows that the quotient-spaces of Fuchsian groups are Riemann surfaces.

Thm 5.9.1 of [JS87]

$\mathbb{H}/\Gamma$ is a connected Riemann surface with $\prod : \mathbb{H} \to \mathbb{H}/\Gamma$ being holomorphic.

In summary, the problem of classifying the Riemann surfaces uniformised by $\mathbb{D}$ involves first classifying Fuchsian groups, and then understanding their properly discontinuous actions.

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Introduction

Now, we turn our attention to interpreting the critical points of a Riemann surface to develop more intuition.

Recall: Critical points

A **critical point** is a point in the domain of the function where the function is either not differentiable or the derivative is equal to zero.

Belyi maps

In the following, "a compact Riemann surface S is defined over $\bar{\mathbb{Q}}$" (Algebraic closure of $\mathbb{Q}$) means the corresponding algebraic curve is defined over $\bar{\mathbb{Q}}$.

Belyi's theorem: Thm 5.1 of [Pos+14] and Sec.2 of [Pér18]

A compact Riemann surface S can be defined over $\bar{\mathbb{Q}}$ if and only if there exists a covering $\beta : S \to \Sigma$ unramified outside of $\{0, 1, \infty\}$.

Belyi map

A rational function $\beta : S \to \Sigma$ which has at most three critical values $\{0, 1, \infty\}$ is called a **Belyi map** where S is a compact Riemann surface. (S, β) is called a *Belyi pair*.

Example: $\beta(z) = f(z) = z^n$ from the slide.11.

Definition of Dessins

In the following, imagining a bipartite graph on a surface would be useful.

For a Belyi map $\beta : S \to \Sigma$, we define the preimages;

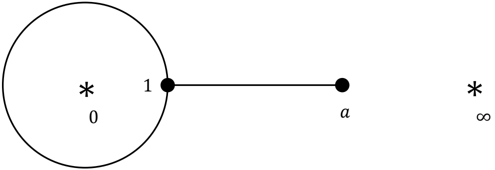
$$\begin{array}{ccc} \beta^{-1}(\{0\}) & \beta^{-1}(\{1\}) & \beta^{-1}(\{[0,1]\}) \\ \downarrow & \downarrow & \downarrow \\ \{\text{black vertices}\} & \{\text{white vertices}\} & \{\text{edges}\} \end{array}$$

The bipartite graph $\Delta_\beta = (V, E)$ with the coloured vertices and the edges (called *Hypermap*) is called a **Dessin d'Enfant**.

Intuitively, we embed the graph on the surface S in 3-dimension to visualise.

Example: Computing a Dessin

Task: Figure out the underlying Belyi map given the following figure with critical points and one unknown (Vertex a). Note that the black vertices correspond to the fibre $f^{-1}(\{0\})$.



Observe that;

- Vertex 1 has degree 3 and Vertex a has degree 1,
- Locations of Vertices $0, \infty$ imply the pole at $x = 0$ or $x = \infty$.

We can infer the form of the underlying Belyi map f as;

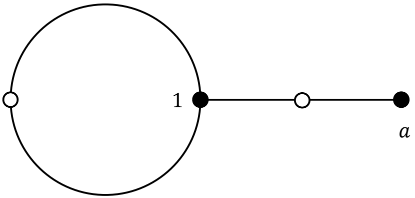
$$f(x) = K\frac{(x-1)^3(x-a)}{x} \text{ for a constant} K.$$

Example: Computing a Dessin

Define the white vertices of f on this graph by observing;

- $f - 1$ forms (a) roots on $f^{-1}(\{1\})$,
- As *bipartite*, two white vertices cannot be on the same edge!

Thus, the placements of white vertices are as follows;



And this implies the *double* roots, so we obtain;

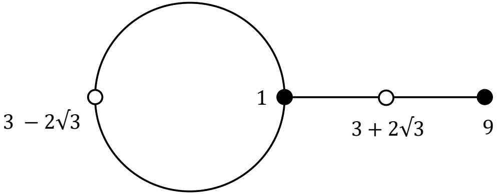
$$f(x) - 1 = K\frac{(x^2 + bx + c)^2}{x} \text{ for } K, b, c \in \mathbb{R}. \tag{1}$$

Example: Computing a Dessin

Substituting the definition of $f(x)$ into (1) and working out the equation, we get $a = 9, b = -6, c = -3$, and $K = -1/64$. Thus,

$$f(x) = -\frac{(x-1)^3(x-9)}{64x}, \quad f(x) - 1 = -\frac{(x^2-6x-3)^2}{64x}$$

The white vertices are the roots of $x^2 - 6x - 3$, that is $3 \pm 2\sqrt{3}$.



A result: Grothendieck correspondence

Let us conclude the talk by introducing one result.
The following result states our intuition from the previous example;

Grothendieck correspondence (Thm 6.7 of [Pos+14])

There is a one-to-one correspondence between Dessins and Belyi pairs.

References

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