What do financial markets say about the exchange rate?*

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First draft: April, 2022

This draft: October 10, 2025

Abstract

We develop a general framework characterizing when exchange rates connect or disconnect from fundamentals across different financial market structures. International portfolio risk sharing governs how tightly exchange rates track households' stochastic discount factors. We introduce a tractable sufficient statistic for connect: the prevalence of globally-traded risks, the component of asset risks that households in both countries can trade through available financial instruments. When globally-traded risks dominate, exchange rate puzzles persist regardless of market structure. In turn, disconnect requires neither extreme segmentation nor thin financial markets; it emerges whenever globally-traded risks are limited. Empirically, cross-country asset return correlations are weak, implying modest amount of globally-traded risks and pervasive disconnect in intermediated market structures. When correlations increase, as observed post-2008 or during high-risk episodes, exchange rates track fundamentals more closely.

JEL classification codes: E44, F31, G15.

Keywords: exchange rates, FX disconnect, market completeness, spanning

*We are grateful to the editor, Benjamin Moll, and three referees for constructive feedback on the paper. We also thank Mark Aguiar, Andy Atkeson, John Campbell, William Cassidy, Lars Hansen, Rohan Kekre, Moritz Lenel, Karen Lewis, Hanno Lustig, Ian Martin, Dmitry Mukhin, Tyler Muir, Stefan Nagel, Fabrizio Perri, Paul Whelan, and Irina Zviadadze for comments on earlier drafts and participants in the seminars and conferences sponsored by the 2024 AFA meeting, Boston University, Columbia GSB, the 2023 NBER Summer Institute, the 2023 Paul Woolley Conference at LSE, Queen Mary University London, the 2023 SED meeting, UC Davis, UCLA, University of Luxembourg, the 2023 Yiran Fan Memorial Conference at University of Chicago, the 2023 UNSW Asset Pricing Workshop, the 2023 Vienna FX Symposium, Warwick Business School, Wharton. We thank Kibeom Lee for excellent research assistance. The latest version is available here.

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Introduction

Current research on exchange rates stands at a crossroads. Historically, exchange rate movements appeared disconnected from fundamentals, at odds with classic equilibrium models (Meese and Rogoff, 1983; Obstfeld and Rogoff, 2001). As Engel (2014) put it, "it is questionable that the models allow us to explain, even after the fact, the movements in major currency rates." The literature answered this challenge by offering a new view on the role of financial markets in exchange rate determination. Gabaix and Maggiori (2015) showed that segmented financial markets can drive exchange rate fluctuations independently of real fundamentals. Itskhoki and Mukhin (2021) produced a quantitative model with segmentation that successfully replicates the disconnect. However, recent studies suggest that a more nuanced view of the role of fundamentals is necessary. For example, Engel and Wu (2024) find that inflation and interest rate differentials, along with global risk measures, explain a significant share of dollar movements in recent years. Reconciling these perspectives is challenging: Lustig and Verdelhan (2019) show that a mild departure from the extreme segmentation of models of disconnect quickly restores the predictions of classic frameworks.

In this paper we bridge the two opposing views of exchange rate determination — connect to fundamentals versus disconnect. We argue that a central feature controlling the extent of disconnect is how much portfolio risk sharing ties the exchange rate to the representative households' stochastic discount factors (SDFs). We provide a general characterization of this relation and its implications across financial market structures — namely, which assets are traded and who trades them. Our general approach shows that for a wide variety of market structures, the extent of disconnect depends on properties of asset returns, without requiring extreme segmentation. We confirm this prediction empirically: when asset returns become more similar across countries as in recent years or during crises, the disconnect weakens.

In classic models, portfolio risk sharing implies that exchange rate depreciation is equal to the relative SDF between the two countries (e.g., Backus and Smith, 1993; Backus, Foresi, and Telmer, 2001). This result is the finance version of the fundamental view. Underlying this relation is the assumption that financial markets are complete and integrated. As many papers show, this view is difficult to reconcile with data, that is, to write a model that generates stochastic processes for the depreciation rate and SDFs that are consistent with this relation, and at the same time match asset prices and exchange rate empirical moments — the volatility (Brandt, Cochrane, and Santa-Clara, 2006), cyclicality (Backus and Smith, 1993), and risk premium (Fama, 1984) puzzles.

Models of disconnect make the polar opposite assumption that financial markets are extremely segmented. Households can only access the risk-free assets of their respective countries, with intermediaries trading across borders. There is no portfolio risk sharing whatsoever. Hence, the exchange rate is not constrained by SDFs. This lack of restrictions allows shocks that do not affect SDFs to impact the exchange rate, resulting in disconnect. For example, such models often introduce disconnected shocks using "noisy" demand for currency.

We characterize what happens in more realistic scenarios in which markets are neither complete and integrated nor extremely segmented. Under these conditions, SDFs typically explain only part of the variation in exchange rates, allowing for fluctuations disconnected from SDFs. Specifically, the relative SDF and the exchange rate move together only in response to shocks that households can trade with each other. This simple characterization is the key result of the paper, extending the complete market relation to a wide class of economies.

To identify these "globally-traded" shocks, it is sufficient to know the financial market structure and the covariance of asset returns. For example, in an economy in which households in both countries trade claims on US inflation (say, using inflation swaps), the response of the exchange rate to an inflation shock must equal that of relative SDFs. Conversely, if households cannot trade claims on output, the exchange rate is not constrained by output shocks. In the extremes, all shocks are globally-traded under complete and integrated markets, leaving no room for disconnect, while none are globally-traded in settings with extreme segmentation.

The benefit of these general results is to characterize how the exchange rate behaves in market structures beyond the extremes without having to completely solve such models. In this regard, we highlight a useful metric for questions of connect and disconnect: the proportion of exchange-rate fluctuations explained by globally-traded shocks. We use this metric to consider three issues.

First, we show that the volatility and cyclicality puzzles of classic models are driven by a large globally-traded component that strongly connects the exchange rate to SDFs. Specifically, given a pair of SDFs, if the puzzles arise under complete markets, they persist in any market structure with a dominant globally-traded component.

One way this situation occurs is when risk-free bonds of both countries are frictionlessly traded by both households even if markets have arbitrary frictions elsewhere. Lustig and Verdelhan (2019) show that the currency puzzles already arise in this minimal setting. This seemingly suggests that the puzzles are unavoidable in any market structure away from extreme segmentation. Our framework shows otherwise: the puzzles in that case stem from a special property of risk-free assets. Risk-free assets permit trading exchange rate risk through the carry trade, which implies that the equilibrium exchange rate is itself a globally-traded risk.

The other route to a large globally-traded component is not as minimal. Markets must allow households to trade risks that nearly span their SDFs. In economies with many shocks, this requires frictionless trading in many assets, which gets close to complete markets.

Second, we show that extreme segmentation is not necessary for disconnect. Our framework reveals that the critical feature of existing models is not segmentation per se, but that it drives the globally-traded component to be small (in fact, zero in the models studied in the literature). Markets can be substantially enriched relative to these models while keeping this component sufficiently small. Intermediaries may be fully sophisticated and trade an arbitrarily large set of assets. Households may access broad local menus of assets, which may be substantially correlated across countries, so long as they cannot be combined to form a rich set of globally-traded risks. Despite featuring many trading opportunities, these market structures still allow the possibility of shocks to the exchange rate that are not present in SDFs.

Third, we show that taking the perspective of these richer structures leads to new predictions relating properties of asset returns with the extent of disconnect. Assume that households trade local stocks and bonds of various maturities. If these two sets of assets exhibit more globally-traded shocks — intuitively, when returns across countries exhibit stronger comovement — disconnect should be weaker. We find that, unconditionally, the amount of globally-traded shocks is small and the corresponding globally-traded proportion of the exchange rate is no more than 25%. This evidence is consistent with the widespread evidence of disconnect (Meese and Rogoff, 1983). Looking across time further confirms the role of financial markets. After 2008, globally-traded shocks become more substantial, and the globally-traded proportion increases, in line with the increased connect documented, e.g., in Engel and Wu (2024). We observe a similar pattern during periods when the VIX is above its median, in line with, e.g., Stavrakeva and Tang (2024).

All these results make the case that portfolio risk sharing is a central determinant of the extent of disconnect, both in theory and in the data. We emphasize, however, that risk sharing on its own is not a complete characterization of disconnect; the nature of shocks also matters. A complete market model could exhibit a disconnected exchange rate if the SDFs themselves have no connection with macroeconomic quantities. Or, a model with extreme segmentation can lead to a connected exchange rate if all shocks in the economy are macroeconomic in nature.

Beyond questions about disconnect, our analysis provides the full menu of constraints on the exchange rate imposed by portfolio risk sharing across market structures. Our findings highlight that many issues with existing models of the exchange rate come from taking stark stylized views of market structure. Departing from these limiting cases and incorporating more realistic features of how financial markets are organized is a promising avenue for understanding the exchange rate.

Contribution to the literature While the literature has explored a number of market structures, general results have been elusive as each case seemingly requires a separate analysis. We are able to make progress by focusing on restrictions on the behavior of the exchange rate, as opposed to fully solving the equilibrium, following the tradition of Hansen and Jagannathan (1991). The distinct feature of the international setting is the preponderant role of the financial market structure in determining these constraints. In the international context, Backus, Foresi, and Telmer (2001) first introduced the notion of a wedge between the relative SDF and the exchange rate in settings departing from complete markets. More recently, Lustig and Verdelhan (2019) characterize this wedge and its implications, focusing exclusively on Euler equations for risk-free assets. We provide a complete characterization that applies to any combination of Euler equations, including cases with Euler equations for risky assets and potential absence of those for risk-free assets (e.g., due to convenience yields on safe assets).

A large body of work maintains the complete and integrated markets assumption of Backus and Smith (1993) and varies assumptions about preferences and aggregate dynamics to obtain a realistic exchange rate. Some prominent examples of this line of work include Verdelhan (2010, habits), Colacito and Croce (2011, long-run risk), and Farhi and Gabaix (2016, disasters), among many others. Hassan, Mertens, and Wang (2024) identify a tension between calibrated single-country economies and the exchange rate risk premium under complete markets.

At the other end of the spectrum, Jeanne and Rose (2002), Gabaix and Maggiori (2015), Itskhoki and Mukhin (2021, 2025), and Kekre and Lenel (2024) take a strongly segmented view of markets with trade in short-term bonds only. Gourinchas, Ray, and Vayanos (2022) and Greenwood, Hanson, Stein, and Sunderam (2022) maintain segmentation but add the term structure. A number of papers consider the exchange rate implications of specific market structures with intermediate amount of frictions. Alvarez, Atkeson, and Kehoe (2002, 2007, 2009), Kocherlakota and Pistaferri (2007, 2008), Zhang (2021) and Marin and Singh (2023) emphasize heterogeneity across households in access to financial markets. Hau and Rey (2006) and Camanho, Hau, and Rey (2022) focus on capital flows in equity markets. Corsetti, Dedola, and Leduc (2008), Engel and Matsumoto (2009), Benigno and Thoenissen (2008) and Lewis and Liu (2022) focus on incomplete markets. Jiang, Krishnamurthy, and Lustig (2021, 2023a,b), Jiang, Krishnamurthy, Lustig, and Sun (2024), and Kekre and Lenel (2023) study the implications of convenience yields of safe assets. We provide a general characterization of portfolio risk-sharing across those approaches without making assumptions about the details of the macroeconomic environment. Doing so allows us to identify the economic forces controlling the extent of disconnect.

Our interest in intermediate market structures is motivated by evidence of weak connection of both asset prices and macroeconomic quantities across countries. Bansal (1997) and Backus, Foresi, and Telmer (2001) find a weak relation between the relative behavior of the yield curve across countries and the exchange rate. Chernov and Creal (2023) highlight that this evidence can be consistent with the absence of arbitrage opportunities.[1] Hau and Rey (2004, 2006) find a stronger, yet incomplete, connection of the exchange rate with cross-country equity returns. Our analysis shows that these facts *per se* do not constitute a test of market structure. However, they are informative about the restrictions that specific market structures impose on the exchange rate.

On the real side, Backus, Kehoe, and Kydland (1992), Backus and Smith (1993) and Kollmann (1995) highlight the low correlation of consumption across countries that cannot be explained by variation in the real exchange rate in standard complete market models. Furthermore, the literature documents a pervasive home bias in portfolios (see e.g. French and Poterba, 1991; Lewis, 1999), providing further evidence suggestive of imperfect risk sharing. Heathcote and Perri (2014) provide an overview of the literature on the efficiency of international risk sharing and Aguiar, Itskhoki, and Mukhin (2025) study the macro-allocative implications implied by the failure of the Backus-Smith condition.

Lastly, we connect to the large literature exploring the sources of shocks to exchange rates. For example, Engel and West (2005) and Chahrour, Cormun, De Leo, Guerrón-Quintana, and Valchev (2023) emphasize news shocks about future macro fundamentals, Gourinchas and Rey (2007) and Pavlova and Rigobon (2007) focus on trade shocks and imbalances, Chen and Rogoff (2003) and Ayres, Hevia, and Nicolini (2020) study commodity shocks, Stavrakeva and Tang (2019), Eichenbaum, Johannsen, and Rebelo (2021) and Fukui, Nakamura, and Steinsson (2023) emphasize monetary shocks and regimes, Adrian, Etula, and Shin (2010) and Lilley, Maggiori, Neiman, and Schreger (2022) focus on financial shocks, and Itskhoki and Mukhin (2024) provide a review. We demonstrate that to understand how a specific source of shocks affect the exchange rate, it is crucial to know whether financial markets allow this shock to be traded globally.

[1] Maurer and Tran (2021) and Sandulescu, Trojani, and Vedolin (2021) consider the problem of recovering SDFs compatible with asset returns expressed in two different currencies.

1 A motivating example

We illustrate how different structures of the financial market affect properties of the exchange rate in a simple risk-sharing world with two countries. Our focus is two-fold. First, we highlight when and to what extent financial flows *disconnected* from macroeconomic dynamics drive the exchange rate. Second, we show the extent of *connect* between macroeconomic quantities and the exchange rate that is due to portfolio risk sharing. While simplistic in many ways, the economies we consider examplify properties that must hold across all models of financial markets, as we demonstrate in the main analysis of the paper.

1.1 Setting

Consider a discrete-time endowment economy in which two countries (*Home* and *Foreign*) exchange goods and financial claims. We assume that Foreign is much larger than Home — e.g., it represents the rest of the world (ROW) — such that Foreign behaves as a representative-agent economy and Home as a small open economy. We denote all quantities for Foreign with an asterisk $*$. Appendix A provides full details of the setup and derivations.

Goods markets There are two goods: a tradable good, denoted by T, and a home non-tradable good, denoted by N. Endowments of the goods — Y_{Nt} and Y_{Tt} at home, and Y^*_{Tt} in ROW — follow exogenous AR(1) processes in logs with a common persistence parameter ρ. The representative agents of each country have logarithmic utility and discount time at rate δ. Specifically, Home's period utility over consumption of the two goods is given by $\log C_{Tt} + \log C_{Nt}$ and Foreign's period utility is $\log C^*_{Tt}$. The large size of Foreign implies that $C^*_{Tt} = Y^*_{Tt}$. For Home, $C_{Nt} = Y_{Nt}$ by non-tradable market clearing, while C_{Tt} can deviate from Y_{Tt} when supported by international risk sharing.

The nominal exchange rate S_t is in units of home currency for one unit of foreign currency. Denote the price of goods in each country's currency by P_{Tt}, P_{Nt} and P^*_{Tt}. The market for tradable goods operates without frictions hence the law of one price holds, $P_{Tt} = P^*_{Tt}S_t$. Monetary policy ensures price stability abroad, $P^*_{Tt} = 1$, and stabilizes the price of non-tradables at home, $P_{Nt} = 1$. This implies that the nominal exchange rate coincides with the price of tradables in home currency, $P_{Tt} = S_t$.[2]

Under these assumptions, the nominal stochastic discount factors (SDFs) of the representative agents are $M_{t+1} = e^{-\delta} Y_{Nt}/Y_{Nt+1}$ at home and $M^*_{t+1} = e^{-\delta} Y^*_{Tt}/Y^*_{Tt+1}$ abroad. The fact that the SDFs are entirely shaped by macroeconomic conditions (endowments) remains unchanged as we vary the structure of international asset markets. While fixed SDFs simplify this example, our results in Section 3 treat SDFs as general and fully endogenous. The extent to which variation in the relative SDF pins down the equilibrium exchange rate is our metric of how much portfolio risk sharing leads to connect.

Financial Markets There are multiple assets in the economy, all in zero net supply; implicitly any asset in positive supply is already embedded in the endowment processes. For each representative agent, Euler equations hold with their SDF for asset returns available to them. In addition to the representative agents, financial markets involve two other market participants located abroad: noise traders and an international intermediary. While these agents' decisions do not necessarily line up with household preferences, they are owned by the Foreign representative agent and transfer the proceeds of their trades at the end of each period.

To simplify the analysis, we assume that noise traders engage exclusively in a currency carry trade using home and foreign risk-free bonds. The size of their position Ψ_t follows an exogenous AR(1) process with persistence ρ, where $\Psi_t > 0$ implies that noise traders are long this value in foreign currency and short the same value in home currency. Fluctuations in this portfolio position are the source of "non-fundamental" shocks in the economy. We are particularly interested in when and how they affect the exchange rate, i.e., lead to disconnect. In contrast, the intermediary can trade every available asset, and chooses its portfolio optimally given a constant absolute risk aversion (CARA) utility with risk-aversion coefficient γ. The intermediary ensures that asset returns remain well-behaved even when representative agents are segmented from accessing certain asset markets.

[2] See Itskhoki and Mukhin (2023) for a discussion of the optimal monetary and exchange rate policy in a similar environment.

Equilibrium Conditions Equilibrium in the model is determined by utility maximization of all agents subject to their budget constraints and market clearing for tradable goods and each asset. For brevity, our description in this section focuses only on the innovations in the linearized system which we denote with lowercase letters. The economy features four primitive sources of variation: macro-fundamental endowments with innovations y_{Nt}, y_{Tt} and y^*_{Tt} and financial noise-trader shocks with innovation ψ_t, with arbitrary correlation structure. The innovations of representative-agent SDFs are $m_t = -y_{Nt}$ and $m^*_t = -y^*_{Tt}$, irrespective of the value of ψ_t. Appendix A derives formally the equilibrium conditions and their linearized counterparts, as well as offers a complete equilibrium characterization in each case of the model.[3]

1.2 Exchange rate under alternative market structures

We characterize equilibrium innovations of the exchange rate under alternative market structures. We start with three standard economies, and then proceed to analyze *intermediated* market structures with alternative sets of assets available to the households.

Benchmark solutions Consider a conventional environment without intermediaries and noise traders. Under complete markets, every agent has access to a full set of state-contingent contracts (Arrow-Debreu securities) that they can mutually trade. In this case, the exchange rate is the ratio of home and foreign nominal SDFs, $S_{t+1}/S_t = M^*_{t+1}/M_{t+1}$. Under financial autarky, the home household is fully segmented from risk sharing in the world asset market. In this case, the exchange rate is determined in the goods market by home endowments, $S_t = Y_{Nt}/Y_{Tt}$. In terms of innovations and dropping the time index through the end of this section, these two benchmark cases result in:

$$s_{\mathbf{C}} = m^* - m = y_N - y^*_T \qquad \text{and} \qquad s_{\mathbf{A}} = y_N - y_T. \tag{1}$$

Both cases tightly link the equilibrium exchange rate to macroeconomic fundamentals, due to portfolio risk sharing in the former case and due to trade balance in the latter.

[3]The portfolio choice decisions are approximated to the second order, in the spirit of the approach in Devereux and Sutherland (2011) and Tille and Van Wincoop (2010), which build on Samuelson (1970).

Finally, in an integrated, but incomplete, financial market in which the home and foreign representative agents can trade a foreign-currency risk-free bond with each other, the equilibrium exchange rate is given by:

$$s = s_{\mathbf{0}}, \qquad \text{where} \qquad s_{\mathbf{0}} = \eta_0 s_{\mathbf{A}} + (1-\eta_0) s_{\mathbf{C}} \tag{2}$$

and $\eta_0 = \frac{1-\beta}{1-\beta\rho}$.[4] Although the agents do not trade risky assets, the risk-free asset allows for a certain degree of risk sharing via consumption-savings decisions of the home agent. When shocks are permanent ($\rho = 1$), a risk-free bond offers no risk-sharing benefit, and $s = s_{\mathbf{A}}$. In contrast, when shocks are transitory ($\rho = 0$) and as $\beta \to 1$, a risk-free bond allows to perfectly smooth consumption risk, and $s = s_{\mathbf{C}}$.

Intermediated financial markets We now turn to economies with intermediated asset markets in which the home and foreign representative agents cannot trade directly, but can participate in international risk sharing via their trades with an intermediary, who in turn also trades with the noise traders. The presence of noise traders does not always matter for the exchange rate. For example, when both households can trade Arrow-Debreu securities with the intermediary, the equilibrium exchange rate is still given by $s = s_{\mathbf{C}}$.

Segmented household trading. We start with the benchmark case in the segmented markets literature: households can only trade their respective local-currency risk-free bonds with the intermediary (Gabaix and Maggiori, 2015; Itskhoki and Mukhin, 2021). Hence, the intermediary must absorb all of the outstanding exchange rate risk. The equilibrium exchange rate innovation in this case is given by:

$$s = s_{\boldsymbol{\kappa}} + \nu_\kappa \psi, \qquad \text{where} \qquad s_{\boldsymbol{\kappa}} = \eta_\kappa s_{\mathbf{A}} + (1-\eta_\kappa) s_{\mathbf{C}}, \tag{3}$$

where $\eta_\kappa \in [0,1]$ and $\nu_\kappa \geq 0$. This showcases that the equilibrium exchange rate features both a component of the fundamental solution $s_{\boldsymbol{\kappa}}$, similar to $s_{\mathbf{0}}$ in equation (2), but also a new non-fundamental component $\nu_\kappa \psi$ which is disconnected from macroeconomic shocks and

[4] In this section, we display only equilibrium innovations s, while Appendix A provides the full solution. Specifically, in this case, the exchange rate follows an ARIMA(1,1,1) process with a predictable component that depends on net foreign assets as a state variable, which themselves follow an endogenous AR(1) process in first differences.

representative-agent SDFs. This latter component can be arbitrarily volatile and account for the bulk of exchange rate fluctuations.

The loading ν_κ on non-fundamental shocks and the coefficient η_κ are determined as a fixed point parametrized by $\kappa \equiv \gamma \text{var}(s) \geq 0$. Exchange rate volatility affects the intermediary's willingness to absorb noise trader demand. In turn, the premium charged by the intermediary to do so affects exchange rate dynamics and, in particular, its volatility. As long as there is fundamental volatility, i.e., $\text{var}(s_{\mathbf{0}}) > 0$, all equilibria also feature non-fundamental volatility, $\kappa > 0$ and $\nu_\kappa > 0$.[5]

This case yields exchange rate fluctuations disconnected from macro fundamentals. However, it does not allow for any portfolio choice by households beyond a simple consumption-savings decision. Thus, a natural question is whether the exchange rate disconnect permitted by (3) is a knife-edge property of extreme segmentation, or alternatively a robust feature of intermediated markets even when they support a non-trivial amount of risk sharing. To address this question we consider economies where agents can trade more assets.

Common trading of risk-free bonds. Assume that each household can trade both home and foreign risk-free bonds with the intermediary. Despite the presence of noise trader shocks, the unique equilibrium features the exchange rate entirely pinned by fundamentals:

$$s = \phi \, s_{\mathbf{0}}, \tag{4}$$

where $s_{\mathbf{0}}$ is the same as in equation (2) and $\phi \equiv \text{cov}(s_{\mathbf{C}}, s_{\mathbf{0}})/\text{var}(s_{\mathbf{0}})$.[6] This result is a manifestation of the general conclusion of Lustig and Verdelhan (2019) that the exchange rate is tightly restricted in settings where every agent trades both currency bonds.

This result appears to rule out disconnected exchange rate fluctuations in economies with even minimal portfolio choice and risk sharing. As it turns out, this case is very special and

[5]Formally, $\nu_\kappa \equiv \frac{\kappa}{1-\rho\lambda_\kappa} \geq 0$ and $\eta_\kappa = \frac{1-\lambda_\kappa}{1-\rho\lambda_\kappa} \in [0,1]$, where $\lambda_\kappa \leq \beta$ is the inverse of the unique root of the dynamic equilibrium system outside the unit circle. In the limit case when $\kappa \to 0$, $\lambda_\kappa \to \beta$, $\nu_\kappa \to 0$ and $\eta_\kappa \to \eta_0 = \frac{1-\beta}{1-\beta\rho}$, and hence $s = s_{\mathbf{0}}$ coincides with the solution (2) for the single-bond integrated economy without intermediation. See Itskhoki and Mukhin (2025) for the discussion of multiplicity of exchange rate equilibria and their properties in frictionally intermediated currency markets.

[6]This value of ϕ ensures that $\text{cov}(s, m^* - m - s) = 0$, which means that the households have attained the best risk sharing that is possible with two bonds. In our fully specified model, this value of ϕ maps into a portfolio position of the home household that is long foreign-currency and short home-currency bonds.

does not generalize beyond trading risk-free bonds to other types of portfolio choice. We demonstrate this in an intermediated economy where each representative agent can trade a common risky asset, in addition to the risk-free bond of their own country.

Common trading of a risky asset. Assume there is a random variable a, say global productivity, and all investors can trade a forward contract on its realization next period paid out in Foreign currency. This is an example of what our general theory calls a globally-traded risk. Any variable x can be decomposed using least-squares projection as $x = \hat{x} + \check{x}$ with $\hat{x} = \beta_x a$, where the loading β_x is such that the residual $\check{x} \perp a$.

In particular, the exchange rate can be represented as $s = \hat{s} + \check{s}$, and we further show that in equilibrium these components take the form:

$$\check{s} = \check{s}_{\boldsymbol{\kappa}} + \nu_\kappa \check{\psi}, \tag{5}$$

$$\hat{s} = \beta_s a = \hat{m}^* - \hat{m}, \tag{6}$$

where $\check{s}_{\boldsymbol{\kappa}} \equiv \eta_\kappa \check{s}_{\mathbf{A}} + (1-\eta_\kappa)\check{s}_{\mathbf{C}}$. The two components of the exchange rate, $\check{s}$ and $\hat{s}$, reflect distinct mechanisms. The economic structure of the residual $\check{s}$ is exactly identical to the simple intermediated economy in equation (3), with suitably defined $\kappa \equiv \gamma \mathrm{var}(\check{s})$.[7] This implies that the exchange rate generically features disconnect. Portfolio risk sharing determines the component $\hat{s}$ by projections of the SDFs onto the globally-traded shock a, that is $\hat{m}^* - \hat{m}$. This term reflects the exchange rate connect due to portfolio risk sharing, generalizing the complete market case of equation (1).

The impact of the degree of risk sharing on the disconnect is continuous. As the globally-traded shock a spans more of the variation in noise trader demand, its residual component $\check{\psi}$ shrinks accordingly, resulting in an increasingly *connected* exchange rate. This stands in stark contrast with the case of trade in two bonds which immediately shuts down the disconnect.[8] Empirically, this implies that changing dynamics of asset returns control the degree of disconnect. We measure this relation in a multi-asset setting in Section 5.

[7] Note that in the simple intermediated economy $s = \check{s}$ and $\hat{s} = 0$, as no part of the exchange rate risk can be traded, and hence $\kappa \equiv \gamma \mathrm{var}(\check{s})$ is a general definition that covers all cases. Furthermore, $\mathrm{var}(\check{s}_{\mathbf{0}}) > 0$ is necessary and sufficient to ensure that $\kappa > 0$ and $\nu_\kappa \equiv \frac{\kappa}{1-\rho\lambda_\kappa} > 0$ in every equilibrium.

[8] As we explain below, the case with two bonds can also be viewed as a special case of equation (6), in which the globally-traded risk is the exchange rate itself, i.e., $a = s$. This results in a unique equilibrium solution that features $\kappa = \gamma \mathrm{var}(\check{s}) = 0$ and $\nu_\kappa = 0$ such that $\check{s} = 0$ and $s = \hat{s} = a$.

Our solutions of the cases of the model in equations (1) through (6) highlight how variation in financial market structure leads to a spectrum of outcomes in terms of connect and disconnect, ranging from complete markets to fully segmented economies. The origin of these possibilities is, perhaps, still mysterious at this point. Our theory below resolves this by providing a set of general principles for exchange rate determination in financial markets.

2 General Framework

This section introduces our representation for a wide range of international financial market structures and the corresponding equilibrium conditions. This allows us to prove general results on the joint restrictions imposed by portfolio risk sharing on the behavior of three endogenous objects: local SDFs, asset returns, and the exchange rate. This level of generality is possible because we can isolate the consequences of portfolio risk sharing without solving, or even specifying, the full equilibrium environment.

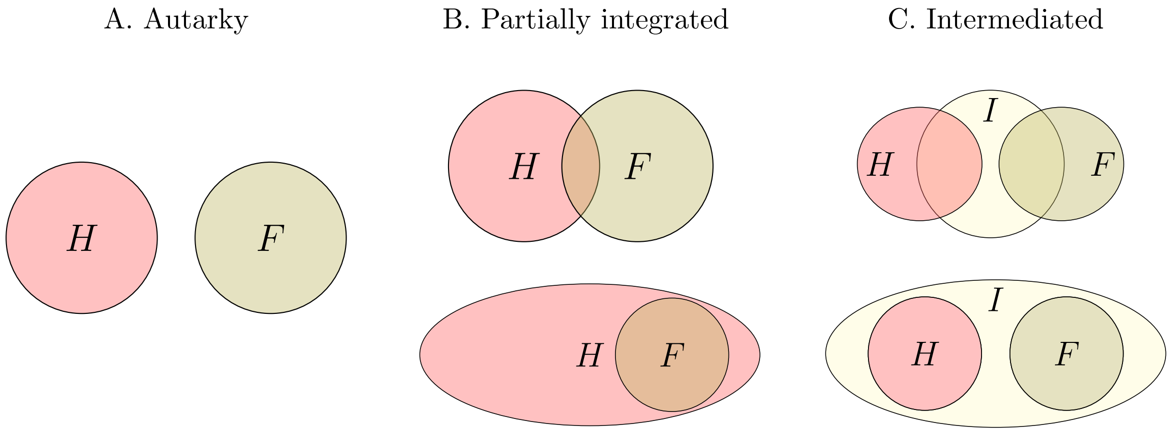
2.1 Market structure

We consider settings with two representative households — home and foreign — each trading a set of assets, H and F, respectively. These sets can contain subsets of local assets and foreign assets converted to local currency that they access frictionlessly. Figure 1 demonstrates some examples. For instance, in autarky H contains domestic stocks and bonds, while F contains the foreign ones, and there is no intermediary that can trade across these sets of assets. Markets are fully integrated when H and F contain the same set of assets, while in partially integrated markets H and F only have a subset of assets in common.[9] Markets are complete when H and F contain (or span) the full set of Arrow-Debreu securities.

Similarly, we consider a set I of assets traded frictionlessly in international markets. Assets can be included in this set for two reasons. First, it could be that home and foreign households trade some assets in common, as in partially or fully integrated cases. Then,

[9] An example of a fully integrated market structure is when H and F both contain a domestic sovereign bond and a foreign equity index. As another example, adding a foreign sovereign bond to F, but not to H makes the market structure partially integrated. We discuss more examples in Appendix C.2.

Figure 1: Examples of Market Structures



The figure illustrates different market structures. H and F are the sets of assets invested in by the home and foreign household. Panel A corresponds to financial autarky. Panel B corresponds to partial integration, symmetric or asymmetric. Panel C corresponds to an intermediated market, with an intermediary I trading some or all assets.

either the home or foreign household can be considered as an international intermediary, with $I = H$ or $I = F$, respectively. In Section 1, such an integrated but incomplete financial market arises when the home household trades risk-free bonds of the two countries, while the foreign household only trades its own risk-free bond. Second, it could be that financial intermediaries (one or many) trade frictionlessly across borders, even if households do not, as in the examples of Figure 1C or the economy with segmented household trading in Section 1. In this case, I contains the assets from H and F that intermediaries can trade frictionlessly. Regardless of its underpinnings, the set I captures the assets that can be used for portfolio risk sharing.

Our main result is that, in this large family of market structures, restrictions on the exchange rate coming from portfolio risk sharing between households are determined by the properties of returns in $H \cap I$ expressed in domestic currency and returns in $F \cap I$ expressed in foreign currency. To continue our examples, if markets are partially integrated and $I = H$, then $H \cap I = H$ are the assets traded by the domestic household and $F \cap I = F \cap H$ are the assets traded by both households. In intermediated markets, $H \cap I$ is the set of assets traded both by the domestic household and the intermediaries; ditto for $F \cap I$.

The base assets in the set $H \cap I$ have log returns $\boldsymbol{r}_{t+1} = (r_{1,t+1}, \ldots, r_{N,t+1})$ in home currency. We assume that this collection includes a risk-free asset with return r_{ft} known at time t. This assumption simplifies notation, but is not necessary. Appendix D proves our main results (Propositions 1 and 2) in the absence of risk-free assets. We consider all feasible portfolios that can be constructed from these assets. The corresponding set of log returns is:

$$\boldsymbol{r}_{p,t+1} = \left\{ r_{p,t+1} = \log\left(\boldsymbol{w}_t' \exp(\boldsymbol{r}_{t+1})\right) \,\middle|\, \exists \boldsymbol{w}_t \in \mathbb{R}^N : \boldsymbol{w}_t'\boldsymbol{\iota} = 1 \right\}.$$

Furthermore, for our analysis in the main text, we assume that asset returns are log-normal, that is $\boldsymbol{r}_{t+1}$ is multivariate normal with mean $\boldsymbol{\mu}_t$ and variance-covariance matrix $\boldsymbol{\Sigma}_t$. Similarly, the returns of base assets in $F \cap I$ are $\boldsymbol{r}^*_{t+1}$ in foreign currency, log-normal of size N^*, and contain a foreign-currency risk-free rate r^*_{ft}. The corresponding set of portfolio returns is $\boldsymbol{r}^*_{p,t+1}$.

Throughout the paper, we distinguish innovations from conditional expectations. We use tilde to denote the innovation (or shock) to any variable x, that is, $\widetilde{x}_{t+1} \equiv x_{t+1} - E_t x_{t+1}$ and hence $var_t(\widetilde{x}_{t+1}) = var_t(x_{t+1})$. Further, we use the Campbell and Viceira (2002) approximation for log portfolio excess returns with the relevant derivations described in Appendix B. In Appendix E, we show how our results generalize to an environment with an arbitrary distribution of returns without portfolio approximation.

2.2 Pricing Assumptions

We introduce two sets of assumptions, which enable us to characterize how risk sharing between households constrains the behavior of the exchange rate.

Local Euler equations We specify valuation mechanisms for each of the households with SDFs m at home and m^* abroad. These SDFs value assets as follows.

Assumption 1. *The domestic (log) stochastic discount factor m_{t+1} prices all assets in H in domestic currency. In particular, it satisfies the Euler equation:*

$$\forall r_{t+1} \in \boldsymbol{r}_{p,t+1} : \quad E_t \exp(m_{t+1} + r_{t+1}) = 1. \tag{7}$$

Similarly, the foreign log SDF m^*_{t+1} prices all assets in F in foreign currency, and

$$\forall r^*_{t+1} \in \boldsymbol{r}^*_{p,t+1} : \quad E_t \exp(m^*_{t+1} + r^*_{t+1}) = 1. \tag{8}$$

These Euler equations imply that the expectation of an excess return is related to its covariance with the respective SDF. In particular, under the log-normal assumption, equations (7) and (8) become:

$$\forall r_{t+1} \in \boldsymbol{r}_{p,t+1}: \qquad E_t\, r_{t+1} - r_{ft} + \frac{1}{2} var_t(r_{t+1}) = -cov_t(m_{t+1}, r_{t+1}), \tag{9}$$

$$\forall r^*_{t+1} \in \boldsymbol{r}^*_{p,t+1}: \qquad E_t\, r^*_{t+1} - r^*_{ft} + \frac{1}{2} var_t(r^*_{t+1}) = -cov_t(m^*_{t+1}, r^*_{t+1}). \tag{10}$$

Recall that $\boldsymbol{r}_{p,t+1}$ and $\boldsymbol{r}^*_{p,t+1}$ are the sets of feasible portfolio returns constructed from assets in $H \cap I$ and $F \cap I$, respectively, which is all we need for the derivation of our formal results.

The Euler equations (7) and (8), or respectively their log-normal versions (9) and (10), act as the point of contact of financial markets with the respective local economies. These conditions hold irrespective of the remainder of the economic environment and connect local asset returns with local SDFs. Our results apply to any admissible pair of home and foreign SDFs that are consistent with equilibrium and observed asset returns. In particular, we are interested in situations when these SDFs are equal to IMRSs of representative households and, thus, reflect local aggregate macroeconomic conditions.[10] Appendix G discusses how our results apply to SDFs constructed using asset returns only without connection to the broader economic environment.

Assumption 1 clarifies what it takes for an asset to be included in H or F. Households have to frictionlessly trade this asset so that the corresponding Euler equation holds. Equilibrium in the financial market may involve borrowing or short-sale constraints, infrequent portfolio adjustment, or convenience yield on certain assets.[11] In all these cases, the Euler equation does not always hold with equality, i.e. it features a wedge. Therefore, such assets are not included in the sets H and F (for a given time period t).

Excluding assets does not imply that households do not trade them, nor that they are irrelevant for the exchange rate. It simply implies that portfolio risk-sharing conditions do not apply to such assets. For example, if investors value the liquidity and safety of holding

[10]For example, with CRRA utility, a representative household's IMRS is $m_{t+1} = -\rho - \Gamma \Delta c_{t+1} - \pi_{t+1}$ where ρ is the preference discount rate, Γ is the coefficient of relative risk aversion, c_t is log aggregate domestic consumption, and π_t is CPI inflation.

[11]Some constraints are not readily observable in the data. For instance, even though currencies may seem easily tradable, households seldom engage with them in practice. This observation suggests the existence of underlying frictions in such investments, perhaps stemming from a lack of sophistication or home bias.

Treasuries (e.g., Krishnamurthy and Vissing-Jorgensen, 2012), the standard Euler equation does not hold even though these assets are widely traded.

International arbitrage So far, none of our analysis involved the exchange rate as we described the constraints imposed on equilibrium by local asset pricing. In order to characterize the interaction between the exchange rate and local finance, one has to take a stand on how international financial markets facilitate portfolio risk-sharing. We assume that there are no arbitrage opportunities for assets in I.

Formally, the set I of international returns that are frictionlessly traded combines the domestic and foreign subsets of returns converted to the domestic currency. Our conclusions are unchanged if we focus on international arbitrage in foreign currency. Following our notation, these international portfolios are generated by the base assets $\boldsymbol{r}^I_{t+1} = (\boldsymbol{r}_{t+1}, \boldsymbol{r}^*_{t+1} + \Delta s_{t+1})$, where Δs_{t+1} is the log home currency depreciation rate. We denote by $\boldsymbol{r}^I_{p,t+1}$ the set of international portfolios generated by these base assets.

Assumption 2. *There are no arbitrage opportunities for assets in I. In the log-normal setting, this assumption is equivalent to:*

$$\forall r_{p,t+1} \in \boldsymbol{r}^I_{p,t+1} : \quad var_t(r_{p,t+1}) = 0 \quad \Rightarrow \quad E_t\, r_{p,t+1} = r_{ft}. \tag{11}$$

In words, any portfolio that has no risk must earn the risk-free rate of return.[12] This condition is equivalent to the existence of an international SDF m^I. For example, this SDF could be the discount factor of one of the households (in markets with partial integration) or of an international arbitrageur (in intermediated markets). In the latter case, m^I need not be related to households' SDFs: intermediary decisions often differ from the motives of their investors (see He and Krishnamurthy, 2018 and Haddad and Muir, 2025).[13] Unlike for households, we do not assume any knowledge of this SDF beyond its existence. This approach reflects our focus on characterizing the set of cross-equation restrictions between (m_{t+1}, m^*_{t+1}) and Δs_{t+1} with minimal assumptions about everything else.

[12] In a log-normal setting, condition (11) is equivalent to the absence of arbitrage opportunities. In more general settings, it is a necessary condition for no arbitrage.

[13] If a household trades claims to the intermediary frictionlessly, it implies that this household's SDF (m or m^*) prices those claims even if m^I is different. These claims would be part of H or F.

While a long tradition imposes no arbitrage as a principle that holds for all possible assets, more recent evidence suggests this is not the case. A prominent violation of no arbitrage in international markets is Covered Interest Parity (CIP) deviations (e.g., Du, Tepper, and Verdelhan, 2018, Ivashina, Scharfstein, and Stein, 2015, among many others): investing in dollars directly does not give the same return as investing abroad and locking in the exchange rate with a forward contract. In models featuring CIP deviations, intermediaries face frictions in trading this arbitrage portfolio. Hence, the portfolio cannot be part of I, but some of its components might be. Variation in I is part of the variation in market structure which is the core interest of this paper.

From the perspective of models with fully integrated markets, the introduction of m^I may appear redundant, as either the home or the foreign households can act as international intermediaries. In such market structures, local Euler equations and a simple currency conversion of asset returns fully characterize risk sharing. This insight does not extend to the much larger class of market structures that we consider: our formalism unifies predictions for models of partial integration and intermediation.

Coming from theories of intermediation, it may also be tempting to make away with local households altogether. This corresponds to replacing both households in Assumption 1 by the arbitrageur from Assumption 2 who prices every asset. Such an approach does not introduce informative risk-sharing restrictions as it effectively considers the same investor twice. Mechanically, the conversion of an intermediary's SDF from domestic to foreign currency is $m^{I*} = m^I + \Delta s$, irrespective of market structure — an accounting relation, not an equilibrium one (see Appendix G for further discussion).

Taking stock, the sets H, F and I together with Assumptions 1 and 2 constitute our representation of portfolio risk sharing. This representation does not rule out any equilibrium model. Instead, any model can be mapped into this structure by appropriately identifying the asset sets that respect the properties postulated in the two assumptions. This includes theories with segmented markets or Euler equation wedges. In such models, when households can freely trade only local risk-free bonds, we typically have $H \cap F \cap I = \emptyset$. Then, risk-sharing forces may impose no constraints on the exchange rate, in which case it is entirely determined by other equilibrium conditions. Of course, we are mainly interested in circumstances when risk sharing imposes some constraints on the equilibrium exchange rate behavior.

2.3 Globally-traded, locally-traded and unspanned shocks

Globally-traded shocks Intuitively, portfolio risk sharing occurs when households can trade the same risks. In the economy of Section 1, when the two households trade a claim to global productivity a, risk sharing pins down part of the exchange rate (the component $\hat{e}$ in equation (6)). With many assets, there can be multiple such dimensions. We formalize this notion with the concept of globally-traded shocks, which will be central to our theory.

Definition 1. *Globally-traded shocks can be traded by local investors in their local currency in both countries. The set of globally-traded shocks is*

$$\boldsymbol{\epsilon}^g_{t+1} = \left\{ \epsilon^g_{t+1} \,\middle|\, \exists \boldsymbol{\lambda} \in \mathbb{R}^N,\, \boldsymbol{\lambda}^* \in \mathbb{R}^{N^*} :\ \epsilon^g_{t+1} = \boldsymbol{\lambda}' \widetilde{\boldsymbol{r}}_{t+1} = \boldsymbol{\lambda}^{*\prime} \widetilde{\boldsymbol{r}}^*_{t+1} \right\}. \tag{12}$$

In the case of common trading of a risky asset in Section 1, the global productivity shock a_{t+1} satisfies this definition immediately: both home and foreign household can trade the forward contract exposed to it, and hence a_{t+1} belongs to both $\widetilde{\boldsymbol{r}}_{t+1}$ and $\widetilde{\boldsymbol{r}}^*_{t+1}$. In more general settings, it is not enough that the global productivity shock affects assets returns traded in each country. Each investor must also be able to combine assets to create a portfolio in local currency that isolates the shock from other sources of risk.

There could be multiple globally-traded shocks: investors might be able to trade exposure to global productivity, inflation, industry-specific risks, etc. Globally-traded shocks need not have a structural interpretation. For example, in integrated markets when investors trade the same assets, $\widetilde{\boldsymbol{r}}_{t+1}$ and $\widetilde{\boldsymbol{r}}^*_{t+1}$ will be similar, often leading to globally-traded risks irrespective of what the assets are linked to.[14] In the extreme case of complete and integrated markets, all possible shocks are globally traded. More generally, globally-traded risks arise as long as investors can implement trading strategies in their respective investment set with identical risk exposure. Appendix C.1 shows how to construct a basis of $\boldsymbol{\epsilon}^g_{t+1}$ using the covariance matrix of $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^*_{t+1}$. Appendix C.2 provides further examples of globally-traded shocks.

[14] Note that an asset traded by both households (e.g., an individual stock) does not immediately constitute a globally-traded risk due to currency conversion. However, we show that a commonly traded *excess* return does result in a globally-traded risk as the exposure to currency risk is eliminated in the construction of the excess return.

Locally-traded shocks We define locally-traded shocks at home by $\boldsymbol{\epsilon}_{t+1} = \{\epsilon_{t+1} | \exists \boldsymbol{\lambda}_\ell \in \mathbb{R}^N : \epsilon_{t+1} = \boldsymbol{\lambda}_\ell' \widetilde{\boldsymbol{r}}_{t+1} \perp \boldsymbol{\epsilon}^g_{t+1}\}$, and their foreign counterpart $\boldsymbol{\epsilon}^*_{t+1}$. Locally-traded shocks are the least-square residuals of return innovations $\widetilde{\boldsymbol{r}}_{t+1}$ and $\widetilde{\boldsymbol{r}}^*_{t+1}$, respectively, after controlling for globally-traded shocks $\boldsymbol{\epsilon}^g_{t+1}$. Locally traded-shocks capture variation in asset returns that is not simultaneously traded in both countries.

In Section 1, the only economy featuring locally-traded shocks is the partially integrated case where both households trade the foreign risk-free bond. Because the foreign household only has access to their own risk-free asset, there is no globally-traded risk. For the home household, investing in the foreign risk-free bond is risky in home currency, providing exposure to exchange rate shocks. Hence, the exchange rate is a locally-traded shock.

Unspanned shocks Finally, we refer to any sources of variation orthogonal to asset returns $(\widetilde{\boldsymbol{r}}_{t+1}, \widetilde{\boldsymbol{r}}^*_{t+1})$ — or equivalently, orthogonal to locally-traded and globally-traded shocks $(\boldsymbol{\epsilon}^g_{t+1}, \boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}^*_{t+1})$ — as unspanned shocks. Unspanned shocks are not traded in either home or foreign financial market. In the intermediated economy with common trading of a risky asset in Section 1, global productivity a_{t+1} is the only traded risk. Therefore, the residuals of macro innovations $(\breve{y}_{N,t+1}, \breve{y}_{T,t+1}, \breve{y}^*_{T,t+1})$ and noise trader demand $\breve{\psi}_{t+1}$ are unspanned shocks.

The classification into globally-traded, locally-traded and unspanned shocks characterizes how risks can be shared in a given financial market structure. Each group of shocks can, in general, contain aggregate or idiosyncratic shocks, as well as macroeconomic or financial shocks. Also, the same shock can belong to different groups when the market structure changes. For example, in Section 1, the residual endowment innovations $(\breve{y}_{N,t+1}, \breve{y}_{T,t+1}, \breve{y}^*_{T,t+1})$ are unspanned in the intermediated economy, and are globally-traded under complete markets.

Exchange rate decomposition We decompose the exchange rate depreciation Δs_{t+1} into four components. First, we partition the depreciation rate into its expectation $E_t \Delta s_{t+1}$ and the depreciation shock $\widetilde{\Delta s}_{t+1} = \Delta s_{t+1} - E_t \Delta s_{t+1}$. Next, we use the taxonomy of shocks introduced above, and decompose the depreciation shock into a globally-traded, locally-traded and unspanned components:

$$\widetilde{\Delta s}_{t+1} = g_{t+1} + \ell_{t+1} + u_{t+1}. \tag{13}$$

We denote with $g_{t+1} \in \boldsymbol{\epsilon}^g_{t+1}$ the component of $\widetilde{\Delta s}_{t+1}$ that is spanned by globally-traded risks, i.e., the predicted value of a least-square projection of the exchange rate on all globally-traded shocks $\text{proj}(\widetilde{\Delta s}_{t+1}|\boldsymbol{\epsilon}^g_{t+1})$. Correspondingly, the local component ℓ_{t+1} is spanned by locally-traded risks, i.e., it is a linear combination of $\boldsymbol{\epsilon}_{t+1}$ and $\boldsymbol{\epsilon}^*_{t+1}$ and orthogonal to $\boldsymbol{\epsilon}^g_{t+1}$. Finally, u_{t+1} is the component of the depreciation shock unspanned by asset returns, i.e., orthogonal to $(\boldsymbol{\epsilon}^g_{t+1}, \boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}^*_{t+1})$.

The decomposition (13) is unique and specific to a given market structure, and it plays a central role in our characterization of restrictions on the behavior of the exchange rate. In particular, it applies across broad classes of market structure as follows.

Lemma 1. (a) *In (partially) integrated markets, the exchange rate is spanned by asset returns, hence* $u_{t+1} = 0$. (b) *In fully integrated markets, the exchange rate and all asset returns are globally-traded risks,* $\widetilde{\Delta s}_{t+1} = g_{t+1}$ *and* $\widetilde{\boldsymbol{r}}_{p,t+1} = \widetilde{\boldsymbol{r}}^*_{p,t+1} = \boldsymbol{\epsilon}^g_{t+1}$. (c) *Intermediated markets can support any decomposition of the exchange rate risk into* g_{t+1}, ℓ_{t+1} *and* u_{t+1} *components.*

To understand this lemma, note that the spanned and unspanned components of the depreciation rate in decomposition (13) can be constructed from asset returns:

$$\widetilde{\Delta s}_{t+1} = \widetilde{r}_{p,t+1} - \widetilde{r}^*_{p,t+1} + u_{t+1}, \tag{14}$$

where $r_{p,t+1} \in \boldsymbol{r}_{p,t+1}$ and $r^*_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}$ are the returns on the pair of portfolios that maximize the R^2 for explaining the exchange rate. In (partially) integrated markets, there exists at least one asset traded in both countries, and hence $r_{i,t+1} = r^*_{i,t+1} + \Delta s_{t+1}$ holds for this asset by simple conversion of currency units. Using this asset in equation (14) in home and foreign currency as $\widetilde{r}_{p,t+1}$ and $\widetilde{r}^*_{p,t+1}$ respectively, we see that $u_{t+1} = 0$, point (a). When markets are fully integrated, all assets are traded across countries and the restriction becomes stronger: innovations to all asset returns and the exchange rate are globally-traded shocks, point (b).[15]

[15] This result relies on the fact that both home- and foreign-currency risk-free assets are available to all investors, which is sufficient to make the exchange rate risk a globally-traded shock by means of a simple carry trade strategy. Without risk-free assets, the exchange rate risk is not necessarily globally traded even in fully integrated markets, whereas all excess returns are still globally traded.

In contrast, intermediated markets generally have no such restrictions and the unspanned component u_{t+1} can play an important role in the exchange rate shock, point (c).

3 The general risk-sharing view of exchange rates

In this section, we characterize the restrictions on the behavior of the exchange rate imposed by international arbitrage given the properties of returns on traded assets, $\boldsymbol{r}$ and $\boldsymbol{r}^*$, and local SDFs m and m^* that price them. We show that Assumptions 1 and 2 impose two sets of necessary restrictions on the depreciation rate: one on the depreciation shocks $\widetilde{\Delta s}_{t+1}$, and another on the expected depreciation $E_t \Delta s_{t+1}$. In Appendix D, we further show that these restrictions are sufficient as well, that is, they characterize all constraints imposed by portfolio risk sharing on the behavior of the exchange rate.

When markets are complete and integrated, these two sets of restrictions lead to the well-known asset market view of exchange rates. Our analysis spells out the implications of these restrictions in a much larger set of market structures. All the proofs are in Appendix D. Appendix E derives exact non-linear versions of the results which do not require any distributional assumptions and hence include the case of disasters.

3.1 Exchange rate shocks

It is natural to think that thanks to risk-sharing, relative marginal utilities of the two households in their respective currencies $m^* - m$ line up with the depreciation rate Δs, as is the case under complete markets. We now show how this logic is altered in a general market structure.

Proposition 1. *Under Assumptions 1 and 2, the globally-traded component of the depreciation shock, g_{t+1} in decomposition (13), must coincide with the least-square projection of the relative SDF on globally-traded shocks:*

$$\operatorname{proj}(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} | \boldsymbol{\epsilon}^g_{t+1}) = \operatorname{proj}(\widetilde{\Delta s}_{t+1} | \boldsymbol{\epsilon}^g_{t+1}) = g_{t+1}. \tag{15}$$

Said differently, start from the pair of local SDFs and regress them on all globally-traded shocks.[16] The predicted value of this regression must equal the globally-traded component of the exchange rate, g_{t+1}:

$$\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} = g_{t+1} + v_{t+1} \qquad \text{with} \;\; v_{t+1} \perp \boldsymbol{\epsilon}^g_{t+1}. \tag{16}$$

Because globally-traded shocks are constructed from asset returns alone, this means that Proposition 1 allows to determine a component of the exchange rate by no arbitrage without any direct knowledge of the exchange rate process. In other words, it is sufficient to know local finance summarized by (m_{t+1}, m^*_{t+1}) and $(\boldsymbol{r}_{t+1}, \boldsymbol{r}^*_{t+1})$ to construct g_{t+1}.

We have already encountered an example of this result in the setting with a commonly traded forward contract in equation (6) of Section 1. There, the only globally-traded shock was the global productivity shock a. Hence, portfolio risk sharing only pins down the projection of the exchange rate on this shock, $g = \beta_s a$, which must also equal the projection of the relative SDF, exactly as Proposition 1 predicts.

In fully integrated markets, $\widetilde{\Delta s}_{t+1} = g_{t+1}$ by Lemma 1, and hence the entire exchange rate shock can be constructed from local finance state by state. This is a powerful result that generalizes the asset market view (AMV) of the exchange rate beyond the case of complete markets (Backus, Foresi, and Telmer, 2001; Brandt, Cochrane, and Santa-Clara, 2006). When the exchange rate risk is a globally-traded shock, Proposition 1 generalizes the familiar complete market relationship $\Delta s_{t+1} = m^*_{t+1} - m_{t+1}$ to:

$$\widetilde{\Delta s}_{t+1} = \text{proj}(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} | \boldsymbol{\epsilon}^g_{t+1}). \tag{17}$$

In this case the exchange rate shock on the right-hand side of (17) is still fully revealed in the local financial market.

However, what is missing from Proposition 1 is just as important as what is there. Risk sharing and no-arbitrage do not impose any restrictions on the local component of the depreciation rate ℓ_{t+1} or its unspanned component u_{t+1}, creating room for exchange rate

[16] Least-square projections of SDFs appear frequently in finance, starting with Hansen and Richard (1987). See also Cochrane (2004), Section 4.1, for a textbook treatment, or Brandt, Cochrane, and Santa-Clara (2006) in the international context.

shocks disconnected from SDFs. In partially integrated or intermediated markets, these two components may dominate the dynamics of the exchange rate over and above the shared component g_{t+1}. For example, the segmented market model in equation (3) of Section 1 has no globally- or locally-traded shocks, $g_{t+1} = \ell_{t+1} = 0$, and hence $\widetilde{\Delta s}_{t+1} = u_{t+1}$. In this case, Proposition 1 implies that the equilibrium exchange rate behavior is unconstrained by portfolio risk sharing.

How does the absence of arbitrage lead to the general result? In fully integrated markets, local and foreign investors must agree on the price of all payoffs after conversion to a common currency, resulting in $cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r_{t+1}) = 0$ for every $r_{t+1} \in \boldsymbol{r}_{p,t+1}$, and indeed $\boldsymbol{\epsilon}^g_{t+1} = \widetilde{\boldsymbol{r}}_{p,t+1}$ in this case by Lemma 1. Proposition 1 is a generalization of this result that holds across the full range of market structures. To preclude arbitrage opportunities, there must be an alignment between home and foreign investors' pricing of risks that they both trade in their respective currencies, which constrains the behavior of the depreciation shock.[17]

Without a change of currency, the argument is standard: an intermediary can buy the globally-traded risk ϵ^g_{t+1} in the home market (valued by m_{t+1}) and sell it in the foreign market (valued by m^*_{t+1}), hence no-arbitrage requires that $cov_t(m^*_{t+1} - m_{t+1}, \epsilon^g_{t+1}) = 0$. This logic extends to the case with currency conversion, and no arbitrage requires the so-called quanto adjustment $cov_t(\Delta s_{t+1}, \epsilon^g_{t+1})$ to expected returns. This implies that the comovement of the depreciation rate with globally-traded shocks must be the same as that of the relative SDFs, $cov_t(m^*_{t+1} - m_{t+1}, \epsilon^g_{t+1}) = cov_t(\Delta s_{t+1}, \epsilon^g_{t+1})$. In other words, if the projection g of Δs onto $\boldsymbol{\epsilon}^g$ is different from that of $m^* - m$, there exists an arbitrage strategy for the international intermediary.[18]

Conversely, for shocks that are not traded by both investors in their respective curren-

[17] In complete markets, the pair of Arrow-Debreu state prices in respective local currencies pins down the value of the exchange rate depreciation in that state by no-arbitrage: any deviation from this value would compel an investor to buy the state where it is cheaper and sell where it is more expensive, after conversion into the same currency. Proposition 1 extends this logic to circumstances with more sparse asset spaces, replacing the concept of state prices with a more general concept of globally-traded shocks ϵ^g. The case of an Arrow-Debreu security corresponds to ϵ^g that is an indicator random variable for a given state of the world.

[18] To prove Proposition 1, in Appendix D we use a zero-cost differential carry trade which is long one unit of a home risk, and short one unit of a foreign risk, with both legs of the trade financed at the respective local risk-free rates. Unlike the conventional carry trade, differential carry eliminates the direct exposure to the exchange rate risk. As a result, this trade acts as the arbitrage strategy for globally-traded risks that pins down the projection of the exchange rate on ϵ^g_{t+1}.

cies, it is impossible to construct candidate arbitrage portfolios that relate the conditional properties of the exchange rate shock to those of the local SDFs (see Appendix D.2). By consequence, risk sharing and no-arbitrage do *not* constrain the components ℓ and u of the exchange rate that are orthogonal to $\boldsymbol{\epsilon}^g$.

Lastly, in Appendix D we demonstrate that Proposition 1 is still valid in the absence of risk-free assets in the trading sets. Thus, prominent frictions in trading of risk-free assets such as convenience yields (Krishnamurthy and Vissing-Jorgensen, 2012; Jiang, Krishnamurthy, and Lustig, 2021) are accommodated in our framework.

3.2 Expected depreciation rate

We turn to restrictions on the behavior of the expected depreciation rate $E_t \Delta s_{t+1}$. Start from the projection of the exchange rate on asset returns, represented by the two portfolios $r_{p,t+1} \in \boldsymbol{r}_{p,t+1}$ and $r^*_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}$ in equation (14). We define δ_t as the difference between the two portfolios' expected returns given by equations (9) and (10):

$$\begin{aligned}\delta_t \equiv E_t\, r_{p,t+1} - E_t\, r^*_{p,t+1} =& \Big[r_{ft} - cov_t(m_{t+1}, r_{p,t+1}) - \frac{1}{2} var_t(r_{p,t+1})\Big] \\ &- \Big[r^*_{ft} - cov_t(m^*_{t+1}, r^*_{p,t+1}) - \frac{1}{2} var_t(r^*_{p,t+1})\Big]. \end{aligned} \tag{18}$$

The following proposition relates the behavior of the expected depreciation rate to spanning of the exchange rate and this quantity δ_t, which only depends on local finance — local asset returns and SDFs.

Proposition 2. *The expected depreciation rate is pinned down by no-arbitrage if the exchange rate is spanned by asset returns, that is, when $u_{t+1} = 0$. In this case:*

$$E_t \Delta s_{t+1} = \delta_t = \underbrace{r_{ft} - r^*_{ft}}_{UIP} - \underbrace{cov_t(m_{t+1}, \Delta s_{t+1})}_{exchange\ rate\ risk\ premium} - \underbrace{\frac{1}{2} var_t(\Delta s_{t+1})}_{convexity} + \theta_t, \tag{19}$$

where $\theta_t \equiv cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r^*_{p,t+1})$, which becomes $\theta_t = 0$ when the exchange rate is spanned by globally-traded shocks and $\widetilde{\Delta s}_{t+1} = g_{t+1}$. Otherwise, if $u_{t+1} \neq 0$, no-arbitrage does not constrain the value of $E_t \Delta s_{t+1}$.

The central implication of Proposition 2 is that it delineates two cases depending on the relation of the exchange rate with asset returns: either local market pricing determines expected depreciation exactly, or it says nothing about it. The expected depreciation rate is closely related to the exchange rate risk premium. Exposure to this risk can be obtained by engaging in the carry trade using the spanning portfolios. This risk premium is pinned down by pricing in local financial markets if and only if the international arbitrageur can use locally traded assets to sell off this risk. Conversely, the absence of arbitrage has no bearing on this quantity if the exchange rate is not spanned by asset returns, that is when $u_{t+1} \neq 0$.

Spanned exchange rate When the exchange rate is spanned by local asset returns, there exists a unique value of the expected depreciation $E_t \Delta s_{t+1}$ given by (19) which is consistent with no international arbitrage. The international arbitrageur uses the two local markets to price the exchange rate risk. Hence, the two local SDFs play a role in the expected depreciation rate. This insight explains the presence of the novel adjustment term θ_t in equation (19) relative to the standard complete market formula with $\theta_t = 0$. It also leads to a symmetric expression to equation (19) which emphasizes the foreign SDF m^*_{t+1}:

$$\delta_t = r_{ft} - r^*_{ft} - cov_t(m^*_{t+1}, \Delta s_{t+1}) + \frac{1}{2} var_t(\Delta s_{t+1}) + \theta^*_t, \tag{20}$$

where $\theta^*_t = cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r_{p,t+1})$.

It is only when the local investors are able to replicate the exchange rate on their own that their individual Euler equations, and hence individual SDFs, are enough to obtain the expected depreciation. If the home (foreign) investor can trade the entire spanning portfolio, then $\theta_t = 0$ ($\theta^*_t = 0$), and the standard complete market formula for the home (foreign) investor applies. For example, this situation occurs in settings in which the home (foreign) investor acts as an international arbitrageur or has access to a currency carry trade.[19] Both home and foreign investors price the exchange rate risk, and hence $\theta_t = \theta^*_t = 0$, when they are both able to trade it — that is, when the exchange rate is a globally-traded risk, $\widetilde{\Delta s}_{t+1} = g_{t+1}$.

[19]Note that by Proposition 1, $\theta_t = 0$ when $r^*_{p,t+1}$ is a globally-traded risk, that is $\widetilde{r}^*_{p,t+1} \in \boldsymbol{\epsilon}^g_{t+1} = \widetilde{\boldsymbol{r}}_{t+1} \cap \widetilde{\boldsymbol{r}}^*_{t+1}$. This means that the home household can trade both $r_{p,t+1}$ and $r^*_{p,t+1}$ risks which span the exchange rate shock, giving the household access to an effective currency carry trade.

Unspanned exchange rate In intermediated markets, the exchange rate shock may not be spanned by traded assets (recall Lemma 1). In this case, the expected depreciation can deviate from δ_t by an arbitrary wedge ζ_t, that is:

$$E_t \Delta s_{t+1} = \delta_t + \zeta_t. \tag{21}$$

Such a wedge is commonly used in the literature as a UIP shock or a convenience yield (see e.g. Itskhoki and Mukhin, 2021; Jiang, Krishnamurthy, Lustig, and Sun, 2024), and it is closely related to the currency demand shock ψ_t in the example of Section 1. The full flexibility afforded by this wedge in condition (21) may lead to implausibly large trading profits for the international investor. One can obtain bounds on the size of these deviations ζ_t by imposing a condition that is stronger than the absence of pure arbitrage in Assumption 2.

Assumption 3 (No quasi-arbitrage)**.** *There is an upper bound B on Sharpe ratios in international markets:*

$$\forall r^I_{p,t+1} \in \boldsymbol{r}^I_{p,t+1} : \quad \left| E_t\, r^I_{p,t+1} - r_{ft} + \frac{1}{2} var_t(r^I_{p,t+1}) \right| \leq B \sqrt{var_t(r^I_{p,t+1})}. \tag{22}$$

This assumption restricts the Sharpe ratio of trades in international markets. Such bounds have a long tradition in finance, going back to Ross (1976), Cochrane and Saa-Requejo (2000), and Kozak, Nagel, and Santosh (2020). Intuitively, it can be motivated by the view that if trades that are too profitable emerged in equilibrium, new financial institutions would step in to take advantage of them. Under this view, we obtain the following condition.

Proposition 3. *Under Assumption 3, the wedge ζ_t in the expected depreciation rate in (21) must satisfy:*

$$\left| \zeta_t + \frac{1}{2} var_t(u_{t+1}) \right| \leq B \sqrt{var_t(u_{t+1})} \equiv B \sqrt{(1 - R^2) var_t(\Delta s_{t+1})}, \tag{23}$$

where R^2 is the R-squared in the regression of Δs_{t+1} on $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^*_{t+1}$.

This result generalizes Proposition 2 by limiting the range of possible expected depreciations in the case of an unspanned exchange rate. It indicates that the deviation ζ_t from the risk premium in the spanned case in (19) is bounded by the volatility of the unspanned shock u_{t+1} in the depreciation rate decomposition (13). In turn, u_{t+1} may also be shaped by the properties of ζ_t as a result of an equilibrium fixed point, as is the case in our example economy in Section 1 (see the full characterization in Appendix A.3).

Relationship between the results When globally-traded risks are the result of common trading of assets in $H \cap F$, e.g., in (partially) integrated markets, Proposition 2 obtains as a standard asset pricing result from local Euler equations in Assumption 1. Furthermore, in this case, Proposition 1 is a direct consequence of Proposition 2. Indeed, any two globally-traded assets $i, j \in H \cap F$ provide two alternative ways to span the exchange rate in Proposition 2, and we can write condition (19) in two ways with respective θ_{it} and θ_{jt}. Hence, it must be that $\theta_{it} - \theta_{jt} = 0$, which corresponds to the projection requirement in Proposition 1 for the globally-traded excess return of asset i over j.[20]

When globally-traded risks arise from spanning of the same shocks by distinct assets in H and F, e.g., in intermediated markets, Propositions 1 and 2 are not directly linked. Proposition 2 remains a standard asset pricing result that links the expected return of a trade to its risk measured by the covariance of its return with SDF, provided the caveat that both home and foreign SDFs are generally required for no-arbitrage pricing of the exchange rate return. Proposition 1 is less conventional, as it characterizes the exchange rate shock rather than its expected return. Furthermore, Proposition 1 applies even when there is no spanning of the exchange rate in Proposition 2 and thus equation (19) does not hold. In other words, the finance exchange rate disconnect (in the sense that $u_{t+1} \neq 0$) does not imply absence of international risk sharing ($g_{t+1} = 0$), as we study next.

[20]By definition of θ_t in (19), $\theta_{it} - \theta_{jt} = cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r_{i,t+1} - r_{j,t+1}) = 0$ where $r_{i,t+1} - r_{j,t+1} = r^*_{i,t+1} - r^*_{j,t+1}$ are the return differentials (excess returns) in home and foreign currency, respectively, forming a globally-traded risk. Note that this condition corresponds to the condition in Lustig and Verdelhan (2019) when i and j are the two risk-free assets. In this case, $r_{i,t+1} - r_{j,t+1} = \Delta s_{t+1}$, and hence $var_t(\Delta s_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1})$.

4 Market structures and disconnect

The general results of the previous section inform our understanding of how finance interacts with the exchange rate beyond the standard market structures studied in the literature. In this section, we apply these results to address two questions. First, does portfolio risk sharing necessarily lead to the currency puzzles that are typically associated with strong connect? Second, is extreme segmentation necessary for financial markets to be a source of disconnected exchange rate volatility? We answer both questions in the negative. Furthermore, we show that the variance of the globally-traded component, $var_t(g_{t+1})$, emerges as an important metric for both exchange rate connect and disconnect implied by any given market structure.

We focus our analysis on specific moments of the exchange rate — namely, its volatility, cyclicality, and the currency risk premium — which have proved puzzling for traditional models in macro-finance. We hold the model of household IMRSs and the aggregate data that inform their properties fixed, hence taking m and m^* as given. We use our general theoretical results and vary the market structure — namely, what assets are traded and who can trade them — to study its implications for the currency puzzles and disconnect. In doing so, we take a complementary approach to a broad literature which has made progress by altering preferences or aggregate dynamics but maintained the assumptions that markets are complete and integrated.

4.1 The currency puzzles

The asset market view (AMV) of the exchange rate under complete and integrated markets results in:

$$\Delta s_{t+1} = m^*_{t+1} - m_{t+1}, \tag{24}$$

which characterizes the entire exchange rate depreciation — both its expectation $E_t \Delta s_{t+1}$ and shocks $\widetilde{\Delta s}_{t+1}$ — from the local SDFs. When the local SDFs are given by the conventional IMRSs of representative households disciplined with macro-data on aggregate consumption growth and inflation, the AMV results in three seminal finance puzzles about the behavior of the exchange rate.

First, consider the variance of the exchange rate depreciation rate implied by (24):

$$\begin{aligned} var_t(\Delta s_{t+1}) &= var_t(m^*_{t+1} - m_{t+1}) \\ &= var_t(m^*_{t+1}) + var_t(m_{t+1}) - 2cov_t(m_{t+1}, m^*_{t+1}). \end{aligned} \tag{25}$$

Brandt, Cochrane, and Santa-Clara (2006) argue that this equation leads to the *volatility puzzle*, with the exchange rate being not volatile enough. Typically observed Sharpe ratios on domestic assets imply highly volatile IMRSs, much more so than exchange rate depreciation. The mild correlation of macroeconomic quantities across countries suggests that the IMRSs are not correlated enough for the last term of equation (25) to offset this high variance and obtain realistic exchange rate risk.[21]

Second, equation (24) also implies:

$$var_t(\Delta s_{t+1}) = cov_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}), \tag{26}$$

and $corr_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}) = 1$. Changes in exchange rates must be perfectly correlated with changes in relative marginal utilities of the domestic and foreign households, that is, the home currency depreciates in relatively good times for home investors. As pointed out by Backus and Smith (1993), this implication is counterfactual for various measures of good times, leading to the *cyclicality puzzle*.

Finally, the expected depreciation rate is:

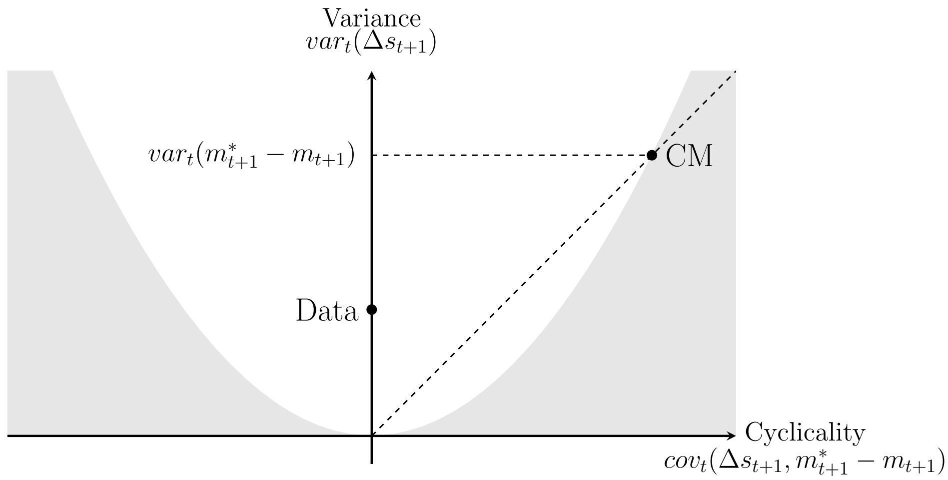
$$E_t \Delta s_{t+1} = r_{ft} - r^*_{ft} \; - \frac{1}{2} var_t(\Delta s_{t+1}) - cov_t(m_{t+1}, \Delta s_{t+1}) \tag{27}$$

The last term, a premium for currency risk, generates deviations from uncovered interest parity (UIP), a well-documented empirical feature. However, standard international models struggle with generating the empirically observed magnitude and dynamics of currency risk premium, resulting in the *risk premium puzzle* (see e.g. Engel, 2014, for a review).[22]

[21] A typical annual standard deviation of the exchange rate is 0.1, while a typical annual Sharpe ratio is of the order of 0.5, which is a lower bound on the standard deviation of SDFs because of the Hansen and Jagannathan (1991) bound. Then, according to (25), the correlation between m and m^* must be at least $1 - \frac{1}{2}\frac{0.1^2}{0.5^2} = 0.98$, which is much in excess of any empirical measures of cross-country comovement.

[22] Even though related, the covered interest parity (CIP) puzzle is an arbitrage-violation puzzle and it is distinct from the other risk-sharing puzzles we emphasize here.

Figure 2: Volatility, cyclicality, and currency puzzles



The grey area represents the infeasible combinations of volatility and cyclicality of depreciation rates due to the Cauchy-Schwarz inequality. The point labeled as CM illustrates the implications of the complete market setting for the properties of the depreciation rate. The point labeled Data is a stylized representation of the exchange rate puzzles summarized in (28).

We assume that the household IMRSs that define m and m^* are such that the exchange rate's cyclicality, volatility, and risk premium are counterfactual under complete and integrated markets, resulting in the currency puzzles. That is, we assume that for given SDFs m and m^* and data on the exchange rate, we have:

$$0 \approx cov_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}) \ll var_t(\Delta s_{t+1}) \ll var_t(m^*_{t+1} - m_{t+1}) \tag{28}$$

and a currency risk premium $r_{ft} - r^*_{ft} - E_t\Delta s_{t+1}$ that considerably exceeds $cov_t(m_{t+1}, \Delta s_{t+1})$ in absolute value — all conflicting with the AMV in (24).

In what follows, we ask how market structures beyond complete and integrated markets constrain the properties of the exchange rate. Propositions 1 and 2 lead us to consider volatility and cyclicality of the exchange separately from the currency premium which we address towards the end of the section.

We use Figure 2 to visualize the exchange rate cyclicality $cov_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1})$ and volatility $var_t(\Delta s_{t+1})$, where we hold the statistical properties of m and m^*—and hence of $var_t(m^*_{t+1} - m_{t+1})$—as given. The point labeled 'CM' shows the prediction of the AMV under complete and integrated markets summarized by (26), hence it lies on the 45-degree line at a point with $var_t(\Delta s_{t+1}) = var_t(m^*_{t+1} - m_{t+1})$. The point labeled 'Data' is a stylized representation of the empirical properties of the exchange rate summarized by (28). The distance between CM and Data represents the first two currency puzzles — volatility on the y-axis and cyclicality on the x-axis.

Finally, the gray area in Figure 2 indicates all combinations of exchange rate volatility and cyclicality that are infeasible due to the mechanical Cauchy-Schwarz inequality:

$$cov_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}) \leq \sqrt{var_t(\Delta s_{t+1}) \cdot var_t(m^*_{t+1} - m_{t+1})}. \tag{29}$$

Correspondingly, the white cone reflects all mathematically feasible combinations before any economic constraints are imposed.

4.2 The volatility-cyclicality tradeoff

Under complete and integrated markets, globally-traded risks span both the SDFs and the exchange rate such that their innovations $\widetilde{\Delta s}_{t+1} = \widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} = g_{t+1}$, and therefore:

$$var_t(\Delta s_{t+1}) = cov_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}) = var_t(g_{t+1}). \tag{30}$$

In words, the volatility and cyclicality of the exchange rate are pinned down by the variance of the globally-traded component g_{t+1}, in manifestation of the strong connect (with no room for disconnect) imposed on the exchange rate by portfolio risk sharing in this case. We next show how this quantity, $var_t(g_{t+1})$, constrains the volatility and cyclicality of the exchange rate across alternative asset market structures.

Proposition 1 provides a general characterization of all restrictions imposed by portfolio risk sharing on the exchange rate shock $\widetilde{\Delta s}_{t+1}$ that, in particular, determine its volatility and cyclicality properties. It has the following implications:[23]

[23]Proposition 4 is a consequence of equation (15) and the Cauchy-Schwarz inequality applied to the non-global components of the exchange rate and SDF differential, $\Delta s_{t+1} - g_{t+1}$ and $m^*_{t+1} - m_{t+1} - g_{t+1}$, respectively. The formal proof is in Appendix D.3.

Proposition 4. *The volatility and cyclicality of the exchange rate must satisfy*

$$\overbrace{var_t(\Delta s_{t+1})}^{volatility} \geq var_t(g_{t+1}) + \frac{\left(\overbrace{cov_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1})}^{cyclicality} - var_t(g_{t+1})\right)^2}{var_t(m^*_{t+1} - m_{t+1}) - var_t(g_{t+1})} \tag{31}$$

when $var_t(g_{t+1}) < var_t(m^*_{t+1} - m_{t+1})$*, and*

$$var_t(\Delta s_{t+1}) \geq var_t(g_{t+1}) = cov_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}). \tag{32}$$

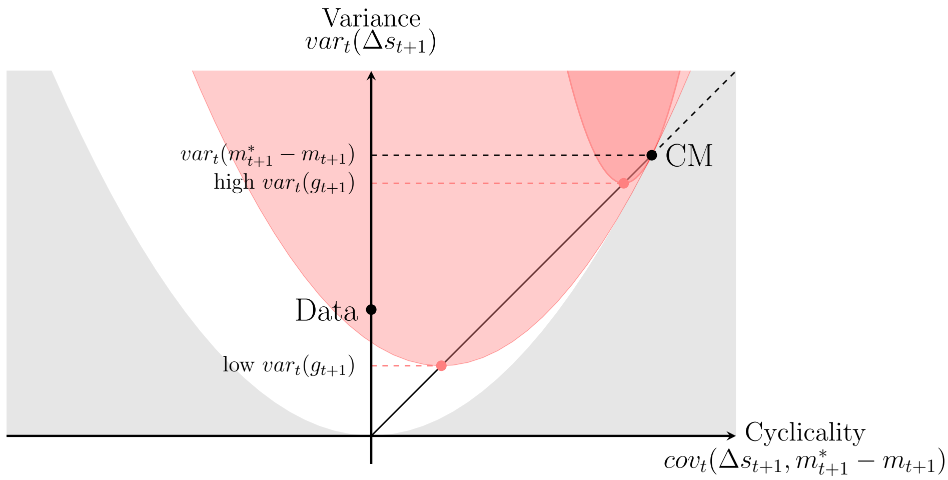
when $var_t(g_{t+1}) = var_t(m^*_{t+1} - m_{t+1})$.

We highlight three properties that follow from Proposition 4. First, for a given value of $var_t(g_{t+1})$, the proposition introduces a tradeoff between volatility and cyclicality. The red cones in Figure 3 illustrate this tradeoff for values of $var_t(g_{t+1})$ away from the boundaries as described by condition (31): $var_t(g_{t+1})$ is larger for the upper dark cone than for the lower light one. The vertex (trough) of the cone corresponds to the minimum level of exchange rate volatility $var_t(\Delta s_{t+1}) = var_t(g_{t+1})$. At this point, volatility coincides with cyclicality, $cov_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}) = var_t(g_{t+1})$. Thus, the vertex is on the 45-degree line segment between the origin and CM. Reducing cyclicality away from this value comes at the cost of increasing volatility — corresponding to the area inside the red cones above their vertex. The vertex reflects the amount of connect imposed by a given market structure, while the area inside the cone reflects the space for disconnect permitted by it.

Second, when the globally-traded component is nil, $var_t(g_{t+1}) = 0$, portfolio risk sharing imposes no additional economic constraints on the exchange rate moments. Specifically, condition (31) in this case recovers the mechanical Cauchy-Schwarz inequality (29) which excludes the gray area in Figures 2 and 3.

Third, the CM point is feasible for any value of $var_t(g_{t+1})$, and hence under any market structure. Furthermore, the range of possible exchange rate moments increases from the measure zero CM point to the full white area permitted by the Cauchy-Schwarz inequality as $var_t(g_{t+1})$ declines from its maximum value equal to $var_t(m^*_{t+1} - m_{t+1})$ to its minimum

Figure 3: The volatility-cyclicality tradeoff



The figure illustrates the trade-off between volatility and cyclicality of the exchange rate as characterized by (31) in Proposition 4. The dark red cone corresponds to a high value of $var_t(g_{t+1})$, while the light red cone corresponds to a low value. See notes to Figure 2.

value of 0.[24] In this sense, reducing the span of globally-traded shocks $var_t(g_{t+1})$ does not rule possibilities out, but instead allows possibilities in — more outcomes in terms of second moments of the exchange rate can be consistent with equilibrium risk sharing and no-arbitrage. Therefore, it is easier to match the data with less dense globally-traded shocks.

Proposition 4 and this discussion make clear that the range of possible exchange rate moments is controlled by the volatility of the globally-traded component of the exchange rate $var_t(g_{t+1})$. In turn, this value is determined by the structure of financial markets. That is, the variance of the globally-traded component serves as a central metric for how the market structure disciplines the volatility and cyclicality of the exchange rate. We explore the interplay between market structures and the globally-traded component next.

4.3 When does risk sharing lead to the currency puzzles?

Proposition 4 suggests that the currency puzzles are not unique to complete and integrated markets, but emerge by continuity in all models with a dominant globally-traded component g_{t+1}, because the exchange rate remains connected to IMRS. Specifically, the puzzles arise

[24]Note that by Proposition 1, and in particular by its corollary in (16), $var_t(g_{t+1})$ cannot exceed $var_t(m^*_{t+1} - m_{t+1})$, and by construction it cannot exceed $var_t(\Delta s_{t+1})$, hence $var_t(g_{t+1}) \leq \min\{var_t(\Delta s_{t+1}), var_t(m^*_{t+1} - m_{t+1})\}$.

when $var_t(g_{t+1})$ is large, imposing tight lower bounds on both volatility and cyclicality of the exchange rate. As the upper bound for $var_t(g_{t+1})$ is given by $\min\{var_t(\Delta s_{t+1}), var_t(m^*_{t+1} - m_{t+1})\}$, we start by exploring two types of market structures which yield the corresponding limiting cases. In these cases the volatility-cyclicality tradeoff takes a degenerate shape and differs from the cones of Figure 3. We then discuss the generic case with a large $var_t(g_{t+1})$.

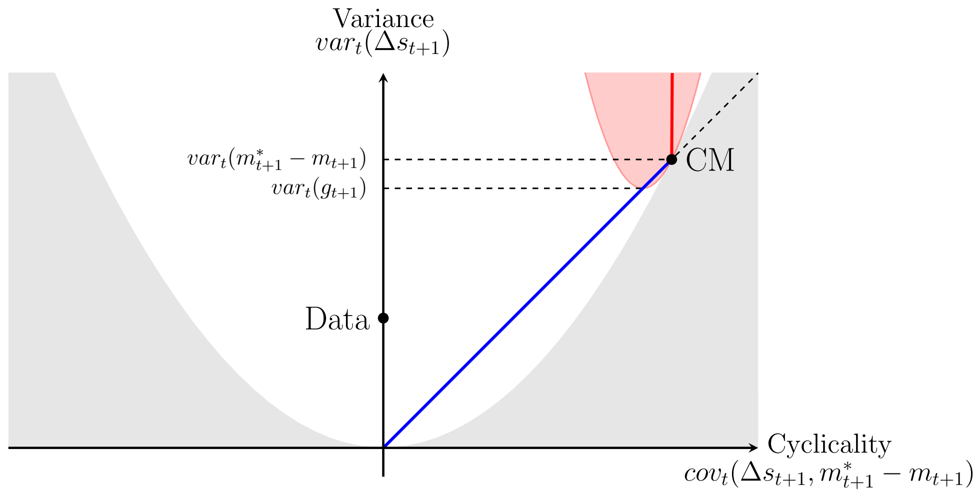
Spanned marginal utilities We consider first the case when the relative IMRS is spanned by globally-traded shocks, $var_t(g_{t+1}) = var_t(m^*_{t+1} - m_{t+1})$. This situation is close to market completeness in the sense that households in each country are able to trade shocks to their marginal utility. This case frequently arises in models with a small number of common macro risks that are traded in both countries, such as international real business cycle (IRBC) models with traded assets spanning productivity shocks. However, this situation does not require integrated markets. For example, it can arise when households cannot trade with each other, $H \cap F = \emptyset$, and intermediaries access all assets ($I = H \cup F$), as long as households in each country have access to a set of assets that is sufficiently rich to span the risks that affect both of their marginal utilities.

By Proposition 4, this case corresponds to equation (32) and yields

$$var_t(\Delta s_{t+1}) \geq var_t(g_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) = cov_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}) \geq 0. \quad (33)$$

The volatility of the relative IMRS puts a lower bound on the volatility and pins down the cyclicality of the exchange rate. Hence, this setting deepens the volatility puzzle and leaves the cyclicality puzzle unchanged relative to complete markets. In Figure 4, the vertical red ray emanating upwards from the CM point represents the volatility-cyclicality tradeoff in this situation. Another way to see this is to notice that in this case, $g_{t+1} = \widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}$, and hence $\widetilde{\Delta s}$ combines the relative IMRS with a component orthogonal to it. Because of this orthogonality, this second component increases volatility but cannot alter cyclicality. In Section 1, this situation occurs with common trading of a risky asset when the traded risk equals the relative endowment, $a_{t+1} = \widetilde{y}_{Nt+1} - \widetilde{y}^*_{Tt+1} = \widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}$. In this case, the component of Δs_{t+1} that is orthogonal to the relative IMRS is driven by noise-trader demand, that is, $\check{\psi}$ in equation (5).

Figure 4: The cyclicality-volatility tradeoff with a large globally-traded component



The red ray depicts all possible volatility-cyclicality combinations when globally-traded shocks span household marginal utility (IMRS) and $var_t(g_{t+1}) = var_t(m^*_{t+1} - m_{t+1})$, as in equation (33). The blue 45° line segment corresponds to the case when the exchange rate is spanned by globally-traded shocks, as in equation (34). The red cone corresponds to condition (31) for a high value of $var_t(g_{t+1})$, but away from the two limiting cases. See notes to Figures 2 and 3.

A testable implication of the spanned IMRS assumption is that a regression of the exchange rate depreciation Δs_{t+1} on the relative IMRS $m^*_{t+1} - m_{t+1}$ yields a coefficient of 1, while the reverse regression yields a coefficient (weakly) less than 1.

Spanned exchange rate The other limiting case arises when the exchange rate risk is globally-traded, $var_t(g_{t+1}) = var_t(\Delta s_{t+1})$. This situation occurs when markets are incomplete as long as they are sufficiently integrated. A prominent example is when risk-free bonds in both currencies are commonly traded as in Lustig and Verdelhan (2019) and in equation (4) in Section 1.

To understand why in this case the exchange rate itself is a globally-traded risk, notice that from the point of view of each investor, the risk-free bond of the other country is risky as its value moves one-to-one with the exchange rate. By engaging in the carry trade, all investors can span the exchange rate shock in their local currency, which implies that exchange rate shocks satisfy the definition of a globally-traded risk (see also Lemma 2 in Appendix B.2).

The implication of condition (31) in Proposition 4 in this case is:

$$0 \leq cov_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}) = var_t(\Delta s_{t+1}) = var_t(g_{t+1}) \leq var_t(m^*_{t+1} - m_{t+1}). \qquad (34)$$

The constraint on the volatility of the exchange rate is weaker than in equation (33), because g can be less volatile than the relative IMRS $m^* - m$. However, just like in the complete and integrated markets case (26), there is a cyclicality puzzle. This constraint on the cyclicality and volatility of the depreciation rate corresponds to the blue 45-degree line segment between the origin and the complete markets point in Figure 4.[25] A testable implication of the spanned exchange rate case is that a regression of $m^*_{t+1} - m_{t+1}$ on Δs_{t+1} yields a coefficient of 1, and weakly less than 1 vice versa. These properties indeed correspond to the conclusions of Lustig and Verdelhan (2019).

In the example of Section 1, when risk-free bonds are commonly traded, the statistical properties of endowments pin down the position of the equilibrium exchange rate on the blue segment of Figure 4. In line with the strong connection to IMRSs in this case, noise-trader shocks cannot affect the exchange rate, since they would push it off the blue segment.

Large globally-traded component The two cases above indicate that the exchange rate continues to be tightly connected to the household IMRSs as long as there is only a single departure from either market completeness or market integration. When we modify who can trade assets by allowing for imperfect market integration or intermediation, as long as the available set of assets is sufficiently rich, we end up in the spanned IMRS scenario. When we limit which assets can be traded while still ensuring market integration for both risk-free bonds, we find ourselves in the spanned exchange rate scenario. Both cases yield tight constraints on the possible properties of the exchange rate, captured respectively by the red ray and the blue line segment in Figure 4, which are measure zero in the space of exchange rate volatility and cyclicality moments, and away from the Data point.

Therefore, relaxing the constraints on the behavior of the exchange rate requires departing

[25] Each point on this segment corresponds to the vertex of a cone for a given value of $var_t(g_{t+1})$, and the tradability of the two risk-free bonds additionally requires that $var_t(\Delta s_{t+1}) = var_t(g_{t+1})$. Constraints imposed by other traded assets create a lower bound on $var_t(g_{t+1})$, restricting the set of feasible exchange rate moments to the sub-segment of the blue line above this value.

from both market completeness and market integration at once. This is necessary, but not sufficient. Even intermediated incomplete markets can feature exchange rate puzzles when the set of traded assets in each county is sufficiently broad to get close to span the relative IMRS $m^* - m$. This situation corresponds to a large value of $var_t(g_{t+1})/var_t(m^*_{t+1} - m_{t+1})$, albeit less than 1. In this case, Proposition 4 still implies tight constraints on the exchange rate moments. This corresponds to the dark red cone in Figure 4 closely surrounding the the red vertical ray. Generalizing the observations from this section, constraints on the exchange rate are continuous in the value of $var_t(g_{t+1})$, and are only sufficiently relaxed to avoid puzzles when $var_t(g_{t+1})$ is small.

4.4 Which market structures are compatible with disconnect?

Models with extreme segmentation imply $var_t(g_{t+1}) = 0$. This property allows them to generate disconnect by allowing shocks that do not affect the relative IMRS to impact the exchange rate. The complementary implication of Proposition 4 is that market structures with a small globally-traded component g_{t+1} can similarly accommodate various exchange rate dynamics and provide ample space for exchange rate disconnect. We depart from the case of $var_t(g_{t+1}) = 0$ and demonstrate that the disconnect mechanism can coexist with a non-trivial amount of portfolio risk sharing, $var_t(g_{t+1}) > 0$, without triggering the exchange rate puzzles. We conclude by providing a concrete example of asset market structures which do not rely on extreme segmentation and yield new predictions for variation in the extent of disconnect.

No globally-traded risks When $var_t(g_{t+1}) = 0$, there are no constraints and as a result all combinations of cyclicality and volatility in the white cone in Figure 2 are compatible with portfolio risk sharing; condition (31) in Proposition 4 becomes equation (29). This situation occurs in many intermediation-based models where households in each country have access to the local risk-free asset only, while intermediaries trade both of these assets and bear the currency risk. Because neither H nor F contain risky assets, there are no globally-traded risks, and hence $g_{t+1} \equiv 0$. Section 1 illustrates this situation through the segmented household trading example, equation (3).

Without globally-traded risks, portfolio risk sharing, or more accurately the absence thereof, always permits the Data point in Figure 2. Attainability, however, depends on primitives outside of Euler equations. In the example of Section 1, the Data point — as well as any other point inside the white cone — is always implementable by an appropriate choice of noise-trader currency demand shocks ψ_{t+1} which in equilibrium are absorbed and priced by the intermediary (see Appendix A.4).

Noise-trader demand is just one of many potential drivers of exchange rate movement that can drive disconnect and help avoid the puzzles in a structure without globally-traded risks. For example, other sources of demand for currencies such as international flows by specific firms, or changes in the risk-bearing capacity of financial intermediaries can also be behind the unspanned movements in exchange rates. Empirically, Avdjiev, Du, Koch, and Shin (2019) relate bank lending and CIP deviations with movements in the dollar, Jiang, Krishnamurthy, and Lustig (2021) emphasize safe asset demand, and Dao, Gourinchas, and Itskhoki (2025) the positions of dealer banks.

Small globally-traded component Extreme forms of market segmentation may not be appealing in practice: households trade more than one asset and intermediaries participate in more than one market. When such trade in assets gives rise to commonly-traded risks across countries, the globally-traded component becomes significant, $var_t(g_{t+1}) > 0$, and portfolio risk sharing imposes constraints on the properties of exchange rate. We show that this does not necessarily introduce the puzzles, nor removes the scope for exchange rate disconnect. If $var_t(g_{t+1})$ is low enough, as in the light red cone in Figure 3, the Data point is still feasible.

The upper bound on the volatility of the globally-traded component $var_t(g_{t+1})$ such that the corresponding red cone contains the Data point is (see Appendix D.3):

$$\frac{var_t(g_{t+1})}{var_t(\Delta s_{t+1})} \leq \frac{1}{1 + var_t(\Delta s_{t+1})/var_t(m^*_{t+1} - m_{t+1})}. \tag{35}$$

This relation highlights an important nuance in how a market structure can address the puzzles. International risk sharing must be sufficiently weak in the sense that $var_t(g_{t+1}) \ll var_t(m^*_{t+1} - m_{t+1})$, but it can still contribute substantially to exchange rate variation,

$var_t(g_{t+1})/var_t(\Delta s_{t+1}) \gg 0$. To see this, notice that under our maintained assumption of a volatility puzzle in complete markets, that is $var_t(\Delta s_{t+1}) \ll var_t(m^*_{t+1} - m_{t+1})$, the right-hand-side of equation (35) is close to 1.[26]

When the Data point is feasible from the perspective of risk-sharing constraints, its attainability still depends on other features of the model. In our example with common trading of a risky asset in Section 1 (equation (6)), where $g_{t+1} = \beta_s a_{t+1}$, any point inside the corresponding red cone in Figure 3 is implementable with a suitable choice of noise-trader demand process ψ_{t+1} (see Appendix A.4).

The currency risk premium We now address the properties of the currency risk premium, and hence the constraints on the expected depreciation $E_t\Delta s_{t+1}$. Proposition 2 provides a sharp delineation. One the one hand, if the exchange rate is spanned by asset returns, expected depreciation is given by equation (19), closely related to the complete market relationship of equation (27). Therefore, market structures that feature connect between the exchange rate and asset returns also feature the currency risk premium puzzle. On the other hand, if the exchange rate is not spanned by asset returns, expected depreciation can deviate arbitrarily from this tight risk-sharing relation.

In integrated markets, as in the blue line in Figure 4, the exchange rate is a globally-traded risk. Because globally-traded shocks are constructed from asset returns, the exchange rate is fully spanned. Therefore, these structures, which already wrestle with the volatility and cyclicality puzzles also face the risk premium puzzle. In market structures with less connect, represented by red cones in Figure 3, the exchange rate may or may not be spanned. For example, the economy with common trading of risky assets in Section 1 generically features an unspanned exchange rate, and hence an unconstrained risk premium. As a consequence, $E_t\Delta s_{t+1}$ in this class of models is determined by forces other than portfolio risk sharing.

If we further impose an upper bound on Sharpe ratios, Proposition 3 applies, and risk premium variations cannot be too large relative to the unspanned exchange rate innovations, constraining the possible range of values of $E_t\Delta s_{t+1}$. Itskhoki and Mukhin (2021) show

[26] Using the same values as in footnote 21 and assuming conservatively a high correlation of the two IMRSs of 0.9, we obtain an upper bound for $var_t(g_{t+1})/var_t(\Delta s_{t+1}) \leq 0.83$.

that small but persistent deviations ζ_t from UIP condition (19) are consistent with both empirical currency premia and substantial exchange rate disconnect, satisfying the upper-bound constraint on the carry trade Sharpe ratio.

A new set of economies We lay out features of economies that have low $var_t(g_{t+1})$ without extreme segmentation. We do so by allowing for a richer set of assets traded in an intermediated setting. The first observation is that the trading opportunities of intermediaries do not affect the set of globally-traded shocks. Intermediaries can be as sophisticated and active in as many markets as is empirically relevant. Formally, one can enrich arbitrarily the set I and price all assets in this set with m^I without affecting the risk-sharing restrictions on the exchange rate that are shaped by the sets $H \cap I$ and $F \cap I$.

Second, one can allow households to trade the assets of their respective countries. Imagine that instead of considering common trading of a risky asset, as in Section 1, households each trade a distinct risky asset. For example, US households trade a US stock index, while UK households trade a UK stock index. If local-currency returns on these assets are perfectly correlated, this situation is exactly equivalent to common trading of a single risky asset. As long as this globally-traded risk explains a small enough fraction of variation in the SDF, it is feasible to avoid the puzzles and match the Data in Figure 3. With weaker correlation between the assets, there are simply no globally-traded risks, and the exchange rate is even more flexible.

With multiple assets in each country, e.g., different stocks and bonds, what matters is how closely related are the risks from combining these two sets of assets, H and F, into portfolios. Our theory defines precisely this proximity by the set of globally-traded risks as in Definition 1. If assets in the two countries are not too tightly related, it is possible to match the Data point and avoid the puzzles. These results highlight the benefit of our framework. While Section 1 explicitly solved specific models, one can characterize the set of globally-traded risks without knowing all the building blocks of a model, relying only on the asset market structure as formalized in Section 2. If one takes a stand on the sets of traded assets H, F and I, the set of globally-traded risks can be constructed empirically directly from the covariance matrix of returns, as we do in Section 5.

Our characterization of the exchange rate in such economies comes from portfolio risk sharing through risky securities traded frictionlessly by households. These equilibrium connections do not hinge on whether no-arbitrage relationships such as CIP hold. For example, trading stocks and bonds can be without frictions, while trading CIP arbitrage portfolios, which also involve derivative positions, is subject to frictions (like, for example, in Garleanu and Pedersen, 2011; Augustin, Chernov, Schmid, and Song, 2024). Therefore, the economies of this section can not only be consistent with the empirical exchange rate properties that we emphasize above, but also with CIP being violated, whether systematically or occasionally (e.g., around the 2008–09 financial crisis).

5 Empirical Analysis

We now turn to the data on asset returns and exchange rates to quantify the role of portfolio risk sharing for connect and disconnect. Our theoretical results show that restrictions between the exchange rate and SDFs depend on the assumed market structure. In particular, given the data on returns, one needs to take a stand on who can trade what assets, that is the sets H, F, and I.[27] For example, when markets are complete, risk sharing implies that the relative household IMRS must equal the observed exchange rate depreciation. Or, in the other extreme of segmented markets, when households only trade their own risk-free asset, there are no restrictions on shocks to household IMRSs from observed asset returns and the exchange rate.

For the purposes of this section, we focus on a market structure where asset returns play a more meaningful role than in these extreme examples. Specifically, we consider the case of intermediated markets in which households of each country trade a broad collection of local bonds and equities, as in Section 4.4. In this market structure, the degree of connect to SDFs cannot be taken as given and needs to be evaluated empirically.

Consistent with the widespread observation of disconnect (Meese and Rogoff, 1983), we find that exchange rates appear to have a large component u_{t+1} unspanned by asset returns,

[27] Appendix G also shows that one has to take a stand on market structure for an economic interpretation of the SDFs recovered from return data.

while globally-traded shocks explain a modest share of exchange rate fluctuations. However, properties of asset returns change with economic conditions. We find that the importance of globally-traded risks increases during periods of high volatility or after 2008, in line with the evidence of connect during these episodes.

5.1 Data

We consider countries corresponding to G10 currencies between 2/1988 and 12/2024. We consider Germany as the representative country for the euro. Prior to the introduction of the euro, we use the Deutsche mark and splice these series together beginning in 1999. Our analysis focuses on the monthly frequency. We obtain exchange rates from WM/Reuters. Government bond yields are from each country's central bank website. Monthly bond returns are computed from bond yields using a second-order Taylor approximation. We obtain equity indices from Morgan Stanley Capital International (MSCI). For each country, 10 different industry indices and 3 different style equity indices (Large + Mid Cap, Value, Growth) are sourced. Risk-free rates are approximated by dividing the 1-year yield by 12.

We perform both unconditional and conditional analysis of spanning, globally-traded risk, and the disconnect. In the conditional case, we use lagged values of the VIX, annual U.S. GDP growth, and the excess bond premium (EBP, Gilchrist and Zakrajsek, 2012).

5.2 Unconditional analysis

Is the exchange rate spanned? Motivated by Proposition 2, we ask whether the depreciation rate is spanned by the combination of domestic and foreign asset returns. We estimate regressions that implement the construction of the maximum spanning portfolio in (14) as follows:

$$\Delta s_{t+1} = \alpha + \beta' \boldsymbol{r}_{t+1} + \beta^{*\prime} \boldsymbol{r}^*_{t+1} + u_{t+1}, \tag{36}$$

where $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^*_{t+1}$ represent asset returns available to home and foreign investors in their respective currencies. Specifically in the case of the assumed market structure, U.S. equities

Table 1: Spanning of depreciation rates by asset returns: R^2

Dependent Variable	AU	CA	DE	JP	NO	NZ	SE	CH	UK
Bonds									
10Y	0.73	-0.15	10.49	7.16	2.99	2.08	7.92	5.17	0.58
All Maturities	8.62	5.74	18.40	11.97	11.34	6.23	16.45	12.75	14.26
Stocks									
Mkt	22.17	26.77	8.31	5.94	12.98	19.59	19.71	12.16	14.38
Mkt + Value/Growth	22.06	28.15	8.13	7.07	14.06	20.03	19.80	12.52	15.10
Mkt + Value/Growth + Ind.	36.45	41.84	19.25	24.86	30.45	30.92	26.90	19.61	27.43
Bond + Equity	39.15	44.58	29.10	31.09	35.77	34.60	34.55	26.43	34.28
N	420	402	410	420	407	352	377	420	420

The table reports the adjusted R^2 of a regression of the depreciation rate on various subsets of asset returns, as in equation (36). Domestic asset returns are in domestic currency; foreign asset returns are in foreign currency. Each column is a different country's currency relative to the U.S. dollar. The first row uses only 10-year bonds, while the second entertains maturities between 2 and 10 years, obtained from various central banks. The next three row consider various stock portfolios: the market (a combination of large and mid-cap stocks), plus value and growth portfolios, plus 10 industry portfolios (all from MSCI). The final row considers all assets simultaneously.

and bond returns are expressed in U.S. dollars, while foreign equities and bond returns are expressed in the currency of that country. The residual u_{t+1} is a direct estimate of the unspanned component of the depreciation rate in equation (13).

Table 1 reports the adjusted R^2 from these regressions for individual countries vis-à-vis the United States. Each row reflects a particular combination of assets used in the regression: we consider bonds and equities separately and in combination. Exact spanning corresponds to an R^2 of 1, and Proposition 3 highlights that this R^2 is an appropriate measure of economic distance to the case of complete spanning.[28]

Our key finding is that these major asset classes do not span the exchange rate. When looking at all assets together, the R^2s range from 26% for Switzerland to 45% for Canada (in each case vis-à-vis the U.S.). Consistent with the evidence in Chernov and Creal (2023), bond returns explain only a modest amount of variation in exchange rates: between 0% and 8% for the 10-year bond alone, and between 6% and 18% for the combination of bonds at

[28] Campbell, Serfaty-De Medeiros, and Viceira (2010) focus on currency hedging of equity and bond portfolios, so they essentially implement reverse regressions with a focus on the sign and significance of the associated betas. The documented insignificant betas for bond portfolios are suggestive of low R^2.

all maturities. Most of the explanatory power comes from equities. While the market alone gets to a substantial amount of variation, the addition of industry returns is particularly informative.

The economic magnitude of the unspanned component $var(u_{t+1})$ is substantial. According to Proposition 3, even the largest R^2 we measure implies a bound for the expected depreciation that is only $\sqrt{1-0.45} \approx 0.74$ of the bound with $R^2 = 0$, not much tighter. We conclude that, under the intermediated market structure we consider, the expected depreciation rate is modestly constrained by the relative local SDFs m and m^*. Conversely, the observed path of $E_t \Delta s_{t+1}$ and the currency risk premium can be consistent with a wide range of conventional household IMRSs.

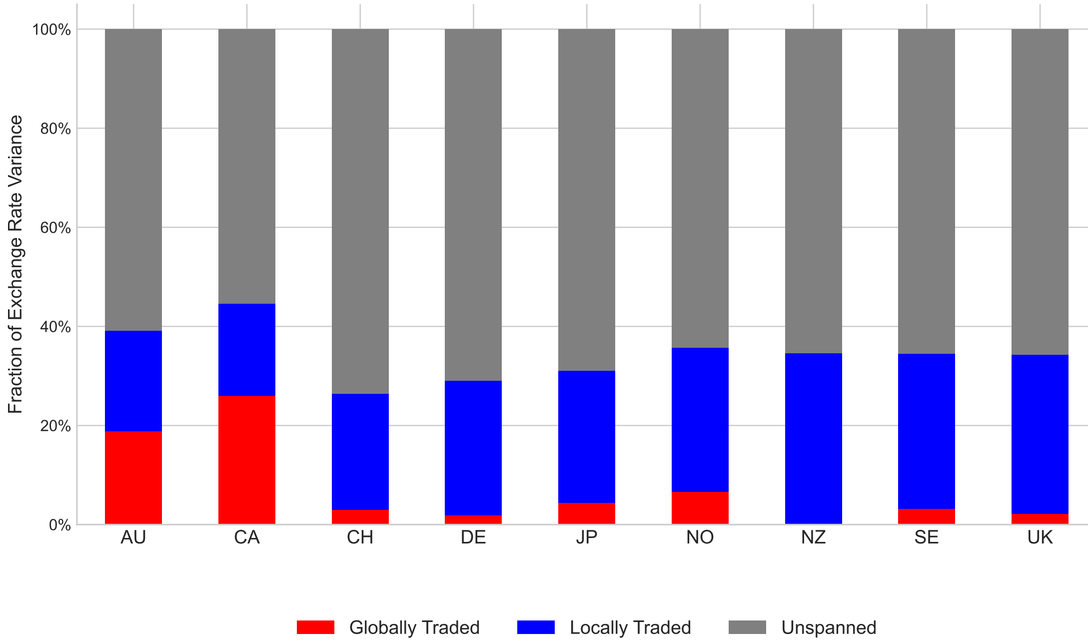
Identifying globally-traded shocks We quantify the importance of globally-traded shocks $\boldsymbol{\epsilon}^g_{t+1}$, which play the key role in Proposition 1. We follow an undirected approach and use canonical correlation analysis (CCA) to identify these shocks from the asset return data. In Appendix F, we also consider a directed approach starting from candidates for globally-traded shocks proposed in the literature, such as global macro and financial variables. The results in that setting are qualitatively similar.

According to Definition 1, globally-traded shocks are innovations to portfolios of asset returns consisting of $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^*_{t+1}$, respectively, with perfect correlation. The CCA procedure constructs US and foreign portfolios with the highest correlation possible in sample. Conditional on finding this pair, the procedure then looks for the next maximally correlated pair of portfolios that are orthogonal to their first pair. And so on. See Appendix C for details.

The values of the largest correlations range from 70% for New Zealand to 90% for Canada. The detailed results are reported in Appendix Table A1. We are generous with interpreting the evidence, and assume that portfolios with a correlation over 60% are sufficiently close to each other to constitute a measure of a globally-traded shock.

We ask how much variation in the depreciation rate is explained by globally-traded shocks. Denote the matrix of foreign portfolio weights by $\boldsymbol{w}^*$ so that $\boldsymbol{w}^{*\prime}\boldsymbol{r}^*_{t+1}$ is our basis of

Figure 5: Decomposition of exchange rate innovations



The figure reports the fraction of variance in exchange rates explained by globally-traded and locally-traded shocks, and shocks that are not spanned by asset returns, under the assumption of an intermediated market structure. Each bar is a different country's currency relative to the U.S. dollar; globally-traded shocks are measured using CCA for stock and sovereign bond returns.

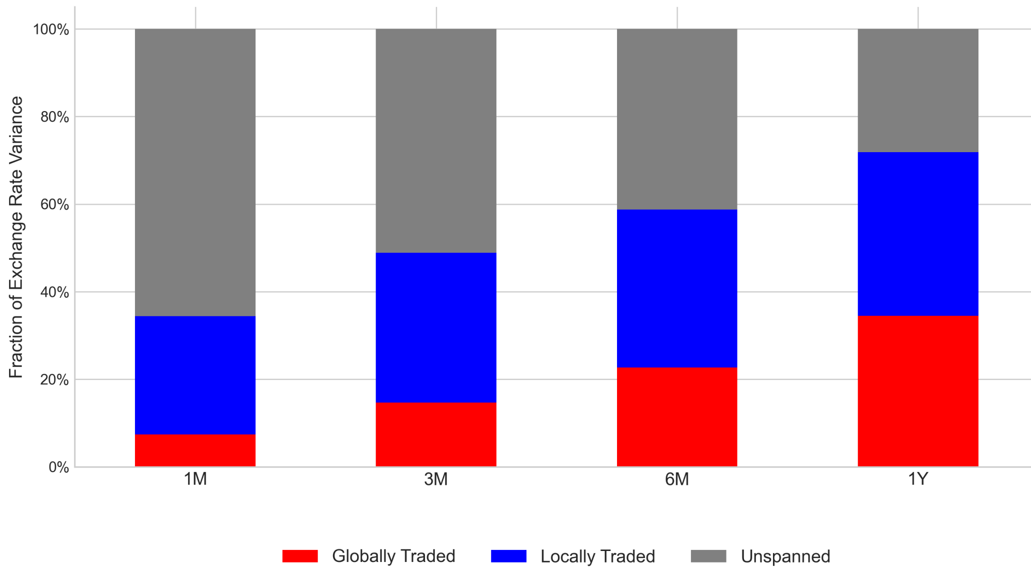
globally-traded risks $\boldsymbol{\epsilon}^g_{t+1}$. We implement regressions of the form:

$$\Delta s_{t+1} = \alpha + \beta^{g\prime}(\boldsymbol{w}^{*\prime}\boldsymbol{r}^*_{t+1}) + \varepsilon_{t+1}. \tag{37}$$

The R^2 of such a regression is the fraction of variance in exchange rate explained by the globally-traded component, $var(g_{t+1})/var(\Delta s_{t+1})$. The regression residual is a direct estimate of the contribution of locally-traded and unspanned shocks to the depreciation rate, $\varepsilon_{t+1} = \ell_{t+1} + u_{t+1}$.

Combining with the results of regression (36), we can decompose variation in the depreciation rate into the contribution of globally-traded, locally-traded, and unspanned shocks. Specifically, we have $var(\beta^{g\prime}(\boldsymbol{w}^{*\prime}\boldsymbol{r}^*_{t+1}))$ for globally-traded shocks, and $var(\varepsilon_{t+1}) - var(u_{t+1})$ for locally-traded shocks, where u_{t+1} was obtained at the previous step from regression (36). Figure 5 reports these quantities as fraction of the variation in depreciation rate; the contributions mechanically add up to 1.

Figure 6: Decomposition of exchange rate innovations by horizon



The figure reports the cross-sectional average fraction of variance in exchange rates explained by globally-traded and locally-traded shocks, and shocks that are not spanned by asset returns, under the assumption of an intermediated market structure. Each bar is computed for a different depreciation horizon; globally-traded shocks are measured using CCA for stock and sovereign bond returns.

For all currencies, at least half of the variation in exchange rates is unspanned by asset returns. Globally-traded risks contribute up to 25% to variation in the depreciation rates (e.g., Australia and Canada), and frequently much less. These estimates should be seen as an upper bound on the role of globally-traded risks; remember that we include any pair of portfolios with correlation above 60%, far from the strict Definition 1.

In light of Proposition 4, the relatively modest role of globally-traded shocks $var(g_{t+1})$ in this intermediated market structure implies weak restrictions between exchange rate risks and IMRSs. Models of this kind are capable of resolving the cyclicality and volatility puzzles, as illustrated in Figure 3 with the lower red cone. The low prevalence of globally-traded shocks provides a justification for the widely documented pattern of disconnect (Meese and Rogoff, 1983; Obstfeld and Rogoff, 2001). Australia and Canada, the only countries with a somewhat larger globally-traded fraction are "commodity currencies" with exchange rates related to the price of oil and other commodities (Chen and Rogoff, 2003).

Some theories posit that Euler equations for US Treasury bonds hold with wedges (e.g., Jiang, Krishnamurthy, and Lustig, 2021). We entertain this possibility by removing US

Table 2: Spanning of depreciation rates by asset returns under different scenarios: R^2

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
Unconditional	39.15	44.58	29.10	31.09	35.77	34.60	34.55	26.43	34.28
Conditional	43.81	44.84	40.28	46.59	39.50	33.33	34.79	32.31	43.97
Before GFC	16.48	31.01	21.87	26.16	16.89	11.53	23.14	28.45	30.26
After GFC	54.98	77.45	56.09	62.74	56.41	50.01	51.23	52.80	52.26
Low VIX	27.50	37.33	34.57	31.31	30.67	21.30	37.74	29.10	42.54
High VIX	61.10	59.69	34.80	41.58	44.87	50.62	48.90	27.94	35.68

The table reports the adjusted R^2 of a regression of the depreciation rate on various subsets of asset returns, as in equation (36). Domestic asset returns are in domestic currency; foreign asset returns are in foreign currency. We use all assets, that is, bonds of maturities between 2 and 10 years, and various stock portfolios: the market (a combination of large and mid-cap stocks), plus value and growth portfolios, plus 10 industry portfolios (all from MSCI). The first row (Unconditional) repeats the results from Table 1. The subsequent rows report variations in conditioning variables.

bonds from the empirical analysis, which corresponds to excluding them from the set H. Appendix F reports the results, which are qualitatively similar to the baseline.

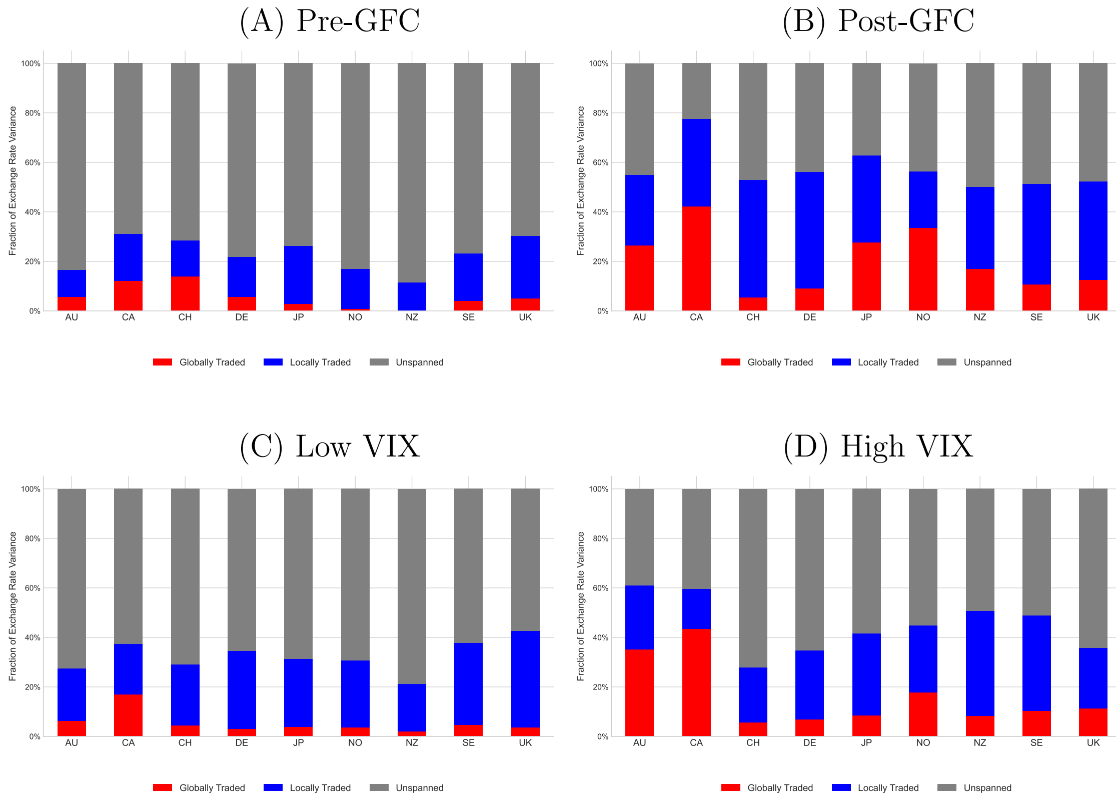
We also consider how the decomposition changes with the trading horizon, i.e., the period over which the depreciation rate and asset returns are computed. To conserve space, Figure 6 reports the cross-country average of the decomposition by horizon. The fraction of exchange rate variation explained by the globally-traded component increases progressively as we go from one month to one year, up to about 35%. This suggests that asset returns are more closely related to each other at longer horizon, potentially opening better risk-sharing opportunities. This observation is also consistent with Kekre and Lenel (2024) who document an increased correlation of exchange rates with interest rates at long horizons.

5.3 Conditional analysis

We evaluate how conditioning information affects the conclusions of the previous section. In particular, we are interested in gauging how a varying degree of similarity of assets across borders affects the degree of connect.

As a first cut, we consider a conditional version of the spanning regression in equation (36) where the betas are allowed to be linear functions of lagged values of the VIX, annual U.S. GDP growth, and EBP. The second row of Table 2 reports the results. Comparing to

Figure 7: Decomposition of exchange rate innovations by subsamples



The figure reports the fraction of variance in exchange rates explained by globally-traded and locally-traded shocks, and shocks that are not spanned by asset returns, under the assumption of an intermediated market structure. Each bar is a different country's currency relative to the U.S. dollar; globally-traded shocks are measured using CCA for stock and sovereign bond returns. Different panels show the results for different subsamples as indicated.

the first row, which summarizes the preceding unconditional results, we see that conditional spanning improves explanatory power, but not by much. The largest relative improvement of 50% occurs for Japan, while countries like Canada, New Zealand, and Sweden see almost no change. The largest adjusted R^2 (for Japan) is 47%, in line with the what we reported for the unconditional analysis.

Next, we isolate two sources of variations over our sample. First, we focus on the role of the global financial crisis (GFC). We implement the spanning regression before (up to November 2007) and after (post July 2009) the crisis. The third and fourth rows of Table 2 report these results. Spanning markedly improves after the GFC: adjusted R^2 range between 10% and 30% before the GFC and between 50% and 80% after. In other words, asset returns are more closely related to the exchange rate in the latter part of the sample. Second, we consider variation in stress in financial markets, which we proxy using VIX values below or above the median. Adjusted R^2 markedly increase from the low to high VIX environments as reported in rows 5 and 6.

To consider the implications for connect, we turn to identifying globally traded components. Our framework predicts a more connected exchange rate when the globally traded component is stronger. We implement the regression of equation (37) in each subsample: before and after GFC, low and high VIX. Figure 7 plots the resulting decomposition of the exchange rate variance into globally traded, locally traded and unspanned components for each of these cases. We observe a dramatic improvement in the importance of the globally-traded component in the second part of the sample, in line with the heightened relation of exchange rate with fundamentals documented in Engel and Wu (2024). Periods of high VIX also go along with an increased globally-traded component, consistent with the conditional connect evidence in Stavrakeva and Tang (2024).

6 Conclusion

In this paper, we propose a general framework for understanding how financial markets determine the behavior of exchange rates. Our theory accommodates many settings: complete or incomplete markets, arbitrary forms of market integration, or situations in which international financial trade happens through intermediaries. We characterize all restrictions on the behavior of exchange rates due to international portfolio risk sharing. These restrictions can be summarized by two conditions that share the simplicity of the complete market result while having richer implications.

We use these results to study different market structures which yields new insights on the interaction of financial markets and the exchange rate. First, classic puzzles of exchange rate volatility and cyclicality stem from large globally-traded risk exposure that connects currencies to stochastic discount factors. These anomalies survive across market configurations whenever this global risk component remains substantial. Second, achieving disconnect doesn't demand financial autarky or extreme forms of segmentation. The critical feature is small globally-traded risks, not segmentation itself. Sophisticated intermediaries can operate vast trading networks, households can hold diversified portfolios with significant return correlations, yet currencies remain disconnected from fundamentals provided these instruments cannot generate extensive globally-traded risk exposure. Third, our theory generates empirically-testable implications across diverse market architectures: currency–fundamental relationships strengthen when international asset returns are more synchronized. Evidence confirms this mechanism: exchange rates track fundamentals more closely during episodes of heightened cross-border return correlation, whether following the global financial crisis or during volatility spikes.

References

Adrian, T., E. Etula, and H. S. Shin (2010): "Risk Appetite and Exchange Rates," *FRB of New York Staff Report*, (361).

Aguiar, M., O. Itskhoki, and D. Mukhin (2025): "How Good is International Risk Sharing? Stepping outside the Shadow of the Welfare Theorems," .

Alvarez, F., A. Atkeson, and P. J. Kehoe (2002): "Money, interest rates, and exchange rates with endogenously segmented markets," *Journal of political Economy*, 110(1), 73–112.

——— (2007): "If exchange rates are random walks, then almost everything we say about monetary policy is wrong," *American Economic Review*, 97(2), 339–345.

——— (2009): "Time-varying risk, interest rates, and exchange rates in general equilibrium," *The Review of Economic Studies*, 76(3), 851–878.

Augustin, P., M. Chernov, L. Schmid, and D. Song (2024): "The term structure of covered interest rate parity violations," *The Journal of Finance*, 79(3), 2077–2114.

Avdjiev, S., W. Du, C. Koch, and H. S. Shin (2019): "The dollar, bank leverage, and deviations from covered interest parity," *American Economic Review: Insights*, 1(2), 193–208.

Ayres, J., C. Hevia, and J. P. Nicolini (2020): "Real exchange rates and primary commodity prices," *Journal of International Economics*, 122, 103261.

Backus, D., and G. Smith (1993): "Consumption and real exchange rates in dynamic economies with non-traded goods," *Journal of International Economics*, 35, 297–316.

Backus, D. K., S. Foresi, and C. I. Telmer (2001): "Affine Term Structure Models and the Forward Premium Anomaly," *Journal of Finance*, 56(1).

Backus, D. K., P. J. Kehoe, and F. E. Kydland (1992): "International real business cycles," *Journal of political Economy*, 100(4), 745–775.

Bansal, R. (1997): "An exploration of the forward premium puzzle in currency markets," *Review of Financial Studies*, 10(2), 369–403.

Benigno, G., and C. Thoenissen (2008): "Consumption and real exchange rates with incomplete markets and non-traded goods," *Journal of International Money and Finance*, 27(6), 926–948.

Blanchard, O. J., and C. M. Kahn (1980): "The solution of linear difference models under rational expectations," *Econometrica*, pp. 1305–1311.

Brandt, M., J. Cochrane, and P. Santa-Clara (2006): "International risk sharing is better than you think, or exchange rates are too smooth," *Journal of Monetary Economics*, 53, 671–698.

Camanho, N., H. Hau, and H. Rey (2022): "Global portfolio rebalancing and exchange rates," *The Review of Financial Studies*, 35(11), 5228–5274.

Campbell, J. Y., K. Serfaty-De Medeiros, and L. M. Viceira (2010): "Global currency hedging," *The Journal of Finance*, 65(1), 87–121.

Campbell, J. Y., and L. Viceira (2002): *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors*. Oxford University Press, Oxford, UK, first edn.

Chahrour, R., V. Cormun, P. De Leo, P. Guerrón-Quintana, and R. Valchev (2023): "Exchange Rate Disconnect Revisited," working paper.

Chen, Y.-c., and K. Rogoff (2003): "Commodity currencies," *Journal of international Economics*, 60(1), 133–160.

Chernov, M., and D. Creal (2023): "International yield curves and currency puzzles," *Journal of Finance*, 78, 209–245.

Chernov, M., M. Dahlquist, and L. Lochstoer (2023): "Pricing currency risks," *The Journal of Finance*, 78(2), 693–730.

Cochrane, J. (2004): *Asset Pricing*. Princeton University Press, Princeton, NJ, revised edn.

Cochrane, J., and J. Saa-Requejo (2000): "Beyond Arbitrage: Good-Deal Asset Price Bounds in Incomplete Markets," *Journal of Political Economy*, 108(1), 79–119.

Colacito, R., and M. M. Croce (2011): "Risks for the long run and the real exchange rate," *Journal of Political Economy*, 119, 153–181.

Corsetti, G., L. Dedola, and S. Leduc (2008): "International risk sharing and the transmission of productivity shocks," *The Review of Economic Studies*, 75(2), 443–473.

Daniel, K., R. J. Hodrick, and Z. Lu (2017): "The carry trade: Risks and drawdowns," *Critical Finance Review*, 6, 211–262.

Dao, M., P.-O. Gourinchas, and O. Itskhoki (2025): "Breaking Parity: Equilibrium exchange rates and currency premia," IMF Working Paper.

Devereux, M. B., and A. Sutherland (2011): "Country portfolios in open economy macromodels," *Journal of the european economic Association*, 9(2), 337–369.

Du, W., A. Tepper, and A. Verdelhan (2018): "Deviations from Covered Interest Rate Parity," *Journal of Finance*, 73, 915–957.

Eichenbaum, M. S., B. K. Johannsen, and S. T. Rebelo (2021): "Monetary policy and the predictability of nominal exchange rates," *The Review of Economic Studies*, 88(1), 192–228.

Engel, C. (2014): "Exchange rates and interest parity," *Handbook of international economics*, 4, 453–522.

Engel, C., and A. Matsumoto (2009): "The international diversification puzzle when goods prices are sticky: it's really about exchange-rate hedging, not equity portfolios," *American Economic Journal: Macroeconomics*, 1(2), 155–188.

Engel, C., and K. D. West (2005): "Exchange Rates and Fundamentals," *Journal of Political Economy*, 113(3), 485–517.

Engel, C., and S. P. Y. Wu (2024): "Exchange Rate Models are Better than You Think, and Why They Didn't Work in the Old Days," Discussion paper, National Bureau of Economic Research.

Fama, E. F. (1984): "Forward and spot exchange rates," *Journal of Monetary Economics*, 14(3), 319–338.

Farhi, E., and X. Gabaix (2016): "Rare disasters and exchange rates," *Quarterly Journal of Economics*, 131(1), 1–52.

French, K. R., and J. M. Poterba (1991): "Investor Diversification and International Equity Markets," *The American Economic Review*, 81(2), 222.

Fukui, M., E. Nakamura, and J. Steinsson (2023): "The Macroeconomic Consequences of Exchange Rate Depreciations," NBER Working Paper No. 31279.

Gabaix, X., and M. Maggiori (2015): " International Liquidity and Exchange Rate Dynamics," *The Quarterly Journal of Economics*, 130(3), 1369–1420.

Garleanu, N., and L. H. Pedersen (2011): "Margin-based Asset Pricing and Deviations from the Law of One Price," *Review of Financial Studies*, 24(6), 1980–2022.

Gilchrist, S., and E. Zakrajsek (2012): "Credit Spreads and Business Cycle Fluctuations," *American Economic Review*, 102(4), 1692–1720.

Gourinchas, P.-O., W. Ray, and D. Vayanos (2022): "A preferred-habitat model of term premia, exchange rates, and monetary policy spillovers," Working Paper.

Gourinchas, P.-O., and H. Rey (2007): "International financial adjustment," *Journal of Political Economy*, 115(4), 665–703.

Greenwood, R., S. G. Hanson, J. C. Stein, and A. Sunderam (2022): "A quantity-driven theory of term premia and exchange rates," forthcoming in *Quarterly Journal of Economics.*

Haddad, V., and T. Muir (2025): "Intermediaries and Asset Prices," Discussion paper, National Bureau of Economic Research.

Hansen, L., and R. Jagannathan (1991): "Implications of security market data for models of dynamic economies," *Journal of Political Economy*, 99, 225–262.

Hansen, L., and S. Richard (1987): "The Role of Conditioning Information in Deducing Testable Restrictions Implied by Dynamic Asset Pricing Models," *Econometrica*, 55, 587–613.

Hassan, T. A., T. Mertens, and J. Wang (2024): "A Currency Premium Puzzle," Working paper.

Hau, H., and H. Rey (2004): "Can portfolio rebalancing explain the dynamics of equity returns, equity flows, and exchange rates?," *American Economic Review*, 94(2), 126–133.

——— (2006): "Exchange rates, equity prices, and capital flows," *The Review of Financial Studies*, 19(1), 273–317.

He, Z., and A. Krishnamurthy (2018): "Intermediary asset pricing and the financial crisis," *Annual Review of Financial Economics*, 10(1), 173–197.

Heathcote, J., and F. Perri (2014): "Assessing international efficiency," in *Handbook of International Economics*, vol. 4, pp. 523–584. Elsevier.

Itskhoki, O., and D. Mukhin (2021): "Exchange Rate Disconnect in General Equilibrium," *Journal of Political Economy*, 129(8), 2183–2232.

——— (2023): "Optimal Exchange Rate Policy," NBER Working Paper No. 31933.

——— (2024): "What Drives the Exchange Rate?," *IMF Economic Review*, forthcoming.

——— (2025): "Mussa puzzle redux," *Econometrica*, 93(1), 1–39.

Ivashina, V., D. S. Scharfstein, and J. C. Stein (2015): "Dollar Funding and the Lending Behavior of Global Banks," *Quarterly Journal of Economics*, 130(3), 1241–1281.

Jeanne, O., and A. K. Rose (2002): "Noise trading and exchange rate regimes," *The Quarterly Journal of Economics*, 117(2), 537–569.

Jiang, Z., A. Krishnamurthy, and H. Lustig (2021): "Foreign safe asset demand and the dollar exchange rate," *The Journal of Finance*, 76(3), 1049–1089.

——— (2023a): "Dollar safety and the global financial cycle," *Review of Economic Studies*, p. rdad108.

——— (2023b): "Implications of Asset Market Data for Equilibrium Models of Exchange Rates," Discussion paper, National Bureau of Economic Research.

Jiang, Z., A. Krishnamurthy, H. Lustig, and J. Sun (2024): "Convenience Yields and

Exchange Rate Puzzles," working paper.

Kekre, R., and M. Lenel (2023): "The Flight to Safety and International Risk Sharing," *American Economic Review*, forthcoming.

——— (2024): "Exchange Rates, Natural Rates, and the Price of Risk," working paper.

Kocherlakota, N. R., and L. Pistaferri (2007): "Household heterogeneity and real exchange rates," *The Economic Journal*, 117(519), C1–C25.

——— (2008): "Inequality and real exchange rates," *Journal of the European Economic Association*, 6(2-3), 597–608.

Kollmann, R. (1995): "Consumption, real exchange rates and the structure of international asset markets," *Journal of International money and finance*, 14(2), 191–211.

Kozak, S., S. Nagel, and S. Santosh (2020): "Shrinking the cross-section," *Journal of Financial Economics*, 135(2), 271–292.

Kremens, L., and I. Martin (2019): "The quanto theory of exchange rates," *American Economic Review*, 109(3), 810–843.

Krishnamurthy, A., and A. Vissing-Jorgensen (2012): "The aggregate demand for treasury debt," *Journal of Political Economy*, 120(2), 233–267.

Lewis, K. K. (1999): "Trying to explain home bias in equities and consumption," *Journal of economic literature*, 37(2), 571–608.

Lewis, K. K., and E. Liu (2022): "How Can Asset Prices Value Exchange Rate Wedges?," Discussion paper, National Bureau of Economic Research.

Lilley, A., M. Maggiori, B. Neiman, and J. Schreger (2022): "Exchange rate reconnect," *Review of Economics and Statistics*, 104(4), 845–855.

Lustig, H., A. Stathopolous, and A. Verdelhan (2019): "The Term Structure of Currency Carry Trade Risk Premia," *American Economic Review*, 109(2), 4142–4177.

Lustig, H., and A. Verdelhan (2019): "Does Incomplete Spanning in International Financial Markets Help to Explain Exchange Rates?," *American Economic Review*, 109(6), 2208–44.

Marin, E. A., and S. R. Singh (2023): "Low Risk Sharing with Many Assets," Discussion paper.

Maurer, T., and N.-K. Tran (2021): "Entangled risks in incomplete FX markets," *Journal of Financial Economics*, 142(1), 146–165.

Meese, R. A., and K. Rogoff (1983): "Empirical exchange rate models of the seventies: Do they fit out of sample?," *Journal of International Economics*, 14(1–2), 3–24.

Miranda-Agrippino, S., and H. Rey (2020): "U.S. Monetary Policy and the Global Financial Cycle," *Review of Economic Studies*, 87, 2754–2776.

Obstfeld, M., and K. Rogoff (2001): "The six major puzzles in international macroeconomics: Is there a common cause?," in *NBER Macroeconomics Annual 2000, Volume 15*, ed. by B. Bernanke, and K. Rogoff, pp. 339–390. MIT Press, Cambridge.

Pavlova, A., and R. Rigobon (2007): "Asset Prices and Exchange Rates," *The Review of Financial Studies*, 20(4), 1139–1180.

Ross, S. A. (1976): "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory*, 13(3), 341–360.

Samuelson, P. (1970): "The Fundamental Approximation Theorem of Portfolio Analysis in terms of Means, Variances and Higher Moments," *The Review of Economic Studies*, 37(4), 537–542.

Sandulescu, M., F. Trojani, and A. Vedolin (2021): "Model-Free International Stochastic Discount Factors," *Journal of Finance*, 76, 935–976.

Stavrakeva, V., and J. Tang (2019): "The dollar during the great recession: US monetary policy signaling and the flight to safety," CEPR Discussion Paper No. DP14034.

——— (2024): "A fundamental connection: Exchange rates and macroeconomic expectations," *Review of Economics and Statistics*, pp. 1–49.

Tille, C., and E. Van Wincoop (2010): "International capital flows," *Journal of international Economics*, 80(2), 157–175.

Verdelhan, A. (2010): "A habit-based explanation of the exchange rate risk premium.," *Journal of Finance*, 65(1), 123–146.

Zhang, S. (2021): "Limited risk sharing and international equity returns," *The Journal of Finance*, 76(2), 893–933.

Online Appendix

A Solution of the Model in Section 1

A.1 Optimization Problems

Foreign maximizes lifetime utility, taking as given the decisions of the intermediary and noise traders

$$\begin{aligned}
&\max_{\{C^*_{Tt}, D^{*\mathcal{F}}_{i,t}\}} E \sum_t e^{-\delta t} \log\left(C^*_{Tt}\right) \\
&\text{s.t. } C^*_{Tt} + \sum_{i\in\mathcal{F}} D^{*\mathcal{F}}_{i,t} + \sum_{i\in\mathcal{I}} D^{*\mathcal{I}}_{i,t} + \sum_{i\in\mathcal{N}} D^{*\mathcal{N}}_{i,t} \\
&\qquad \le Y^*_{Tt} + \sum_{i\in\mathcal{H}} D^{*\mathcal{F}}_{i,t-1} R^*_{i,t} + \sum_{i\in\mathcal{I}} D^{*\mathcal{I}}_{i,t-1} R^*_{i,t} + \sum_{i\in\mathcal{N}} D^{*\mathcal{N}}_{i,t-1} R^*_{i,t}, \\
&\qquad D^{*\mathcal{F}}_{i,-1} = 0,
\end{aligned}$$

where $D^{*\mathcal{C}}_i$ denotes demand of various constituencies ($\mathcal{C} = \mathcal{F}$, or $\mathcal{H}$, or $\mathcal{N}$, or $\mathcal{I}$) for foreign asset i. Because Foreign is very large relative to other market participants, positions of intermediaries and noise traders, as well as any trade with the home country becomes negligible relative to the endowment.

Because of its size, Foreign behaves as if it were in a single-agent endowment economy. The first-order conditions of this problem imply the Euler equations:

$$\forall i \in \mathcal{F}, \quad E_t\left[M^*_{t+1} R^*_{i,t+1}\right] = 1, \quad M^*_{t+1} = e^{-\delta} \frac{C^*_{Tt}}{C^*_{Tt+1}}, \tag{A1}$$

and

$$C^*_{Tt} = Y^*_{Tt}.$$

Home maximizes lifetime utility

$$\begin{aligned}
&\max_{\{C_{Tt}, C_{Nt}, D_{i,t}\}} E \sum_t e^{-\delta t} \left(\log\left(C_{Tt}\right) + \log\left(C_{Nt}\right)\right) \\
&\text{s.t. } C_{Tt} + \frac{1}{S_t} C_{Nt} + \sum_{i\in\mathcal{H}} D^{*\mathcal{H}}_{i,t} \le Y_{Tt} + \frac{1}{S_t} Y_{Nt} + \sum_{i\in\mathcal{H}} D^{*\mathcal{H}}_{i,t-1} R^*_{i,t}, \qquad \text{(A2)} \\
&\qquad D^{*\mathcal{H}}_{i,-1} = 0.
\end{aligned}$$

First, focus on consumption of non-tradables. The within period marginal rate of substitution between tradables and non-tradables have to equal their ratio of prices, the exchange rate. Second, in equilibrium, the home agent consumes their endowment of nontradables because they have nobody to exchange it with. This gives the two conditions:

$$C_{Nt} = Y_{Nt}, \text{ and } C_{Nt} = S_t C_{Tt}. \tag{A3}$$

Lastly,

$$\forall i \in \mathcal{H}, \quad E_t\left[e^{-\delta}\frac{C_{Tt}}{C_{Tt+1}}R^*_{i,t+1}\right] = 1. \tag{A4}$$

The intermediary enters enters the period with wealth $W_{0,t}$, and chooses its optimal portfolio to maximize expected utility of final wealth $W_{1,t+1}$. it solves:

$$\max_{D^{*\mathcal{I}}_{i,t}} E_t\left(\frac{1}{1-\gamma_t}W^{*1-\gamma_t}_{1.t+1}\right) \tag{A5}$$

$$\text{s.t. } W^*_{1.t+1} = \sum_{i\in\mathcal{I}} D^{*\mathcal{I}}_{i,t}R^{*\mathcal{I}}_{i,t+1} \text{ and } \sum_{i\in\mathcal{I}} D^{*\mathcal{I}}_{i,t} = W^*_{0,t}. \tag{A6}$$

Finally, consider market clearing for the assets. For any asset in $\mathcal{F}$, because the Foreign is much larger than all the other players, the price is pinned down by the foreign Euler equation. The Foreign effectively provides the asset perfectly elastically at this price, so there is no need for a market clearing condition. For all other assets, the condition is:

$$D^{*\mathcal{H}}_{i,t} + D^{*\mathcal{I}}_{i,t} + D^{*\mathcal{N}}_{i,t} = 0. \tag{A7}$$

A.2 Simplified Economy

At Home, Equation (A3) implies static trade-off across goods:

$$y_{Nt} - c_{Tt} = s_t.$$

The "savings" Euler equation arises from applying (A4) to the risk-free asset

$$r_{ft} = E_t\left[\Delta y_{Nt+1}\right] - \frac{1}{2}var_t(\Delta y_{Nt+1}) + \delta.$$

For any other asset i, we get

$$E_t\left[r_{i,t+1}\right] - r_{ft} + \frac{1}{2}var_t(r_{i,t+1}) = -cov_t\left(-\Delta y_{Nt+1}, r_{i,t+1}\right), \forall i \in \mathcal{H}$$

Equations (A2) and (A3) imply the budget constraint. Indeed, in foreign currency we get:

$$C_{Tt} + \sum_{i\in\mathcal{H}} D^{*\mathcal{H}}_{i,t} = Y_{Tt} + \sum_{i\in\mathcal{H}} D^{*\mathcal{H}}_{i,t-1}R^*_{i,t}$$

(we use equality because it binds).

First, replace all returns but for the first asset (which is home risk-free) in terms of excess returns:

$$C_{Tt} + \sum_{i\in\mathcal{H}} D^{*\mathcal{H}}_{i,t} = Y_{Tt} + \left(\sum_{i\in\mathcal{H}} D^{*\mathcal{H}}_{i,t-1}\right)R^*_{1t} + \sum_{i\in\mathcal{H}, i\neq 1} D^{*\mathcal{H}}_{i,t-1}\left(R^*_{i,t} - R^*_{1,t}\right).$$

Denote net foreign assets by $B_t = \sum_{i\in\mathcal{H}} D^{*\mathcal{H}}_{i,t}$. Then,

$$C_{Tt} + B_t = Y_{Tt} + B_{t-1}R^*_{1t} + \sum_{i\in\mathcal{H}, i\neq 1} D^{*\mathcal{H}}_{i,t-1}\left(R^*_{i,t} - R^*_{1,t}\right).$$

We linearize this equation around $\bar{B} = 0$, $\overline{R^*_{i,t} - R^*_{1,t}} = 0$, and $\bar{C}_T = \bar{Y}_T$ and rearrange:

$$dB_t - dB_{t-1}\bar{R}^*_1 - \sum_{i\in\mathcal{H}, i\neq 1} \bar{D}^{*\mathcal{H}}_i d\left(R^*_{i,t} - R^*_{1,t}\right) = dY_{Tt} - dC_{Tt}.$$

Use $\beta = \left(\bar{R}^*_1\right)^{-1} \approx e^{-\delta}$ and define the scaled saving positions $b_t = dB_t/\bar{Y}_T$ and $d^{*\mathcal{H}}_i = \bar{D}^{*\mathcal{H}}_i/\bar{Y}_T$, and replace the excess return by the log-difference:

$$b_t - \beta^{-1}b_{t-1} - \sum_{i\in\mathcal{H}, i\neq 1} d^{*\mathcal{H}}_i\left(r^*_{i,t} - r^*_{1,t}\right) = y_{Tt} - c_{Tt}.$$

In Foreign the Euler equations are

$$r^*_{ft} = E_t\left[\Delta y^*_{Tt+1}\right] - \frac{1}{2}var_t\left(\Delta y^*_{Tt+1}\right) + \delta,$$
$$E_t\left[r^*_{i,t+1}\right] - r^*_{ft} + \frac{1}{2}var_t\left(r^*_{i,t+1}\right) = -cov_t\left(-\Delta y^*_{Tt+1}, r^*_{i,t+1}\right), \forall i\in\mathcal{F}$$

using Equation (A1).

We can log-linearize the intermediary optimization problem (A5) and (A6) as in Campbell and Viceira (2002); see Appendix B.1. The optimization problem becomes:

$$\max_{r_{p,t+1}} E_t\left[r^*_{p,t+1}\right] + \frac{1}{2}\left(1-\gamma_t\right)var_t\left(r^*_{p,t+1}\right).$$

The optimal portfolio position in risky asset is

$$\frac{\mathbf{D}^{*\mathcal{I}}_t}{W^*_t} = \frac{1}{\gamma_t}\mathbf{\Sigma}^{-1}_t\left[E_t\left(\mathbf{r}^*_{t+1}\right) - r^*_{ft} + diag(\mathbf{\Sigma}_t)/2\right],$$

where boldfaced notation is used for vectors and matrices. We assume that each period, intermediaries start with wealth $W^*_{0t} = \bar{Y}_T$ and define the scaled investments as $d^{*\mathcal{I}}_{it} = D^{*\mathcal{I}}_{it}/\bar{Y}_T$. Then the equation becomes:

$$\mathbf{d}^{*\mathcal{I}}_t = \frac{1}{\gamma_t}\mathbf{\Sigma}^{-1}_t\left[E_t\left(\mathbf{r}^*_{t+1}\right) - r^*_{ft} + diag(\mathbf{\Sigma}_t)/2\right].$$

Finally we scale the demand from noise traders by $\bar{Y}_T$ as well, define $d^{*\mathcal{N}}_{it} = D^{*\mathcal{N}}_{it}/\bar{Y}_T$, and assume that it follows a set of autonomous AR(1). Markets clear:

$$d^{*\mathcal{H}}_i + d^{*\mathcal{I}}_{it} + d^{*\mathcal{N}}_{it} = 0$$

(when the foreign country does not participate).

To summarize, we solve the following set of equations:

$$
\begin{aligned}
y_{Nt} - c_{Tt} &= s_t, \\
r_{ft} &= E_t\left[\Delta y_{Nt+1}\right] - \frac{1}{2} var_t(\Delta y_{Nt+1}) + \delta, \\
E_t\left[r_{i,t+1}\right] - r_{ft} + \frac{1}{2} var_t(r_{i,t+1}) &= -cov_t\left(-\Delta y_{Nt+1}, r_{i,t+1}\right), \forall i \in \mathcal{H}, \\
r^*_{ft} &= E_t\left[\Delta y^*_{Tt+1}\right] - \frac{1}{2} var_t\left(\Delta y^*_{Tt+1}\right) + \delta, \\
E_t\left[r^*_{i,t+1}\right] - r^*_{ft} + \frac{1}{2} var_t\left(r^*_{i,t+1}\right) &= -cov_t\left(-\Delta y^*_{Tt+1}, r^*_{i,t+1}\right), \forall i \in \mathcal{F}, \\
r_{i,t+1} - \Delta s_{t+1} &= r^*_{i,t+1}, \\
b_t - \beta^{-1} b_{t-1} - \sum_{i \in \mathcal{H}, i \neq 1} d_i^{*\mathcal{H}}\left(r^*_{i,t} - r^*_{1,t}\right) &= y_{Tt} - c_{Tt}, \\
\lim_{t \to \infty} \beta^t b_t &= 0, \\
\mathbf{d}_t^{*\mathcal{I}} &= \frac{1}{\gamma_t} \mathbf{\Sigma}_t^{-1}\left[E_t\left(\mathbf{r}^*_{t+1}\right) - r^*_{ft} + diag(\mathbf{\Sigma}_t)/2\right], \\
d_{it}^{*\mathcal{H}} + d_{it}^{*\mathcal{I}} + d_{it}^{*\mathcal{N}} &= 0 \text{ if } i \notin \mathcal{F}
\end{aligned}
$$

for $d_{it}^{*\mathcal{I}}$, c_{Tt}, and s_t. We assume that the three endowments and noise-trader demand follow AR(1) processes with persistence ρ.

A.3 Solving various cases of the model

We specialize the system of equations above to particular market structures and solve for the exchange rate s_t. Noise traders only demand the carry trade, hence $d_t^{*\mathcal{N}}$ only has one dimension.

Segmented household trading

Each household trades their own risk-free bond. The intermediary trades both bonds.

Home budget constraint:

$$b_{t-1} = \beta b_t - \beta\left(y_{Tt} - c_{Tt}\right).$$

Home Euler:

$$r_{ft} = E_t\left[\Delta c_{Tt+1} + \Delta s_{t+1}\right] - \frac{1}{2} var_t(\Delta c_{Tt+1} + \Delta s_{t+1}) + \delta.$$

Foreign Euler:

$$r^*_{ft} = E_t\left[\Delta y^*_{Tt+1}\right] - \frac{1}{2} var_t\left(\Delta y^*_{Tt+1}\right) + \delta.$$

Optimal intermediary position:

$$\mathbf{d}_t^{*\mathcal{I}} = \frac{1}{\gamma} \mathbf{\Sigma}_t^{-1}\left[E_t\left(\mathbf{r}^*_{t+1}\right) - r^*_{ft} + diag(\mathbf{\Sigma}_t)/2\right].$$

The intermediary's only risky asset is the home risk-free bond: $r^*_{t+1} = r_{ft} - \Delta s_{t+1}$, so $\mathbf{d}_t^{*\mathcal{I}}$ is a scalar. This leads to:

$$d_t^{*\mathcal{I}} = \frac{1}{\gamma var_t\left(\Delta s_{t+1}\right)}\left[r_{ft} - r_{ft}^{*} - E_t\left[\Delta s_{t+1}\right] + var_t\left(\Delta s_{t+1}\right)/2\right].$$

Market-clearing for home risk-free bond:

$$b_t + d_t^{*\mathcal{I}} + d_t^{*\mathcal{N}} = 0.$$

To solve the model take the difference of Euler equations (without second order terms):

$$E_t\left[\Delta c_{Tt+1}\right] = r_{ft} - r_{ft}^{*} - E_t\left[\Delta s_{t+1}\right] + E_t\left[\Delta y_{Tt+1}^{*}\right].$$

Then notice that the right-hand-side coincides with intermediary demand (if we remove second-order terms in there too):

$$E_t\left[\Delta c_{Tt+1}\right] = \gamma var_t\left(\Delta s_{t+1}\right) d_t^{*\mathcal{I}} + E_t\left[\Delta y_{Tt+1}^{*}\right].$$

As an aside, note that this relation ties down intermediary demand to the UIP deviation ζ_t:

$$\zeta_t = E_t\left[\Delta s_{t+1}\right] - (r_{ft} - r_{ft}^{*}) = \gamma var_t(\Delta s_{t+1}) d_t^{*\mathcal{I}}$$

Going back to the previous relation, we can use market-clearing:

$$\begin{aligned} E_t\left[\Delta c_{Tt+1}\right] &= -\gamma var_t\left(\Delta s_{t+1}\right) d_t^{*\mathcal{N}} - \gamma var_t\left(\Delta s_{t+1}\right) b_t + E_t\left[\Delta y_{Tt+1}^{*}\right] \\ &= -\gamma\kappa d_t^{*\mathcal{N}} - \gamma\kappa b_t + E_t\left[\Delta y_{Tt+1}^{*}\right], \end{aligned}$$

where we define $\kappa = var_t\left(\Delta s_{t+1}\right)$ (and conjecture that this variance is constant). Write the budget constraint from t to $t+1$:

$$b_t = \beta E_t\left[b_{t+1}\right] - \beta E_t\left[y_{Tt+1} - c_{Tt+1}\right].$$

These equations lead to a system

$$\begin{aligned} b_t &= \beta E_t\left[b_{t+1}\right] + \beta E_t[c_{Tt+1}] - \beta E_t\left[y_{Tt+1}\right], \\ c_{Tt} - \gamma\kappa b_t &= E_t[c_{Tt+1}] + \gamma\kappa d_t^{*\mathcal{N}} - E_t\left[\Delta y_{Tt+1}^{*}\right]. \end{aligned}$$

Define $V_t = (b_t, c_{Tt})'$, then

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ -\gamma\kappa & 1 \end{pmatrix} V_t &= \begin{pmatrix} \beta & \beta \\ 0 & 1 \end{pmatrix} E_t\left[V_{t+1}\right] + Z_t, \\ E_t\left[V_{t+1}\right] &= \begin{pmatrix} \beta & \beta \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -\gamma\kappa & 1 \end{pmatrix} V_t - \begin{pmatrix} \beta & \beta \\ 0 & 1 \end{pmatrix}^{-1} Z_t, \\ &= \begin{pmatrix} \frac{1}{\beta} + \gamma\kappa & -1 \\ -\gamma\kappa & 1 \end{pmatrix} V_t - \begin{pmatrix} \beta & \beta \\ 0 & 1 \end{pmatrix}^{-1} Z_t, \\ E_t\left[V_{t+1}\right] &= AV_t + \hat{Z}_t \end{aligned}$$

with

$$Z_t = \begin{pmatrix} -\beta E_t\left[y_{Tt+1}\right] \\ \gamma\kappa d_t^{*\mathcal{N}} - E_t\left[\Delta y_{Tt+1}^*\right] \end{pmatrix} = \begin{pmatrix} -\beta\rho y_{Tt} \\ \gamma\kappa d_t^{*\mathcal{N}} - (\rho-1)\, y_{Tt}^* \end{pmatrix},$$

$$\hat{Z}_t = -\begin{pmatrix} \beta & \beta \\ 0 & 1 \end{pmatrix}^{-1} Z_t = -\begin{pmatrix} \frac{1}{\beta} & -1 \\ 0 & 1 \end{pmatrix} Z_t,$$

$$\hat{Z}_t = \begin{pmatrix} \rho y_{Tt} + \gamma\kappa d_t^{*\mathcal{N}} - (\rho-1) y_{Tt}^* \\ -\gamma\kappa d_t^{*\mathcal{N}} + (\rho-1) y_{Tt}^*. \end{pmatrix}$$

Next, we follow Blanchard and Kahn (1980) and compute eigenvalues and left eigenvectors of the matrix A:

$$\lambda_1 = \frac{\left(\frac{1}{\beta}+\gamma\kappa+1\right)+\sqrt{\left(\frac{1}{\beta}+\gamma\kappa+1\right)^2-\frac{4}{\beta}}}{2} \geq \frac{1}{\beta} > 1, \qquad v_1 = \begin{pmatrix} 1-\lambda_1 \\ 1 \end{pmatrix}, \tag{A8}$$

$$\lambda_2 = \frac{\left(\frac{1}{\beta}+\gamma\kappa+1\right)-\sqrt{\left(\frac{1}{\beta}+\gamma\kappa+1\right)^2-\frac{4}{\beta}}}{2} \leq 1, \qquad v_2 = \begin{pmatrix} 1-\lambda_2 \\ 1 \end{pmatrix}.$$

Therefore, $\lambda_i = \lambda_i(\kappa)$ depend on κ such that $\lambda_1(0) = 1/\beta$ and $\lambda_2(0) = 1$, and $\lambda_1(\kappa) > 1/\beta$ and $\lambda_2(\kappa) < 1$ for $\kappa > 0$. Note that in the notation of the main text in Section 1, $\lambda_\kappa = 1/\lambda_1(\kappa) \leq \beta$.

Define $x_t = v_1' V_t$, the linear combination corresponding to the exploding root. Then:

$$E_t\left[x_{t+1}\right] = \lambda_1 x_t + v_1'\hat{Z}_t,$$
$$x_t = \frac{1}{\lambda_1}E_t\left[x_{t+1}\right] - \frac{1}{\lambda_1}v_1'\hat{Z}_t.$$

There is a unique stationary solution to this equation, satisfing the transversality condition, which is:

$$\begin{aligned} x_t &= \sum_{k=0}^{\infty}\left(\frac{1}{\lambda_1}\right)^k\left[-\frac{1}{\lambda_1}v_1'E_t\left[\hat{Z}_{t+k}\right]\right] \\ &= \sum_{k=0}^{\infty}\left(\frac{\rho}{\lambda_1}\right)^k\left[-\frac{1}{\lambda_1}v_1'\hat{Z}_t\right] \\ &= \frac{-1}{1-\frac{\rho}{\lambda_1}}\frac{1}{\lambda_1}v_1'\hat{Z}_t = -\frac{1}{\lambda_1-\rho}v_1'\hat{Z}_t, \\ (1-\lambda_1)\, b_t + c_{Tt} &= -\frac{1}{\lambda_1-\rho}v_1'\hat{Z}_t, \\ b_t &= -\frac{1}{1-\lambda_1}\frac{1}{\lambda_1-\rho}v_1'\hat{Z}_t - \frac{1}{1-\lambda_1}c_{Tt}. \end{aligned}$$

We plug this expression for b_t back into the budget constraint $\beta^{-1}b_{t-1} = b_t + c_{Tt} - y_{Tt}$:

$$\frac{1}{\beta}b_{t-1} = \left(1 - \frac{1}{1-\lambda_1}\right) c_{Tt} - \frac{1}{1-\lambda_1}\frac{1}{\lambda_1 - \rho} v_1' \hat{Z}_t - y_{Tt}$$
$$\frac{1}{\beta}b_{t-1} = -\frac{\lambda_1}{1-\lambda_1} c_{Tt} - \frac{\lambda_1}{\lambda_1-\rho} y_{Tt} - \frac{\lambda_1(\rho-1)}{(1-\lambda_1)(\lambda_1-\rho)} y^*_{Tt} + \frac{\lambda_1 \gamma\kappa}{(1-\lambda_1)(\lambda_1-\rho)} d_t^{*\mathcal{N}}$$
$$c_{Tt} = \frac{1-\lambda_1^{-1}}{\beta} b_{t-1} + \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}} y_{Tt} + \frac{\lambda_1^{-1}(1-\rho)}{1-\rho\lambda_1^{-1}} y^*_{Tt} + \frac{\lambda_1^{-1}\gamma\kappa}{1-\rho\lambda_1^{-1}} d_t^{*\mathcal{N}}$$
$$s_t = y_{Nt} - c_{Tt} = -\frac{1-\lambda_1^{-1}}{\beta} b_{t-1} + \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}} (y_{Nt} - y_{Tt}) + \frac{\lambda_1^{-1}(1-\rho)}{1-\rho\lambda_1^{-1}} (y_{Nt} - y^*_{Tt}) - \frac{\lambda_1^{-1}\gamma\kappa}{1-\rho\lambda_1^{-1}} d_t^{*\mathcal{N}}.$$

Finally, there is a condition for $\kappa = var_t(\Delta s_{t+1})$:

$$\kappa = var\left(\frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}} \tilde{s}_{A,t} + \frac{\lambda_1^{-1}(1-\rho)}{1-\rho\lambda_1^{-1}} \tilde{s}_{C,t} - \frac{\lambda_1^{-1}\gamma\kappa}{1-\rho\lambda_1^{-1}} \tilde{d}_t^{*\mathcal{N}}\right).$$

We can re-write the solution for s_t in terms of innovations, e.g., $\tilde{s}_t = s_t - E_{t-1}s_t$:

$$\tilde{s}_t = \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}} (\tilde{y}_{Nt} - \tilde{y}_{Tt}) + \frac{\lambda_1^{-1}(1-\rho)}{1-\rho\lambda_1^{-1}} (\tilde{y}_{Nt} - \tilde{y}^*_{Tt}) - \frac{\lambda_1^{-1}\gamma\kappa}{1-\rho\lambda_1^{-1}} \tilde{d}_t^{*\mathcal{N}}.$$

Define $\tilde{s}_{C,t} = (\tilde{y}_{Nt} - \tilde{y}^*_{Tt})$ and $\tilde{s}_{A,t} = (\tilde{y}_{Nt} - \tilde{y}_{Tt})$, which are the complete market and (financial) autarky exchange rate shocks respectively. Then

$$\tilde{s}_t = \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}} \tilde{s}_{A,t} + \frac{\lambda_1^{-1}(1-\rho)}{1-\rho\lambda_1^{-1}} \tilde{s}_{C,t} - \frac{\lambda_1^{-1}\gamma\kappa}{1-\rho\lambda_1^{-1}} \tilde{d}_t^{*\mathcal{N}}, \tag{A9}$$

which is rearranged as equation (3) in the main text (recall that in the text, $\lambda_\kappa \equiv \lambda_1^{-1} \leq \beta$ and $\lambda_1 \equiv \lambda_1(\kappa)$ as defined above). Innovations to the exchange rate are comprised of a linear combination of complete-markets and autarky scenarios plus disconnected fluctuations arising from financial markets. We denote $\psi_t \equiv -\tilde{d}_t^{*\mathcal{N}}$ in the text in Section 1.

Combining all derivations together, we have:

$$\psi_t = -\tilde{d}_t^{*\mathcal{N}},$$
$$u_t = \frac{\lambda_1 - 1}{\lambda_1 - \rho}(\tilde{y}_{Nt} - \tilde{y}_{Tt}) + \frac{1-\rho}{\lambda_1-\rho}(\tilde{y}_{Nt} - \tilde{y}^*_{Tt}) + \frac{\gamma\kappa}{\lambda_1-\rho}\psi_t,$$
$$\zeta_t = -\gamma\kappa(b_t + d_t^{*\mathcal{N}}),$$
$$b_t = \frac{1}{\beta\lambda_1} b_{t-1} + \frac{1-\rho}{\lambda_1-\rho}(y_{Tt} - y^*_{Tt}) - \frac{\gamma\kappa}{\lambda_1-\rho} d_t^{*\mathcal{N}}$$

where we solved for $b_t = \beta^{-1}b_{t-1} + y_{Tt} - c_{Tt}$ using the solution for c_{Tt}, and recall that $\kappa = var_t(u_{t+1})$ is an equilibrium constant. Note that ψ_t is (negative of) the exogenous innovation to the noise trader positions $d_t^{*\mathcal{N}}$, and it is an important component of u_t, while $d_t^{*\mathcal{N}}$ is an important component of the UIP wedge ζ_t. This is the sense in which ψ_t, u_t and ζ_t are all tied together — noise-trader shocks drive both ζ_t and u_t — as we stated in the text in Section 3.

Common trading of risk-free bonds

Both households can trade both bonds. Home Euler equations:

$$r_{ft} = E_t\left[\Delta c_{Tt+1} + \Delta s_{t+1}\right] - \frac{1}{2}var_t(\Delta c_{Tt+1} + \Delta s_{t+1}) + \delta,$$
$$r^*_{ft} + E_t\left[\Delta s_{t+1}\right] - r_{ft} + \frac{1}{2}var_t(\Delta s_{t+1}) = -cov_t\left(-\Delta y_{Nt+1}, \Delta s_{t+1}\right)$$

The foreign Euler equations:

$$r_{ft} = E_t\left[\Delta y^*_{Tt+1} + \Delta s_{t+1}\right] - \frac{1}{2}var_t\left(\Delta y^*_{Tt+1} + \Delta s_{t+1}\right) + \delta,$$
$$r_{ft} + E_t\left[-\Delta s_{t+1}\right] - r^*_{ft} + \frac{1}{2}var_t\left(\Delta s_{t+1}\right) = -cov_t\left(-\Delta y^*_{Tt+1}, -\Delta s_{t+1}\right)$$

Equalizing the first and the third equations and ignoring variance terms eliminates r_{ft} and gives

$$E_t\left[\Delta c_{Tt+1}\right] = E_t\left[\Delta y^*_{Tt+1}\right].$$

The budget constraint is:

$$-b_{t-1} = -\beta b_t + \beta\left(y_{Tt} - c_{Tt}\right) + \beta d^{*\mathcal{H}}\left(r^*_{f,t-1} - r_{ft-1} + \Delta s_t\right).$$

Using the second (or the fourth) Euler equation at first order, we get: $r^*_{ft} - r_{ft} + E_t\left[\Delta s_{t+1}\right] = 0$, a UIP condition.We can plug it into the budget constraint:

$$-b_{t-1} = -\beta b_t + \beta\left(y_{Tt} - c_{Tt}\right) + \beta d^{*\mathcal{H}}\left(\Delta s_t - E_{t-1}\left[\Delta s_t\right]\right).$$

These equations lead to a system

$$\begin{aligned} b_t &= \beta E_t\left[b_{t+1}\right] + \beta E_t[c_{Tt+1}] - \beta E_t\left[y_{Tt+1}\right], \\ c_{Tt} &= E_t[c_{Tt+1}] - E_t\left[\Delta y^*_{Tt+1}\right]. \end{aligned}$$

Define $V_t = (b_t, c_{Tt})'$, then

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} V_t &= \begin{pmatrix} \beta & \beta \\ 0 & 1 \end{pmatrix} E_t\left[V_{t+1}\right] + Z_t, \\ E_t\left[V_{t+1}\right] &= \begin{pmatrix} \beta & \beta \\ 0 & 1 \end{pmatrix}^{-1} V_t - \begin{pmatrix} \beta & \beta \\ 0 & 1 \end{pmatrix}^{-1} Z_t, \\ &= \begin{pmatrix} \frac{1}{\beta} & -1 \\ 0 & 1 \end{pmatrix} V_t - \begin{pmatrix} \beta & \beta \\ 0 & 1 \end{pmatrix}^{-1} Z_t, \\ E_t\left[V_{t+1}\right] &= AV_t + \hat{Z}_t \end{aligned}$$

with

$$Z_t = \begin{pmatrix} -\beta E_t\left[y_{Tt+1}\right] \\ -E_t\left[\Delta y^*_{Tt+1}\right] \end{pmatrix} = \begin{pmatrix} -\beta\rho y_{Tt} \\ -(\rho-1)\, y^*_{Tt} \end{pmatrix},$$

$$\hat{Z}_t = -\begin{pmatrix} \beta & \beta \\ 0 & 1 \end{pmatrix}^{-1} Z_t = -\begin{pmatrix} \frac{1}{\beta} & -1 \\ 0 & 1 \end{pmatrix} Z_t,$$

$$\hat{Z}_t = \begin{pmatrix} \rho y_{Tt} - (\rho-1)y^*_{Tt} \\ (\rho-1)y^*_{Tt} \end{pmatrix}.$$

Next, we follow Blanchard and Kahn (1980) and compute eigenvalues and left eigenvectors of the matrix A:

$$\lambda_1 = \frac{1}{\beta} > 1, \quad v_1 = \begin{pmatrix} 1-\lambda_1 \\ 1 \end{pmatrix},$$

$$\lambda_2 = 1, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Define $x_t = v_1' V_t$, the linear combination corresponding to the exploding root. Then:

$$E_t\left[x_{t+1}\right] = \lambda_1 x_t + v_1'\hat{Z}_t,$$

$$x_t = \frac{1}{\lambda_1} E_t\left[x_{t+1}\right] - \frac{1}{\lambda_1} v_1'\hat{Z}_t.$$

There is a unique stationary solution to this equation, satisfing the transversality condition, which is:

$$\begin{aligned} x_t &= \sum_{k=0}^{\infty}\left(\frac{1}{\lambda_1}\right)^k \left[-\frac{1}{\lambda_1} v_1' E_t\left[\hat{Z}_{t+k}\right]\right] \\ &= \sum_{k=0}^{\infty}\left(\frac{\rho}{\lambda_1}\right)^k \left[-\frac{1}{\lambda_1} v_1'\hat{Z}_t\right] \\ &= \frac{-1}{1-\frac{\rho}{\lambda_1}}\frac{1}{\lambda_1} v_1'\hat{Z}_t = -\frac{1}{\lambda_1-\rho} v_1'\hat{Z}_t, \\ (1-\lambda_1)\, b_t + c_{Tt} &= -\frac{1}{\lambda_1-\rho} v_1'\hat{Z}_t, \\ b_t &= -\frac{1}{1-\lambda_1}\frac{1}{\lambda_1-\rho} v_1'\hat{Z}_t - \frac{1}{1-\lambda_1} c_{Tt}. \end{aligned}$$

We plug this expression for b_t back into the budget constraint $\beta^{-1}b_{t-1} = b_t + c_{Tt} - y_{Tt} - d^{*\mathcal{H}}\widetilde{\Delta s}_t$:

$$\begin{aligned} \frac{1}{\beta} b_{t-1} &= \left(1 - \frac{1}{1-\lambda_1}\right) c_{Tt} - \frac{1}{1-\lambda_1}\frac{1}{\lambda_1-\rho} v_1'\hat{Z}_t - y_{Tt} - d^{*\mathcal{H}}\widetilde{\Delta s}_t \\ \frac{1}{\beta} b_{t-1} &= -\frac{\lambda_1}{1-\lambda_1} c_{Tt} - \frac{\lambda_1}{\lambda_1-\rho} y_{Tt} - \frac{\lambda_1(\rho-1)}{(1-\lambda_1)(\lambda_1-\rho)} y^*_{Tt} - d^{*\mathcal{H}}\widetilde{\Delta s}_t \\ c_{Tt} &= \frac{1-\lambda_1^{-1}}{\beta} b_{t-1} + \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}} y_{Tt} + \frac{\lambda_1^{-1}(1-\rho)}{1-\rho\lambda_1^{-1}} y^*_{Tt} + (1-\lambda_1^{-1}) d^{*\mathcal{H}}\widetilde{\Delta s}_t \\ s_t = y_{Nt} - c_{Tt} &= -\frac{1-\lambda_1^{-1}}{\beta} b_{t-1} + \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}\left(y_{Nt} - y_{Tt}\right) + \frac{\lambda_1^{-1}(1-\rho)}{1-\rho\lambda_1^{-1}}\left(y_{Nt} - y^*_{Tt}\right) - (1-\lambda_1^{-1}) d^{*\mathcal{H}}\widetilde{\Delta s}_t. \end{aligned}$$

In innovations, we obtain:

$$\begin{aligned}\tilde{s}_t &= \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}\left(\tilde{y_{Nt}}-\tilde{y_{Tt}}\right)+\frac{\lambda_1^{-1}\left(1-\rho\right)}{1-\rho\lambda_1^{-1}}\left(\tilde{y_{Nt}}-\tilde{y_{Tt}^{*}}\right)-\left(1-\lambda_1^{-1}\right)d^{*\mathcal{H}}\tilde{s}_t \\ &= \frac{1-\beta}{1-\beta\rho}\left(\tilde{y_{Nt}}-\tilde{y_{Tt}}\right)+\frac{\beta\left(1-\rho\right)}{1-\beta\rho}\left(\tilde{y_{Nt}}-\tilde{y_{Tt}^{*}}\right)-\left(1-\beta\right)d^{*\mathcal{H}}\tilde{s}_t.\end{aligned}$$

Solving for $\tilde{s}_t$, we obtain

$$\begin{aligned}\tilde{s}_t &= \frac{1}{1+\left(1-\lambda_1^{-1}\right)d^{*\mathcal{H}}}\left[\frac{\lambda_1^{-1}\left(1-\rho\right)}{1-\rho\lambda_1^{-1}}\tilde{s}_{C,t}+\frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}\tilde{s}_{A,t}\right] \\ &= \frac{1}{1+\left(1-\beta\right)d^{*\mathcal{H}}}\left[\frac{\beta\left(1-\rho\right)}{1-\beta\rho}\tilde{s}_{C,t}+\frac{1-\beta}{1-\beta\rho}\tilde{s}_{A,t}\right] \\ &= \phi\left[\alpha\tilde{s}_{C,t}+\left(1-\alpha\right)\tilde{s}_{A,t}\right].\end{aligned}$$

where $\phi=\frac{1}{1+(1-\beta)d^{*\mathcal{H}}}$ and $\alpha=\frac{\beta(1-\rho)}{1-\beta\rho}$. The expression in the first line resembles the one in the case of intermediated bonds, but the value of λ_1 is different, there is no role for disconnected fluctuations, and the expression is rescaled (by ϕ). This result is equation (4) in the main text.

Adding the second and fourth equations removes r_{ft}^{*} :

$$\begin{aligned}var_t\left(\Delta s_{t+1}\right) &= -cov_t\left(-\Delta y_{Nt+1}+\Delta y_{Tt+1}^{*},\Delta s_{t+1}\right), \\ 0 &= -cov_t\left(-\Delta y_{Nt+1}+\Delta y_{Tt+1}^{*}+\Delta s_{t+1},\Delta s_{t+1}\right).\end{aligned}$$

This expression implies

$$\begin{aligned}0 &= cov_{t-1}\left(\tilde{s}_t-\tilde{s}_{C,t},\tilde{s}_t\right)=cov\left[\left(\phi\alpha-1\right)\tilde{s}_{Ct}+\phi\left(1-\alpha\right)\tilde{s}_{A,t},\phi\left[\alpha\tilde{s}_{C,t}+\left(1-\alpha\right)\tilde{s}_{A,t}\right]\right] \\ &= \left(\phi\alpha-1\right)\phi\alpha var\left(\tilde{s}_{Ct}\right)+\phi^2\left(1-\alpha\right)^2 var(\tilde{s}_{At})+\left(\left(\phi\alpha-1\right)\phi\left(1-\alpha\right)+\phi^2\alpha\left(1-\alpha\right)\right)cov(\tilde{s}_{Ct},\tilde{s}_{At}).\end{aligned}$$

The solutions are $\phi=0$ (which is not a solution because it would map to $d^{*\mathcal{H}}=\infty$) and:

$$\phi=\frac{\alpha\cdot\mathrm{var}(\tilde{s}_{Ct})+(1-\alpha)\cdot\mathrm{cov}(\tilde{s}_{Ct},\tilde{s}_{At})}{\alpha^2\cdot\mathrm{var}(\tilde{s}_{Ct})+(1-\alpha)^2\cdot\mathrm{var}(\tilde{s}_{At})+2\alpha\left(1-\alpha\right)\cdot\mathrm{cov}(\tilde{s}_{Ct},\tilde{s}_{At})},$$

which implies

$$d^{*\mathcal{H}}=\frac{\alpha(\alpha-1)\cdot\mathrm{var}(\tilde{s}_{Ct})+(1-\alpha)^2\cdot\mathrm{var}(\tilde{s}_{At})+\left(2\alpha-1\right)\left(1-\alpha\right)\cdot\mathrm{cov}(\tilde{s}_{Ct},\tilde{s}_{At})}{\left(1-\beta\right)\cdot\left[\alpha\cdot\mathrm{var}(\tilde{s}_{Ct})+(1-\alpha)\cdot\mathrm{cov}(\tilde{s}_{Ct},\tilde{s}_{At})\right]}.$$

If $var\left(\tilde{s}_{A,t}\right)=0$ then $\phi=1/\alpha$ and the exchange rate is the same as complete markets.

Common trading of a risky asset

Assume there is a global productivity variable a_t which follows an AR(1) with persistence ρ and is such that each endowment can be written as:

$$
\begin{aligned}
y_{Nt} &= \beta_N a_t + \check{y}_{Nt}, \\
y_{Tt} &= \beta_T a_t + \check{y}_{Tt}, \\
y^*_{Tt} &= \beta^*_T a_t + \check{y}^*_{Tt}, \\
d^{*\mathcal{N}}_t &= \beta^*_{\mathcal{N}} a_t + \check{d}^{*\mathcal{N}}_t,
\end{aligned}
$$

where, for any variable x, the residual $\check{x} \perp a$.

Guess that the exchange rate innovation takes the form:

$$\tilde{s}_{t+1} = \beta_s \tilde{a}_{t+1} + u_{t+1}.$$

We assume that all investors can trade a forward on global productivity next period paid out in foreign currency. In addition, each household trades their own bond, and there is an intermediary that trades both bonds. Noise trader demand for the carry trade has innovation $\tilde{d}^{*\mathcal{N}}_t = -\psi_t$.

The home and foreign "savings" Euler equations are the same as before. Noticing that the forward is an excess return we get

$$E_t\left[r^*_{fwd,t+1}\right] + \frac{1}{2} var_t(r^*_{fwd,t+1}) = -cov_t\left(-\Delta c_{Tt+1}, r^*_{fwd,t+1}\right) = -cov_t\left(-\Delta c_{Tt+1}, \tilde{a}_t\right).$$

The foreign Euler

$$E_t\left[r^*_{fwd,t+1}\right] + \frac{1}{2} var_t(r^*_{fwd,t+1}) = -cov_t\left(-\Delta y^*_{Tt+1}, r^*_{fwd,t+1}\right) = -cov_t\left(-\Delta y^*_{Tt+1}, \tilde{a}_t\right).$$

Home budget constraint is

$$b_{t-1} = \beta b_t - \beta\left(y_{Tt} - c_{Tt}\right) - \beta d^*_{fwd} r^*_{fwd,t} = \beta b_t - \beta\left(y_{Tt} - c_{Tt}\right) - \beta d^*_{fwd} \tilde{a}_t.$$

The optimal intermediary portfolio is

$$\mathbf{d}^{*\mathcal{I}}_t = \frac{1}{\gamma} \mathbf{\Sigma}^{-1}_t \left[E_t\left(\mathbf{r}^*_{t+1}\right) - r^*_{ft} + diag(\mathbf{\Sigma}_t)/2\right].$$

There are two risky assets: the home risk-free bond with expected returns $(r_{ft} - E_t[\Delta s_{t+1}])$ and the forward. The covariance matrix between these assets is given by:

$$\mathbf{\Sigma} = \begin{pmatrix} \beta_s^2 var_t(a_{t+1}) + var_t(u_{t+1}) & \beta_s var_t(a_{t+1}) \\ \beta_s var_t(a_{t+1}) & var_t(a_{t+1}) \end{pmatrix},$$

with inverse:

$$\mathbf{\Sigma}^{-1} = \begin{pmatrix} \dfrac{1}{\mathrm{var}_t(u_{t+1})} & -\dfrac{\beta_s}{\mathrm{var}_t(u_{t+1})} \\ -\dfrac{\beta_s}{\mathrm{var}_t(u_{t+1})} & \dfrac{\beta_s^2\, \mathrm{var}_t(a_{t+1}) + \mathrm{var}_t(u_{t+1})}{\mathrm{var}_t(a_{t+1})\, \mathrm{var}_t(u_{t+1})} \end{pmatrix}.$$

Focusing on the position in home risk-free bond, which we denote $d_t^{*\mathcal{I}}$ to economize on notation, we obtain:

$$d_t^{*\mathcal{I}} = \frac{1}{\gamma var_t\left(u_{t+1}\right)}\left[r_{ft} - r_{ft}^* - E_t\left[\Delta s_{t+1}\right] + var_t\left(u_{t+1}\right)/2 + \beta_s E_t\left[r_{fwd,t+1}^*\right]\right].$$

Note that this expression can also be interpreted as the univariate optimal position in a portfolio combining the home risk-free asset with a hedging position of $-\beta_s$ in the forward, yielding a return purely exposed to u_{t+1}.

Take the difference between home and foreign savings Euler equations:

$$E_t\left[\Delta c_{Tt+1}\right] = r_{ft} - r_{ft}^* - E_t\left[\Delta s_{t+1}\right] + E_t\left[\Delta y_{Tt+1}^*\right].$$

We can then replace the "expected carry return" with $d_t^{*\mathcal{I}}$, and remove all second-order terms (variances and covariances):

$$\begin{aligned} E_t\left[\Delta c_{Tt+1}\right] &= \gamma var_t\left(u_{t+1}\right) d_t^{*\mathcal{I}} - var_t\left(u_{t+1}\right)/2 - \beta_s E_t\left[r_{fwd,t+1}^*\right] + E_t\left[\Delta y_{Tt+1}^*\right] \\ &= \gamma var_t\left(u_{t+1}\right) d_t^{*\mathcal{I}} + E_t\left[\Delta y_{Tt+1}^*\right] \end{aligned}$$

Finally, substitute $d_t^{*\mathcal{I}}$ out using the market clearing condition:

$$E_t\left[\Delta c_{Tt+1}\right] = -\gamma var_t\left(u_{t+1}\right) d_t^{*\mathcal{N}} - \gamma var_t\left(u_{t+1}\right) b_t + E_t\left[\Delta y_{Tt+1}^*\right],$$

and denote $\kappa = var_t\left(u_{t+1}\right)$ (constant variance is a conjecture to be verified).

We can now setup our Blanchard-Kahn system by taking the budget constraint in expectation:

$$\begin{aligned} b_t &= \beta E_t\left[b_{t+1}\right] + \beta E_t[c_{Tt+1}] - \beta E_t\left[y_{Tt+1}\right], \\ c_{Tt} - \gamma\kappa b_t &= E_t[c_{Tt+1}] + \gamma\kappa d_t^{*\mathcal{N}} - E_t\left[\Delta y_{Tt+1}^*\right]. \end{aligned}$$

The system is exactly the same as in the case of intermediated bond trading, and the solution is

$$b_t = -\frac{1}{1-\lambda_1}\frac{1}{\lambda_1-\rho}v_1'\hat{Z}_t - \frac{1}{1-\lambda_1}c_{Tt}.$$

Plug this back into the budget constraint, which is different from the case of intermediated bond trading, to get:

$$\begin{aligned} \frac{1}{\beta}b_{t-1} &= \left(1-\frac{1}{1-\lambda_1}\right)c_{Tt} - \frac{1}{1-\lambda_1}\frac{1}{\lambda_1-\rho}v_1'\hat{Z}_t - y_{Tt} - d_{fwd}^*\tilde{a}_t \\ \frac{1}{\beta}b_{t-1} &= -\frac{\lambda_1}{1-\lambda_1}c_{Tt} - \frac{\lambda_1}{\lambda_1-\rho}y_{Tt} - \frac{\lambda_1(\rho-1)}{(1-\lambda_1)(\lambda_1-\rho)}y_{Tt}^* + \frac{\lambda_1\gamma\kappa}{(1-\lambda_1)(\lambda_1-\rho)}d_t^{*\mathcal{N}} - d_{fwd}^*\tilde{a}_t \\ c_{Tt} &= \frac{1-\lambda_1^{-1}}{\beta}b_{t-1} + \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}y_{Tt} + \frac{\lambda_1^{-1}\left(1-\rho\right)}{1-\rho\lambda_1^{-1}}y_{Tt}^* + \frac{\lambda_1^{-1}\gamma\kappa}{1-\rho\lambda_1^{-1}}d_t^{*\mathcal{N}} + \left(1-\lambda_1^{-1}\right)d_{fwd}^*\tilde{a}_t \\ s_t &= y_{Nt} - c_{Tt} \\ &= -\frac{1-\lambda_1^{-1}}{\beta}b_{t-1} + \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}\left(y_{Nt}-y_{Tt}\right) + \frac{\lambda_1^{-1}\left(1-\rho\right)}{1-\rho\lambda_1^{-1}}\left(y_{Nt}-y_{Tt}^*\right) \\ &- \frac{\lambda_1^{-1}\gamma\kappa}{1-\rho\lambda_1^{-1}}d_t^{*\mathcal{N}} - \left(1-\lambda_1^{-1}\right)d_{fwd}^*\tilde{a}_t. \end{aligned}$$

We can re-write the expression for the exchange rate in terms of innovations:

$$\tilde{s}_t = \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}\tilde{s}_{A,t} + \frac{\lambda_1^{-1}(1-\rho)}{1-\rho\lambda_1^{-1}}\tilde{s}_{C,t} - \frac{\lambda_1^{-1}\gamma\kappa}{1-\rho\lambda_1^{-1}}\tilde{d}_t^{*\mathcal{N}} - \left(1-\lambda_1^{-1}\right) d_{fwd}^{*}\tilde{a}_t.$$

Compared to the intermediated bond case, there is an additional disconnected fluctuation arising from the risky-asset market.

We still have to determine κ and d_{fwd}^{*}. First, κ solves

$$\kappa = var\left(\frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}\check{s}_{A,t} + \frac{\lambda_1^{-1}(1-\rho)}{1-\rho\lambda_1^{-1}}\check{s}_{C,t} - \frac{\lambda_1^{-1}\gamma\kappa}{1-\rho\lambda_1^{-1}}\check{d}_t^{*\mathcal{N}}\right),$$

which does not depend on d_{fwd}^{*}.

Second, equal pricing for the forward implies

$$\begin{aligned} cov_t\left(-\Delta c_{Tt+1}, \Delta\tilde{a}_t\right) &= cov_t\left(-\Delta y_{Tt+1}^{*}, \Delta\tilde{a}_t\right), \\ cov_t\left(c_{Tt+1} - \Delta y_{Tt+1}^{*}, \Delta\tilde{a}_t\right) &= 0, \\ cov_t\left(\Delta y_{Nt+1} - \Delta y_{Tt+1}^{*} - \Delta s_{t+1}, \Delta\tilde{a}_t\right) &= 0. \end{aligned}$$

Substituting c_{Tt+1} into the middle equation, we obtain:

$$\begin{aligned} 0 &= cov_t\left(c_{Tt+1} - \Delta y_{Tt+1}^{*}, \Delta\tilde{a}_t\right) \\ &= \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}\beta_T + \frac{\lambda_1^{-1}(1-\rho)}{1-\rho\lambda_1^{-1}}\beta_T^{*} + \frac{\lambda_1^{-1}\gamma\kappa}{1-\rho\lambda_1^{-1}}\beta_{\mathcal{N}}^{*} + \left(1-\lambda_1^{-1}\right) d_{fwd}^{*} - \beta_T^{*} \\ -\left(1-\lambda_1^{-1}\right) d_{fwd}^{*} &= \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}\beta_T + \left[\frac{\lambda_1^{-1}(1-\rho)}{1-\rho\lambda_1^{-1}} - 1\right]\beta_T^{*} + \frac{\lambda_1^{-1}\gamma\kappa}{1-\rho\lambda_1^{-1}}\beta_{\mathcal{N}}^{*} \\ &= \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}\beta_T + \frac{\lambda_1^{-1}-1}{1-\rho\lambda_1^{-1}}\beta_T^{*} + \frac{\lambda_1^{-1}\gamma\kappa}{1-\rho\lambda_1^{-1}}\beta_{\mathcal{N}}^{*} \\ -\left(1-\lambda_1^{-1}\right) d_{fwd}^{*} &= \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}\left(\beta_T - \beta_T^{*}\right) + \frac{\lambda_1^{-1}\gamma\kappa}{1-\rho\lambda_1^{-1}}\beta_{\mathcal{N}}^{*}. \end{aligned}$$

Substituting in the global-local representation of the endowments, we obtain:

$$\begin{aligned} \tilde{s}_t &= \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}\check{s}_{A,t} + \frac{\lambda_1^{-1}(1-\rho)}{1-\rho\lambda_1^{-1}}\check{s}_{C,t} \\ &+ \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}(\beta_N - \beta_T)\tilde{a}_t + \frac{\lambda_1^{-1}(1-\rho)}{1-\rho\lambda_1^{-1}}(\beta_N - \beta_T^{*})\tilde{a}_t \\ &+ \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}\left(\beta_T - \beta_T^{*}\right)\tilde{a}_t + \frac{\lambda_1^{-1}\gamma\kappa}{1-\rho\lambda_1^{-1}}\beta_{\mathcal{N}}^{*}\tilde{a}_t \\ &- \frac{\lambda_1^{-1}\gamma\kappa}{1-\rho\lambda_1^{-1}}\tilde{d}_t^{*\mathcal{N}} \\ &= \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}\check{s}_{A,t} + \frac{\lambda_1^{-1}(1-\rho)}{1-\rho\lambda_1^{-1}}\check{s}_{C,t} - \frac{\gamma\kappa\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}\check{d}_t^{*\mathcal{N}} + \left[\beta_N - \beta_T^{*}\right]\tilde{a}_t. \end{aligned} \tag{A10}$$

This expression coincide with equations (5) and (6) in the main text (recall again that in the text, $\lambda_\kappa \equiv \lambda_1^{-1} \leq \beta$ and $\lambda_1 \equiv \lambda_1(\kappa)$ as defined above). The last term is the global component of the complete-market solution for the exchange rate (difference in the SDFs projected on the globally traded component), equation (6). The rest of the terms reflect exposure to local shocks, equation (5). Note that we denote $\check{\psi}_t = -\check{d}_t^{*\mathcal{N}}$ in the text of Section 1.

A.4 Attainability of points in cones

We prove that, for the models with either segmented household trading, or with common trading of a risky asset, all points in the corresponding cones in terms of cyclicality and volatility are attainable for certain noise trader dynamics. We proceed constructively.

Segmented household trading In this setting there are no globally-traded risk, hence all combinations of volatility and cyclicality should be feasible. Consider a candidate exchange rate shock $\tilde{s}_t$ with a given combination, denoting $\kappa = var(\tilde{s})$. This implies an eigenvalue $\lambda_1 = \lambda_1(\kappa)$ from equation (A8). If one assumes that noise trader shocks are given by:

$$\psi_t \equiv -\tilde{d}_t^{*\mathcal{N}} = \frac{1-\rho\lambda_1^{-1}}{\lambda_1^{-1}\gamma\kappa}\left(\tilde{s}_t - \frac{1-\lambda_1^{-1}}{1-\rho\lambda_1^{-1}}\tilde{s}_{A,t} - \frac{\lambda_1^{-1}(1-\rho)}{1-\rho\lambda_1^{-1}}\tilde{s}_{C,t}\right) \tag{A11}$$

The conjectured exchange rate is the solution to the economy because it satisfies equation (A9), which is just a rearranging of the previous condition.

Common trading of a risky asset In this case, a_t is the only globally-traded shock, and the globally-traded component of the exchange rate is $g_t = (\beta_N - \beta_T^*)\tilde{a}_t$. The corresponding cone is covered by all shocks that can be written as $\tilde{s}_t = g_t + u_t$ with $u_t \perp g_t$. Consider one such candidate exchange rate, with the corresponding u_t. Then, one can repeat exactly all the process of the previous paragraph with the residual components $\check{s}_t = u_t$ and equation (A10) to find a residual component of noise trader demand $\check{\psi}_t \equiv -\check{d}_t^{*\mathcal{N}}$ that justifies the exchange rate. One can choose any value for the loading of noise trader on a_t because it is irrelevant for the equilibrium.

B Portfolio algebra

B.1 Portfolio approximation

To maintain tractability, we follow Campbell and Viceira (2002) and approximate the log portfolio excess returns relative to a risk-free rate r_{ft}:

$$\begin{aligned} r_{p,t+1} - r_{ft} &= \log\left(\boldsymbol{w}_t' \exp(\boldsymbol{r}_{t+1} - r_{ft})\right) \\ &\approx \boldsymbol{w}_t'(\boldsymbol{r}_{t+1} - r_{ft}) + \frac{1}{2}\boldsymbol{w}_t' \operatorname{diag}(\boldsymbol{\Sigma}_t) - \frac{1}{2}\boldsymbol{w}_t'\boldsymbol{\Sigma}_t\boldsymbol{w}_t, \end{aligned} \tag{A12}$$

where $\boldsymbol{\Sigma}_t$ is the $N \times N$ variance-covariance matrix of log returns and $\boldsymbol{w}_t$ is the vector of portfolio weights such that $\boldsymbol{w}_t'\boldsymbol{\iota} = \sum_{i=1}^N w_{it} = 1$. Note that by convention we use the vector notation where, for example, $\boldsymbol{w}_t' \exp(\boldsymbol{r}_{t+1} - r_{ft}) = \sum_{i=1}^N w_{it} R_{i,t+1}/R_{ft}$ and $R_{i,t+1}/R_{ft} = \exp(r_{i,t+1} - r_{ft})$.

The approximation in (A12) allows us to represent portfolios returns as linear combination of log returns. Importantly, it is stable by recombination, leading to the same result when applied in two steps or all at once for a portfolio of portfolios. The approximation becomes exact as time becomes continuous and the underlying data-generating process for returns converges to a purely diffusive stochastic process.

B.2 Two international portfolios

Two international portfolios are useful for the derivation of our main results.

Carry trade. One zero-cost portfolio, often referred to as carry, entails taking long and short positions in related assets:

$$R_{\text{carry},t+1} = R_{t+1} - R^*_{t+1} \cdot S_{t+1}/S_t, \tag{A13}$$

where R_{t+1} and R^*_{t+1} denote asset returns in levels and S_t denotes the level of the exchange rate.

Traditionally, traded assets are taken to be domestic and foreign risk-free (one-period) bonds. But carry does not have to be limited to that. For instance, Lustig, Stathopolous, and Verdelhan (2019) consider long-term bonds. More generally, one could use any pair of assets, e.g. risky assets that are close to each other with $corr_t(r_{t+1}, r^*_{t+1}) \approx 1$. The key characteristic of the carry trade is that it exposes the arbitrageur to currency risk.

Lemma 2. *The conversion from foreign to home returns in the carry portfolio introduces exposure to currency risk,* $\widetilde{r}_{\text{carry},t+1} = \widetilde{r}_{t+1} - \widetilde{r}^*_{t+1} - \widetilde{\Delta s}_{t+1}$.

Proof. To apply the log approximation in equation (A12), we convert the zero-cost portfolio (A13) to a funded portfolio by adding a unit position in the risk-free asset:

$$R_{p,t+1} \equiv R_{\text{carry},t+1} + R_{f,t} = R_{t+1} - R^*_{t+1} \cdot S_{t+1}/S_t + R_{f,t}.$$

The portfolio $R_{p,t+1}$ corresponds to the weights $w_1 = 1$ in the domestic risky asset R_{t+1}, $w_2 = -1$ in the foreign risky asset converted to local currency, $R^*_{t+1} \cdot S_{t+1}/S_t$, and $w_3 = 1$ in the domestic risk-free asset with $\boldsymbol{w}_t = (w_1, w_2, w_3)'$. These weights lead to an expression for the log gross return relative to the risk-free rate $R_{p,t+1}/R_{f,t}$:

$$\begin{aligned} r_{\text{carry},t+1} &\equiv r_{p,t+1} - r_{ft} \\ &= r_{t+1} - r^*_{t+1} - \Delta s_{t+1} + cov_t(r_{t+1} - r^*_{t+1} - \Delta s_{t+1}, r^*_{t+1} + \Delta s_{t+1}). \end{aligned} \tag{A14}$$

This confirms the claim in Lemma 2 about the carry return innovation $\widetilde{r}_{\text{carry},t+1}$ as the covariance term is part of the expected return at time t. ■

Thus, we conclude that the carry portfolio return is exposed to the exchange rate risk. Note that in the special case of carry based on risk-free assets, $r_{\text{carry},t+1} = r_{ft} - r^*_{ft} - \Delta s_{t+1} - var_t(\Delta s_{t+1})$, and thus $\widetilde{r}_{\text{carry},t+1} = -\widetilde{\Delta s}_{t+1}$, that is the carry risk equals the negative of the exchange rate risk. This property holds for any carry with risky assets such that $\widetilde{r}_{t+1} = \widetilde{r}^*_{t+1}$.

Differential carry. The fact that carry is exposed to currency risk prompts us to consider another zero-cost portfolio, labeled as differential carry, which is long one unit of the domestic asset, and short one unit of the foreign asset, financed at the respective risk-free rates:

$$R_{\text{diff},t+1} = (R_{t+1} - R_{ft}) - (R^*_{t+1} - R^*_{ft}) \cdot S_{t+1}/S_t. \tag{A15}$$

Intuitively, this portfolio does not introduce additional currency exposure because, in contrast to carry, only the foreign excess return is converted to the home currency. We demonstrate this formally in the following lemma.

Lemma 3. *The conversion from foreign- to home-currency returns in the differential carry does not introduce additional exposure to currency risk,* $\widetilde{r}_{\text{diff},t+1} = \widetilde{r}_{t+1} - \widetilde{r}^*_{t+1}$.

Proof. To apply the log approximation in equation (A12), we convert the zero-cost portfolio (A15) to a funded portfolio by adding a unit position in the risk-free asset:

$$R_{p,t+1} \equiv R_{\text{diff},t+1} + R_{f,t} = R_{t+1} - (R^*_{t+1} - R^*_{ft}) \cdot S_{t+1}/S_t.$$

The portfolio $R_{p,t+1}$ corresponds to the weights $w_1 = 1$ in the domestic risky asset R_{t+1}, $w_2 = -1$ in the foreign risky asset converted to local currency, $R^*_{t+1} \cdot S_{t+1}/S_t$, and $w_3 = 1$ in the foreign risk-free asset converted to local currency, $R^*_{ft} \cdot S_{t+1}/S_t$, with $\boldsymbol{w}_t = (w_1, w_2, w_3)'$. These weights lead to an expression for the relative log return:

$$\begin{aligned} r_{\text{diff},t+1} &\equiv r_{p,t+1} - r_{ft} \\ &= (r_{t+1} - r_{ft}) - (r^*_{t+1} - r^*_{ft}) + cov_t(r^*_{t+1}, r_{t+1} - r^*_{t+1} - \Delta s_{t+1}). \end{aligned} \tag{A16}$$

Thus, only the covariance of the foreign return with the exchange rate has a material impact on portfolio performance, not the shocks to the exchange rate. ■

The disappearance of exchange rate risk for the differential carry return is in part due to our portfolio approximation. In Appendix Section H, we confirm that this approximation is very tight empirically. We compare the excess returns on various stock portfolios and sovereign bonds in their origin currency, $R^*_{t+1} - R^*_{ft}$, and after conversion to home currency (USD), $(R^*_{t+1} - R^*_{ft})S_{t+1}/S_t$. The correlation between the two monthly series is always around 99.9%. Also, see Daniel, Hodrick, and Lu (2017, Online Appendix C) and Chernov, Dahlquist, and Lochstoer (2023, Internet Appendix II). Appendix E derives exact versions of our results which do not rely on the portfolio return approximation in (A12).

C Globally-traded shocks

We first provide a technical construction of globally-traded shocks. To see specific examples of globally-traded shocks in various market structure, skip to Appendix C.2.

C.1 Identification and construction

We show how to identify a basis for the set of globally-traded shocks $\boldsymbol{\epsilon}^g_{t+1}$ from the base returns $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^*_{t+1}$. We drop time indices and tildes for parsimony.

First, recall what canonical correlation analysis does.

Definition 2. *Canonical correlation analysis identifies pairs* $(\boldsymbol{\lambda}_i, \boldsymbol{\lambda}^*_i)$ *for* $i = 1, .., K$ *for some* K *such that:*

1. $\forall i \quad var(\boldsymbol{\lambda}'_i \boldsymbol{r}) \neq 0$;
2. $\forall i \quad \boldsymbol{\lambda}'_i \boldsymbol{r} = \boldsymbol{\lambda}^{*\prime}_i \boldsymbol{r}^*$;

3. $\forall i \neq j \;\; \boldsymbol{\lambda}_i' \boldsymbol{r} \perp \boldsymbol{\lambda}_j' \boldsymbol{r}$;

4. $\forall r \in span(\boldsymbol{r}), \forall r^* \in span(\boldsymbol{r}^*)$: *if* $\forall i \; r \perp \boldsymbol{\lambda}_i' \boldsymbol{r}$ *and* $r^* \perp \boldsymbol{\lambda}_i^{*\prime} \boldsymbol{r}^*$, *then* $r \neq r^*$.

Condition 1 says that each canonical component is non-degenerate, that is, has non-zero variance. Condition 2 says that each component can be expressed using only local returns and only foreign returns. Condition 3 indicates that the various components must be orthogonal to each other. Condition 4 indicates that the analysis exhausts all possible components: one cannot find pairs of portfolios satisfying condition 2 in the space orthogonal to the canonical components.

We show that this procedure identifies a basis of $\boldsymbol{\epsilon}^g$.

Lemma 4. *The collection* $(\boldsymbol{\lambda}_1' \boldsymbol{r}, .., \boldsymbol{\lambda}_K' \boldsymbol{r})$ *identified by canonical correlation analysis is a basis of* $\boldsymbol{\epsilon}^g$.

Proof. By Definition 1 and by point 2 of Definition 2, all $\boldsymbol{\lambda}_i' \boldsymbol{r}$ are in $\boldsymbol{\epsilon}^g$. Thus, $span(\boldsymbol{\lambda}_1' \boldsymbol{r}, .., \boldsymbol{\lambda}_K' \boldsymbol{r}) \subset \boldsymbol{\epsilon}^g$.

Let us show the other direction. Assume that $\exists r \in \boldsymbol{\epsilon}^g$ such that $r \notin span(\boldsymbol{\lambda}_1' \boldsymbol{r}, .., \boldsymbol{\lambda}_K' \boldsymbol{r})$. We can orthogonalize r to all the $\boldsymbol{\lambda}_i' \boldsymbol{r}$ and obtain $\hat{r}$. Because $\hat{r}$ is a linear combination of r and $\boldsymbol{\lambda}_i' \boldsymbol{r}$ which are all in $\boldsymbol{\epsilon}^g$, it is also in $\boldsymbol{\epsilon}^g$, and therefore in $span(\boldsymbol{r})$ and $span(\boldsymbol{r}^*)$. By substituting $\hat{r}$ for both r and r^* in point 4 of Definition 2, we immediately obtain a contradiction. Therefore, $span(\boldsymbol{\lambda}_1' \boldsymbol{r}, .., \boldsymbol{\lambda}_K' \boldsymbol{r}) \supset \boldsymbol{\epsilon}^g$, and the two sets are equal. By point 3 of Definition 2, $dim(span(\lambda_1' r, \ldots \lambda_K' r)) = K$, so $(\lambda_1' r, \ldots \lambda_K' r)$ is indeed a basis of $\boldsymbol{\epsilon}^g$. ■

Furthermore, we relate the dimension of $\boldsymbol{\epsilon}^g$ to the rank of covariance matrices of $\boldsymbol{r}$, $\boldsymbol{r}^*$, and the two combined.

Lemma 5. *The dimension of* $\boldsymbol{\epsilon}^g$ *is:*

$$dim(\boldsymbol{\epsilon}^g) = rank(var(\boldsymbol{r})) + rank(var(\boldsymbol{r}^*)) - rank(var(\boldsymbol{r}, \boldsymbol{r}^*)).$$

Proof. Observe that, by construction,

$$dim(span(\boldsymbol{r}, \boldsymbol{r}^*)) = \underbrace{dim(span(\boldsymbol{\epsilon}^g)) + dim(span(\boldsymbol{\epsilon}))}_{=dim(span(\boldsymbol{r}))} + \underbrace{dim(span(\boldsymbol{\epsilon}^*))}_{=dim(span(\boldsymbol{r}^*)) - dim(\boldsymbol{\epsilon}^g)}.$$

Therefore,

$$dim(\boldsymbol{\epsilon}^g) = dim(span(\boldsymbol{r})) + dim(span(\boldsymbol{r}^*)) - dim(span(\boldsymbol{r}, \boldsymbol{r}^*)),$$

which yields the result. ■

C.2 Examples of globally-traded, locally-traded and unspanned risks

C.2.1 Alternative market structures with bonds and equities

Consider a world with four assets:

1. a home risk-free bond with return r_{ft} in home currency;

2. a foreign risk-free bond with return r_{ft}^* in foreign currency;

3. a home equity index with return $r_{H,t+1}$ in home currency;

4. a foreign equity index with return $r^*_{F,t+1}$ in home currency,

and a nominal exchange rate depreciation rate Δs_{t+1}.

We consider a variety of international market structures with various subsets of these assets, assumingthat an international intermediary can trade all of them so that:

$$\boldsymbol{r}^I_{t+1} = (r_{ft}, r^*_{ft} + \Delta s_{t+1}, r_{H,t+1}, r^*_{F,t+1} + \Delta s_{t+1}). \tag{A17}$$

Case 1 Consider the non-integrated market where H contains the home bond and home equity index, while F contains the foreign bond and foreign equity index. In this case the sets of return in $H \cap I$ and $F \cap I$ in local currency are respectively given by:

$$\boldsymbol{r}_{t+1} = (r_{ft}, r_{H,t+1}) \qquad \text{and} \qquad \boldsymbol{r}^*_{t+1} = (r^*_{ft}, r^*_{F,t+1}).$$

By Definition 1, assuming the two equity indexes do not have perfectly correlated returns in their respective currencies, $|corr_t(r_{H,t+1}, r^*_{F,t+1})| < 1$, the set of globally-traded shocks in this case is empty, $\boldsymbol{\epsilon}^g_{t+1} = \emptyset$. Otherwise, there exists $\lambda \neq 0$ such that $\widetilde{r}_{H,t+1} = \lambda \widetilde{r}^*_{F,t+1}$, and thus $\boldsymbol{\epsilon}^g_{t+1} = \widetilde{r}_{H,t+1} = \lambda \widetilde{r}^*_{F,t+1}$. This may be the case when both $r_{H,t+1}$ and $r^*_{F,t+1}$ are driven by the same fundamental shock (e.g., relative productivity). Note importantly that these are returns expressed in different currencies. If $\widetilde{\Delta s}_{t+1} = \delta r_{H,t+1}$, then the exchange rate is spanned by the globally-traded shock, $\widetilde{\Delta s}_{t+1} \in \boldsymbol{\epsilon}^g_{t+1}$; otherwise, it is not.[29]

Case 2 Now allow both households to trade both risk-free bonds so that:

$$\boldsymbol{r}_{t+1} = (r_{ft}, r^*_{ft} + \Delta s_{t+1}, r_{H,t+1}) \qquad \text{and} \qquad \boldsymbol{r}^*_{t+1} = (r^*_{ft}, r_{ft} - \Delta s_{t+1}, r^*_{F,t+1}).$$

In this case, independently of the statistical properties of the equity returns $(r_{H,t+1}, r^*_{F,t+1})$, the exchange rate is a globally-traded risk, $\widetilde{\Delta s}_{t+1} \in \boldsymbol{\epsilon}^g_{t+1}$. Therefore, $\widetilde{\Delta s}_{t+1} = g_{t+1}$ and $u_{t+1} = \ell_{t+1} = 0$ according to decomposition (13). This is because both households can trade the exchange rate risk.

If, additionally, the exchange rate is spanned by $(r_{H,t+1}, r^*_{F,t+1})$, then $\boldsymbol{\epsilon}^g_{t+1} = (\widetilde{\Delta s}_{t+1}, \widetilde{r}_{H,t+1}, \widetilde{r}^*_{F,t+1})$ irrespective of the correlation between $r_{H,t+1}$ and $r^*_{F,t+1}$. This is because using $r^*_{ft} + \Delta s_{t+1}$ and $r_{H,t+1}$, the home households can construct a portfolio that spans $r^*_{F,t+1}$, making it a globally-traded risk; and symmetrically for $r_{H,t+1}$. This situation arises naturally when $r_{H,t+1}$ reflects home productivity, $r^*_{F,t+1}$ reflects foreign productivity, and Δs_{t+1} in proportional to relative productivity. However, the presence of additional shocks may disrupt such spanning.

Case 3 Consider now that both households can trade every asset such that

$$\boldsymbol{r}_{t+1} = (r_{ft}, r^*_{ft} + \Delta s_{t+1}, r_{H,t+1}, r^*_{F,t+1} + \Delta s_{t+1}), \tag{A18}$$

$$\boldsymbol{r}^*_{t+1} = (r^*_{ft}, r_{ft} - \Delta s_{t+1}, r^*_{F,t+1}, r_{H,t+1} - \Delta s_{t+1}). \tag{A19}$$

This is the case of a fully integrated market. According to Definition 1, such a case always features a full set of globally-traded shocks $\boldsymbol{\epsilon}^g_{t+1} = (\widetilde{\Delta s}_{t+1}, \widetilde{r}_{H,t+1}, \widetilde{r}^*_{F,t+1})$ irrespective of statistical properties of the returns. This is because excess returns for every asset are a globally traded. Therefore, fully integrated markets imply $\widetilde{\Delta s}_{t+1} = g_{t+1}$ and $u_{t+1} = \ell_{t+1} = 0$ according to decomposition (13).

[29] If $\nexists \lambda : \widetilde{r}_{H,t+1} \neq \lambda \widetilde{r}^*_{F,t+1}$ and $\widetilde{\Delta s}_{t+1}$ is spanned by $(\widetilde{r}_{H,t+1}, \widetilde{r}^*_{F,t+1})$, then $\widetilde{\Delta s}_{t+1} = \ell_{t+1}$ is a locally-traded shock.

Case 4 Consider now three asymmetric partially integrated scenarios. In all of these cases, the foreign household only trades the foreign bond and the foreign equity index, $\boldsymbol{r}^*_{t+1} = (r^*_{ft}, r^*_{F,t+1})$.

(i) The home household trades both bonds and the home equity, $\boldsymbol{r}_{t+1} = (r_{ft}, r^*_{ft} + \Delta s_{t+1}, r_{H,t+1})$. In this case, assuming $r^*_{F,t+1}$ is not spanned by $(\Delta s_{t+1}, r_{H,t+1})$, there are no globally-traded shocks, $\boldsymbol{\epsilon}^g_{t+1} = \emptyset$. However, the exchange rate is spanned by local returns, as the home household has an access to a carry trade with risk-free bonds, and thus $\widetilde{\Delta s}_{t+1} = \ell_{t+1}$ and $u_{t+1} = g_{t+1} = 0$ according to decomposition (13).

(ii) The home household trades the home bond and both equity indexes, $\boldsymbol{r}_{t+1} = (r_{ft}, r_{H,t+1}, r^*_{F,t+1} + \Delta s_{t+1})$. Similarly, assuming $r^*_{F,t+1}$ is not spanned by $(\Delta s_{t+1}, r_{H,t+1})$, there are no globally-traded shocks, $\boldsymbol{\epsilon}^g_{t+1} = \emptyset$. This case is more interesting, however, because the same risky asset (the foreign equity index) is traded by both households. Nonetheless, there are still no globally-traded risks as $r^*_{F,t+1}$ is not traded in local currency, but instead is converted to home currency using the exchange rate, which itself is not a globally-traded risk. Furthermore, the exchange rate is not spanned in this case even for the home household as they do not have access to a carry trade with risk-free bonds. However, it is spanned by the joint set of asset returns $\boldsymbol{r}^I_{t+1} = (\boldsymbol{r}_{t+1}, \boldsymbol{r}^*_{t+1})$, and therefore it is a locally-traded risk, $\widetilde{\Delta s}_{t+1} = \ell_{t+1}$ and $g_{t+1} = u_{t+1} = 0$ according to decomposition (13).

(iii) The home households can trade all asset as in (A18). Assuming $|corr_t(r_{H,t+1}, r^*_{F,t+1})| < 1$, the set of globally-traded shocks in this case is $\boldsymbol{\epsilon}^g_{t+1} = \widetilde{r}_{F,t+1}$. The home households can trade the exchange rate risk using the carry trade as in (i), making it a locally-traded risk ($\widetilde{\Delta s}_{t+1} = g_{t+1} + \ell_{t+1}$ and $u_{t+1} = 0$), and hence they can also trade the $r^*_{F,t+1}$ risk in home currency by combining the foreign equity and the carry trade.

Note that any partial integration makes the exchange rate risk locally traded, but generally not globally-traded without both risk-free bonds being globally-traded, as we illustrate further in the next case.

Case 5 Finally, we consider the case of a symmetrically partially integrated markets where each household can hold both equities, but not the bond of the other country:

$$\boldsymbol{r}_{t+1} = (r_{ft}, r_{H,t+1}, r^*_{F,t+1} + \Delta s_{t+1}) \qquad \text{and} \qquad \boldsymbol{r}^*_{t+1} = (r^*_{ft}, r^*_{F,t+1}, r_{H,t+1} - \Delta s_{t+1}).$$

Irrespective, of the statistical properties of the risky returns, $\widetilde{r}_{H,t+1} - \widetilde{r}^*_{F,t+1} - \Delta s_{t+1} \in \boldsymbol{\epsilon}^g_{t+1}$. Furthermore, if $(r_{H,t+1}, r^*_{F,t+1}, \Delta s_{t+1})$ has statistically a full rank (i.e., there is no linear dependence between these random variables), then $\boldsymbol{\epsilon}^g_{t+1} = (\widetilde{r}_{H,t+1} - \widetilde{r}_{F,t+1} - \Delta s^*_{t+1})$, and the exchange rate is not spanned by globally-traded shocks. However, it is locally-traded due to partial integration with risky assets. That is: $\widetilde{\Delta s}_{t+1} = g_{t+1} + \ell_{t+1}$ and $u_{t+1} = 0$ according to decomposition (13). This is because each household can construct a carry trade using the two risky assets, and the return on this carry trade is the globally-traded shock (see Appendix B). No other random variable is spanned by both $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^*_{t+1}$. By induction, the presence of n risky assets traded in common by home and foreign households will introduce $n-1$ globally-traded risks (excess returns), but in general do not make the exchange rate a globally-traded risk. This contrasts with the cases 2 and 3, where the exchange rate risk was immediately globally traded due to a carry trade strategy with two risk-free bonds.

C.2.2 Partial integration: a commonly traded asset

Consider an asset i with return $R_{i,t+1}$ in home currency and corresponding return $R^*_{i,t+1} = R_{i,t+1}\frac{S_t}{S_{t+1}}$ after conversion to foreign currency. The corresponding log returns are $r_{i,t+1} = \log R_{i,t+1}$ and $r^*_{i,t+1} = r_{i,t+1} - \Delta s_{t+1}$. Therefore, when $R_{i,t+1}$ is available to the home household and $R^*_{i,t+1}$ is available to the foreign household (that is, $i \in H \cap F$), then a simple one-asset pair of portfolios $r_{p,t+1} = r_{i,t+1}$ and $r^*_{p,t+1} = r^*_{i,t+1}$ spans the exchange: $\Delta s_{t+1} = r_{p,t+1} - r^*_{p,t+1}$ and, therefore, $u_{t+1} = 0$ in equation (14).[30]

If asset i is the only asset traded in common (that is, $\{i\} = H \cap F$), then there is no globally-traded risk (assuming $\widetilde{\Delta s}_{t+1} \neq 0$), that is $\boldsymbol{\epsilon}^g_{t+1} = \emptyset$ and $g_{t+1} = 0$ in equation (13). Indeed, in this case, the only risk that can be spanned in H is $\widetilde{r}_{i,t+1}$, and the only risk that can be spanned in F is $\widetilde{r}_{i,t+1} - \widetilde{\Delta s}_{t+1} \neq \widetilde{r}_{i,t+1}$.

Traded excess return Consider now a traded excess return $R_{i,t+1} - R_{j,t+1}$ on a zero-cost portfolio, with the foreign currency excess return given by $R^*_{i,t+1} - R^*_{j,t+1} = (R_{i,t+1} - R_{j,t+1})S_t/S_{t+1}$.[31] Following the same steps as in the proof of Lemma 3 in Appendix B.2, one can show that the risk of the log excess return is given by $\widetilde{r}_{i,t+1} - \widetilde{r}_{j,t+1}$ in the home currency and by $\widetilde{r}^*_{i,t+1} - \widetilde{r}^*_{j,t+1} = (\widetilde{r}_{i,t+1} - \widetilde{\Delta s}_{t+1}) - (\widetilde{r}_{j,t+1} - \widetilde{\Delta s}_{t+1}) = \widetilde{r}_{i,t+1} - \widetilde{r}_{j,t+1}$ in the foreign currency. Therefore, unlike a traded asset, a traded excess return is a globally-traded risk.

One way to construct an excess return for a traded asset i is to subtract a risk-free rate, $R_{j,t+1} = R_{ft}$. However, to obtain a globally-traded risk, the risk-free rate should be in the same currency in both markets, so that $R^*_{j,t+1} = R_{ft}S_t/S_{t+1}$. The same works with a risk-free rate in foreign currency, in which case $R_{j,t+1} = R^*_{ft}S_{t+1}/S_t$ and $R^*_{j,t+1} = R^*_{ft}$. In the former case, the globally-traded risk is $\widetilde{r}_{i,t+1}$, and in the latter it is $\widetilde{r}_{i,t+1} - \widetilde{\Delta s}_{t+1}$. Note that this requires that a risk-free bond (either in one or the other currency) is also traded by both households, in addition to risky asset i.

C.2.3 Globally-traded risks vs. common risks

It may be intuitively appealing to think about sources of common variation in domestic and foreign assets as globally-traded shocks. There is a critical difference between such intuition and the formal definition of globally-traded shocks, which requires replication of the exposure to such shock solely using assets of either country.

As an example, consider economies with N risky assets each, with all of these assets having exposure to a shock ϵ_{t+1}: $\widetilde{r}_{i,t+1} = \alpha_i\epsilon_{t+1} + \beta_i\epsilon_{i,t+1}$, and $\widetilde{r}^*_{i,t+1} = \alpha^*_i\epsilon_{t+1}$, and ϵ_{t+1} and all the $\epsilon_{i,t+1}$ are orthogonal to each other. If $\beta_i = 0$ for at least one domestic asset i, then ϵ_{t+1} is a globally-traded shock. If none of the β_i are equal to zero, then ϵ_{t+1} is not a globally-traded shock because

[30] Note that it is of no significance whether asset i is the same stock traded by both households or there are two assets with identical returns (in a common currency) traded separately in the home and foreign markets (e.g., as might be the case with ADRs or stocks like Royal Dutch Shell), as long as there is an intermediary with access to both assets in the latter case.

[31] One example of such return can be a commodity forward. In general, the return is defined as $R_{i,t+1} = (P_{i,t+1} + D_{i,t+1})/P_{it}$, where P is the price of the asset and D is the dividend. Then the pay-out on a commodity forward is given by the following excess return: $R_{i,t+1} - R_{j,t+1} = P_{i,t+1} - P_{j,t+1}$, where $P_{it} = P_{jt} = 1$, $D_{i,t+1} = D_{j,t+1} = 0$, and $P_{i,t+1}$ is the realized spot commodity price next period and $P_{j,t+1} = F_{it}$ is the forward commodity price.

it cannot be isolated from $\widetilde{\boldsymbol{r}}_{t+1}$. It is only when $N \to \infty$ than one can construct a portfolio of $\widetilde{r}_{i,t+1}$ to isolate ϵ_{t+1} via diversification.

D Derivation of the main results

D.1 Proof of propositions in Section 3

Proof of Proposition 1 Consider one of the globally-traded shocks, ϵ^g_{t+1}. By definition 1, there exist two portfolios $r_{p,t+1} \in \boldsymbol{r}_{p,t+1}$ and $r^*_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}$ such that $\epsilon^g_{t+1} = \widetilde{r}_{p,t+1} = \widetilde{r}^*_{p,t+1}$.

The differential carry portfolio of Lemma 3 is in $\boldsymbol{r}^I_{p,t+1}$. In this case, the portfolio has no risk because $\widetilde{r}_{p,t+1} = \widetilde{r}^*_{p,t+1}$. The shocks to foreign and domestic return perfectly offset each other. By assumption 2, the portfolio must have expected returns equal to the risk-free rate. That is:

$$0 = E_t[r_{p,t+1} - r_{ft}] - E_t[r^*_{p,t+1} - r^*_{ft}] - cov_t(r^*_{p,t+1}, \Delta s_{t+1}) + cov_t(r^*_{p,t+1}, r_{p,t+1} - r^*_{p,t+1}).$$

The last term is equal to 0 because $r_{p,t+1} - r^*_{p,t+1}$ has no risk. We can replace the first two terms by covariances with the SDFs using the domestic and foreign Euler equations (9) and (10),

$$\begin{aligned}0 = &-cov_t(m_{t+1}, r_{p,t+1}) - \frac{1}{2}var_t(r_{p,t+1}) + cov_t(m^*_{t+1}, r^*_{p,t+1}) + \frac{1}{2}var_t(r^*_{p,t+1}) \\ &- cov_t(r^*_{p,t+1}, \Delta s_{t+1}).\end{aligned}$$

Remembering that both portfolio shocks are equal to ϵ^g_{t+1}, this expression simplifies to:

$$cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, \epsilon^g_{t+1}) = 0.$$

This equation is equivalent to

$$cov(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} - \widetilde{\Delta s}_{t+1}, \epsilon^g_{t+1}) = 0,$$

which implies equation (15). Furthermore, under log-normality, this condition is equivalent to the equality of respective conditional expectation, $E\big(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} | \epsilon^g_{t+1}\big) = E\big(\widetilde{\Delta s}_{t+1} | \epsilon^g_{t+1}\big)$.

Because this result holds for any globally-traded shock, it must also hold in terms of multivariate projections on all globally-traded shocks $\boldsymbol{\epsilon}^g_{t+1}$. ■

Proposition 1 without risk-free assets The differential carry portfolio from Lemma 3 that we use in the proof of Proposition 1 relies on the availability of both risk-free rates in the intermediaries set of returns r^I_{t+1}. Proposition 1 generalizes to environments without risk-free assets. A globally-traded risk requires the existence of a jointly spanned excess return in $H \cap I$ and $F \cap I$ in respective local currencies. In our baseline setting, we obtained excess returns by subtracting the respective local risk-free rates from a given spanned return, forming a leg of the differential carry portfolio. More generally, we need to focus on excess returns of zero-cost portfolios which we denote with $r_{z,t+1}$ and $r^*_{z,t+1}$, respectively. By analogy with the definition of $\boldsymbol{r}_{p,t+1}$, we have:

$$\boldsymbol{r}_{z,t+1} = \big\{ r_{z,t+1} = \log\big(\boldsymbol{w}'_t \exp(\boldsymbol{r}_{t+1})\big) \;\big|\; \exists \boldsymbol{w}_t \in \mathbb{R}^N : \boldsymbol{w}'_t \boldsymbol{\iota} = 0 \big\}. \tag{A20}$$

Then a globally-traded shock is defined as $\boldsymbol{\epsilon}^g_{t+1} \equiv \widetilde{\boldsymbol{r}}_{z,t+1} \cap \widetilde{\boldsymbol{r}}^*_{z,t+1}$, where formally $\widetilde{\boldsymbol{r}}_{z,t+1}$ is the set of all spanned risks of zero-cost portfolios $\widetilde{r}_{z,t+1}$. Proposition 1 then holds under this generalized definition of globally-traded shocks $\boldsymbol{\epsilon}^g_{t+1}$. Note that in the presence of risk-free assets, this definition coincides with Definition 1. In the absence of risk-free assets, we need to find a pair of assets in each set $H \cap I$ and $F \cap I$ that have a perfectly correlated excess return in respective local currencies (see examples in Appendix C.2).

Proof of Proposition 2 Consider the carry portfolio of Lemma 2 constructed with a pair of portfolios $r_{p,t+1} \in \boldsymbol{r}_{p,t+1}$ and $r^*_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}$ which span the exchange rate (equation (14)). In this case, the portfolio has no risk because $\widetilde{r}_{p,t+1} - \widetilde{r}^*_{p,t+1} = \widetilde{\Delta s}_{t+1}$. The shocks to foreign and domestic return perfectly offset exchange rate risk. By assumption 2, the portfolio must have expected returns equal to the risk-free rate. This corresponds to

$$0 = E_t[r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}] + cov_t(r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}, r^*_{p,t+1} + \Delta s_{t+1}).$$

The covariance term is equal to 0, because $r_{p,t+1} = r^*_{p,t+1} - \Delta s_{t+1}$ has no risk. We can replace expected returns using the domestic and foreign Euler equations (9) and (10):

$$\begin{aligned} E_t \Delta s_{t+1} = r_{ft} - cov_t(m_{t+1}, r_{p,t+1}) - \frac{1}{2} var_t(r_{p,t+1}) \\ - r^*_{ft} + cov_t(m^*_{t+1}, r^*_{p,t+1}) + \frac{1}{2} var_t(r^*_{p,t+1}) = \delta_t. \end{aligned}$$

We replace $\widetilde{r}_{p,t+1} = \widetilde{r}^*_{p,t+1} + \widetilde{\Delta s}_{t+1}$:

$$\begin{aligned} E_t \Delta s_{t+1} &= r_{ft} - r^*_{ft} - cov_t(m_{t+1}, \Delta s_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, r^*_{p,t+1}) \\ &\quad + \frac{1}{2} var_t(r^*_{p,t+1}) - \frac{1}{2} var_t(\Delta s_{t+1}) - \frac{1}{2} var_t(r^*_{p,t+1}) - cov_t(\Delta s_{t+1}, r^*_{p,t+1}) \\ &= r_{ft} - r^*_{ft} - cov_t(m_{t+1}, \Delta s_{t+1}) - \frac{1}{2} var_t(\Delta s_{t+1}) \\ &\quad + cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r^*_{p,t+1}). \end{aligned}$$

This proves part b) of Proposition 2. If markets are fully integrated, all asset returns are globally-traded shocks, and proposition 1 implies that the last term in the equation above is equal to 0, part a) of the proposition. If the exchange rate is not spanned by asset returns, it is impossible to construct a trade with expected returns involving the expected depreciation rate that is risk-free. Therefore, no-arbitrage imposes no restriction on the expected depreciation rate. ■

Proposition 2 without risk-free assets Note that no step in the proof requires the existence of risk-free assets, as the carry trade in Lemma 2 builds on a pair of arbitrary portfolios $r_{p,t+1} \in \boldsymbol{r}_{p,t+1}$ and $r^*_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}$ that spans the exchange rate risk. The rates r_{ft} and r^*_{ft} in the definition of δ_t in (14) (and in asset pricing equations (9) and (10)) can in general be replaced by shadow risk-free rates defined as $r_{ft} \equiv -E_t m_{t+1} - \frac{1}{2} var_t(m_{t+1})$ and similarly for r^*_{ft}, even when risk-free assets are not available.

One situation must be handled separately: when the replicating portfolio features $\widetilde{r}_{p,t+1} = \widetilde{\Delta s}_{t+1}$ and hence $\widetilde{r}^*_{p,t+1} = 0$ when $r^*_{p,t+1} = r^*_{ft}$ is unavailable (or, symmetrically, without r_{ft}). Note, however, that $r_{p,t+1}$ with $\widetilde{r}_{p,t+1} = \widetilde{\Delta s}_{t+1}$ is equivalent to a foreign-currency risk-free asset traded by the domestic household, and therefore we can use this asset to define a shadow foreign risk-free rate: $r_{p,t+1} = r^*_{ft} + \Delta s_{t+1}$, where r^*_{ft} can be backed out using the home household's asset pricing condition (9):

$$r^*_{ft} + E_t \Delta s_{t+1} \equiv E_t\, r_{p,t+1} = r_{ft} - \frac{1}{2} var_t(\Delta s_{t+1}) - cov_t(m_{t+1}, \Delta s_{t+1}),$$

which coincides with the prediction of Proposition 2. Note that, again, in the absence of risk-free assets, r_{ft} and r^*_{ft} are shadow rates. The contrast with the general case is that in this case both of them are defined by the properties of domestic SDF m_{t+1}.

Proof of Proposition 3 Recall our decomposition of the depreciation rate into spanned and unspanned components, $\Delta s_{t+1} = E_t \Delta s_{t+1} + g_{t+1} + \ell_{t+1} + u_{t+1}$. Because $g_{t+1} + \ell_{t+1}$ is spanned by asset returns, there exists $r_{p,t+1} \in \boldsymbol{r}_{p,t+1}$ and $r^*_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}$ such that $\widetilde{r}_{p,t+1} - \widetilde{r}^*_{p,t+1} = g_{t+1} + \ell_{t+1}$. Using Lemma 2, we see that the risk of this portfolio is equal to $var_t(u_{t+1})$. We apply Assumption 3 to relate this risk to the expected return of the carry trade.

$$\begin{aligned}&\left| E_t[r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}] + cov_t(r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}, r^*_{p,t+1} + \Delta s_{t+1}) + \frac{1}{2} var_t(u_{t+1}) \right| \\ &\quad \leq B\sqrt{var_t(u_{t+1})}.\end{aligned}$$

Examining the terms in the left-hand-side, we have:

$$\begin{aligned} E_t[r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}] &= \delta_t - E_t[\Delta s_{t+1}] = -\psi_t, \\ cov_t(r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}, r^*_{p,t+1} + \Delta s_{t+1}) &= cov(-u_{t+1}, \widetilde{r}_{p,t+1} + u_{t+1}) \\ &= -var(u_{t+1}), \end{aligned}$$

where the last equality uses the fact that $u_{t+1} \perp (\boldsymbol{\epsilon}^g_{t+1}, \boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}^*_{t+1}) \ni \widetilde{r}_{p,t+1}$. Plugging these two results on the left-hand side of the inequality above, we obtain:

$$\left| \psi_t + \frac{1}{2} var_t(u_{t+1}) \right| \leq B\sqrt{var_t(u_{t+1})}.$$

Finally, by construction, the pair of portfolios $(r_{p,t+1}, r^*_{p,t+1})$, maximizes $R^2 = 1 - \frac{var_t(\Delta s_{t+1} - r_{p,t+1} + r^*_{p,t+1})}{var_t(\Delta s_{t+1})} = 1 - \frac{var_t(u_{t+1})}{var_t(\Delta s_{t+1})}$, and hence $var_t(u_{t+1}) = (1 - R^2) var_t(\Delta s_{t+1})$. ■

D.2 Propositions 1 and 2 are sufficient for no-arbitrage

We show that the results of Propositions 1 and 2 are not only necessary for the absence of international arbitrage — Assumption 2 — but also sufficient. Specifically we show the following.

Proposition 5. *If:*

1. *Assumption 1 holds,*
2. $E\left(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} | \epsilon^g_{t+1}\right) = E\big(\widetilde{\Delta s}_{t+1} | \epsilon^g_{t+1}\big),$
3. (a) *either* $\exists r^s_{p,t+1} \in \boldsymbol{r}_{p,t+1}, r^{s*}_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}$ *such that* $\widetilde{\Delta s}_{t+1} = \widetilde{r}^s_{p,t+1} - \widetilde{r}^{s*}_{p,t+1}$ *and*

$$\begin{aligned} E_t \Delta s_{t+1} = r_{ft} - r^*_{ft} - cov_t(m^*_{t+1}, \Delta s_{t+1}) + \frac{1}{2} var_t(\Delta s_{t+1}) \\ + cov_t\left(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r_{p,t+1}\right), \end{aligned}$$

(b) or $\forall r^s_{p,t+1} \in \boldsymbol{r}_{p,t+1}, r^{s*}_{p,t+1} \in \boldsymbol{r}^*_{p,t+1}, \widetilde{\Delta s}_{t+1} \neq \widetilde{r}^s_{p,t+1} - \widetilde{r}^{s*}_{p,t+1}$,

then there are no arbitrage opportunities in international markets, that is Assumption 2 holds.

Proof: We proceed by contradiction. Assume that there exists an international arbitrage:

$$\exists r^I_{p,t+1} \in \boldsymbol{r}^I_{p,t+1} : \; var_t(r^I_{p,t+1}) = 0 \;\text{ and }\; E_t r^I_{p,t+1} \neq r_{ft},$$

and denote $\boldsymbol{w}$ and $\boldsymbol{w}^*$ the set of weights of such a portfolio on $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^*_{t+1} + \Delta s_{t+1}$. Remember that $\boldsymbol{\iota}'_N \boldsymbol{w} + \boldsymbol{\iota}'_{N^*}\boldsymbol{w}^* = 1$. We consider the cases of 3a and 3b in turn.

Assume condition 3a holds. As a preliminary, note that this condition is equivalent to saying that a carry portfolio constructed with $r^s_{p,t+1}$ and $r^{s*}_{p,t+1}$ has no risk and no average excess return. Consider the following portfolio: long $\boldsymbol{w}'\boldsymbol{r}_{t+1}$, long $(\boldsymbol{\iota}'_{N^*}\boldsymbol{w}^*)\, r^s_{p,t+1}$, long $\boldsymbol{w}^{*\prime}\left(\boldsymbol{r}^*_{t+1} + \Delta s_{t+1}\right)$, short $(\boldsymbol{\iota}'_{N^*}\boldsymbol{w}^*)\,(r^{s*}_{p,t+1} + \Delta s_{t+1})$. Because we have added and subtracted the same total weights, the new weights still add up to 1, so this is still a portfolio. Because this portfolio combines two risk-free portfolios — our assumed arbitrage and the risk-free carry trade — its expected return is the sum of the two expected returns, that is $E_t\, r^I_{p,t+1}$. The total weight on foreign returns in the portfolio are $\boldsymbol{\iota}'_{N^*}\boldsymbol{w}^* - \boldsymbol{\iota}'_{N^*}\boldsymbol{w}^* = 0$. Therefore, this trade is a differential carry portfolio (defined in Appendix B.2). Because it has no risk, its home and foreign legs offset each other. They form a globally-traded shock. Applying condition 1 in the proposition and Lemma 3 leads immediately to the result that the portfolio return must equal the risk-free rate. This contradicts the assumption that $E_t\, r^I_{p,t+1} \neq r_{ft}$.

Now assume that condition 3b holds. If $\boldsymbol{\iota}'_{N^*}\boldsymbol{w}^* \neq 0$, then the arbitrage portfolio has a non-zero loading on Δs_{t+1} in addition to the home and foreign returns. Because the portofolio is riskless this implies that we can find a pair of home and foreign returns that spans the depreciation rate, a contradiction of condition 3b. If $\boldsymbol{\iota}'_{N^*}\boldsymbol{w}^* = 0$, then the two legs of the portfolio in their home currency perfeclty offset each other. Their innovations constitute a globally-traded shock and applying condition 1 in the proposition jointly with Lemma 3 implies that the arbitrage portfolio has 0 expected return, a contradiction as well. ■

D.3 Proofs of the results of Section 4

Proof of Proposition 4 Apply the Cauchy-Schwarz inequalty to $\widetilde{\Delta s}_{t+1} - g_{t+1}$ and $(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}) - g_{t+1}$:

$$cov_t(\widetilde{\Delta s}_{t+1} - g_{t+1}, (\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}) - g_{t+1})^2 \leq var_t(\widetilde{\Delta s}_{t+1} - g_{t+1}) var_t((\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}) - g_{t+1}). \tag{A21}$$

Proposition 1 implies that both $\widetilde{\Delta s}_{t+1} - g_{t+1}$ and $(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}) - g_{t+1}$ are orthogonal to g_{t+1}. Therefore:

$$var_t(\widetilde{\Delta s}_{t+1} - g_{t+1}) = var_t(\Delta s_{t+1}) - var_t(g_{t+1})$$
$$var_t((\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}) - g_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) - var_t(g_{t+1})$$
$$cov_t(\widetilde{\Delta s}_{t+1} - g_{t+1}, (\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}) - g_{t+1}) = cov_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}) - var_t(g_{t+1})$$

When $var_t((\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}) - g_{t+1}) > 0$, plugging in and rearranging the terms in equation (A21) gives equation (31). When $var_t((\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}) - g_{t+1}) = 0$, we get $var_t(g_{t+1}) = cov_t(\Delta s_{t+1}, m^*_{t+1} - m_{t+1})$, with the first term being no greater than $var_t(\Delta s_{t+1})$ by definition of g_{t+1}, yielding (32). ■

Maximum $var_t(g_{t+1})$ without puzzles We ask what is the largest value of $var_t(g_{t+1})$ so that the Data point falls within the red cone in Figure 3. This is the value such that the frontier of the cone reaches exactly that point. The parabola is defined by taking condition (31) with equality. The Data point is characterized by the empirical value of $var_t(\Delta s_{t+1})$ and cyclicality 0. Plugging in, this corresponds to solving:

$$var_t(\Delta s_{t+1}) = var_t(g_{t+1}) + \frac{var_t(g_{t+1})^2}{var_t(m^*_{t+1} - m_{t+1}) - var_t(g_{t+1})} \tag{A22}$$

Dividing by $var_t(\Delta s_{t+1})$ gives:

$$\begin{aligned}
1 = \frac{var_t(g_{t+1})}{var_t(\Delta s_{t+1})} + \left(\frac{var_t(g_{t+1})}{var_t(\Delta s_{t+1})}\right)^2 / \left(\frac{var_t(m^*_{t+1} - m_{t+1})}{var_t(\Delta s_{t+1})} - \frac{var_t(g_{t+1})}{var_t(\Delta s_{t+1})}\right)\\
\frac{1 - \frac{var_t(g_{t+1})}{var_t(\Delta s_{t+1})}}{\left(\frac{var_t(g_{t+1})}{var_t(\Delta s_{t+1})}\right)^2} = 1/\left(\frac{var_t(m^*_{t+1} - m_{t+1})}{var_t(\Delta s_{t+1})} - \frac{var_t(g_{t+1})}{var_t(\Delta s_{t+1})}\right)\\
\frac{\left(\frac{var_t(g_{t+1})}{var_t(\Delta s_{t+1})}\right)^2}{1 - \frac{var_t(g_{t+1})}{var_t(\Delta s_{t+1})}} = \frac{var_t(m^*_{t+1} - m_{t+1})}{var_t(\Delta s_{t+1})} - \frac{var_t(g_{t+1})}{var_t(\Delta s_{t+1})}\\
\frac{\frac{var_t(g_{t+1})}{var_t(\Delta s_{t+1})}}{1 - \frac{var_t(g_{t+1})}{var_t(\Delta s_{t+1})}} = \frac{var_t(m^*_{t+1} - m_{t+1})}{var_t(\Delta s_{t+1})}.
\end{aligned}$$

Because $x/(1-x) = a \Leftrightarrow x = a/(1+a)$, this gives:

$$\frac{var_t(g_{t+1})}{var_t(\Delta s_{t+1})} = \frac{\frac{var_t(m^*_{t+1} - m_{t+1})}{var_t(\Delta s_{t+1})}}{1 + \frac{var_t(m^*_{t+1} - m_{t+1})}{var_t(\Delta s_{t+1})}}. \tag{A23}$$

Notice that the deeper the volatility puzzle in complete markets, that is, the larger the value of $var_t(m^*_{t+1} - m_{t+1})/var_t(\Delta s_{t+1})$ is, the larger the possible contribution of globally-traded risks to the exchange rate $var_t(g_{t+1})/var_t(\Delta s_{t+1})$.

E Exact non-linear version of the propositions

Our proofs rely on returns being log-normal and a log-linearization of portfolio returns as described in Appendix B.1. In this section we address the question of how the propositions change without distributional assumption and approximation.

E.1 A version of Proposition 1

Consider two portfolios, domestic with returns $R_{p,t+1}$ and foreign with returns $R^*_{p,t+1}$ such that their innovations coincide with one of the globally-traded shocks, that is, they can be represented as $R_{p,t+1} = \alpha_t + R^*_{p,t+1}$. The local Euler equations imply:

$$\begin{aligned}
E_t(M_{t+1}R_{p,t+1}) &= 1,\\
E_t(M^*_{t+1}R^*_{p,t+1}) &= 1.
\end{aligned}$$

The local Euler equations can be re-written as

$$E_t(R_{p,t+1}) = R_{ft} - cov_t\left(\frac{M_{t+1}}{E_t(M_{t+1})}, R_{p,t+1}\right) \tag{A24}$$

$$E_t(R^*_{p,t+1}) = R^*_{ft} - cov_t\left(\frac{M^*_{t+1}}{E_t(M^*_{t+1})}, R^*_{p,t+1}\right). \tag{A25}$$

Now consider an intermediary whose SDF expressed in the units of domestic currency, M^I_{t+1}, satisfies the following Euler equations:

$$E_t(M^I_{t+1}R_{ft}) = 1, \tag{A26}$$

$$E_t(M^I_{t+1}R^*_{ft}S_{t+1}/S_t) = 1, \tag{A27}$$

$$E_t(M^I_{t+1}(R_{p,t+1} - R_{ft})) = 0, \tag{A28}$$

$$E_t(M^I_{t+1}(R^*_{p,t+1} - R^*_{ft})S_{t+1}/S_t) = 0. \tag{A29}$$

The intermediary trades the zero-cost differential carry portfolio:

$$\begin{aligned}
0 &= E_t\left(M^I_{t+1}\left[(R_{p,t+1} - R_{ft}) - (R^*_{p,t+1} - R^*_{ft}) \cdot S_{t+1}/S_t\right]\right) \\
&= E_t\left(M^I_{t+1}\left[(R_{p,t+1} - R_{ft}) - (R_{p,t+1} - \alpha_t - R^*_{ft}) \cdot S_{t+1}/S_t\right]\right) \\
&= E_t\left(M^I_{t+1}\left[(R_{p,t+1} - R_{ft})(1 - S_{t+1}/S_t) + (\alpha_t + R^*_{ft} - R_{ft}) \cdot S_{t+1}/S_t\right]\right).
\end{aligned}$$

Replace the risk-free rates by the expressions from the local Euler equations (A24) and (A25), divide the equation by $E_t(M^I_{t+1})$, and define

$$\begin{aligned}
cov^I_t\left(\frac{S_{t+1}}{S_t}, R^*_{p,t+1}\right) \equiv\ & E_t\left(\frac{M^I_{t+1}}{E_t(M^I_{t+1})}(R^*_{p,t+1} - R^*_{ft})\frac{S_{t+1}}{S_t}\right) \\
& - \underbrace{E_t\left(\frac{M^I_{t+1}}{E_t(M^I_{t+1})}(R^*_{p,t+1} - R^*_{ft})\right)}_{0} \cdot E_t\left(\frac{M^I_{t+1}}{E_t(M^I_{t+1})}\frac{S_{t+1}}{S_t}\right), \\
E^I_t\left(\frac{S_{t+1}}{S_t}\right) \equiv\ & E_t\left(\frac{M^I_{t+1}}{E_t(M^I_{t+1})}\frac{S_{t+1}}{S_t}\right) = \frac{R_{ft}}{R^*_{ft}}.
\end{aligned}$$

Then

$$0 = -cov^I_t\left(\frac{S_{t+1}/S_t}{E^I_t(S_{t+1}/S_t)}, R^*_{p,t+1}\right) + cov_t\left(\frac{M^*_{t+1}}{E_t(M^*_{t+1})} - \frac{M_{t+1}}{E_t(M_{t+1})}, R^*_{p,t+1}\right).$$

(We replace $R_{p,t+1}$ with $R^*_{p,t+1}$ in the cov^I_t term because of our assumption about $R_{p,t+1}$ and $R^*_{p,t+1}$.)

This expression implies

$$\begin{aligned}
& cov_t\left(\frac{M^*_{t+1}}{E_t(M^*_{t+1})} - \frac{M_{t+1}}{E_t(M_{t+1})} - \frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)}, R^*_{p,t+1}\right) \\
&= \underbrace{cov^I_t\left(\frac{S_{t+1}/S_t}{E^I_t(S_{t+1}/S_t)}, R^*_{p,t+1}\right) - cov_t\left(\frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)}, R^*_{p,t+1}\right)}_{W}
\end{aligned} \tag{A30}$$

As we noted in section B.1, the log approximation that we use in Proposition 1 becomes exact if time is continuous and the data-generating process converges to a pure diffusion. Under such scenario, the covariance in the equation above is observable, and, thus, has the same value with and without risk adjustment (via M^I_{t+1}). As a result, $W = 0$. Also, each Arrow-Debreu claim makes the corresponding state globally-traded. For such a globally-traded risk $W = 0$.

Further, the projection result depends on the knowledge of intermediary's SDF, M^I_{t+1} via the term with cov^I_t. The log approximation relies only on the existence of such SDF, due to Assumption 2, and allows us to be agnostic about its actual values.

The first term in the second line is equal to $R^*_{ft} \cdot QRP_t$, where QRP_t is the quanto-implied risk premium of Kremens and Martin (2019). Its role in our paper is different from that of these authors. They use it to approximate the currency risk premium assigned by the intermediary, $R^*_{ft} E_t(S_{t+1}/S_t) - R_{ft} = -R_{ft} cov_t(M^I_{t+1}, R^*_{ft} \cdot S_{t+1}/S_t)$. Here it measures the gap in projections of the relative discount factor and the depreciation rate on globally-traded risks.

E.2 A version of Proposition 2

Consider two portfolios, domestic with returns $R_{p,t+1}$ and foreign with returns $R^*_{p,t+1}$ such that their innovations span the exchange rate, that is, they can be represented as $R_{p,t+1} = \alpha_t + R^*_{p,t+1} S_{t+1}/S_t$. The local Euler equations (A24) and (A25) hold for these portfolios. Also, we consider a (domestically funded) intermediary whose SDF, M^I_{t+1}, satisfies the following Euler equations:

$$E_t(M^I_{t+1} R_{p,t+1}) = 1,$$
$$E_t(M^I_{t+1} R^*_{p,t+1} S_{t+1}/S_t) = 1.$$

First, we show that $\alpha_t = 0$. The intermediary can form a zero-cost carry portfolio:

$$0 = E_t\left(M^I_{t+1}\left[R_{p,t+1} - R^*_{p,t+1} S_{t+1}/S_t\right]\right) = \alpha_t E_t\left(M^I_{t+1}\right).$$

Therefore, the expected return of the carry portfolio is equal to zero:

$$\begin{aligned}
0 &= E_t\left(R_{p,t+1} - R^*_{p,t+1} S_{t+1}/S_t\right) \\
&= R_{ft} - cov_t\left(\frac{M_{t+1}}{E_t(M_{t+1})}, R_{p,t+1}\right) \\
&- E_t(R^*_{p,t+1}) E_t(S_{t+1}/S_t) - cov_t(R^*_{p,t+1}, S_{t+1}/S_t) \\
&= R_{ft} - cov_t\left(\frac{M_{t+1}}{E_t(M_{t+1})}, R_{p,t+1}\right) - cov_t(R^*_{p,t+1}, S_{t+1}/S_t) \\
&- \left[R^*_{ft} - cov_t\left(\frac{M^*_{t+1}}{E_t(M^*_{t+1})}, R^*_{p,t+1}\right)\right] E_t(S_{t+1}/S_t),
\end{aligned}$$

where we substituted the local Euler equations (A24) and (A25) in lines 2 and 5, respectively. This equation implies the currency risk premium:

$$
\begin{aligned}
R^*_{ft} E_t\left(\frac{S_{t+1}}{S_t}\right) - R_{ft} &= -cov_t\left(\frac{M_{t+1}}{E_t(M_{t+1})}, R^*_{p,t+1}\frac{S_{t+1}}{S_t}\right) \\
&+ cov_t\left(\frac{M^*_{t+1}}{E_t(M^*_{t+1})}, R^*_{p,t+1}\right) E_t\left(\frac{S_{t+1}}{S_t}\right) - cov_t\left(R^*_{p,t+1}, \frac{S_{t+1}}{S_t}\right) \\
&= -R^*_{ft} cov_t\left(\frac{M_{t+1}}{E_t(M_{t+1})}, \frac{S_{t+1}}{S_t}\right) - cov_t\left(\frac{M_{t+1}}{E_t(M_{t+1})}, [R^*_{p,t+1} - R^*_{ft}]\frac{S_{t+1}}{S_t}\right) \\
&+ cov_t\left(\frac{M^*_{t+1}}{E_t(M^*_{t+1})} E_t\left(\frac{S_{t+1}}{S_t}\right) - \frac{S_{t+1}}{S_t}, R^*_{p,t+1}\right) \\
&= \underbrace{-R_{ft} cov_t\left(M_{t+1}, R^*_{ft}\frac{S_{t+1}}{S_t}\right)}_{\text{complete markets}} \\
&+ \underbrace{cov_t\left(\frac{M^*_{t+1}}{E_t(M^*_{t+1})} - \frac{M_{t+1}}{E_t(M_{t+1})} - \frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)}, R^*_{p,t+1}\right) E_t\left(\frac{S_{t+1}}{S_t}\right)}_{A} \\
&- \underbrace{cov_t\left(\frac{M_{t+1}}{E_t(M_{t+1})}, [R^*_{p,t+1} - R^*_{ft}]\left[\frac{S_{t+1}}{S_t} - E_t\left(\frac{S_{t+1}}{S_t}\right)\right]\right)}_{B},
\end{aligned}
$$

where in the first line we take advantage of spanning and replace $R_{p,t+1}$ with $R^*_{p,t+1}S_{t+1}/S_t$; the third line is obtained from the first by adding and subtracting the leading term in line 3; the fourth line is obtained by combining the two terms in the second line; the 6th and 7th lines are obtained by adding and subtracting $cov_t(M_{t+1}/E_t(M_{t+1}), R^*_{p,t+1})$.

The term B in the seventh line is the domestic household's risk premium for quanto exposure and disappears in the log-normal approximation. Also, $B = 0$ if $R^*_{p,t+1}$ happens to be R^*_{ft}, that is, domestic household can trade foreign risk-free bond. The term A in the sixth line is equal to zero in this case as well.

Next, if financial markets are integrated then the innovation to $R^*_{p,t+1}$ is a globally-traded shock. Then, equation (A30) from the non-linear version of Proposition 1 implies that

$$A = W \cdot E_t(S_{t+1}/S_t).$$

As is the case for Proposition 1, the log approximation treats this term as close to zero.

It might appear that the departure from log-normality in the case of integrated markets leads to two extra terms, A and B. In fact, when markets are integrated $A - B$ can be simplified to a term with a single source of departures from zero. Indeed, we obtain

$$
\begin{aligned}
\frac{A-B}{E_t(S_{t+1}/S_t)} &= cov_t^I\left(\frac{S_{t+1}/S_t}{E_t^I(S_{t+1}/S_t)}, R^*_{p,t+1}\right) - cov_t\left(\frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)}, R^*_{p,t+1}\right) \\
&- cov_t\left(\frac{M_{t+1}}{E_t(M_{t+1})} - 1, [R^*_{p,t+1} - R^*_{ft}]\left[\frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)} - 1\right]\right) \\
&= E_t\left(\frac{M^I_{t+1}}{E_t(M^I_{t+1})}[R^*_{p,t+1} - R^*_{ft}]\left[\frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)} - 1\right]\right) \\
&- E_t\left(\frac{M_{t+1}}{E_t(M_{t+1})}[R^*_{p,t+1} - R^*_{ft}]\left[\frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)} - 1\right]\right).
\end{aligned}
$$

Thus, $A - B$ is close to zero when the intermediary prices the globally-traded (quanto) risk the same way as the domestic household.

If there is no spanning, $R_{p,t+1} \neq \alpha_t + R^*_{p,t+1}S_{t+1}/S_t$, then it is impossible to find a risk-free strategy and derive restrictions on the currency risk premium.

F Empirical analysis: additional results

CCA analysis for the undirected approach Table A1 reports the results. Each column represents a foreign country. For a given country, each row reports the canonical correlation between the assets of that country and the US assets, reported in order of importance, starting from the largest.

The values of the largest correlations range from 64% for New Zealand to 90% for Canada. In some cases lower ranked correlations are similar to the largest one, like for Canada or the UK. In other cases, the magnitude of correlation drops off quickly, e.g., for New Zealand or Norway. Strictly speaking, the evidence suggests that there are no globally-traded shocks amongst the assets that we consider.

Excluding US bonds There are theories that posit that Euler equations for US Treasury bonds hold with wedges (e.g., Jiang, Krishnamurthy, and Lustig, 2021). If that is the case our framework would exclude such assets from the set H. In this appendix we evaluate how such exclusion would affect our empirical results. Table A2 reports the spanning regression. Figure A1 displays the decomposition of the variance into the globally-traded, locally-traded, and unspanned components. The results are similar to their counterparts when US bonds are included.

Directed approach Instead of being agnostic about the nature of globally-traded shocks we rely on macroeconomic research and assume that they are known. Specifically, we take VIX, GFC (Miranda-Agrippino and Rey, 2020), and EBP (Gilchrist and Zakrajsek, 2012) as such shocks. This approach requires a strong assumption that portfolios of traded assets in each economy can span these shocks.

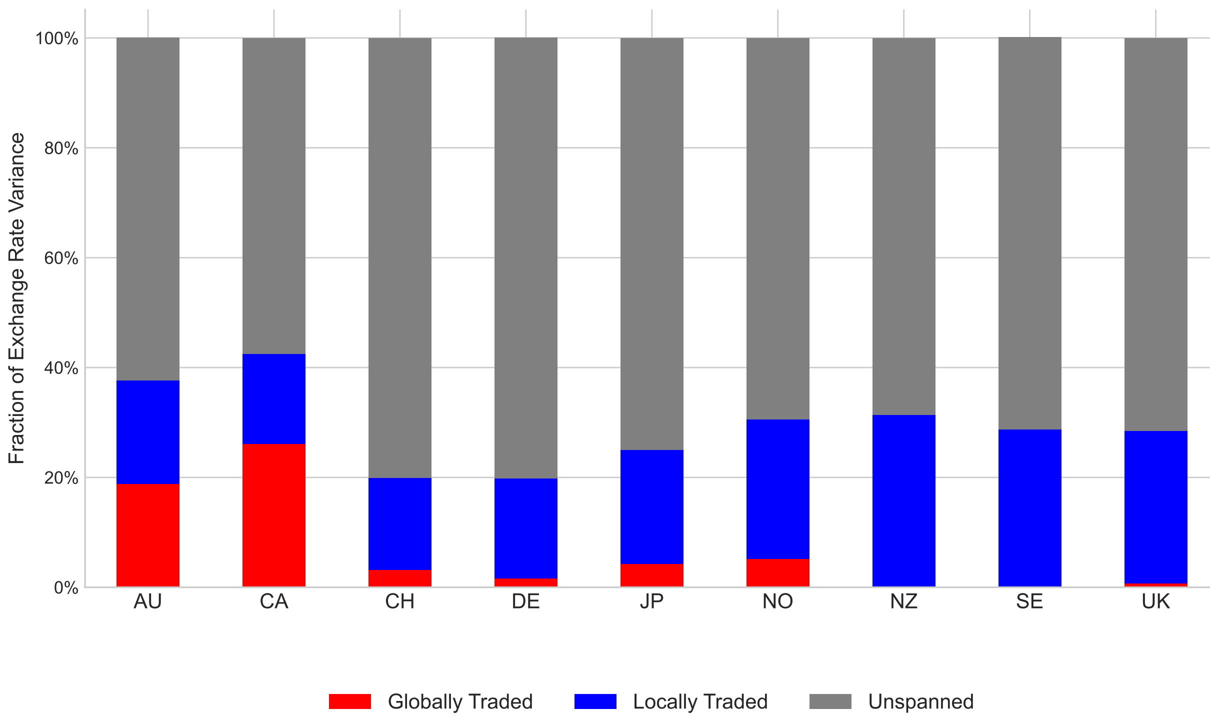
For each country, we regress its depreciation rate vs USD on these measures of globally-traded shocks. The R^2 from such a regression produce the fraction of the exchange rate variation due to globally-traded shocks. Next, we implement the regression in Equation (36) where the set of returns is complemented by the three global shocks to obtain the unspanned component. Naturally,

Table A1: Maximally correlated shocks across asset markets

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
Rank 1	75.50	90.00	83.10	75.75	78.16	69.67	81.44	82.92	85.91
Rank 2	65.65	85.26	75.74	62.39	64.33	58.05	68.62	63.00	77.94
Rank 3	59.05	84.89	66.96	58.74	57.01	50.57	59.96	58.92	72.16
Rank 4	56.10	79.36	64.41	53.59	47.12	38.31	54.42	56.17	69.19
Rank 5	51.60	77.21	53.18	48.02	40.49	30.51	47.03	51.68	67.32
Rank 6	39.59	70.27	47.29	45.41	33.95	27.36	42.87	46.62	60.87
Rank 7	34.57	61.37	43.67	40.72	30.73	22.34	39.54	41.15	55.18
Rank 8	32.10	56.77	40.76	39.08	26.67	21.75	35.73	34.96	50.76
N	420	402	410	420	407	352	377	420	420

The table reports the correlation in % between the maximally correlated portfolios of asset returns between the U.S. and each country. The successive pairs of portfolio are orthogonal to each other, and obtained by canonical correlation analysis. Domestic asset returns are in domestic currency; foreign asset returns are in foreign currency. Each column is for a different country's assets relative to the U.S. assets. The assets include government bonds of maturities between 2 and 10 years (obtained from various central banks) and various stock portfolios: the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios (from MSCI).

Figure A1: Decomposition of exchange rate innovations excluding US bonds



The figure reports the fraction of variance in exchange rates explained by globally-traded shocks, locally-traded shocks, and shocks that are not spanned by asset returns, under the assumption of an intermediated market structure described in Section 5. Each bar is a different country's currency relative to the U.S. dollar; globally-traded shocks are measured using CCA for stock and sovereign bond returns. US bonds are excluded from the analysis.

Table A2: Spanning of depreciation rates by asset returns – R^2 (excluding US Bonds)

Dependent Variable	AU	CA	DE	JP	NO	NZ	SE	CH	UK
Bonds									
10Y	-0.11	0.09	0.39	1.03	0.87	-0.29	1.47	-0.23	0.13
All Maturities	2.02	-0.47	2.12	1.41	1.08	0.80	3.46	0.30	3.07
Stocks									
Mkt	22.17	26.77	8.31	5.94	12.98	19.59	19.71	12.16	14.38
Mkt + Value/Growth	22.06	28.15	8.13	7.07	14.06	20.03	19.80	12.52	15.10
Mkt + Value/Growth + Ind.	36.45	41.84	19.25	24.86	30.45	30.92	26.90	19.61	27.43
Bond + Equity	37.65	42.49	19.81	25.05	30.60	31.41	28.61	19.97	28.52
N	420	402	410	420	407	352	377	420	420

The table reports the adjusted R^2 of a regression of the depreciation rate on various subsets of asset returns, as in equation (36). Domestic asset returns are in domestic currency; foreign asset returns are in foreign currency. We exclude US bonds from consideration. Each column is a different country's currency relative to the U.S. dollar. The first row uses only 10-year bonds, while the second entertains maturities between 2 and 10 years, obtained from various central banks. The next three row consider various stock portfolios: the market (a combination of large and mid-cap stocks), plus value and growth portfolios, plus 10 industry portfolios (all from MSCI). The final row considers all assets simultaneously.

it is going to be smaller than that in the previous section. The knowledge of the variation due to global and unspanned shocks delivers the variation due to locally-traded shocks.

Figure A2 reports the resulting decomposition of the variation in the exchange rate into the three types of shocks. The directed approach delivers somewhat larger contribution of global shocks, but qualitatively the conclusions are unchanged. The unspanned shocks represent the largest share of shocks. Contribution of the global shocks is the largest for Australia and Canada, which approach 50%.

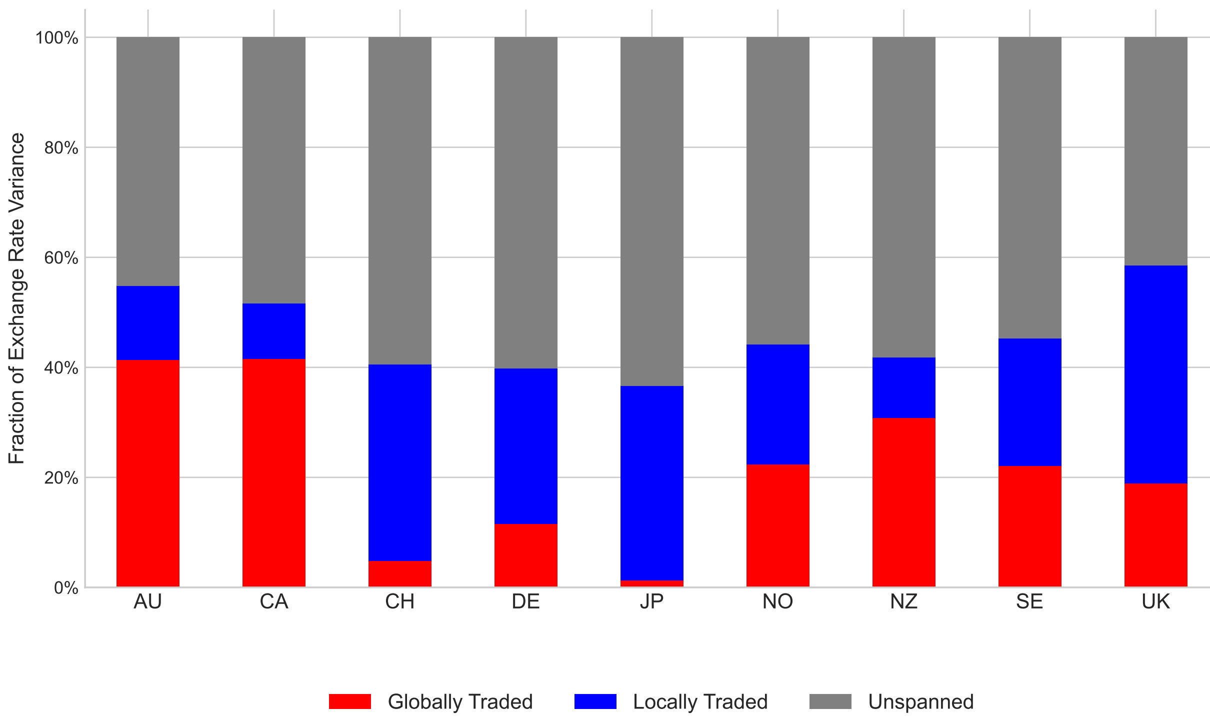
G What do return data say about the exchange rate?

Our propositions have implications for various exercises using return data and the assumption of no-arbitrage only. We first show how to use our formal framework in this context, then discuss these implications and their relation to the literature.

An alternative interpretation of the framework Instead of representing the assets that investors in each country can access like in Section 2.1, the sets H and F can simply represent different subsets of return data expressed in different currencies. In this interpretation H is a set of assets for which we have data on returns $\boldsymbol{r}_{t+1}$ in the home currency, F is a set of assets for which we have data on returns $\boldsymbol{r}^*_{t+1}$ in the foreign currency. If we assume that neither of these sets of returns feature arbitrage opportunities, one can construct minimum (log) variance SDFs by combining asset returns in the spirit of Hansen and Jagannathan (1991): $m_{t+1} = \lambda' \boldsymbol{r}_{t+1}$ and $m^*_{t+1} = \lambda^{*\prime} \boldsymbol{r}^*_{t+1}$. This setting is equivalent to Assumption 1.

Furthermore, one might want to assume that there are no arbitrage opportunities when a

Figure A2: Decomposition of exchange rate innovations: directed approach



The figure reports the fraction of variance in exchange rates explained by globally-traded shocks, locally-traded shocks, and shocks that are not spanned by asset returns, under the assumption of an intermediated market structure described in Section 5. Each bar is a different country's currency relative to the U.S. dollar. Globally-traded shocks are measured using exposures to changes in VIX, GFC, and EBP.

combination of these assets is traded together. We call I this joint set of assets, so that this lack of arbitrage across datasets coincide with Assumption 2. For example, if one assumes that there are no arbitrage opportunities between the two sets of returns, this corresponds to $I = H \cup F$.

In this interpretation of the framework, Propositions 1 and 2 convey all restrictions between the return data, the corresponding minimum variance SDFs, and the exchange rate. Some combinations of assumptions about which data are observed coincide with assumptions about the market structures we study in the paper, and hence we can use the implications of our propositions for these structure. Still, note that this equivalence is only mathematical and the interpretation is different: here we are simply isolating different datasets as opposed to making assumptions about what different groups of investors have access to.

Recovering the exchange rate using local currency returns only. A first classic question is whether the exchange rate can be recovered from return data on local assets only. For example, one might want to estimate SDFs for the dollar yield curve and the pound yield curve, respectively, and use their ratio to recover the exchange rate. This exercise, followed by Bansal (1997) and Backus, Foresi, and Telmer (2001) should work if markets are complete and local returns in each country span each state of the world. However, their estimates are at odds with the empirical behavior of the exchange rate. Chernov and Creal (2023) propose a model of SDFs including shocks that are not spanned by the yield curves that can price both yield curves and be consistent with the exchange rate.

We consider the general version of this exercise. H and F each contain distinct local assets in their local currency and by m and m^* are the minimum variance SDFs constructed from each set of asset returns. The assumption that there are no international arbitrage opportunities between all these assets corresponds to $I = H \cup F$.

Mathematically, this situation coincides exactly with the intermediated models we consider in Section 5. In this case, globally-traded shocks can be constructed using CCA on the local asset returns $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}^*_{t+1}$. Proposition 1 immediately says that only the global component of the exchange rate g_{t+1} is pinned down. If the local asset returns do not have common shocks, nothing can be said about exchange rate movements. With globally-traded shocks, the projection of $m^*_{t+1} - m_{t+1}$ on these shocks reveals the projection of Δs_{t+1} on these shocks. Furthermore, the local component ℓ_{t+1} and u_{t+1} can be arbitrary, so the exchange rate can have any amount of excess volatility above this projection-based component g_{t+1}.

Chernov and Creal (2023) find that at most 10% of exchange rate variation is explained by common shocks. Our empirical results suggest that, even after adding stocks to sovereign bonds, the variance of the globally-traded component g_{t+1} is small relative to the variance of the depreciation rate. This implies that a modest component of the exchange rate can be recovered by observing local returns and using the assumption of no-arbitrage alone.

This conclusion does not rule out that the depreciation rate might exhibit substantial correlation with specific assets, as long as it is through locally-traded shocks. Finally, to the extent that the exchange rate features an unspanned component — like we find empirically for stocks and bonds — Proposition 2 indicates that no-arbitrage does not pin down expected depreciation.

Finally, this conclusion also does not rule out that markets might be complete and integrated. Throughout this exercise we maintain the assumption that there exists an SDF pricing all assets (Assumption 2), which might coincide with the IMRSs of both home and foreign households.

Constructing pairs of SDFs satisfying the AMV. Alternatively, one might want to find SDFs that price the same assets in each currency so that $m^*_{t+1} - m_{t+1} = \Delta s_{t+1}$.[32] The minimum variance log SDFs solve this question, and our framework demonstrates why.

In this exercise, we have $H = F = I$, and $\boldsymbol{r}_{t+1} = \boldsymbol{r}^*_{t+1} + \Delta s_{t+1}$. We denote by m_{t+1} the minimum variance SDF pricing $\boldsymbol{r}_{t+1}$, and symmetrically m^*_{t+1} prices $\boldsymbol{r}^*_{t+1}$. Mathematically, this case coincides with a situation of fully integrated markets. Therefore, by Lemma 1, $\text{proj}(\widetilde{\Delta s}_{t+1}|\boldsymbol{\epsilon}^g_{t+1}) = \widetilde{\Delta s}_{t+1}$. Furthermore, because all asset returns are globally-traded shocks and hence the minimum variance SDFs are spanned by asset returns by construction, we also have $\text{proj}(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}|\boldsymbol{\epsilon}^g_{t+1}) = \widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}$. So, Proposition 1 implies that shocks to the relative minimum variance SDF coincide with shocks to the depreciation rate. Similarly, Proposition 2 implies that the means are equalized as well. Therefore, $\Delta s_{t+1} = m^*_{t+1} - m_{t+1}$.

Interestingly, Sandulescu, Trojani, and Vedolin (2021) show that, away from the log-normal case, this recovery result generalizes by focusing on minimum entropy SDFs: in a log-normal setting, mimimizing the entropy of the SDF is equivalent to minimize the variance of the log SDFs.[33]

Of course, this is not the only way to construct pairs of SDFs pricing the assets in two currencies. Under the assumption of no arbitrage, for any SDF m_{t+1} that prices $\boldsymbol{r}_{t+1}$, the SDF $m_{t+1} + \Delta s_{t+1}$ prices $\boldsymbol{r}^*_{t+1} = \boldsymbol{r}_{t+1} - \Delta s_{t+1}$.

This observation highlights that these recovery exercises do not necessarily lead to economically meaningful SDFs. For example, the world might be well-described by an intermediated market structure where a global intermediary trades all asset but households face arbitrary types of frictions. Then, all that this exercise is doing is recovering the projection of m^I_{t+1} and $m^{I*}_{t+1} = m^I_{t+1} + \Delta s_{t+1}$ on asset returns as opposed to the IMRSs of local investors. This observation echoes the conclusion of our main theoretical analysis: data on returns and the depreciation rate alone are in general not enough to identify the financial market structure.

H Evaluating the portfolio approximation

We report the correlation (in %) between the excess return on various stock portfolios —Table A3— and bonds of different maturities —Table A5— in their origin currency and converted to U.S. dollars. Tables A4 and A6 start from the U.S. version of these portfolios and converts them to foreign currency. These correlations are pervasively extremely high, almost all over 99.9%.

[32] Constructing the exchange rate using observation of the same asset returns in their original currency and in another currency is in general trivial. Naturally, if one has "labels" on the return data, the depreciation rate is simply the ratio of a return in its own and in foreign currency. Even without labels, one can generically recover a unique depreciation rate so that the ratio of returns in home and foreign currency are equal across asset pairs.

[33] Sandulescu, Trojani, and Vedolin (2021) also study cases with asymmetric data observations across countries, but with both risk-free assets observed in each country. Mathematically, this corresponds to the case of partial integration in which the exchange rate is spanned by globally-traded shocks.

Table A3: Correlation between excess returns converted in different currencies: foreign stocks

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
Market	99.88	99.91	99.93	99.96	99.88	99.89	99.91	99.94	99.94
Value	99.92	99.94	99.93	99.96	99.89	99.85	99.92	99.93	99.94
Growth	99.82	99.88	99.93	99.96	99.9	99.93	99.92	99.95	99.94
Oil, Gas, Coal	99.89	99.93	NA	99.96	99.92	99.92	99.93	NA	99.96
Basic Material	99.84	99.94	99.94	99.95	99.88	99.91	99.91	99.96	99.91
Consumer Discretionary	99.91	99.95	99.93	99.96	99.92	99.94	99.94	99.93	99.96
Consumer Products, Services	99.88	99.96	99.97	99.95	NA	NA	99.94	99.93	99.98
Industrials	99.90	99.91	99.94	99.95	99.89	99.92	99.92	99.94	99.94
Health Care	99.91	99.97	99.96	99.96	NA	99.91	99.93	99.96	99.97
Financials	99.92	99.95	99.94	99.96	99.89	99.93	99.91	99.93	99.92
TeleCom	99.92	99.95	99.96	99.96	99.92	99.84	99.93	99.94	99.96
Technology	99.91	99.88	99.96	99.96	99.86	NA	99.94	99.95	99.95
Utilities	99.93	99.91	99.94	99.97	NA	99.93	NA	99.95	99.97

The table reports the correlation (in %) between the excess return on various stock indices expressed in their home currency and converted to U.S. dollar. The portfolios include the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios, all from MSCI. Each column corresponds to a different country.

Table A4: Correlation between excess returns converted in different currencies: U.S. stocks

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
US Market	99.88	99.94	99.95	99.96	99.87	99.90	99.92	99.94	99.94
US Value	99.90	99.95	99.96	99.96	99.87	99.91	99.92	99.95	99.95
US Growth	99.87	99.93	99.94	99.96	99.88	99.90	99.92	99.94	99.94
US Oil, Gas, Coal	99.90	99.96	99.97	99.98	99.92	99.92	99.94	99.96	99.96
US Basic Material	99.81	99.90	99.92	99.95	99.85	99.88	99.90	99.93	99.93
US Consumer Discretionary	99.91	99.95	99.95	99.96	99.9	99.91	99.92	99.95	99.95
US Consumer Products, Services	99.93	99.97	99.97	99.97	99.92	99.93	99.94	99.96	99.96
US Industrials	99.86	99.93	99.94	99.96	99.84	99.90	99.90	99.94	99.94
US Health Care	99.90	99.96	99.95	99.96	99.88	99.93	99.93	99.95	99.96
US Financials	99.91	99.95	99.95	99.94	99.87	99.93	99.91	99.92	99.94
US TeleCom	99.87	99.93	99.95	99.95	99.9	99.91	99.93	99.96	99.95
US Technology	99.88	99.93	99.94	99.96	99.89	99.91	99.92	99.94	99.94
US Utilities	99.84	99.92	99.94	99.96	99.85	99.88	99.91	99.96	99.94

The table reports the correlation (in %) between the excess return on various stock indices expressed in the U.S. dollars and converted to foreign currency. The portfolios include the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios, all from MSCI. Each column corresponds to a different country.

Table A5: Correlation between excess returns converted in different currencies: foreign bonds

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
2Y Bond	99.86	99.97	99.92	99.97	NA	99.85	99.91	99.91	99.95
3Y Bond	99.86	99.97	99.92	99.97	99.91	NA	NA	99.93	99.96
4Y Bond	NA	99.97	99.93	99.97	NA	NA	NA	99.94	99.96
5Y Bond	99.87	99.97	99.93	99.97	99.91	99.85	99.91	99.93	99.96
6Y Bond	NA	99.96	99.93	99.97	NA	NA	NA	99.92	99.96
7Y Bond	NA	99.96	99.93	99.96	NA	NA	99.91	99.91	99.96
8Y Bond	NA	99.96	99.92	99.96	NA	NA	NA	99.90	99.96
9Y Bond	NA	99.96	99.92	99.96	NA	NA	NA	99.89	99.96
10Y Bond	99.87	99.96	99.93	99.96	99.91	99.88	99.91	99.88	99.96

The table reports the correlation (in %) between the excess return on government bonds of different maturity expressed in their home currency and converted to U.S. dollars. Bond returns are constructed from yields obtained from each country's central bank. Each column corresponds to a different country.

Table A6: Correlation between excess returns converted in different currencies: U.S. bonds

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
US 2Y Bond	99.9	99.95	99.95	99.97	99.91	99.93	99.95	99.93	99.96
US 3Y Bond	99.91	99.96	99.95	99.97	99.92	99.93	99.95	99.92	99.96
US 4Y Bond	99.92	99.96	99.94	99.96	99.92	99.94	99.95	99.91	99.96
US 5Y Bond	99.91	99.97	99.93	99.96	99.91	99.94	99.95	99.89	99.95
US 6Y Bond	99.91	99.97	99.93	99.96	99.89	99.94	99.94	99.88	99.95
US 7Y Bond	99.9	99.96	99.92	99.96	99.88	99.94	99.94	99.86	99.95
US 8Y Bond	99.89	99.96	99.91	99.96	99.86	99.93	99.93	99.85	99.95
US 9Y Bond	99.88	99.96	99.9	99.96	99.85	99.93	99.93	99.84	99.95
US 10Y Bond	99.88	99.96	99.9	99.96	99.84	99.93	99.92	99.83	99.94

The table reports the correlation (in %) between the excess return on U.S. government bonds of different maturity expressed in U.S.. dollars and converted to foreign currency. Bond returns are constructed from yields obtained from the Federal Reserve. Each column corresponds to a different country.