2022 Canadian Open Mathematics Challenge

Canadian Mathematical Society
Société mathématique du Canada

Student Identification

Please print clearly and complete all information below. Failure to print legibly or provide complete information may result in your exam being disqualified.

1. Given/First Name: (Required)

2. Sur-/Last Name: (Required)

3. Are you currently registered in full-time attendance at an elementary, secondary or Cégep school, or home schooled and have been since at least September 15th of this year? (Required for qualification)

☐ **Yes** ☐ No

4. Are you a Canadian Citizen or a Permanent Resident of Canada (regardless of your current address)? (Required for qualification)

☐ **Yes** ☐ No

5. Do you want to be considered (if eligible) to be invited to Canada's European Girls' Mathematics Olympiad (EGMO) team in Slovenia in April 2023 (only Canadian citizens or permanent residents who identify as female are eligible)

☐ **Yes** ☐ No

6. Date of Birth: (Required)

Y Y Y Y M M D D

7. Grade: (Required)

☐ 8 ☐ 10 ☐ 12
☐ 9 ☐ 11 ☐ Cégep
other: ______________

8. Email Address: (Required)

DO NOT PHOTOCOPY BLANK EXAMS!
Each page of each copy is uniquely pre-coded to facilitate computer-assisted marking.

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Question A1 (4 points)

John had a box of candies. On the first day he ate exactly half of the candies and gave one to his little sister. On the second day he ate exactly half of the remaining candies and gave one to his little sister. On the third day he ate exactly half of the remaining candies and gave one to his little sister, at which point no candies remained. How many candies were in the box at the start?

Your solution:

Your final answer:

[A correct answer here earns full marks]

Question A2 (4 points)

A palindrome is a whole number whose digits are the same when read from left to right as from right to left. For example, 565 and 7887 are palindromes. Find the smallest six-digit palindrome divisible by 12.

Your solution:

Your final answer:

[A correct answer here earns full marks]

Question A3 (4 points)

Initially, there are four red balls, seven green balls, eight blue balls, ten white balls, and eleven black balls on a table. Every minute, we may repaint one of the balls into any of the other four colours. What is the minimum number of minutes after which the number of balls of each of the five colours is the same?

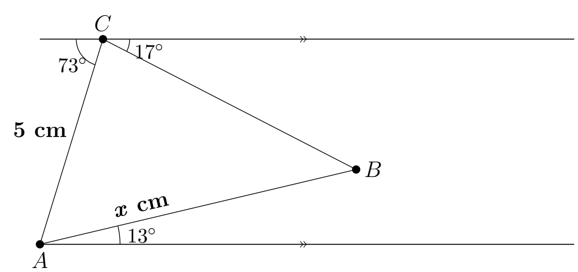
Your solution:

Your final answer:

[A correct answer here earns full marks]

Question A4 (4 points)

In the diagram, triangle ABC lies between two parallel lines as shown. If segment AC has length 5 cm, what is the length (in cm) of segment AB?



Your solution:

Your final answer:

[A correct answer here earns full marks]

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Question B1 (6 points)

The floor function of any real number a is the integer number denoted by $\lfloor a \rfloor$ such that $\lfloor a \rfloor \leq a$ and $\lfloor a \rfloor > a - 1$. For example, $\lfloor 5 \rfloor = 5$, $\lfloor \pi \rfloor = 3$ and $\lfloor -1.5 \rfloor = -2$.

Find the difference between the largest integer solution of the equation $\lfloor x/3 \rfloor = 102$ and the smallest integer solution of the equation $\lfloor x/3 \rfloor = -102$.

Your solution:

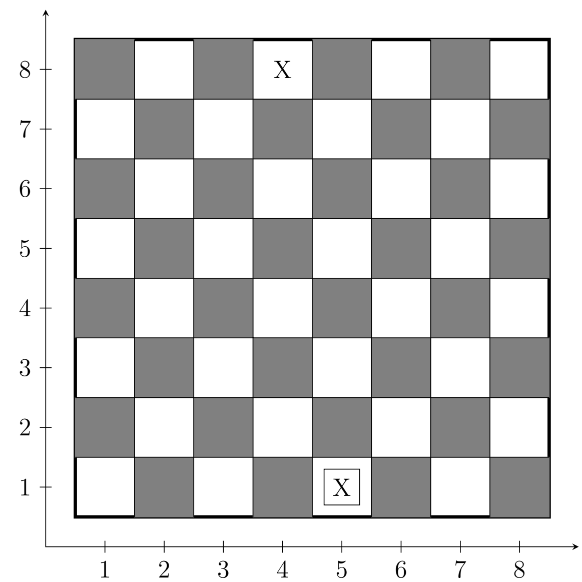
Your final answer:

[A correct answer here earns full marks]

Question B2 (6 points)

A stone general is a chess piece that moves one square diagonally upward on each move; that is, it may move from the coordinate (a, b) to either of the coordinates $(a-1, b+1)$ or $(a+1, b+1)$.

How many ways are there for a stone general to move from (5, 1) to (4, 8) in seven moves on a standard 8 by 8 chessboard?



Your solution:

Your final answer:

[A correct answer here earns full marks]

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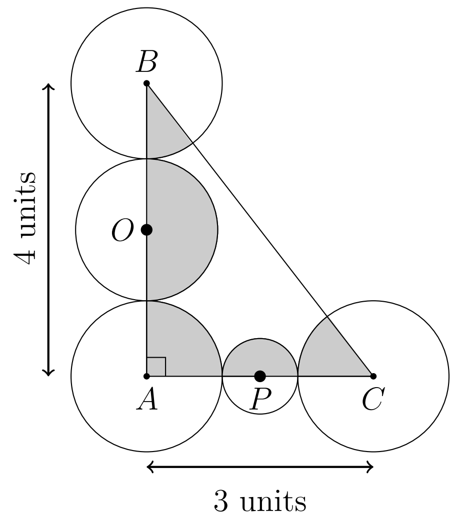
In the diagram shown, right triangle ABC has side lengths $AC = 3$ units, $AB = 4$ units, and $BC = 5$ units. Circles centred around the corners of the triangle all have the same radius, and the circle with centre O has area 4 times that of the circle with centre P. The shaded area is $k\pi$ square units.

What is k?



Your solution:

Your final answer:

[A correct answer here earns full marks]

Question B4 (6 points)

Determine all integers a for which $\dfrac{a}{1011 - a}$ is an even integer.

Your solution:

Your final answer:

[A correct answer here earns full marks]

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Question C1 (10 points)

(a) Find all integer values of a such that equation $x^2 + ax + 1 = 0$ does not have real solutions in x.

(b) Find all pairs of integers (a, b) such that both equations

$$x^2 + ax + b = 0 \quad \text{and} \quad x^2 + bx + a = 0$$

have no real solutions in x.

(c) How many ordered pairs (a, b) of positive integers satisfying $a \leq 8$ and $b \leq 8$ are there, such that each of the equations

$$x^2 + ax + b = 0 \quad \text{and} \quad x^2 + bx + a = 0$$

has two unique real solutions in x?

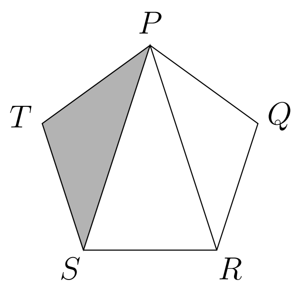
Your solution:

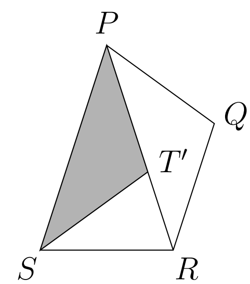
You **must** show all your work.

Question C2 (10 points)

(a) Show that the two diagonals drawn from a vertex of a regular pentagon trisect the angle.

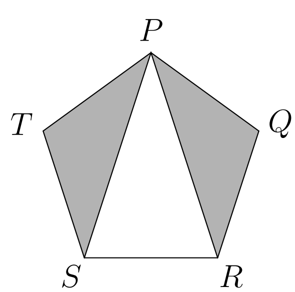
(b) Since the diagonals trisect the angle, if regular pentagon $PQRST$ is folded along the diagonal SP, the side TP will fall on the diagonal PR, as shown on the right. Here T' is the position of vertex T after the folding.

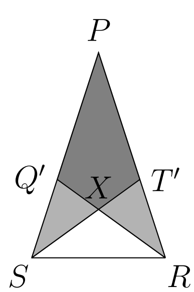




Find the ratio $\frac{PT'}{T'R}$. Express your answer in the form $\frac{a+\sqrt{b}}{c}$, where a, b, c are integers.

(c) Regular pentagon $PQRST$ has an area of 1 square unit. The pentagon is folded along the diagonals SP and RP as shown on the right. Here, T' and Q' are the positions of vertices T and Q respectively, after the foldings. The segments ST' and RQ' intersect at X.





Determine the area (in square units) of the uncovered triangle XSR. Express your answer in the form $\frac{a+\sqrt{b}}{c}$, where a, b, c are integers.

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Your solution:

You **must** show all your work.

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Question C3 (10 points)

Yana and Zahid are playing a game. Yana rolls her pair of fair six-sided dice and draws a rectangle whose length and width are the two numbers she rolled. Zahid rolls his pair of fair six-sided dice, and draws a square with side length according to the rule specified below.

(a) Suppose that Zahid always uses the number from the first of his two dice as the side length of his square, and ignores the second. Whose shape has the larger average area, and by how much?

(b) Suppose now that Zahid draws a square with the side length equal to the minimum of his two dice results. What is the probability that Yana's and Zahid's shapes will have the same area?

(c) Suppose once again that Zahid draws a square with the side length equal to the minimum of his two dice results. Let $D = \text{Area}_{\text{Yana}} - \text{Area}_{\text{Zahid}}$ be the difference between the area of Yana's figure and the area of Zahid's figure. Find the expected value of D.

Your solution:

You **must** show all your work.

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Question C4 (10 points)

An integer container (x, y, z) is a rectangular prism with positive integer side lengths x, y, z, where $x \leq y \leq z$. A stick has $x = y = 1$; a flat has $x = 1$ and $y > 1$; and a box has $x > 1$. There are 5 integer containers with volume 30: one stick (1, 1, 30), three flats (1, 2, 15), (1, 3, 10), (1, 5, 6) and one box (2, 3, 5).

(a) How many sticks, flats and boxes are there among the integer containers with volume 36?

(b) How many flats and boxes are there among the integer containers with volume 210?

(c) Suppose $n = p_1^{e_1} \cdots p_k^{e_k}$ has k distinct prime factors $p_1, p_2, \ldots, p_k$, each with integer exponent $e_1 \geq 1, e_2 \geq 1, \ldots, e_k \geq 1$ and $k \geq 3$. How many boxes are there among the integer containers with volume n? Express your answer in terms of $e_1, e_2, \ldots, e_k$. How many boxes with volume $n = 8!$ are there?

Your solution:

You **must** show all your work.

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