

A generalization of parallel addition

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Summary. Parallel addition of positive operators, a concept introduced by W. N. Anderson and R. J. Duffin [1] in connection with network theory, has already been studied by several authors. We specifically mention W. N. Anderson and G. E. Trapp [2] and [3], T. Ando [4], K. Nishio [12], E. L. Pekarev and J. L. Smul'jan [13], as well as the article [10] by the present authors.

The purpose of this note is to study a generalization of parallel addition. In particular, it will be shown (Theorem 3.2) that the corresponding quasi-units, a concept introduced in [10], are again the extreme points of the convex sets, formed by the positive operators less than or equal to some fixed operator.

1. Parallel addition

Let H denote a real or complex Hilbert space with scalar product $(\cdot, \cdot)$ and $\mathcal{H}$ the class of bounded, linear Hermitian operators on H . An operator $A \in \mathcal{H}$ is said to be positive if $(Ax, x) \geq 0$ for all $x \in H$. In what follows we shall only consider the set $\mathcal{H}^+$ of positive Hermitian operators. For operators $A, B \in \mathcal{H}^+$ we write $A \leq B$ if $B - A$ is positive.

DEFINITION. The parallel sum $A : B$ of two positive operators A and B is defined by the formula

$$((A : B)x, x) = \inf\{(Ay, y) + (Bz, z); y + z = x, y, z \in H\} \quad (x \in H).$$

If the operator $A + B$ has a bounded inverse then

$$A : B = A(A + B)^{-1}B; \tag{1.1}$$

AMS (1980) subject classification: Primary 47D25. Secondary 39B50, 46C10, 31D05.

Manuscript received January 10, 1989.

the general case is reduced to this one, observing that

$$A : B = \inf_{\varepsilon > 0} (A + \varepsilon I) : B,$$

where I denotes the identity operator on H .

Among the properties of parallel addition we only mention the following ones: For all $A, B, C \in \mathcal{H}^+$ and $\lambda \in \mathbb{R}^+$,

$$A : B = B : A, \quad (1.2)$$

$$(A : B) : C = A : (B : C), \quad (1.3)$$

$$\lambda(A : B) = (\lambda A) : (\lambda B), \quad (1.4)$$

$$A : B \leq A, \quad (1.5)$$

$$A \leq B \Rightarrow A : C \leq B : C, \quad (1.6)$$

$$A : (\lambda A) = \frac{\lambda}{\lambda + 1} A, \quad (1.7)$$

For proofs we refer, e.g., to [2] or [13].

2. Generalization of parallel addition

Formula (1.7) suggests that we replace the factor $\lambda/(\lambda + 1)$ occurring on the right hand side by an arbitrary function $g(\lambda)$. In order to achieve more flexibility, we exchange the set $\mathcal{H}^+$ of positive Hermitian operators on H by an arbitrary cone of such operators, i.e. by a subset $\mathfrak{A} \subset \mathcal{H}^+$ with the property that

$$A \in \mathfrak{A}, \quad \lambda \in \mathbb{R}^+ \Rightarrow \lambda A \in \mathfrak{A}. \quad (2.1)$$

In particular we consider a cone $\mathfrak{A}$ of positive operators admitting a binary operation $(A, B) \rightarrow A \circ B$ such that, with respect to some given strictly positive function g on $\mathbb{R}^+$, the following properties hold: For all $A, B, C \in \mathfrak{A}$, and all $\lambda \in \mathbb{R}^+$,

$$A \circ B = B \circ A, \quad (2.2)$$

$$(A \circ B) \circ C = A \circ (B \circ C), \quad (2.3)$$

$$\lambda(A \circ B) = (\lambda A) \circ (\lambda B), \quad (2.4)$$

$$A \circ B \leq A, \quad (2.5)$$

$$A \leq B \Rightarrow A \circ C \leq B \circ C, \quad (2.6)$$

$$A \circ (\lambda A) = g(\lambda)A. \quad (2.7)$$

Note however, that

THEOREM 2.1. *Under the hypotheses (2.2)–(2.7), the function g must necessarily be of the form*

$$g(\lambda) = (1 + \lambda^{-p})^{-1/p} \quad (\lambda \in \mathbb{R}^+) \quad (2.8)$$

for some $0 < p \leq \infty$, provided that in case $p = \infty$ we set $g(\lambda) = \min(1, \lambda)$.

Proof. Introduce the function

$$f(\lambda, \mu) = \frac{\lambda}{g(\lambda/\mu)} \quad (\lambda, \mu > 0) \quad (2.9)$$

and observe first that on account of (2.4) and (2.7)

$$\left(\frac{1}{\lambda} A\right) \circ \left(\frac{1}{\mu} A\right) = \frac{1}{f(\lambda, \mu)} A. \quad (2.10)$$

Then according to the relations (2.2)–(2.7) the function f must have the following properties: For all $\alpha, \lambda, \mu, \nu \in \mathbb{R}^+ \setminus \{0\}$,

$$f(\alpha\lambda, \alpha\mu) = \alpha f(\lambda, \mu), \quad (2.11)$$

$$f(\lambda, f(\mu, \nu)) = f(f(\lambda, \mu), \nu), \quad (2.12)$$

$$f(\lambda, \mu) = f(\mu, \lambda), \quad (2.13)$$

$$\lambda_1 \leq \lambda_2, \quad \mu_1 \leq \mu_2 \Rightarrow f(\lambda_1, \mu_1) \leq f(\lambda_2, \mu_2), \quad (2.14)$$

$$f(1, \lambda) \geq 1. \quad (2.15)$$

A careful inspection of Bohnenblust's article [9] then shows that the conditions (2.11)–(2.15) imply that

$$f(\lambda, \mu) = (\lambda^p + \mu^p)^{1/p} \quad (2.16)$$

for some $0 < p \leq \infty$, where in case $p = \infty$ one puts $f(\lambda, \mu) = \max(\lambda, \mu)$. Consequently $g(\lambda) = \lambda/f(\lambda, 1)$ must have the form (2.8).

REMARKS. (1) Notice that the proof of Theorem 2.1 does not depend on the fact that the elements are operators. (2) A similar argument had been used in [5] in order to characterize parallel addition.

EXAMPLES. (1) The case $p = 1$ corresponds to parallel addition. (2) Let A_0 be any positive Hermitian operator on H and $\mathfrak{A}$ the second commutant of A_0 (see, e.g., [11], p. 382). Choose $\mathfrak{A} = \mathfrak{A} \cap \mathcal{H}^+$ and $0 < p < \infty$. Then the binary operation

$$A \circ B = (A^p : B^p)^{1/p}, \quad (2.17)$$

defined in terms of parallel addition, has the required properties (2.2)–(2.7). Furthermore

$$A \circ (\lambda A) = (1 + \lambda^{-p})^{-1/p} A.$$

In case $p = \infty$, we take $\mathfrak{A} = \mathcal{H}^+$, where H is finite dimensional, and set

$$A \circ B = A \wedge B.$$

Then, $A \circ (\lambda A) = \min(\lambda, 1)A$.

REMARK. We do not know if the binary operation (2.17) is monotone on the whole set $\mathcal{H}^+$.

3. Quasi-units

In [10] the concept of an E -quasi-unit has been introduced in order to characterize the extreme points of the convex set of positive operators less than or equal to some given operator $E \in \mathcal{H}^+$. The terminology was taken over from potential theory (see remark at the end of this section). In [3] it has been observed that the quasi-units are the same as the so-called *shorted operators*.

In this section we are going to study quasi-units in the general setting introduced in the previous section. However, we shall assume that, in addition, the cone $\mathfrak{A}$ has the following properties:

$$\mathfrak{A} \text{ is convex, i.e., } A, B \in \mathfrak{A} \Rightarrow A + B \in \mathfrak{A}, \quad (3.1)$$

$$\mathfrak{A} \text{ is (upper) Dedekind } \sigma\text{-complete, i.e. every bounded increasing sequence } (A_n)_{n \in \mathbb{N}} \text{ admits a supremum,} \quad (3.2)$$

$$\text{for all } A, B \in \mathfrak{A}: A \leq B \Rightarrow B - A \in \mathfrak{A}. \quad (3.3)$$

Note that all examples, given in Section 2, satisfy (3.1)–(3.3).

In the sequel we shall always assume that $(\mathfrak{A}, p)$ denotes a convex cone of positive operators satisfying conditions (3.1)–(3.3) and endowed with a binary operation $(A, B) \rightarrow A \circ B$ having the properties (2.2)–(2.7), with respect to $g(\lambda) = (1 + \lambda^{-p})^{-1/p}$, $0 < p \leq \infty$. Analogously to (4.1) of [10] we introduce — relative to a fixed $E \in \mathfrak{A}$ — the mapping $\mathbb{P}_{E,p}: \mathfrak{A} \rightarrow \mathfrak{A}$ defined by

$$\mathbb{P}_{E,p} A = \sup_{n \in \mathbb{N}} E \circ (nA), \quad \text{for all } A \in \mathfrak{A}. \quad (3.4)$$

For reference, we list the following elementary properties:

$$\mathbb{P}_{E,p} A \leq E, \quad (3.5)$$

$$\mathbb{P}_{E,p}(\lambda A) = \mathbb{P}_{E,p} A, \quad \text{for all } \lambda \in \mathbb{R}^+, \quad (3.6)$$

$$A \leq E \Rightarrow A \leq \mathbb{P}_{E,p} A, \quad (3.7)$$

$$A \leq B \Rightarrow \mathbb{P}_{E,p} A \leq \mathbb{P}_{E,p} B. \quad (3.8)$$

We now introduce the following notation:

DEFINITION. An operator $A \in \mathfrak{A}$ is called an (E, p) -quasi-unit if

$$\mathbb{P}_{E,p} A = A. \quad (3.9)$$

The following lemma yields a first characterization of (E, p) -quasi-units.

LEMMA 3.1. Let $0 < p < \infty$ and let $(\mathfrak{A}, p)$ be defined as above with respect to $g(\lambda) = (1 + \lambda^{-p})^{-1/p}$. Fix $E \in \mathfrak{A}$. Then for any $A \in \mathfrak{A}$ with $A \leq E$ the following conditions are equivalent:

- (i) A is an (E, p) -quasi-unit,
- (ii) $E \circ (\lambda A) = g(\lambda)A$ for all $\lambda > 0$,
- (iii) $E \circ (\lambda A) = g(\lambda)A$ for some $\lambda > 0$,
- (iv) $E \circ (\lambda A) = (\lambda E) \circ A$ for all $\lambda > 0$,
- (v) $E \circ (\lambda A) = (\lambda E) \circ A$ for some $\lambda > 0$, $\lambda \neq 1$,
- (vi) $A \circ (E - A) = 0$.

Proof. (i) $\Rightarrow$ (ii): Suppose that A is an (E, p) -quasi-unit. Set

$$B = \frac{1}{g(\lambda)} [E \circ (\lambda A)]$$

for an arbitrary $\lambda > 0$. Then, by (2.6) and (2.7), $E \geq A$ implies that $B \geq A$ and similarly $B \leq E$. From $B \leq E$ and (3.7) we then conclude that

$$\begin{aligned} B &\leq \mathbb{P}_{E,p} B = \sup_{n \in \mathbb{N}} E \circ (nB) = \sup_{n \in \mathbb{N}} E \circ \left[\frac{n}{g(\lambda)} (E \circ (\lambda A)) \right] \\ &\leq \sup_{n \in \mathbb{N}} E \circ \left(\frac{n\lambda}{g(\lambda)} A \right) = \mathbb{P}_{E,p} A. \end{aligned}$$

Hence

$$A \leq B \leq \mathbb{P}_{E,p} A = A,$$

showing that $B = A$.

(ii) $\Rightarrow$ (iii): trivial

(iii) $\Rightarrow$ (i): Set

$$f(x) = \frac{x}{g(x)} = (1 + x^p)^{1/p}, \quad (x > 0).$$

Then, by hypothesis,

$$E \circ \lambda A = \frac{\lambda}{f(\lambda)} A. \quad (3.10)$$

Applying this equality twice we conclude that

$$\begin{aligned} \frac{1}{f^2(\lambda)} A &= \left(\frac{1}{f(\lambda)} A \right) \circ \left(\frac{1}{\lambda f(\lambda)} E \right) = \left(A \circ \frac{1}{\lambda} E \right) \circ \left(\frac{1}{\lambda f(\lambda)} E \right) \\ &= A \circ \frac{1}{\lambda} \left(E \circ \frac{1}{f(\lambda)} E \right) = A \circ \frac{1}{\lambda f(f(\lambda))} E. \end{aligned}$$

Since $f^2(\lambda) = f(\lambda f(f(\lambda)))$ we thus have shown that, if (3.10) is valid for some $\lambda > 0$, it also holds for $\mu = \lambda f(f(\lambda))$. Hence defining the sequence $(\lambda_k)_{k \in \mathbb{N}}$ inductively by

$$\lambda_{k+1} := \lambda_k f(f(\lambda_k)) \quad \text{for } k \in \mathbb{N}, \text{ with } \lambda_1 := \lambda$$

it satisfies the relation

$$E \circ \lambda_k A = \frac{\lambda_k}{f(\lambda_k)} A, \quad \text{for } k \in \mathbb{N}. \quad (3.11)$$

Noting that $f(f(x)) = (2 + x^p)^{1/p} \geq 2^{1/p}$ we conclude from

$$\lambda_{k+1} \geq \lambda_k \cdot 2^{1/p} \quad (k \in \mathbb{N})$$

that $\lambda_k \rightarrow \infty$ for $k \rightarrow \infty$.

Consequently, by (3.11),

$$\mathbb{P}_{E,p}A = \sup E \circ (\lambda_k A) = \sup \frac{\lambda_k}{f(\lambda_k)} A = \sup_{k \rightarrow \infty} g(\lambda_k)A = A,$$

i.e., A is an (E, p) -quasi-unit.

(ii) $\Rightarrow$ (iv): By hypothesis $E \circ (\lambda A) = g(\lambda)A$ for all $\lambda > 0$.

Hence

$$(\lambda E) \circ A = \lambda \left[E \circ \left(\frac{1}{\lambda} A \right) \right] = \lambda g\left(\frac{1}{\lambda}\right) A = g(\lambda)A = E \circ (\lambda A),$$

for all $\lambda > 0$.

(iv) $\Rightarrow$ (v): trivial.

(v) $\Rightarrow$ (i): Suppose that

$$E \circ (\lambda A) = (\lambda E) \circ A \quad (3.12)$$

for some $\lambda > 0$, $\lambda \neq 1$. W.l.o.g we may assume that $\lambda > 1$, since otherwise we simply divide both sides of (3.12) by λ .

Introducing again the function $f(x) = [x/g(x)]$ ($x > 0$), we denote by μ the unique positive real number with $f(\mu) = \lambda^2$. From

$$E = f(x) \left[\left(\frac{1}{x} E \right) \circ E \right],$$

valid for all $x > 0$, and (3.12) we then conclude that

$$\begin{aligned} E \circ (\lambda A) &= \left[f(\mu) \left(\frac{1}{\mu} E \circ E \right) \right] \circ (\lambda A) = \left(\frac{\lambda^2}{\mu} E \right) \circ \lambda (E \circ A) \\ &= \left(\frac{\lambda^2}{\mu} E \right) \circ \lambda (E \circ \lambda A) = (\lambda^2 A) \circ \lambda \left(E \circ \frac{\lambda}{\mu} E \right) = (\lambda^2 A) \circ \left(\frac{\lambda}{f(\mu/\lambda)} E \right) \\ &= \frac{\lambda}{f(\mu/\lambda)} \left[\left(\lambda f\left(\frac{\mu}{\lambda}\right) A \right) \circ E \right]. \end{aligned}$$

Analogously, we find

$$A \circ (\lambda E) = \frac{\lambda}{f(\mu/\lambda)} \left[\left(\lambda f\left(\frac{\mu}{\lambda}\right) E \right) \circ A \right].$$

Hence

$$E \circ \left[\lambda f\left(\frac{\mu}{\lambda}\right) A \right] = \left[\lambda f\left(\frac{\mu}{\lambda}\right) E \right] \circ A.$$

We see that if $\lambda > 1$ satisfies (3.12), so does the number $\lambda f(\mu/\lambda)$ where μ is given by $f(\mu) = \lambda^2$. This suggests that we define the sequences $(\lambda_k)_{k \in \mathbb{N}}$ and $(\mu_k)_{k \in \mathbb{N}}$ inductively by

$$\begin{cases} f(\mu_k) = \lambda_k^2 & (k \in \mathbb{N}) \\ \lambda_{k+1} = \lambda_k f(\mu_k/\lambda_k), & \lambda_1 = \lambda. \end{cases}$$

Then we have

$$E \circ (\lambda_k A) = (\lambda_k E) \circ A, \quad (3.13)$$

for all $k \in \mathbb{N}$.

The inequality $\lambda f(\mu/\lambda) \geq f(\mu)$ for all $\lambda > 1$ and $\mu > 0$ leads to $\lambda_k f(\mu_k/\lambda_k) \geq f(\mu_k) = \lambda_k^2$. Thus $\lambda_{k+1} \geq \lambda_k^2$ for all $k \in \mathbb{N}$ and therefore $\lambda_k \rightarrow \infty$, if $k \rightarrow \infty$. Taking into account (3.13), we get

$$\mathbb{P}_{E,p} A = \sup_{k \rightarrow \infty} E \circ (\lambda_k A) = \sup_{k \rightarrow \infty} [(\lambda_k E) \circ A] \leq A.$$

Since the reverse inequality follows from $A \leq E$ by (3.7), we conclude that $\mathbb{P}_{E,p} A = A$.

(ii) $\Rightarrow$ (vi): Define a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of positive real numbers inductively by

$$\lambda_1 = g(1) + 1, \quad \lambda_{n+1} = \frac{g(\lambda_n)}{g(1)} \quad (n \in \mathbb{N}).$$

Clearly $\lambda_1 \geq 1$. Assume now that $\lambda_n \geq 1$ for some $n \in \mathbb{N}$. Since g is increasing we have

$$\lambda_{n+1} = \frac{g(\lambda_n)}{g(1)} \geq \frac{g(1)}{g(1)} = 1.$$

Then, by induction, $\lambda_n \geq 1$ for all $n \in \mathbb{N}$. On the other hand $g(\alpha x) \leq \alpha g(x)$ for all $\alpha \geq 1$. Applying this result with $\lambda_n \geq 1$ we see that

$$\lambda_{n+1} = \frac{g(\lambda_n)}{g(1)} \leq \frac{\lambda_n g(1)}{g(1)} = \lambda_n$$

for all $n \in \mathbb{N}$. Hence (λ_n) is a decreasing sequence of real numbers (≥ 1) and therefore has a limit when $n \rightarrow \infty$. We write $x = \lim_{n \rightarrow \infty} \lambda_n$.

From the definition of λ_n and g we obtain the equation

$$x = \frac{g(x)}{g(1)} = \frac{x}{g(1)(1+x^p)^{1/p}}.$$

The solution of this equation is $x = 1$. Thus $\lambda_n \rightarrow 1$ when $n \rightarrow \infty$.

We claim that

$$A + A \circ (E - A) \leq \lambda_n A \quad (3.14)$$

holds for all $n \in \mathbb{N}$. The case $n = 1$ follows from

$$A + A \circ (E - A) \leq A + A \circ E = (g(1) + 1)A.$$

Assuming (3.14) to hold for some $k \in \mathbb{N}$ we find that

$$(A + A \circ (E - A)) \circ (A + A \circ (E - A)) \leq (\lambda_k A) \circ E.$$

An application of (ii) leads to

$$A + A \circ (E - A) \leq \frac{g(\lambda_k)}{g(1)} A = \lambda_{k+1} A.$$

Therefore, by induction, (3.14) holds for all $n \in \mathbb{N}$. When n tends to infinity in (3.14) we conclude

$$A \circ (E - A) = 0.$$

(vi) $\Rightarrow$ (iii): Let $A \circ (E - A) = 0$. Set $B = [1/g(1)](A \circ E) - A$. Using the assumption $A \leq E$, we obtain

$$B \leq \frac{1}{g(1)} (E \circ E) - A = E - A < 2^{1/p} (E - A).$$

Since we also have $B \leq 2^{1/p}A$ we infer that

$$g(1)B = B \circ B \leq 2^{1/p}[A \circ (E - A)] = 0.$$

Consequently,

$$A \circ E = g(1)A.$$

Q.E.D.

We are now in a position to prove the following.

THEOREM 3.2. *Let $(\mathfrak{A}, p)$, $0 < p \leq \infty$, be the convex cone of operators defined above, and $E \in \mathfrak{A}$. Then an operator $A \in \mathfrak{A}$ is an (E, p) -quasi-unit if and only if it is an extreme point of the convex set $\mathcal{B}_E = \{B \in \mathfrak{A}: B \leq E\}$.*

Proof. Case 1: $0 < p < \infty$.

Suppose $A \in \mathcal{B}_E$ is an extreme point. Setting $\alpha = 2^{1/p} = [1/g(1)] > 1$ and $\beta = \alpha - 1$ we conclude from $A \leq E$ that

$$\alpha(A \circ E) \leq \frac{1}{g(1)}(E \circ E) = E$$

and

$$\alpha A \leq \beta E + A = \beta E + \alpha(A \circ A) \leq \beta E + \alpha(A \circ E).$$

Hence

$$\frac{\alpha}{\beta}(A - A \circ E) \leq E.$$

Introducing $B := \alpha(A \circ E)$ and $C := \alpha/\beta(A - A \circ E)$, we obtain the decomposition

$$A = \frac{1}{\alpha}B + \left(1 - \frac{1}{\alpha}\right)C, \quad \text{with } B, C \in \mathcal{B}_E.$$

By hypothesis this implies that $A = B = C$, and in particular $A \circ E = g(1)A$. Hence condition (iii) in Lemma 3.1 is satisfied and consequently A is an (E, p) -quasi-unit. Conversely, assume that A is an (E, p) -quasi-unit and

$$A = \lambda B + (1 - \lambda)C,$$

for some $B, C \in \mathcal{B}_E$ and some $0 < \lambda < 1$. Then

$$E \circ (nA) \geq E \circ (n\lambda B) \geq B \circ (n\lambda B) = g(n\lambda)B$$

for all $n \in \mathbb{N}$ and hence

$$A = \mathbb{P}_{E,p} A = \sup_{n \in \mathbb{N}} E \circ (nA) \geq \sup_{n \in \mathbb{N}} g(n\lambda)B = B.$$

Similarly, $A \geq C$, and thus $A = B = C$.

Case 2. $p = \infty$. Here $g(\lambda) = \min(1, \lambda)$ ($\lambda > 0$).

If $A \in \mathcal{B}_E$ is an extreme point of $\mathcal{B}_E$, set

$$B := E \circ (2A) \text{ and } C := 2A - E \circ (2A).$$

Then $A = \frac{1}{2}(B + C)$. Moreover, $B \leq E$ and $2A - (E \circ 2A) \leq 2A - (A \circ 2A) = A \leq E$. Hence, by hypothesis, $A = B = C$. In particular $E \circ (2A) = A = g(2)A$ and therefore, as the proof of Lemma 3.1 shows, A is an (E, ∞) -quasi-unit. The converse direction is proved the same way as in case 1.

REMARK. Note that the case $p = \infty$ is a special case of the axiomatic setting investigated in [6] and [7]. It had been introduced in order to characterize axiomatically the cone of harmonic, resp. superharmonic, functions, an object of potential theory. The proof of Theorem 3.2 therefore agrees — in case $p = \infty$ — with the one to be found in [6].

Acknowledgement

The authors gratefully acknowledge financial support by the Academy of Finland for their research during the second author's stay at the University of Joensuu.

REFERENCES

- [1] ANDERSON, W. N. and DUFFIN, R. J., *Series and parallel addition of matrices*. J. Math. Anal. Appl. 26, 576–594 (1969).
- [2] ANDERSON, W. N. and TRAPP, G. E., *Shorted operators, II*. SIAM J. Appl. Math. 28 (1975), 60–71.
- [3] ANDERSON, W. N. and TRAPP, G. E., *The extreme points of a set of positive semidefinite operators*. (Preprint).
- [4] ANDO, T., *Lebesgue-type decomposition of positive operators*. Acta Sci. Math. 38 (1976), 253–260.
- [5] ANDO, T. and NISHIO, K., *Characterizations of operations derived from network connections*. J. Math. Analysis and Appl. 53 (1976), 539–549.

- [6] ARSOVE, M. G. and LEUTWILER, H., *Infinitesimal generators and quasi-units in potential theory*. Proc. Nat. Acad. Sci. U.S.A. 72 (1975), 2498–2500.
- [7] ARSOVE, M. G. and LEUTWILER, H., *Algebraic potential theory*. Mem. Amer. Math. Soc. 23 (1980), no. 226.
- [8] ARSOVE, M. G. and LEUTWILER, H., *A unified theory of harmonic measures and capacitary potentials*. Math. Z. 183 (1983), 419–442.
- [9] BOHNENBLUST, F., *An axiomatic characterization of L_p spaces*. Duke Math. J. 6 (1940), 627–640.
- [10] ERIKSSON, S. L. and LEUTWILER, H., *A potential-theoretic approach to parallel addition*. Math. Ann. 274 (1986), 301–317.
- [11] LUXEMBURG, W. A. J. and ZAAANEN, A. C., *Riesz spaces, Vol. I*. North-Holland, American Elsevier, New York, 1971.
- [12] NISHIO, K., *Characterization of Lebesgue-type decomposition of positive operators*. Acta Sci. Math. (Szeged) 42 (1980), 145–152.
- [13] PEKAREV, E. L. and SMUL'JAN, J. L., *Parallel addition and parallel subtraction of operators*. Math. U.S.S.R. Izv. 10 (1976), 351–370.

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