Simple Sets

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These are notes I wrote up some time in 80s, based on two papers and on a lecture by Michael Stob at MIT.

1 Effectively Simple Sets and the Recursion Theorem

This section is based on two papers:

- C. G. Jockusch, Jr. and R. I. Soare, "Degrees of members of Π^0_1 classes", *Pacific J. Math.*, v.40 (1972), pp.605–616.
- D. A. Martin, "Completeness, the recursion theorem, and effectively simple sets", *Proc. Amer. Math. Soc.*, v.17 (1966) pp.838–842.

Definition: An r.e. set S is **effectively simple** if it is coinfinite, and there is a recursive function f such that

$$|W_x| \geq f(x) \Rightarrow W_x \cap S \neq \varnothing.$$

Remarks: Clearly, any effectively simple set is simple. If S is simple, then $W_x \cap S = \varnothing$ implies that W_x is finite, so the effectivity requirement says that $|W_x|$ can be recursively bounded. Post's simple set (Rogers, Theorem 8-II p.106) is effectively simple, with $f(x) = 2x + 2$.

Martin proved that any effectively simple set is complete.

Definition: A (total) function F **dominates** a partial function ψ if $F(x) \geq \psi(x)$ whenever $\psi(x)$ is defined.

Lemma: Suppose A is a set with the following property: for every partial recursive function ψ, there is a (total) function F, recursive in A, that dominates ψ. Then A is complete.

Proof: Suppose B is r.e., and let b be a recursive function enumerating B: $B = \{b(n) | n \in \mathbb{N}\}$. Let ψ be the **computation function** of B: $\psi(x) = (\mu s) b(s) = x$ (so $\psi(x)$ is the number of steps it takes to conclude that $x \in B$, and $\psi(x)$ diverges if $x \notin B$). Let F be a function recursive in A that dominates ψ. Then

$$x \in B \Leftrightarrow x \in \{b(s) | 0 \leq s \leq F(x)\}.$$

So $B \leq_T F \leq_T A$. ▌

Theorem (Martin): All effectively simple sets are complete.

Proof: Suppose S is effectively simple, and f is a recursive function such that $|W_x| \geq f(x)$ implies $W_x \cap S \neq \varnothing$. Also let S be enumerated without repetition by $s(x)$, and let

$$S_t = \{s(x) | x < t\}.$$

Let ψ be an arbitrary partial recursive function. Using the recursion theorem, let $g(x)$ be a recursive function such that

$$W_{g(x)} = \begin{cases} \varnothing \text{ if } \psi(x) \uparrow \\ \{\text{the first } f(g(x)) \text{ elements of } \mathbb{N} \setminus S_{\psi(x)}\} \text{ if } \psi(x) \downarrow . \end{cases}$$

If $\psi(x)$ converges, then $|W_{g(x)}| = f(g(x))$, so $W_{g(x)} \cap S \neq \varnothing$. However, by construction, $W_{g(x)} \cap S_{\psi(x)} = \varnothing$. Thus $S_{\psi(x)}$ is "inaccurate approximation" of S, in a certain sense. More precisely, let

$$F(x) = (\mu t) \left[\begin{array}{l} \{\text{the first } f(g(x)) \text{ elements of } \mathbb{N} \setminus S\} = \\ \{\text{the first } f(g(x)) \text{ elements of } \mathbb{N} \setminus S_t\} \end{array} \right].$$

(So by step $F(x)$ in the enumeration of S, $S_{F(x)}$ is "accurate".) Observe that if

$$\{\text{the first } f(g(x)) \text{ elements of } \mathbb{N} \setminus S\} = \{\text{the first } f(g(x)) \text{ elements of } \mathbb{N} \setminus S_t\}$$

and $t' > t$, then this also holds with $S_{t'}$ replacing S_t. Clearly (i) F is total; (ii) F is recursive in S; (iii) F dominates ψ (because $S_{\psi(x)}$ is "inaccurate"). So by the lemma, S is complete. ▌

The Yates "permitting" method enables us to construct a simple set that is recursive in an arbitrary nonrecursive r.e. set. Jockusch and Soare observed, by a modification of Martin's argument, that the resulting simple set is in fact recursively equivalent to the give r.e. set. (Also Yates' simple set is, like Post's, not hypersimple. See remarks at end.) Hence there is a simple set in every r.e. degree $> \mathbf{0}$. (Yates proved this first by a different method.)

Theorem (Yates): Let A by a nonrecursive r.e. set. Then there exists a simple set $S \leq_T A$.

Proof: Let f be a 1–1 recursive function enumerating A. Also, let $W_{x,t}$ be the part of W_x enumerated by t steps in the enumeration of W_x.

We enumerate S in an infinite number of steps. At step t, each W_x with $x < t$ has the opportunity to contribute one element to S. Specifically, if $W_{x,t}$ is disjoint from the part of S enumerated before step t, and if there is a y such that

1. $y \in W_{x,t}$, and

2. $y > 2x$, and
3. $y > f(t)$

then we add the least such y to S.

Let S be the r.e. set produced by this procedure.

- S is coinfinite. For each W_x contributes at most one element, and for all k, $W_0,\ldots,W_{k-1}$ are the only possible contributors of an element in $\{0,\ldots,2k\}$.
- $S \leq_T A$. Since f is 1–1,
$$(\forall y)(\exists t_0)(\forall t > t_0)f(t) \geq y.$$
Now, t_0 may be found from y recursively in A. If y is not added to S by step t_0, then y will never be added afterwards.
- W_x infinite $\Rightarrow W_x \cap S \neq \varnothing$. Suppose W_x is infinite and $W_x \cap S = \varnothing$. We show that A is recursive, contrary to assumption. Let $e \in \mathbb{N}$. Let y be such that $y < 2x$, $y \in W_x$, and $y > e$. (Such y can be found recursively.) Let t_0 be greater than x and sufficiently large so that $y \in W_{x,t_0}$. Then for all $t \geq t_0$, we must have $f(t) \geq y > e$, else $W_x \cap S \neq \varnothing$. But then
$$e \in A \Leftrightarrow e \in \{f(x)|s < t_0\}.$$

∎

Porism (Jockusch and Soare): The simple set constructed above is recursively equivalent to A.

Proof: Let ψ be the computable function of A: $\psi(x) = (\mu s)f(s) = x$. (Of course, ψ is partial recursive.) The idea is to try to dominate ψ with a function recursive in S, as in the proof of Martin's theorem. Let S_t be the part of S that is enumerated before step t in Yates' contruction.

Using the recursion theorem, let $g(x)$ be a 1–1 recursive function such that

$$W_{g(x)} = \begin{cases} \varnothing \text{ if } \psi(x) \uparrow \\ \{\text{the first } 2g(x)+2 \text{ elements of } \mathbb{N} \smallsetminus S_{\psi(x)}\} \text{ if } \psi(x) \downarrow . \end{cases}$$

(Note: if $\psi(x)$ converges, then $W_{g(x)}$ contains a $y > 2g(x)$. This is not quite as strong as in Martin's proof, where we could conclude that $W_{g(x)} \cap S \neq \varnothing$.)

Still following Martin's proof, define

$$F(x) = (\mu t) \left[\begin{array}{l} \{\text{the first } 2g(x)+2 \text{ elements of } \mathbb{N} \smallsetminus S\} = \\ \{\text{the first } 2g(x)+2 \text{ elements of } \mathbb{N} \smallsetminus S_t\} \end{array} \right].$$

Clearly, F is total and recursive in S; also, if $t' \geq F(x)$, then

$$\{\text{the first } 2g(x)+2 \text{ elements of } \mathbb{N} \smallsetminus S\} = \{\text{the first } 2g(x)+2 \text{ elements of } \mathbb{N} \smallsetminus S_{t'}\}.$$

Let $\widehat{A} = \{x \in A | F(x) < \psi(x)\}$. (In Martin's proof, $\widehat{A} = \varnothing$.) If $\widehat{A}$ is finite, then clearly A is recursive in S. Suppose $\widehat{A}$ is infinite.

Assume $x \in \widehat{A}$. Then $W_{g(x)} \cap S = \varnothing$, even though $W_{g(x)}$ contains a $y > 2g(x)$. How does $W_{g(x)}$ manage to stay disjoint from S? Answer: each $y \in W_{g(x)}$ that is $> 2g(x)$ must run afoul of the "permission" clause (3)—$y > f(t)$—in Yates' construction.

Precisely, suppose $x \in \widehat{A}$, $y \in W_{g(x)}$, $y > 2g(x)$, and $t_0 > g(x)$ and is sufficiently large so that $y \in W_{g(x),t_0}$ Then for all $t \geq t_0$ we must have $f(t) \geq y$. Thus if $e < y$, we may conclude

$$e \in A \Leftrightarrow e \in \{f(s) | s < t_0\}.$$

Hence to determine if $e \in A$, it is enough to find a pair (y, t_0) as above such that $e < y$.

A is r.e., thus $\widehat{A}$ is r.e. in S. Also g is 1–1 and $\widehat{A}$ is infinite by hypothesis, thus there is an $x \in \widehat{A}$ such that $g(x) > e$ (a fortiori, $y > 2g(x) > e$). Hence the required pair (y, t_0) can be found recursively in S. ▮

Remark: the above proof is not uniform. Jockusch and Soare assert that it may be made uniform by using the recursion theorem to modify Yates' construction so as to make $\widehat{A} = \varnothing$. (Presumably, one ends up with recursive functions s, h, and k such that $W_{s(x)}$ is simple if W_x is nonrecursive, and

$$\chi_{W_x} = \varphi_{h(x)}^{W_{s(x)}}, \qquad \chi_{W_{s(x)}} = \varphi_{k(x)}^{W_x}$$

so $W_x \equiv_T W_{s(x)}$ "effectively".)

Final remark: Of course, Dekker's theorem (Rogers, §9.5, p.140) yields immediately hypersimple (and hence simple) sets of every nonrecursive r.e. degree. However, Yates' simple set in not hypersimple. Proof: as noted, only $W_0, \ldots, W_{k-1}$ can contribute to $\{0, \ldots, 2k\}$. Therefore the $(k+1)$-st element of $\overline{S}$ is at most $2k$.

2 Promptly simple sets and the splitting property

Partial notes on a lecture by M. Stob in the 1980 MIT Summer Logic Seminar, reporting on joint work by W. Maas, M. Stob, and R. Shore.

The part herein reproduced gives an example of the "shiny black box" technique. Stob's lecture dealt primarily with the following problem: what do the structural properties of an r.e. set imply about its Turing degree?

Example of a structural property: M is maximal.

Theorem (Martin): $\mathbf{d}$ is the degree of a maximal set iff $\mathbf{d}$ is **high**, i.e., $\mathbf{d}' = \mathbf{0}''$.

Definition: $\mathcal{E}$ is the lattice (under $\subseteq$) of r.e. sets. $\mathcal{F}$ is the sublattice of finite sets. $\mathcal{E}^* = \mathcal{E}/\mathcal{F}$. $A \equiv_{\mathcal{E}^*} B$ (A is **automorphic** to B) iff there is an automorphism of $\mathcal{E}^*$ carrying the equivalence class of A, $A/\mathcal{F}$, to the equivalence class of B.

Theorem (Soare): If M_1 and M_2 are maximal, then $M_1 \equiv_{\mathcal{E}^*} M_2$.

We might regard automorphic sets as "structurally equivalent". Then any two maximal sets are structurally equivalent.

Definition: $\mathcal{L}(A) = \{B \in \mathcal{E} | B \supseteq A\}$. ($A$ is r.e., of course.) $\mathcal{L}^*(A) = \mathcal{L}(A)/\mathcal{F}$.

A is maximal iff $\mathcal{L}^*(A) = 2 = \{0, 1\}$, the two element boolean algebra.

Soare's theorem can be phrased:

$$\mathcal{L}^*(M_1) \cong \mathcal{L}^*(M_2) = 2 \Rightarrow M_1 \equiv_{\mathcal{E}^*} M_2.$$

Conjecture:

$$\mathcal{L}^*(A) \cong \mathcal{L}^*(B) \Rightarrow A \equiv_{\mathcal{E}^*} B.$$

This conjecture is false, with $\mathcal{L}^*(A) \cong \mathcal{L}^*(A) \cong$ the countable atomless boolean algebra.

Other false conjectures were noted.

Definition: S is **promptly simple** if S is coinfinite, r.e., and there is a recursive enumeration of S and a recursive function p such that

$$W_e \text{ infinite } \Rightarrow (\exists x, s)(x \in [(W_e^{(s+1)} \setminus W_e^{(s)}) \cap S^{(p(s))}]).$$

I.e., x is enumerated in W_e during step s, and appears in S by step $p(s)$. Thus $W_e \cap S \neq \varnothing$, and S is simple. Most constructions of simple sets yield promptly simple sets, since the witness to $W_e \cap S \neq \varnothing$, $x \in W_e \cap S$, is usually added to S shortly after it appears in W_e. However, simple sets exist that are not promptly simple.

The following is an "enumeration free" definition of prompt simplicity.

Lemma: S is promptly simple iff there is a recursive f such that, for all e

$$\begin{aligned}
&W_{f(e)} \subseteq W_e \\
&W_{f(e)} \cap \overline{S} = W_e \cap \overline{S} \\
&W_e \text{ infinite } \Rightarrow W_e \setminus W_{f(e)} \neq \varnothing \\
&S \text{ is coinfinite and r.e.}
\end{aligned}$$

Proof: $\Rightarrow$: $W_{f(e)} = \{x | (\exists s) x \in W_e^{(s+1)} \setminus W_e^{(s)} \setminus S^{(p(s))}\}$. That is, we put into $W_{f(e)}$, all elements of $W_e \cap S$ that, for some s, appear in W_e during step s, but don't appear in S by step $p(s)$.

$\Leftarrow$: exercise.

Friedberg's splitting theorem: If A is r.e. but not recursive, then there are r.e. sets A_1, A_2, such that

$$\begin{aligned}
&A = A_1 \cup A_2 \\
&A_1 \cap A_2 = \varnothing \\
&A_1 \text{ and } A_2 \text{ are not recursive.}
\end{aligned}$$

Definition: S has the **splitting property** if, for every r.e. A, there are r.e. sets A_1, A_2, such that

$$\begin{aligned}
&A = A_1 \cup A_2 \\
&A_1 \cap A_2 = \varnothing \\
&A_1 \text{ and } A_2 \text{ are not recursive if } A \text{ is not recursive} \\
&A_1 \subseteq S.
\end{aligned}$$

Main theorem: Every promptly simple set has the splitting property.

Proof of Friedberg's theorem:

Enumerate A without repetition (so $|A^{(s+1)} \setminus A^{(s)}| = 1$ for all s). We wish to satisfy the following requirements:

$$R(e,i): \qquad W_e \neq \overline{A_i} \qquad i = 1,2$$

Suppose x is to be added to A during step s. We will add x either to A_1 or to A_2; thus $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \varnothing$. Find the smallest (highest priority) $R(e,i)$ such that

$$x \in W_e^{(s)} \text{ and } W_e^{(s)} \cap A_i^{(s)} = \varnothing$$

Add x to A_i. We now prove three things.

(1) If $W_e \cap A_i = \varnothing$ for $i = 1$ or 2, then $W_e \setminus A$ is r.e.

Proof: Suppose $W_e \setminus A$ is not r.e. Then

$$\{x | (\exists s)(x \in [A^{(s+1)} \setminus A^{(s)}] \cap W_e^{(s)})\} \text{ is infinite.}$$

(If not, $W_e \setminus A$ is equal, except for a finite set, to $\{x | (\exists s) x \in W_e^{(s)} \setminus A^{(s)}\}$, which is r.e.)

We show that $W_e \cap A_1 \neq \varnothing$. (By a parallel argument, $W_e \cap A_2 \neq \varnothing$.) Wait until all higher priority requirements than $R(e,1)$ that are going to be satisfied, have been. I.e., let t be such that

$$(\forall R(d,j) < R(e,1))[W_d \cap A_j \neq \varnothing \Rightarrow W_d^{(t)} \cap A_j^{(t)} \neq \varnothing].$$

Let x and s be such that

$$\begin{aligned} &s \geq t \\ &x \in A^{(s+1)} \setminus A^{(s)} \\ &x \in W_e^{(s)}. \end{aligned}$$

Then x is added to A_1 at step s, so $W_e \cap A_1 \neq \varnothing$.

(2) If W is r.e. and $W \setminus A_i$ is r.e., for $i = 1$ or 2, then $W \setminus A$ is r.e.

Proof: $W \setminus A_i = W_e$. $W_e \cap A_i = \varnothing$, so $W_e \setminus A$ is r.e. But $W_e \setminus A = W \setminus A_i \setminus A = W \setminus A$.

(3) If A_1 or A_2 is recursive then A is recursive.

Proof: Let $W = \mathbb{N}$ in (2). ▌

We now modify this proof to show that if S is promptly simple then S has the splitting property. Let f be a recursive function such that

$$\begin{aligned}
&W_{f(e)} \subseteq W_e \\
&W_{f(e)} \cap \overline{S} = W_e \cap \overline{S} \\
&W_e \text{ infinite } \Rightarrow W_e \setminus W_{f(e)} \neq \varnothing.
\end{aligned}$$

The proof uses the recursion theorem in the form of the "shiny black box". That is, the construction given generates a sequence of auxiliary r.e. sets, V_e; we assume, even before the construction is complete, that we have indices for these sets: $V_e = W_{g(e)}$ with g recursive. We justify this after the proof.

Enumerate A without repetition, as before. We have the same requirements:

$$R(e,i): \qquad W_e \neq \overline{A_i} \qquad i = 1, 2.$$

As before, we let x be the element added to A during step s, and let $R(e,i)$ be the highest priority requirement such that

$$x \in W_e^{(s)} \text{ and } W_e^{(s)} \cap A_i^{(s)} = \varnothing.$$

If $i = 2$, then we still add x to A_2, but if $i = 1$, we must insure that $A_1 \subseteq S$. In this case, we put x into V_e, the auxiliary set.

Let

$$V_e = W_{g(e)},$$

V_e being the r.e. set obtained during the entire course of the construction (i.e., $\bigcup_{t=1}^{\infty} V_e^{(t)}$, not $V_e^{(s)}$).

Enumerate $W_{f(g(e))}$ and S simultaneously. More precisely, interleave the enumerations so that elements appear alternately from $W_{f(g(e))}$ and S. If x appears first in S, then add x to A_1. If x appears first in $W_{f(g(e))}$, then add x to A_2. (Note: x must appear in one or the other, since $W_{f(g(e))} \cap \overline{S} = W_{g(e)} \cap \overline{S}$, and we know that $x \in W_{g(e)}$.)

The following are immediate: $A = A_1 \cup A_2$, $A_1 \cap A_2 = \varnothing$, $A_1 \subseteq S$. We now look at the three facts from Friedberg's proof.

(1) If $W_e \smallsetminus A$ is not r.e., then $W_e \cap A_i = \varnothing$ for $i = 1$ and 2.

Proof: For A_2, this is just as before.

If $W_e \smallsetminus A$ is not r.e., then

$$\{x|(\exists s)(x \in [A^{(s+1)} \smallsetminus A^{(s)}] \cap W_e^{(s)})\} \text{ is infinite}$$

so, by waiting until higher priority requirements have been settled, we may conclude that either $W_e \cap A_1 \neq \varnothing$, or else that V_e is infinite. But if $V_e = W_{g(e)}$ is infinite, then there is an $x \in W_{g(e)} \smallsetminus W_{f(g(e))}$, and we may be sure that this x will appear in S before it appears in $W_{f(g(e))}$, so this x will be added to A_1. Thus $W_e \cap A_1 \neq \varnothing$, and (1) is proved.

(2) and (3) follow as before.

Finally, we justify the hypothesis $V_e = W_{g(e)}$. Let W_k be an arbitrary r.e. set. Run the construction as given, except that instead of using $W_{g(e)}$, use

$$W_{h(e,k)} = \{x|\langle x, e\rangle \in W_k\}$$

where $\langle \cdot, \cdot \rangle$ is a recursive pairing function.

The construction will generate r.e. sets V_e as before. (The construction may "stall" at a step for which $x \notin W_{f(h(e,k))} \cup S$, in which case almost all the V_e end up being finite, but so what.)

Let

$$W_{r(k)} = \{\langle x, e\rangle | x \in V_e\}$$

Clearly r is recursive. Let k be such that

$$W_k = W_{r(k)}$$

by the recursion theorem. For this k, $W_{h(e,k)} = V_e$, and we are done. ∎