Lie Groups Notes

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I'm mainly looking at three books: Stillwell [4], Hall [2], and Duistermaat & Kolk [1]. These are in increasing order of sophistication. Page references to Hall in the form p.40[36], meaning p.40 in the 2nd edition and p.36 in the 1st.

1 Basics

The core concepts are:

- Lie algebra $\mathfrak{g}$ associated with Lie group G.
- exp and log, the local homeomorphisms between G and $\mathfrak{g}$.
- The brackets and the adjoints: $[X, Y]$, $\mathrm{Ad}_x(Y)$, $\mathrm{ad}_X(Y)$.

I will generally use x, y, z for elements of G and X, Y, Z for elements of $\mathfrak{g}$.

Here I just want to list how these concepts are handled in the three books. Both Hall and Stillwell restrict themselves (mainly) to matrix groups. Duistermaat & Kolk treat the general concept. The order in which the concepts are introduced differs among the three, for logical reasons.

1.1 $\mathfrak{g}$ from G

Stillwell: $X \in \mathfrak{g}$ iff $X = x'(0)$ for some smooth curve $x : \mathbb{R} \to G$ with $x(0) = 1$. So basically the tangent space to G at 1. (p.93)

Hall: $X \in \mathfrak{g}$ iff $e^{tX} \in G$ for all $t \in \mathbb{R}$. Obviously any X satisfying this also satisfies the Stillwell condition. (p.56[39])

DK: $\mathfrak{g} = T_1(G)$, the tangent space to G at 1. The differential geometry background is assumed. (p.3)

First order of business: $\mathfrak{g}$ is a vector space over $\mathbb{R}$, i.e., rX and $X + Y$ belong to $\mathfrak{g}$ for all $X, Y \in \mathfrak{g}$ and $r \in \mathbb{R}$.

Stillwell: Differentiate $x(rt)$ and $x(t)y(t)$ (pp.103–104).

Hall: For rX, it's trivial that $e^{rtX} \in G$ for all t. For $X + Y$, he uses the Lie product formula (p.40[36]) plus the assumption that G is closed in $\mathrm{GL}(n; \mathbb{C})$. (p.56[44])

DK: Automatic.

1.2 exp and log

Stillwell and Hall first treat the special case where $G = \mathrm{GL}(n; \mathbb{C})$ and $\mathfrak{g} = \mathrm{gl}(n; \mathbb{C})$. Besides defining $\exp(X)$ and $\log(x)$, we also need the identities $\exp(\log(x)) = x$ and $\log(\exp(X)) = X$ for a neighborhood of 1 in $\mathrm{GL}(n; \mathbb{C})$ and of 0 in $\mathrm{gl}(n; \mathbb{C})$.

Stillwell: Power series. He proves $\log(\exp(X)) = X$ for $|\exp(X) - 1| < 1$ by comparing coefficients with the complex-valued (or real-valued) case. He does not explicitly prove $\exp(\log(x)) = x$ (for $|x - 1| < 1$) but he does use it. (pp.140–141)

Hall: Power series. He explicitly identifies the neighborhoods: $|X| < \log 2 \Rightarrow |\exp(X) - 1| < 1$, so we have a local homeomorphism between the disk of radius log 2 about 0 and its image under exp, a neighborhood of 1 (see §2 and fig.1). He proves the identities by diagonalizing the matrices, or taking limits if not diagonalizable. (pp.31,38[27,34])

Hall has to include some of this material before his definition of $\mathfrak{g}$, since that depends on $\exp(tX)$.

For the case where G is any closed subgroup of $\mathrm{GL}(n;\mathbb{C})$, Stillwell and Hall have to show that exp maps a neighborhood of 0 into $\mathfrak{g}$ and a neighborhood of 1 into G.

Stillwell: For the exp mapping, let $X = x'(0)$. Now $x'(0)$ is a limit of difference quotients. Comparing this limit with the power series for log, and using facts about exp and log, Stillwell shows that $\exp(X)$ is a limit of elements of G; since G is closed in $\mathrm{GL}(n;\mathbb{C})$, we're done. (pp.143–144). The log mapping argument is essentially the same as in Hall. (pp.145–149) (Incidentally, Stillwell thanks Hall for an important observation.)

Hall: It's immediate from Hall's definition of $\mathfrak{g}$ that exp maps all of $\mathfrak{g}$ into G. For the log mapping, the key step is to decompose $\mathrm{gl}(n;\mathbb{C})$ into $\mathfrak{g}$ and its orthogonal complement. That enables us to write $\log(x) = X + Y$ with $X \in \mathfrak{g}$ and Y in the orthogonal complement. The problem is to show that $Y = 0$. Hall does this using what we've already learned about log and exp between $\mathrm{GL}(n;\mathbb{C})$ and $\mathrm{gl}(n;\mathbb{C})$, the inverse function theorem, and some topology (compactness, G being closed in $\mathrm{GL}(n;\mathbb{C})$). (pp.68–70[49–50])

Hall uses these results to show that his definition of $\mathfrak{g}$ is equivalent to the tangent space definition (p.71).

These arguments rely heavily on G being a matrix group. DK thus take a completely different approach.

DK: They first define left and right invariant vector fields. Using these they show that for any $X \in \mathfrak{g}$, there is a unique one-parameter subgroup (i.e., smooth homomorphism $h : \mathbb{R} \to G$) with $h'(0) = X$, and then they define

$\exp(X) = h(1)$. They get the local inverse log via the inverse function theorem; they call it the open mapping theorem. (pp.16–19)

DK follow this general developement with a section looking at the special case of matrix groups.

1.3 Adjoints and Bracket

We have $\mathrm{ad}_X(Y) = [X, Y]$. For any $x \in G$, $\mathrm{Ad}_x : \mathfrak{g} \to \mathfrak{g}$ is a linear mapping, as is ad_X for any $X \in \mathfrak{g}$.

Stillwell: He defines $[X, Y] = XY - YX$. He shows that $\mathfrak{g}$ is closed under the bracket by differentiating $x(s)y(t)x(s)^{-1}$, first with respect to t (giving the adjoint $\mathrm{Ad}_x(Y) = xYx^{-1}$) and then with respect to t. (pp.98,104)

Hall: First he shows that $\mathfrak{g}$ is closed under the adjoint $Y \mapsto xYx^{-1}$ (p.56[43]). He shows that $\mathfrak{g}$ is closed under the bracket by differentiating $e^{tX}Ye^{-tX}$, and using the fact that $\mathfrak{g}$ is a closed subset of $\mathrm{gl}(n; \mathbb{C})$. (p.57[44]). Hall also includes material about the abstract definition of a Lie algebra, more in the second edition than the first. He defines $\mathrm{ad}_X(Y)$ as an alternate notation for $[X, Y]$.

DK: They begin with the smooth map $\mathrm{Ad}_x : y \mapsto xyx^{-1}$ sending G to G; it's derivative at $y = 1$ is $\mathrm{Ad}_x : \mathfrak{g} \to \mathfrak{g}$, since $\mathfrak{g}$ is defined to be the tangent space at 1. Note that Ad_x belongs to $\mathrm{GL}(\mathfrak{g})$. Next step: $x \mapsto \mathrm{Ad}_x$ is smooth and sends 1 to 1 and G to $\mathrm{GL}(\mathfrak{g})$. So its derivative at 1 is a linear mapping from $\mathfrak{g}$ to $\mathrm{gl}(\mathfrak{g})$, denoted ad. In other words, $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$, and we can define $[X, Y] = \mathrm{ad}_X(Y)$. Bilinearity follows from the linearity of ad as a mapping from $\mathfrak{g}$ to $\mathrm{gl}(\mathfrak{g})$. (pp.2–3)

With the definition $[X, Y] = XY - YX$, the antisymmetry of the bracket is immediate and the Jacobi identity is a routine computation. DK can't use that, since XY is not defined in their more general setting. I won't go through the argument they do use (p.4).

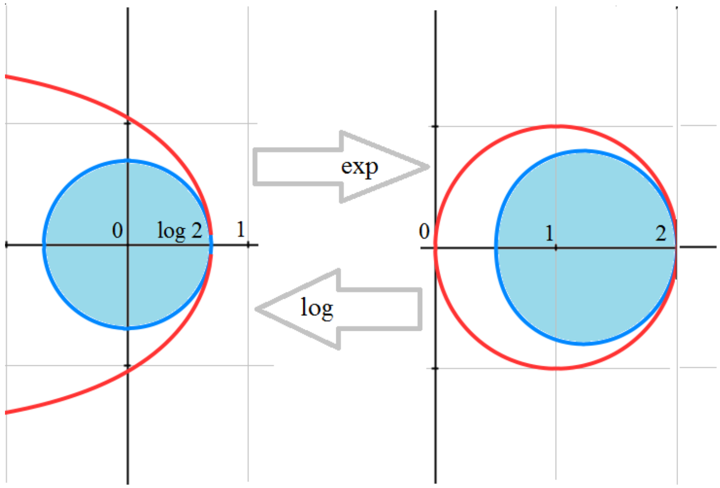

Figure 1: Exp and log neighborhoods

As a side note, suppose $x(t)$ and $y(t)$ are smooth curves in $\mathrm{GL}(n;\mathbb{C})$ (or $\mathrm{GL}(n;\mathbb{R})$), passing through 1 at $t=0$. Differentiating $x(t)y(t)x(t)^{-1}y(t)^{-1}$ at $t=0$ yields 0. But if we look at the commutator $c(s,t)=x(s)y(t)x(s)^{-1}y(t)^{-1}$ and compute $\partial_s\partial_t c(s,t)$, evaluating at $s=t=0$ at the final step, you get $XY-YX$. (Likewise for $\partial_t\partial_s c(s,t)$.) This shows how the bracket is basically a "curvature", a second-order effect.

Kirillov [3, §3.5] explains this in the context of smooth vector fields on a differential manifold. The bracket becomes the Lie derivative. Kirillov defines the bracket via the CBH formula, i.e., as the factor making this equation work:

$$\exp(X)\exp(Y)=\exp\left(X+Y+\tfrac{1}{2}[X,Y]+\cdots\right)$$

2 Exp and log neighborhoods

In the complex plane, exp and log provide homeomorphisms between neighborhoods of 0 and 1, as shown in fig.1. The neighborhood of 0 is the blue disk

$|z| < \log 2$; the neighborhood of 1 is the blue oval on the right, which lies inside the disk $|w - 1| < 1$.

The series for $\log w$ converges inside the disk $|w - 1| < 1$ (red circle), and its image is outlined by the red curve looking something like a parabola. It's not; it has horizontal asymptotes $y = \pm\pi/2$, as you can see from the correspondence

$$\log r + i\theta \leftrightarrow re^{i\theta}$$

plus a little geometry. (Here $w = re^{i\theta}$ is a point inside the disk $|w - 1| < 1$.)

The series for exp converges everywhere, but the disk $|z| < \log 2$ is the largest origin-centered disk whose exp-image lies inside the red circle on the right; i.e., $|z| < \log 2$ implies $|e^z - 1| < 1$.

All this continues to hold if we regard points in the two neighborhoods as representing matrices X in $\mathrm{gl}(n;\mathbb{C})$ (on the left) and x in $\mathrm{GL}(n;\mathbb{C})$ (on the right). That's because $|XY| \leq |X||Y|$ (Cauchy-Schwarz), so the convergence works out the same way.

Hall lays all this out in §2.1, but without the figure.

3 Components and Covers

Let G be a (real) Lie group, let G_1 be the component of the identity, and let $\widetilde{G}_1$ be the universal cover of G_1. So $\widetilde{G}_1$ is a connected simply connected Lie group. As such, it has a "1–1 relationship" with its Lie algebra $\mathfrak{g}$: closed normal subgroups of $\widetilde{G}_1$ correspond 1–1 with ideals of $\mathfrak{g}$, homomorphisms $\widetilde{G}_1 \to \widetilde{H}_1$ correspond 1–1 with homomorphisms $\mathfrak{g} \to \mathfrak{h}$, and connected simply connected real Lie groups correspond 1–1 with real Lie algebras. (Of course, *proving* all these assertions takes work.)

Important but obvious fact: the Lie algebra of G is the same as the Lie algebra of $\widetilde{G}_1$, because open sets surround the identities of G and $\widetilde{G}_1$, with a diffeomorphism between them that is also a local group isomorphism: $\varphi : U \leftrightarrow V$ with $\varphi(xy) =$

$\varphi(x)\varphi(y)$ for all x, y sufficiently close to 1 (i.e., for all x, y in some neighborhood $U' \subseteq U$ of 1).

In this section, I will look at what we need to reconstruct G from $\widetilde{G}_1$. Let me spell out what I mean by "reconstruct". *Given* that $\widetilde{G}_1$ is the universal cover of the identity component of a Lie group G, *what additional data* do we need, besides $\widetilde{G}_1$, to determine G up to (Lie group) isomorphism? A related but somewhat harder question: given a connected simply connected Lie group $\widetilde{G}_1$, how do we find all the associated Lie groups G, up to isomorphism? (We'll see the difference between the two questions as we proceed. Let's call them the "reconstruction" and "find-all" questions.)

Useful and evocative terminology: if $p : E \to B$ is a surjective function, we also call it a *fibration*. The fibers are the $p^{-1}(b)$'s, so E is a disjoint union of its fibers.

3.1 G from G_1

First, topology.

Lemma: Let E be a topological space and let $\pi_0(G)$ be the set of all its components. Let $p : E \to \pi_0(G)$ be the map sending a point to the component that contains it. Give $\pi_0(G)$ the quotient topology. Then $\pi_0(G)$ is discrete. Furthermore, suppose that all the components of E are homeomorphic, and that E is locally connected (components are open). Let F be a component. Then E is homeomorphic to $F \times \pi_0(G)$.

Remark: G and G_1 satisfy all the hypotheses, since for any $x \in G$, left translation by x is a homeomorphism from G_1 to xG_1, and all components are of this form. (Also G, as a manifold, is locally connected.) So we can reconstruct the topology of G, just knowing the topology of G_1 and the set of components as a discrete set of points. In other words, all we need is the cardinality of $\pi_0(G)$.

Proof: To show that $\pi_0(G)$ is discrete, we just need to show that the inverse image of any $b \in \pi_0(G)$ is closed in E. But $p^{-1}(b)$ is just a component of E, and standard general topology tells us that all components are closed.

For any $b \in \pi_0(G)$, let $\varphi_b : F \to b$ be a homeomorphism. Define $\varphi : F \times \pi_0(G) \to E$ by $(x, b) \mapsto \varphi_b(x)$. As $F \times \pi_0(G)$ is the disjoint union of all the $F \times \{b\}$ as b ranges over $\pi_0(G)$ (i.e., all the fibers), and E is the disjoint union of its components, this is a bijection.

Since $\pi_0(G)$ is discrete, sets of form $U \times \{b\}$ with U open in F and $b \in \pi_0(G)$ form a basis for $F \times \pi_0(G)$. If E is locally connected, then any open $V \subseteq E$ is the disjoint union of open sets $\bigsqcup_{b\in\pi_0(G)} V \cap b$. So open sets contained in components form a basis for E (i.e., V such that $V \subseteq b$ for some $b \in \pi_0(G)$). It's obvious that φ and φ^{-1} send basic open sets to basic open sets, so φ is a homeomorphism. ▌

Another way to describe this: G is a trivial fiber bundle with fiber G_1 and base space $\pi_0(G)$. So G_1 plus $|\pi_0(G)|$ determines G as a topological space, i.e., up to homeomorphism. We've solved both the reconstruction and find-all questions, up to homeomorphism.

Next, group structure. Since G_1 is a closed normal[1] subgroup of G, we have a short exact sequence

$$1 \to G_1 \to G \to G/G_1 \to 1$$

So G is an extension of G/G_1 by G_1. (Some would say, an extension of G_1 by G/G_1. Terry Tao has a discussion of this.)

Note that G/G_1 as a set is just $\pi_0(G)$: the cosets of G_1 are precisely the components of G. We conclude that $\pi_0(G)$ inherits a group structure from G. Finding G as a topological group becomes an instance of the so-called *group extension problem*: given groups N and Q, find all groups G containing N as a normal subgroup with $G/N \cong Q$. In other words, find all short exact sequences

$$1 \to N \to G \to Q \to 1$$

with given N and Q. Two solutions with G and G' are considered equivalent if there is an isomorphism $G \leftrightarrow G'$ making the diagram in fig.2 commute. The group extension problem has an enormous literature, but it's primarily algebraic. To avoid stepping into this, let's just assume we are given *extension data*, sufficient to determine G as an extension of $\pi_0(G)$ by G_1 (up to equivalence).

[1] If $g \in G$ and $t \mapsto x(t)$ is a path in G_1 from $x(0) = 1$ to $x(1) = x$, then $gx(t)g^{-1}$ is a path in G_1 from 1 to gxg^{-1}.

$$\begin{tikzcd}
& & G \arrow[dr] \arrow[dd, leftrightarrow] & & \\
1 \arrow[r] & N \arrow[ur] \arrow[dr] & & Q \arrow[r] & 1 \\
& & G' \arrow[ur] & &
\end{tikzcd}$$

Figure 2: Equivalent Extensions

Finally, differential structure. We start with an atlas on G and restrict it to G_1. We can also restrict it to any component xG_1. Conversely, the atlas on all of G is clearly determined by the collection of all these restricted atlases. Since left translations are smooth maps, and invertible, the restricted atlas on G_1 determines the restricted atlas on xG_1. Conclusion: the differential structure of G_1 plus the group structure of G is enough to reconstruct the differential structure of G.

Summarizing, the Lie group structure of G_1 plus $\pi_0(G)$ as a group plus extension data for G is enough to reconstruct (i.e., it determines completely) the Lie group G.

I believe we've also solved the find-all problem, modulo the extension problem. In other words, if we are given a group G extending $\pi_0(G)$ by G_1, and a differential atlas on G_1 making G_1 into a Lie group, then the procedure sketched above will always extend this to an atlas making G into a Lie group with identity component G_1.

3.2 G_1 from $\widetilde{G}_1$

First topology. We have a standard construction of the universal cover $\widetilde{X}$ of a connected topological manifold X. (Or more generally, of any "nice" space X, but we have no use for the additional generality.) We pick a basepoint $x_0 \in X$. Points of $\widetilde{X}$ are (fixed-endpoint) homotopy classes of paths from the basepoint to points of X. If α is a path from x_0 to x, then the projection map $p : \widetilde{X} \to X$ sends $[\alpha]$ to $\alpha(1) = x$. Standard theorems tell us that $\widetilde{X}$ is a regular covering space of X, that $\pi_1(\widetilde{X}) = 1$, and that $\pi_1(X)$ is isomorphic to the group of deck transformations of $\widetilde{X}/X$. There is a unique deck transformation taking x_0 to any point in the fiber over x_0.

In the reverse direction, given $\widetilde{X}$ and the group T of deck transformations, we reconstruct X as the quotient $\widetilde{X}/T$.

Next, differential structure. The projection map is a *local homeomorphism*, i.e., if $p(\tilde{x}) = x$, then there is a neighborhood $\widetilde{U} \ni \tilde{x}$ with $p \restriction \widetilde{U}$ a homeomorphism onto $p(\widetilde{U})$. This makes it trivial to lift a differential atlas from X to $\widetilde{X}$, or conversely to "collapse" an atlas from $\widetilde{X}$ to X. Because deck transformations are smooth, it doesn't matter which covering patch we use, when going from $\widetilde{X}$ to X.

Finally, group structure. Let the manifold X be a (connected) Lie group G_1. We always use the identity 1 as the basepoint.

We lift the group structure uniquely from G_1 to $\widetilde{G}_1$ like so. Let $[\alpha]$ and $[\beta]$ be elements of $\widetilde{G}_1$. Then $t \mapsto \alpha(t) \cdot \beta(t)$ is a path from the basepoint to $\alpha(1) \cdot \beta(1)$. I'll denote it $\alpha \cdot \beta$; $\alpha\beta$ as usual denotes the concatenation of paths. The homotopy class $[\alpha \cdot \beta]$ doesn't depend on the choice of α and β, because the group operation is continuous. Let $[\alpha] \cdot [\beta]$ be $[\alpha \cdot \beta]$. It's also easy to see that the projection is a group epimorphism.

Once $\widetilde{G}_1$ has been constructed, it's easiest to reason about it using just these facts: $\widetilde{G}_1$ is connected and simply connected, $p : \widetilde{G}_1 \to G_1$ is a covering map, and p is a continuous epimorphism. As an illustration, we show that the fiber $p^{-1}(1)$ is (a) a normal subgroup of $\widetilde{G}_1$, (b) lying in its center, and (c) isomorphic to $\pi_1(G_1)$. (Note that these imply that $\pi_1(G_1)$ is abelian. This holds for any topological group—see below. Also, (b) implies the normality of $p^{-1}(1)$.)

For (a), observe that $p^{-1}(1)$ is just the kernel of p, so it's a normal subgroup of $\widetilde{G}_1$. For (b), suppose $p(\tilde{x}) = 1$, and let $\tilde{g}$ be an arbitrary element of $\widetilde{G}_1$. Let $t \mapsto \tilde{g}(t)$ be a path connecting $\tilde{g}(0) = 1$ to $\tilde{g}(1) = \tilde{g}$. Let $p(\tilde{g}(t)) = g(t)$. Then $\tilde{g}(t)\tilde{x}\tilde{g}(t)^{-1}$ is a path connecting $\tilde{x}$ to $\tilde{g}\tilde{x}\tilde{g}^{-1}$. But

$$p(\tilde{g}(t)\tilde{x}\tilde{g}(t)^{-1}) = g(t)1g(t)^{-1} = 1;$$

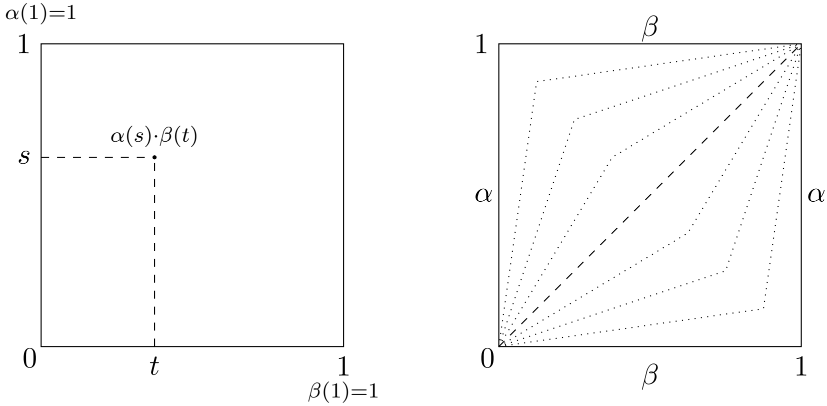

Figure 3: Commuting Paths

covering space theory tells us that constant paths lift to constant paths, so $\tilde{x} = \tilde{g}\tilde{x}\tilde{g}^{-1}$, i.e., $\tilde{x}$ lies in the center.

An interlude before (c): π_1 of any topological group G is abelian. Let α and β be loops based at the identity. Map $I \times I \to G$ by the rule $(s,t) \mapsto \alpha(s) \cdot \beta(t)$ (see the left diagram in fig.3). Since $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = 1$, the vertical edges of the square are both α and the horizontal edges both β. As the right diagram clearly shows, $\alpha\beta$ is homotopic to $\beta\alpha$. Done.

With the same diagram, but re-interpreted, we can prove (c). Lift the entire square to $\widetilde{G}_1$, starting with the $(0,0)$ corner at the identity. Note that $\widetilde{\alpha}(1)$ and $\widetilde{\beta}(1)$ both belong to the fiber $p^{-1}(1)$. Let's write $\widetilde{\alpha}(1) = x$, $\widetilde{\alpha}(1) = y$. In fact if we start with arbitrary $x, y \in p^{-1}(1)$, we can create the lifted square: just pick a path $\widetilde{\alpha}$ from 1 to x and a path $\widetilde{\beta}$ from 1 to y, and map $I \times I \to \widetilde{G}_1$ via $(s,t) \mapsto \widetilde{\alpha}(s) \cdot \widetilde{\beta}(t)$. Because $\widetilde{G}_1$ is simply connected, it doesn't matter what paths $\widetilde{\alpha}$ and $\widetilde{\beta}$ we choose: any other choices will be homotopic, inducing a homotopy of the entire square (i.e., we get homotopic maps $I \times I \to \widetilde{G}_1$).

OK, start with arbitrary x and y in the fiber over 1, construct the square as described, and use p to project it down to G_1, getting fig.3. We define a bijection ι from $p^{-1}(1)$ to $\pi_1(G_1)$ this way: given any element of the fiber, choose a path in

$\widetilde{G}_1$ from 1 to the element, project the path down to G_1, and take the homotopy class of the resulting loop. Example: $\widetilde{\alpha}$ is a path from 1 to x, the projection α is a loop, and $\iota(x) = [\alpha] \in \pi_1(G_1)$. Because $\widetilde{G}_1$ is simply connected, $\iota(x)$ doesn't depend on the choice of $\widetilde{\alpha}$. The inverse ι^{-1} is easily constructed: given $[\alpha] \in \pi_1(G_1)$, pick a loop $\alpha \in [\alpha]$, lift α to $\tilde{\alpha}$ in $\widetilde{G}_1$ starting at 1, and set $\iota^{-1}([\alpha]) = \widetilde{\alpha}(1)$. This doesn't depend on the choice of α. It's easily verified that ι and ι^{-1} are inverses. So ι is a bijection.

Finally, ι is an isomorphism. Proof: recall that $\widetilde{\alpha}$ is the left edge of the lifted square, $\widetilde{\beta}$ the bottom edge, and that $\widetilde{\alpha}$ and $\widetilde{\beta}$ go from 1 to x and y respectively. The $(1,1)$ corner of the lifted square is $x \cdot y$, by construction. To find $\iota(x \cdot y)$ we can use any path from 1 to $x \cdot y$; use $\widetilde{\alpha}\widetilde{\beta}_{\text{top}}$, where $\widetilde{\beta}_{\text{top}}$ is the *top* edge of the square[2]. Note that $p(\widetilde{\beta}_{\text{top}}) = p(\widetilde{\beta})$, because $p(x) = 1$. So:

$$
\begin{aligned}
\iota(x) &= [\alpha] \\
\iota(y) &= [\beta] \\
\iota(x \cdot y) &= [p(\widetilde{\alpha})p(\widetilde{\beta}_{\text{top}})] = [\alpha][\beta]
\end{aligned}
$$

The upshot is a short exact sequence

$$1 \to \pi_1(G_1) \to \widetilde{G}_1 \to G_1 \to 1$$

This is known as a *central extension* (of G_1 by $\pi_1(G_1)$), because $\pi_1(G_1)$ lies in the center of $\widetilde{G}_1$. Central extensions occupy a prominent place in group cohomology theory.

In the reverse direction, the group structure on $\widetilde{G}_1$, plus the group T of deck transformations of $\widetilde{G}_1$ over G_1, determines the group structure on G_1. That's because the projection p is just the quotient map from $\widetilde{G}_1$ to $\widetilde{G}_1/T$, and p is an epimorphism. This is a "reconstruction" anwer, not a "find-all" answer. You can't start with any old group of diffeomorphisms of $\widetilde{G}_1$ onto itself and take the quotient. You won't always get a covering space, nor will the quotient map always give a well-defined group operation on the quotient space. But if your group is the group of deck transformations, then everything works out.

[2] It's interesting to recall that the diagonal $t \mapsto \widetilde{\alpha}(t) \cdot \widetilde{\beta}(t)$ was our original *definition* of the product on $\widetilde{G}_1$.

I mentioned that the group of deck transformations is also isomorphic to $\pi_1(G_1)$. I don't want to go into much detail, but here's the gist. Any $[\alpha] \in \pi_1(G_1)$ induces a permutation on the fiber $p^{-1}(1)$: given $x \in p^{-1}(1)$, lift α to $\widetilde{\alpha}$ starting at x, and set $\sigma_\alpha(x) = \widetilde{\alpha}(1)$. Then σ_α is permutation of the fiber that depends only on the homotopy class $[\alpha]$. For a so-called *regular* covering, the map $[\alpha] \mapsto \sigma_\alpha$ is an isomorphism of $\pi_1(G_1)$ onto a permutation group, and the deck transformations induce this same permuation group. (Coverings by simply connected spaces are always regular.) So we get three isomorphic groups: the permutation group, the deck transformations, and the fundamental group.

In the case of a topological group, we have more. Fig.3 can be adapted to show that the deck transformations are precisely left translations by elements of the fiber over the identity. In symbols, if $x \in p^{-1}(1)$, then $y \mapsto x \cdot y$ is a deck transformation, and all deck transformations are obtained this way. Equally well you can use right translations. Even more: a deck transformation is determined by its action on a single element. Since the left and right translations by x both send 1 to x, it follows that they are identical. This applies, of course, only to x's belonging to the fiber $x \in p^{-1}(1)$.

4 SU(2), SO(3), and O(3)

I quickly recap the double-covering story, as a warm-up for the next section on $\mathrm{O}(1,3)$, i.e., the Lorentz group. We begin with generalities about the unitary groups. I refer to Hall (or other authors) to justify some non-evident facts.

1. $\mathrm{U}(n)$ defining equation: $gg^* = 1$. Differentiating, the Lie group is $\mathrm{u}(n)$ defined by $X + X^* = 0$, i.e., anti-hermitian matrices. X is anti-hermitian iff iX is hermitian. (We'll write $i\mathrm{u}(n)$ for the hermitian matrices.) For $g \in \mathrm{U}(n)$, we have $|\det(g)| = 1$.

2. $\mathrm{SU}(n)$: in addition, $\det(g) = 1$. So for $\mathrm{su}(n)$, $\mathrm{trace}(X) = 0$. Also, $\mathrm{trace}(X) = 0$ iff $\mathrm{trace}(iX) = 0$.

3. $\mathrm{U}(n)$ has the adjoint action on $\mathrm{u}(n)$, $X \mapsto gXg^{-1}$. This is the same as $X \mapsto gXg^*$ because $g^{-1} = g^*$.

4. The adjoint action preserves both determinants and traces: $\det(X) = \det(gXg^{-1})$, $\operatorname{trace}(X) = \operatorname{trace}(gXg^{-1})$. Indeed, this holds for any $g \in \mathrm{GL}(n;\mathbb{C})$ and any $X \in \mathrm{gl}(n;\mathbb{C})$.

5. Both $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are connected for all $n \geq 1$ (Hall p.18[14]). We have a short exact sequence

$$1 \to \mathrm{SU}(n) \to \mathrm{U}(n) \to \mathrm{U}(1) \to 1$$

with the epimorphism given by $g \mapsto \det(g)$; as we just noted, $|\det(g)| = 1$ for $g \in \mathrm{U}(n)$. (Elements of $\mathrm{U}(1)$ are *phases* in physics lingo.)

Topologically, this short exact sequence exhibits $\mathrm{U}(n)$ as a fiber bundle with base space $U(1)$ and fiber $\mathrm{SU}(n)$ (Kirillov [3, p.8]).

6. $\mathrm{SU}(n)$ is simply connected, and $\pi_1(\mathrm{U}(n)) = \mathbb{Z}$, for all $n \geq 1$ (Hall p.378[16,335]).

Now we turn to the special case $n = 2$.

7. Any hermitian 2×2 matrix can be written uniquely as:

$$\begin{aligned}\begin{bmatrix} t+z & x+iy \\ x-iy & t-z \end{bmatrix} &= t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + z\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + x\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \\ &= t1 + x\sigma_x + y\sigma_y + z\sigma_z\end{aligned}$$

where the σ_i's are the Pauli matrices.

8. The determinant of $X = t1 + x\sigma_x + y\sigma_y + z\sigma_z$ is $t^2 - x^2 - y^2 - z^2$. The trace is $2t$.

9. Hence we can identify $i\mathrm{u}(2)$ with Minkowski space: $t1 + x\sigma_x + y\sigma_y + z\sigma_z \leftrightarrow (t, x, y, z)$. This also identifies $i\mathrm{su}(2)$ with Euclidean 3-space $t = 0$. The adjoint action of $\mathrm{U}(2)$ on $i\mathrm{u}(2)$ preserves both the Minkowski metric and t, and so in particular preserves the subspace $i\mathrm{su}(2)$ and its Euclidean metric.

10. The map $g \mapsto \mathrm{Ad}_g$, with Ad_g acting on $i\mathrm{u}(2)$, is a homomorphism from $\mathrm{U}(2)$ to the Lorentz group $\mathrm{O}(1,3)$. The image is *not* the full Lorentz group. Specifically:

- t is preserved, and so the Euclidean subspace $t = 0$ is invariant (as already noted).
- Since U(2) is connected, so is the image, and hence lies in the component of the identity, i.e., in the proper Lorentz group $\mathrm{SO}^+(1,3)$.

It follows that the image is contained in SO(3), regarded as a subgroup of $\mathrm{SO}(1,3)$. In fact, even if we restrict the domain to SU(2), the image is SO(3) (Hall p.24).

As our main protagonist has just entered the stage, it's worth recapping. I will treat SO(3) as the subgroup of t-preserving elements of $\mathrm{SO}(1,3)$ ($\mathrm{SO}(3) \subset \mathrm{SO}(1,3)$). Note that the action of a t-preserving linear transformation of Minkowski space is completely determined by its action on the $t = 0$ Euclidean subspace.

The map $g \mapsto \mathrm{Ad}_g$ determines an epimorphism $\mathrm{U}(2) \to \mathrm{SO}(3)$. If we restrict the domain to SU(2), we still get an epimorphism. We now look at the kernels.

11. The kernel of the map $\mathrm{U}(2) \to \mathrm{SO}(3)$ is the set of scalar multiples of the identity by numbers of norm 1, i.e., $\{e^{i\theta}1 : \theta \in \mathbb{R}\}$. Proof: obviously $\{e^{i\theta}1 : \theta \in \mathbb{R}\}$ is contained in the kernel. Say g is in the kernel, so $gXg^{-1} = X$, i.e., $gX = Xg$ for all $X \in i\mathrm{su}(2)$. Let $X = \sigma_x$, the first Pauli matrix; a straightfoward computation shows that g must be diagonal. A similar computation with $X = \sigma_y$ shows that the two diagonal entries are equal. Finally, $g \in \mathrm{U}(2)$ implies that these entries have norm 1.
12. The kernel of its restriction to SU(2) is $\{1, -1\}$, as these are the only two scalar multiples of the identity with determinant 1. Thus for any $g \in \mathrm{SU}(2)$, $\pm g$ both map to the same element of SO(3).
13. As noted earlier, U(2) has fundamental group $\mathbb{Z}$ but SU(2) is simply connected.

It will be useful to know about the flips around the three coordinate axes:

14. The matrices

$$f_x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \quad f_y = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}; \quad f_z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{SU}(2) \arrow[r, hook] \arrow[d, "\pi"'] & \mathbb{Z}_2 \oplus \mathrm{SU}(2) \arrow[r] \arrow[d, "\widetilde{\pi}"] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathrm{SO}(3) \arrow[r, hook] & \mathrm{O}(3) \cong \mathbb{Z}_2 \oplus \mathrm{SO}(3) \arrow[r] & \mathbb{Z}_2 \arrow[r] & 1
\end{tikzcd}$$

Figure 4: Direct Product Double Cover

are elements of SU(2) representing 180° rotations around the x, y, and z axes. (To check this, just compute how they act on the Pauli matrices.) We have

$$f_x^2 = f_y^2 = f_z^2 = -1$$

Next we look at SO(3) as a subgroup of O(3).

15. SO(3) is the component of the identity of O(3).
16. $\mathrm{O}(3)/\mathrm{SO}(3) = \mathbb{Z}_2$.
17. The 3×3 matrix -1 is traditionally used as the representative element of the non-identity component, and denoted P (for *parity operator*). So the coset $P\mathrm{SO}(3)$ is the non-identity component. (I make an exception to the rule "small letters for Lie group elements, capitals for Lie algebra elements".)
18. $\mathrm{O}(3) \cong \mathbb{Z}_2 \oplus \mathrm{SO}(3)$: every element can be written uniquely as Pg with $g \in \mathrm{SO}(3)$, and P lies in the center.
19. SO(3) is a simple group (Stillwell p.33).

Can we complete this diagram?

$$\begin{tikzcd}
\mathrm{SU}(2) \arrow[r, hook] \arrow[d, "\pi"'] & G \arrow[d, "\widetilde{\pi}"] \\
\mathrm{SO}(3) \arrow[r, hook] & \mathrm{O}(3)
\end{tikzcd}$$

We can, in two ways. G must have an element $\widetilde{P}$ with $\tilde{\pi}(\widetilde{P}) = P$. Assume that $\widetilde{P}$ lies in the center of G. Then G is set-theoretically the product $\{1, \widetilde{P}\} \times \mathrm{SU}(2)$; write elements of G as $1g$ or $\widetilde{P}g$, with $g \in \mathrm{SU}(2)$. Multiplication is defined by:

$$(1g)(1h) = 1(gh)$$
$$(1g)(\widetilde{P}h) = \widetilde{P}(gh)$$
$$(\widetilde{P}g)(1h) = \widetilde{P}(gh)$$
$$(\widetilde{P}g)(\widetilde{P}h) = \widetilde{P}^2(gh)$$

Now $\widetilde{P}$ is in the center of G, so $\widetilde{P}^2$ is also in the center. But $\widetilde{P}^2 \in \mathrm{SU}(2)$, and the center of $\mathrm{SU}(2)$ consists of $\{\pm 1\}$. Therefore the two possibilities are $\widetilde{P}^2 = \pm 1$. Either choice defines a G completing the diagram above.

20. If $\widetilde{P}^2 = 1$, then G is isomorphic to the direct product $\mathbb{Z}_2 \oplus \mathrm{SU}(2)$. We have the commutative diagram of fig.4, where both short exact sequences split on the right. (Note that $\{1, \widetilde{P}\}$ and $\{1, -\widetilde{P}\}$ are subgroups of order 2 in the double cover of Ø(3); we can use either to right-split the top exact sequence.)

21. $\widetilde{P}^2 = -1$. G is a semidirect product of $\mathrm{SU}(2)$ and $\mathbb{Z}_2$, $G \cong \mathrm{SU}(2) \rtimes \mathbb{Z}_2$. Now, $\widetilde{P}$ has order 4, so we can't use $\{1, \widetilde{P}\}$ to right-split the sequence. But $\widetilde{P}f$ has order 2 with f being any of the flips f_x, f_y, f_z of item14. (Note that $\widetilde{P}f_x$ maps to a mirror reflection along the x-axis, ditto y and z axes.) Any element of G can be written uniquely in the form $\widetilde{P}fg$ with $g \in \mathrm{SU}(2)$. $\mathrm{SU}(2)$ is normal in G (subgroup of index 2). That's all we need for G to be the internal semidirect product of $\mathrm{SU}(2)$ and $\{1, \widetilde{P}f\}$. ([tbd] Is conjugation by $\widetilde{P}f$ given by this formula, for $g \in \mathrm{SU}(2)$: $g \mapsto -g$?)

5 The Lorentz Group

In the previous section, we learned that there is a natural bijection between $i\mathrm{u}(2)$ and Minkowski space, with the determinant giving the metric. Also, the adjoint action of $\mathrm{U}(2)$ on $i\mathrm{u}(2)$ preserves t and the metric, because it preserves the trace and the determinant (respectively). So the map $g \mapsto \mathrm{Ad}_g$ determines a homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(1,3)$; we saw that this has image $\mathrm{SO}(3)$ and kernel $\{\pm 1\}$.

Now we want to enlarge the domain beyond $\mathrm{SU}(2)$ to get other elements of the Lorentz group, like boosts. We want the trace *not* to be preserved, since that's what preserves t, but det must be left invariant to preserve the Minkowski metric. Also, we still want $i\mathrm{u}(2)$ to be an invariant subspace.

The action $X \mapsto gXg^{-1}$, for g an arbitrary element of $\mathrm{GL}(2;\mathbb{C})$, preserves the trace and hence t; on the other hand, it doesn't preserve "being hermitian". If g belongs to $\mathrm{U}(2)$, then we can also write gXg^{-1} as gXg^*. This *will* preserve "being hermitian" even if g is any invertible matrix:

$$(gXg^*)^* = g^{**}X^*g^* = gXg^*.$$

If we demand that g belong to $\mathrm{SL}(2;\mathbb{C})$, then this also preserves the determinant:

$$\det(gXg^*) = \det(g)\det(X)\det(g)^* = \det(X)$$

Thus we have a double covering of the proper Lorentz group. Continuing our list:

22. The action $X \mapsto gXg^*$ defines a double covering $\mathrm{SL}(2;\mathbb{C}) \to \mathrm{SO}^+(1,3)$ of the proper Lorentz group.
23. Let T stand for time inversion; we still have the parity operator $P(t,x,y,z) = (t,-x,-y,-z)$. The full Lorentz group $\mathrm{O}(1,3)$ has four components

$$\mathrm{SO}^+(1,3) \sqcup P\mathrm{SO}^+(1,3) \sqcup T\mathrm{SO}^+(1,3) \sqcup PT\mathrm{SO}^+(1,3)$$

24. The elements $\{1, P, T, PT\}$ form a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

[tbd: double covers of the full Lorentz group; boosts; commutators.]

References

[1] J. J. Duistermaat and J. A. C. Kolk. *Lie Groups.* Springer, 2000.

[2] Brian C. Hall. *Lie Groups, Lie Algebras, and Representations.* Springer, 2nd edition, 2015.

[3] Alexander Kirillov Jr. *An Introduction to Lie Groups and Lie Algebras.* Cambridge University Press, 2008.

[4] John Stillwell. *Naive Lie Theory.* Springer, 2008.