Constructive mathematics without any choice

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Abstract

Richman has argued that constructive mathematics without countable choice is preferable [18]. In some settings, it is beneficial to go one step further and remove even unique choice. This allows e.g. interesting studies about the arithmetical hierarchy of classical logic as carried out by Akama et al. [1] and synthetic computability under the assumption of classical logic [6, 21].

Concretely, we work in a minimal variant of type theory with a syntactic universe of propositions, without (propositional or functional) extensionality, proof irrelevance (PI), or the axiom of unique choice (AUC). Rocq's type theory CIC is an example of such a system, and all our proofs are mechanised in Rocq. We show (1) the partition principle (PP), which follows from the axiom of choice (AC), and PI suffice to prove excluded middle for equalities (ELEM) and, consequently, PP and propositional extensionality suffice to prove full excluded middle (LEM), (2) for a variant of Coquand's result that LEM implies PI, ELEM suffices, (3) under the assumption of PI, a dual of AC stating "every injection has an inverting surjection" (iIF) is equivalent to the law of excluded middle and the axiom of unique choice (AUC) and furthermore equivalent to the proposition that for every type U which is a homotopy proposition, $U + \neg U$ holds (hLEM), (4) a form of the Cantor-Bernstein theorem (CB) is equivalent to LEM and AUC in the presence of functional extensionality, and (5) AUC makes the arithmetical hierarchy of classical logic collapse.

1. A stronger Diaconescu-Goodman-Myhill-Werner theorem AC is equivalent to the statement that every surjection has an inverting injection (sIF), in both (ZF) set theory and type theory. The statement that every surjection induces an injection (which does not have to be inverting) is called the partition principle (PP). Its status is open: it is not known whether it is strictly weaker than AC.

Over intuitionistic ZF, the Goodman-Myhill-Diaconescu theorem states that AC implies the law of excluded middle [4, 11]. It is well-known that Diaconescu's theorem can be replicated in type theory assuming propositional and functional extensionality. Benjamin Werner proved that proof irrelevance alone suffices to prove a variant of excluded middle for propositions of the form $x_1 = x_2$ at arbitrary types (ELEM) [20]. We show that already PP and PI imply ELEM, strengthening the result further.

$$\mathsf{PP} := \forall XY : \mathbb{T}.\ \forall s : X \to Y.\ \mathsf{surjective}\ s \to \exists i : Y \to X.\ \mathsf{injective}\ i$$

$$\mathsf{PI} := \forall P : \mathbb{P}.\ \forall p_1 p_2 : P.\ p_1 = p_2 \qquad \mathsf{PE} := \forall PQ : \mathbb{P}.\ (P \leftrightarrow Q) \to P = Q$$

$$\mathsf{ELEM} := \forall X : \mathbb{T}.\ \forall x_1 x_2 : X.\ x_1 = x_2 \vee x_1 \neq x_2 \qquad \mathsf{LEM} := \forall P : \mathbb{P}.\ P \vee \neg P$$

Theorem 1. $\mathsf{PP} \wedge \mathsf{PI} \longrightarrow \mathsf{ELEM}$

Note that from AC, one can deduce the stronger $\mathsf{inhabited}\ (\forall X : \mathbb{T}. \forall x_1 x_2 : X.\ x_1 = x_2 + x_1 \neq x_2)$, where $\mathsf{inhabited}\ T$ is a proposition for any type T. We do no see that PP implies the same principle.

Since PE implies both PI and turns ELEM into full LEM, this is a strenthening of Diaconescu's theorem. Jean Abou Samra discovered a direct proof that AC and PE imply full LEM [19].

Corollary 1. $\mathsf{PP} \wedge \mathsf{PE} \longrightarrow \mathsf{LEM}$

Proof. PE (already for $Q := \top$) implies PI because assuming a proof p_1 means $P = \top$, and all proofs of $\top$ are equal. Furthermore, PE and ELEM imply LEM by doing a case analysis on $P = \top$. ☐

Relatedly, Swan proved that (using PE), if every type can be imposed with an irreflexive relation with at most one minimum (a very weak form of the well-ordering principle), then LEM holds [14, Fact 10.32]. We are not aware that PP alone would imply this principle but know that LEM does.

2. A variant of Coquand's irrelevance theorem Coquand proved that LEM implies PI [3, p. 19] for Leibniz equality, exploiting Girard's paradox [10]. Using inductive equality rather than Leibniz equality, we refine his result to show that assuming ELEM suffices. The proof relies on Hedberg's theorem [13] and Lawvere's theorem [15]. Note that thus under PP, ELEM and PI are equivalent.

Theorem 2. $\mathsf{ELEM} \longrightarrow \mathsf{PI}$

Proof. Consider an inductive (or impredicatively encoded) proposition B with two proofs T and F. Note that for PI it is enough to show $T = F$. We define $U := \forall P.\ P \to B$ together with $s : U \to (U \to B)$ by $su := uU$. This function is a surjection as with ELEM we can define its inverse $i : (U \to B) \to U$ by

$$i\ H\ P := \begin{cases} H^e & \text{, if } e : U = P \text{ and } H^e \text{ is the transport of } H \text{ along } e \\ \lambda\_.F & \text{, otherwise} \end{cases}$$

and then show that $s(iH) = iHU = H^e = H$ where in the last step we use Hedberg's theorem to argue that $e : U = U$ is trivial and therefore transport along e is the identity. But now Lawvere's theorem applied to the surjection $s : U \to (U \to B)$ yields a fixed point for every function $B \to B$, so in particular there is one for the negation defined by $NT := F$ and $NF := T$, from which $T = F$ follows. $\square$

This proof combines observations by Barbanera and Berardi [2] as well as Harper and Mitchell [12].

3. Invertibility of injective functions Recall that sIF is equivalent to the axiom of choice, and PP is a consequence of AC unknown to be equivalent. We now study their duals coPP and iIF and show that coPP still implies LEM under PI, while iIF is equivalent to the conjunction of LEM and AUC.

$$\begin{aligned}
\mathsf{iIF} &:= \forall XY : \mathbb{T}. \forall x_0 : X. \forall i : X \to Y. \text{ injective } i \to \exists s : Y \to X. \forall x. s(ix) = x \\
\mathsf{coPP} &:= \forall XY : \mathbb{T}. \forall x_0 : X. \forall i : X \to Y. \text{ injective } i \to \exists s : Y \to X. \text{ surjective } s \\
\mathsf{AUC} &:= \forall XY : \mathbb{T}. \forall R : X \to Y \to \mathbb{P}. (\forall x. \exists! y. Rxy) \to \exists f : X \to Y. \forall x. Rx(fx)
\end{aligned}$$

When propositions are treated semantically as proof irrelevant types like in univalent mathematics, AUC holds for free. Note that with a syntactic universe of propositions, AUC does not follow from PI.

Lemma 1. $\mathsf{coPP} \land \mathsf{PI} \longrightarrow \mathsf{LEM}$

Proof. Define $i : P + 1 \to \mathbb{B}$ which sends proofs of P to true and the element of 1 to false. This is clearly an injection assuming PI. Now coPP yields a surjection $s : \mathbb{B} \to P + 1$. The only interesting case is $s\ \mathsf{true} = s\ \mathsf{false} = \mathsf{inr}\ \star$, which proves $\neg P$ by surjectivity of s. $\square$

The proof is due to Pradic and Brown [17], slightly refined to the explicit use of PI.

Theorem 3. $\mathsf{iIF} \land \mathsf{PI} \longleftrightarrow \mathsf{LEM} \land \mathsf{AUC}$

Proof. $\mathsf{iIF} \land \mathsf{PI}$ imply LEM because of the last lemma. LEM implies PI due to the last theorem. We need to prove that (1) $\mathsf{iIF} \land \mathsf{PI}$ imply AUC, and that (2) $\mathsf{LEM} \land \mathsf{AUC}$ implies iIF.

For (1), we may assume LEM as well. To prove AUC, assume $\forall x. \exists! y.\ Rxy$. First, we use LEM to do a case analysis on whether X has an element. If it does not have one, we are done. If it does, we may assume $x_0 : X$. We define $i : (\Sigma x : X. \Sigma y : Y.\ Rxy) \to X$ by projection. By iIF we obtain an inverting $s : X \to (\Sigma x : X. \Sigma y : Y.\ Rxy)$, which allows defining a choice function $f : X \to Y$ by projection on sx.

For (2), assume an injection $i : X \to Y$ and $x_0 : X$. We define a surjection by applying AUC to the relation which maps y to x if $ix = y$ and to x_0 otherwise. $\square$

This result can be extended to the variant of LEM treating propositions semantically:

$$\mathsf{hprop}\ (U : \mathbb{T}) := \forall xy : U.\ x = y \qquad \mathsf{hLEM} := \mathsf{inhabited}\ (\forall U : \mathbb{T}.\ \mathsf{hprop}\ U \to U + (U \to \bot))$$

Corollary 2. *Assuming* PI*,* $\mathsf{iIF} \longleftrightarrow \mathsf{hLEM}$*.*

Proof. With the last theorem, the only interesting part is how hLEM implies AUC. Given $\forall x. \exists! y.\ Rxy$, we need to construct $X \to Y$. We use hLEM for $\Sigma y.\ Rxy$, return the projection in one case and have a contradiction in the other case. $\square$

4. Cantor-Bernstein theorem The Cantor-Bernstein theorem (CB) states that if there are two injections between types, the types are in bijection. Pradic and Brown showed that CB implies excluded middle [17] while Escardó has shown that the usual proof using LEM translates to HoTT [5]. We extend their result to a system without AUC, for a variant of CB stating that the bijection can be *constructed*:

$$\begin{aligned}
\mathsf{CB} := \mathsf{inhabited}\ (&\forall XY : \mathbb{T}. \forall f : X \to Y. \forall g : Y \to X. \text{ injective } f \to \text{injective } g \to \\
&\Sigma f' g'. (\forall x. g'(f'x) = x) \land (\forall y.\ f'(g'y) = y))
\end{aligned}$$

Theorem 4. *Assuming functional extensionality,* $\mathsf{CB} \longleftrightarrow \mathsf{hLEM}$ *and thus* $\mathsf{CB} \land \mathsf{PI} \longleftrightarrow \mathsf{LEM} \land \mathsf{AUC}$*.*

Proof. The same proofs as in the above mentioned literature just go through. $\square$

For a constructive variant of CB see Forster, Jahn, and Smolka [7], for generalisations see Pradic [16].

5. The arithmetical hierarchy of classical logic This hierarchy has been introduced by Akama, Berardi, Hayashi, and Kohlenbach [1] and studied by Fujiwara [9] and Forster, Kirst, and Mück [8] in systems that do not prove the axiom of unique choice. There is an immediate technical reason for this:

Theorem 5. $\mathsf{AUC} \longrightarrow \forall n \geq 1.\ \Sigma_n\text{-}\mathsf{LEM} \longleftrightarrow \Sigma_1\text{-}\mathsf{LEM}$*.*

Proof. Under LPO and AUC, Σ_1 predicates are decidable by a function and thus expressible without quantifiers. Consequently, $\Sigma_1 = \Sigma_0 = \Pi_0$, making the whole hierarchy collapse. $\square$

Open Questions To date, we cannot find a proof that PP implies AUC, nor any principle the dual of CB implies, though it trivially follows from PP and CB. It would be interesting to understand whether if ELEM is strictly weaker than LEM, in particular whether $\mathsf{AC} \land \mathsf{PI} \land \mathsf{ELEM} \land \mathsf{CT} \land \neg\mathsf{LPO}$ is consistent.

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