On the logical structure of choice, bar induction, maximality and well-foundedness principles

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We report on the current state of an ongoing classification and formalisation effort to constructively relate various logical principles classically equivalent to choice axioms.

Generalised dependent choice and generalised bar induction. It was shown in [BH21] that weak Kőnig's lemma, the fan theorem, bar induction, dependent choice, the Boolean prime ideal theorem and general choice have respectively dual generalisations *Generalised Dependent Choice* (GDC_{ABT}) and *Generalised Bar Induction* (GBI_{ABT}), each dependent on a domain A, a codomain B and a predicate T. Both general schemes relate an observational (extensional) with an effective/(intensional/(co-)inductively defined) statement about the approximability of a function from A to B, with the predicate T constraining the set of valid finite approximations. For GDC_{ABT} the implication goes from coinductive to observational

$$\underbrace{T \text{ } \textit{coinductively A-B-approximable}}_{\textit{effective}} \Longrightarrow \underbrace{T \text{ } \textit{has an A-B-choice function}}_{\textit{observational}}$$

while for GBI_{ABT} it goes from observational to inductive

$$\underbrace{T \text{ } \textit{A-B-barred}}_{\textit{observational}} \Longrightarrow \underbrace{T \text{ } \textit{A-B-inductively barred}}_{\textit{effective}}$$

with

$$\begin{array}{lcl} T & : & (A \times B)^* \to \mathbb{P} \\ \uparrow T & \triangleq & \text{upwards closure of } T \text{ up to permutation and duplication} \\ \downarrow T & \triangleq & \text{downwards closure of } T \text{ up to permutation and duplication} \\ v \subset \alpha & \triangleq & \forall (a,b)\, ((a,b) \in v \Rightarrow \alpha(a) = b) \end{array}$$

$$\begin{array}{lcl} T \text{ coinductively } A\text{-}B\text{-approximable} & \triangleq & \langle\rangle \in \nu P.\lambda v.(v \in \downarrow T \ \wedge \ \forall a^A \notin dom(v)\, \exists b^B\, (v \star (a,b) \in P)) \\ T \text{ has an } A\text{-}B\text{-choice function} & \triangleq & \exists \alpha^{A \to B}\, \forall v^{(A\times B)^*}\, (v \subset \alpha \Rightarrow v \in T) \\ T \ A\text{-}B\text{-barred} & \triangleq & \forall \alpha^{A \to B}\, \exists v^{(A\times B)^*}\, (v \subset \alpha \wedge v \in T) \\ T \ A\text{-}B\text{-inductively barred} & \triangleq & \langle\rangle \in \mu P.\lambda v.(v \in \uparrow T \ \vee \ \exists a^A \notin dom(v)\, \forall b^B\, (v \star (a,b) \in P)) \end{array}$$

In this setting, GDC_{ABT} intuitionistically has the strength of

- the general axiom of choice when T derives pointwise from a relation R on $A \times B$ without introducing further constraints, that is when $T(u)$ is defined as $\forall (a,b) \in u\, R(a,b)$,
- the Boolean prime ideal theorem if B is the two-element set $\mathbb{B}$ (for a constructive definition of prime filter),
- the axiom of dependent choice if $A = \mathbb{N}$,
- weak Kőnig's lemma if $A = \mathbb{N}$ and $B = \mathbb{B}$ (up to weak classical reasoning).

GBI_{ABT} intuitionistically has the strength of

- Gödel's completeness theorem in the form validity implies provability for entailment relations if $B = \mathbb{B}$ (for a constructive definition of validity),
- bar induction if $A = \mathbb{N}$ and T is decidable
- the weak fan theorem if $A = \mathbb{N}$ and $B = \mathbb{B}$.

The content of [BH21] has been formalised in Rocq by the 2nd, 5th and 7th authors with corrections and refinements that have been integrated into an updated version.

Generalised Update Induction and Existence of a Maximal Partial Choice Function. It was shown in [HK24] that U. Berger's update induction [Ber04] (UI_{BT} for T the underlying formula of an open predicate $P(\alpha) \triangleq \exists n T(\alpha_{|n})$, where $\alpha_{|n}$ is the restriction of a sequence α to its first n elements) can be seen as a countable variant of the contrapositive of the Teichmüller-Tukey lemma (TTL_{AT} for T the underlying predicate of a family of sets of "finite character", that is, for a family of sets of the form $P(X) \triangleq \forall l\,(l \subset X \Rightarrow T(l))$, for l ranging over finite sequence of elements in A), a maximality principle close to Zorn's lemma classically known to be equivalent to the general axiom of choice. This correspondence can be expressed by introducing a new scheme *Existence of a Maximal Partial Choice Function* ($\exists\mathsf{MPCF}_{ABT}$) and its contrapositive, *Generalised Update Induction* (GUI_{ABT}), such that:

- UI_{BT} arises as the restriction $\mathsf{GUI}_{\mathbb{N}BT}$ of GUI_{ABT}
- TTL_{AT} arises as the restriction $\exists\mathsf{MPCF}_{A\mathbb{1}T}$ of $\exists\mathsf{MPCF}_{ABT}$ for $\mathbb{1}$ a singleton type
- $\exists\mathsf{MPCF}_{ABT}$ is classically equivalent to GDC_{ABT} when T derives pointwise from a relation
- $\exists\mathsf{MPCF}_{\mathbb{N}BT}$ is implied by $\mathsf{GDC}_{\mathbb{N}BT}$, with the converse direction still to be studied
- $\exists\mathsf{MPCF}_{A\mathbb{B}T}$ implies $\mathsf{GDC}_{A\mathbb{B}T}$ when A has decidable equality, with the converse direction still to be studied

This work has been partially formalised in Rocq by the 8th author, uncovering an omitted use of the excluded middle in one of the proofs.

Degrees of freedom in the formulation of GUI_{ABT} and $\exists\mathsf{MPCF}_{ABT}$. It is shown in a current work in progress [AH25] that GUI_{ABT} and $\exists\mathsf{MPCF}_{ABT}$ have many different ways to be constructively formulated, depending on how much classical logic and unique choice are involved in the formulation. For instance, partiality of a function from A to B can be expressed in a weak way as a (non-necessarily total) functional relation, but it can also be formulated in a stronger way, using classical logic and unique choice, as a total function from A to B extended with an extra element denoting undefinedness. The formulation of $\exists\mathsf{MPCF}$ has then 4 degrees of freedom taking 3 different values and 1 degree of freedom with 2 values, that is 162 variants in total, forming a kite of implications.

Generalised Bar Induction and Continuity induction. In a similar spirit, the 2nd and 5th author and their co-authors have analysed and formalised various continuity principles from the perspective of constructive reverse mathematics [BFM+25]. There are various ways of expressing that a function is continuous which constructively are not necessarily equivalent. In this context, they introduce the notion of *continuity induction* CI and study its relation to GBI.

Outline. The proposed talk will survey these schemes and connections, opening a way to compute with the general axiom of choice and its variants knowing that (generalised) update induction, as a well-foundedness principle, is realisable using a simple while loop [Ber04].

References

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