Toposes with enough points as categories of étale spaces

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Goal of this talk

Show a recent result on the interaction between

categorical logic and topology.

Specifically, our result expands upon the similarity between

ultraproducts and ultraconvergence,

first cast in categorical form by Makkai (1987).

Warm-up : classical propositional logic.

The free Boolean algebra on a countable set,

$$A,$$

admits an embedding into the power set algebra

$$\mathcal{P}(2^\omega),$$

which sends a logical formula φ to

$$\hat{\varphi} := \{ x \in 2^\omega \mid [\![\varphi]\!]_x = 1 \}.$$

The injectivity of $\hat{(\ )}$ is completeness of truth table semantics,

and the image of $\hat{(\ )}$ consists of the clopens of 2^ω.

In the first-order setting, topology on 2^ω is replaced by ultraproducts.

Background: Compact Hausdorff spaces as β-algebras.

For any set I, the set

$$\beta I,$$

of ultrafilters on I, carries a topology: for each $U \subseteq I$, take

$$\hat{U} := \{ x \in \beta I \mid U \in x \}$$

as a basic open.

Thm. (Manes) The functor $\beta : \underline{\mathrm{Set}} \to \underline{\mathrm{KHaus}}$ is left adjoint to $U : \underline{\mathrm{KHaus}} \to \underline{\mathrm{Set}}$

and $\underline{\mathrm{KHaus}} \simeq$ algebras for the monad β on $\underline{\mathrm{Set}}$.

$$\begin{tikzcd} & \beta I \arrow[dr, dashed, "\tilde{f}"] & \\ \underline{\mathrm{Set}} \ni I \arrow[ur, hook] \arrow[rr, "f"'] & & K \in \underline{\mathrm{KHaus}} \end{tikzcd}$$

$$\boxed{\lim_{i \to \mu} f_i := \tilde{f}(\mu).}$$

Background: Topological spaces as lax β-algebras.

For an (arbitrary) topological space, X, if $x \in X$, $(x_i) \in X^I$, $\mu \in \beta I$,

write $\quad x \rightsquigarrow \lim\limits_{i \to \mu} x_i$

if x is a limit point of the ultrafamily $((x_i), \mu)$.

By considering $\mu \in \beta X$ and $(x_i) = \mathrm{id}_X \in X^X$, we get a relation

$$\rightsquigarrow \lim \subseteq X \times \beta X,$$

which fully describes the space X:

Thm (Barr) The category $\underline{\mathrm{Top}}$ is equivalent to the category of lax algebras for the oplax monad β on $\underline{\mathrm{Rel}}$.

Representing a topos via its points.

models of $\mathcal{E}$ in Set
$\shortparallel$
geometric morphisms $\mathbf{Set} \to \mathcal{E}$

$\downarrow$ def

Let $\mathcal{E}$ be a (Grothendieck) topos, X a full subcategory of $\mathrm{pt}\,\mathcal{E}$.

For any object φ of $\mathcal{E}$, we have an evaluation map,

$$\hat{\varphi} : X \longrightarrow \underline{\mathbf{Set}}$$

$$x \longmapsto x^*(\varphi), \text{ also written } [\![\varphi]\!]_x,$$

which yields a functor

$$\hat{(\ )} : \mathcal{E} \longrightarrow \underline{\mathbf{Set}}^X.$$

If this functor reflects isos, we say that X is separating, and that $\mathcal{E}$ has enough points, or $\mathcal{E}$ is complete w.r.t. its Set-models.

Question. How to characterize the image of $\hat{(\ )}$?

Main result: reconstructing a topos from continuous maps to Set.

Thm. Let $\mathcal{E}$ be a Grothendieck topos, and let X be a separating class of points of $\mathcal{E}$. Then

$$\widehat{(\ )} : \mathcal{E} \longrightarrow \mathrm{Cont}(X, \underline{\mathrm{Set}})$$

is an equivalence.

definition of continuity follows on next slide!

- Proved more or less simultaneously by A. Hamad; G. Saadia; and us.
- Generalizes from coherent topoi to arbitrary Gr. topoi with enough points:

Thm. (Makkai 1987; Lurie 2017) The functor $\widehat{(\ )}$ is an equivalence for any coherent topos (taking $X := \mathrm{pt}\,\mathcal{E}$).

- Of course, not all topoi have enough points.
- A constructive version is work in progress. (Saadia & Tarantino)

Analogy with Stone duality

Boolean algebra A	frame F	topos $\mathcal{E}$
Boolean space of ultrafilters X	sober space of points X	ultraconvergence space of points X
clopen set in X $=$ continuous map to discrete 2	open set in X $=$ continuous map to Sierpinski $\mathbb{S}$	étale space over X $=$ continuous map to $\underline{\mathrm{Set}}$
the embedding $\hat{(\ )} : A \hookrightarrow \mathcal{P}(X)$	the morphism $\hat{(\ )} : F \to \mathcal{O}(X)$	the lex left adjoint functor $\hat{(\ )} : \mathcal{E} \to \underline{\mathrm{Set}}^X$
always injective	injective iff F spatial	conservative iff X separating

Continuity and ultrastructure

- A function $f: X \to Y$ between topological spaces is continuous iff
  for any $(x_i)_i \in X^I$, $\mu \in \beta I$, and $x \in X$,
  if $x \rightsquigarrow \lim_{i \to \mu} x_i$, then $f(x) \rightsquigarrow \lim_{i \to \mu} f(x_i)$.

- Key definition. For any $(x_i)_{i \in I} \in \underline{\mathrm{Set}}^I$, $\mu \in \beta I$, $x \in \underline{\mathrm{Set}}$,
  an ultra-arrow from x to $((x_i)_i, \mu)$
  is a function $x \longrightarrow \prod_i x_i / \mu$ $\xleftarrow{\text{def}}$ ultraproduct of sets

  "the ultraconvergence structure on Set"

- The ultra-arrows turn $\underline{\mathrm{Set}}$ into an ultraconvergence space.
  ↳ a.k.a. "virtual ultracategory".

- A continuous functor $X \longrightarrow \underline{\mathrm{Set}}$ is a mapping of objects and ultraarrows,
  subject to certain axioms.

  an arbitrary ultraconvergence space, precise def. in two slides!

Ultra-arrows between points

Let X a separating class of points of a topos $\mathcal{E}$.

- For any $(x_i) \in X^I$, $\mu \in \beta I$, the functor

$$\begin{aligned} x_\mu : \mathcal{E} &\longrightarrow \underline{\mathrm{Set}} \\ \varphi &\longmapsto \prod_i x_i^*(\varphi)/\mu \end{aligned}$$

is coherent, but ⚠ not necessarily of the form y^* for any $y \in \mathrm{pt}\,\mathcal{E}$: it may fail to preserve infinite colimits.

- We define an ultra-arrow from $x \in X$ to $(x_i)_i, \mu$ to be any natural transformation $x \Rightarrow x_\mu$. Notation: $x \rightsquigarrow \lim_{i \to \mu} x_i$.
- In Makkai & Lurie, $\mathcal{E}$ is assumed coherent, so that ⚠ does not occur.

Ultraconvergence spaces

the ultracompletion pseudomonad $\beta : \mathrm{Cat} \to \mathrm{Cat}$ (Garner, Rosolini, Shulman).

An ultraconvergence space is a tuple $(X, \mathrm{Hom}_{\rightsquigarrow}, \mathrm{id}, \cdot, [-])$

where
- $\mathrm{Hom}_{\rightsquigarrow} : X \times \beta X \to \underline{\mathrm{Set}}$ are ultra-arrows;
- $\mathrm{id}_x : x \rightsquigarrow \lim_{* \to 1} x$ an identity;
- $\cdot$: composition, $r : x \rightsquigarrow \lim_{i \to \mu} y_i$ and $s_i : y_i \rightsquigarrow \lim_{j \to \nu_i} z_{i,j}$

  gives $(s_i)_i \cdot r : x \rightsquigarrow \lim_{(i,j) \to \sum_\mu \nu_i} z_{i,j}$;
- $[-]$: reindexing, $r : x \rightsquigarrow \lim_{i \to \mu} y_i$, $h : \nu \to \mu$

  gives $r[h] : x \rightsquigarrow \lim_{j \to \nu} y_{hj}$;

subject to axioms.

Thm (Aristote & Tarantino, 2026) U.c. spaces $\simeq$ lax profunctorial β-algebras.

↳ See Q. Aristote's talk on Friday!

A Grothendieck construction for ultraconvergence

"specialization"

Let B an ultraconvergence space with underlying category $\mathrm{Sp}(B)$.

Def. A continuous map $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$ is **étale** if

$$\begin{array}{l} \forall\ e \overset{\exists!}{\rightsquigarrow} \lim\limits_{i\to\mu} e_i \\ \quad\ \ \downarrow p \\ \forall\ r:\ x \rightsquigarrow \lim\limits_{i\to\mu} x_i \end{array}$$

Proposition. The category of continuous $\underline{\mathrm{Set}}$-valued maps

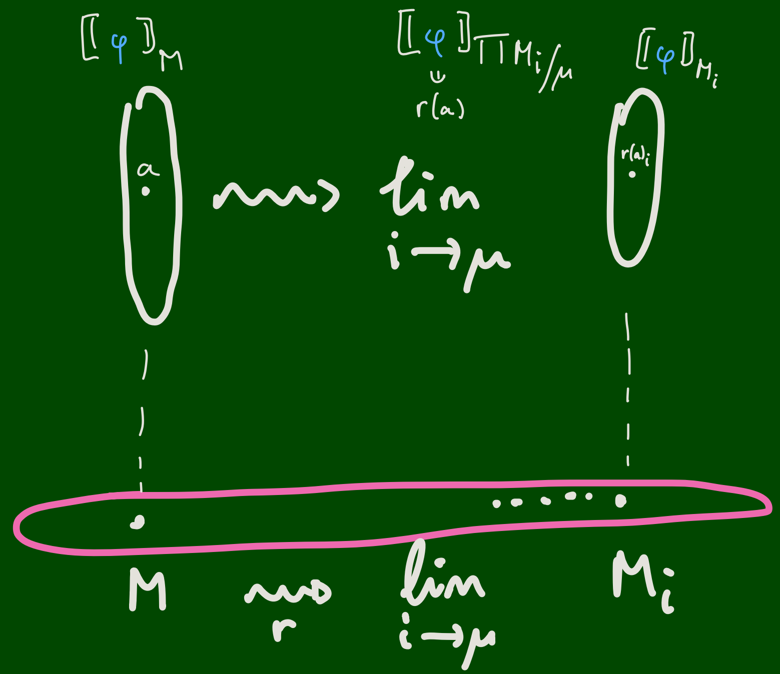
$$\mathrm{Cont}(B, \underline{\mathrm{Set}})$$

is equivalent to the category $\mathrm{Et}(B)$ of étale spaces over B:

$$\begin{array}{ccc} \mathrm{Et}(B) & \simeq & \mathrm{Cont}(B, \underline{\mathrm{Set}}) \\ \downarrow & & \downarrow \\ \mathrm{coFib}_{\mathrm{disc}}(\mathrm{Sp}\,B) & \simeq & \underline{\mathrm{Set}}^{\mathrm{Sp}(B)} \end{array}$$

The semantic étale space

When $\mathcal{E}$ is a topos and X is a separating set of $\underline{\mathrm{Set}}$-models of $\mathcal{E}$, we get, for any object φ of $\mathcal{E}$, an étale space $[\![\varphi]\!]$ over X:



$$[\![\varphi]\!] \xrightarrow{\text{étale}} X \subseteq \mathrm{Mod}_{\underline{\mathrm{Set}}}(\mathcal{E})$$

Our main theorem says that every étale space over X arises in this way.

"Conceptual completeness": all geometric concepts are definable.

Conclusion & further work

- Our proof of conceptual completeness is from first principles.

  Q. How does it relate to topological groupoid approach of Butz & Moerdijk, Joyal & Tierney, Awodey & Forssell?

- I drew an analogy with canonical extensions (see also yesterday's talk by J. Wrigley)

  Q. How to make this analogy precise?

- There is a close connection to Garner's ionads, which would be worth investigating in more depth.

- Ultraconvergence spaces which are not of the form pt $\mathcal{E}$ exist, e.g., complete metric spaces. Q. What is the associated 'topos'-like structure?

For more, see the question list & talks from our recent workshop on "Ultracategories and Stone duality".

Collaborations and more questions are welcome!

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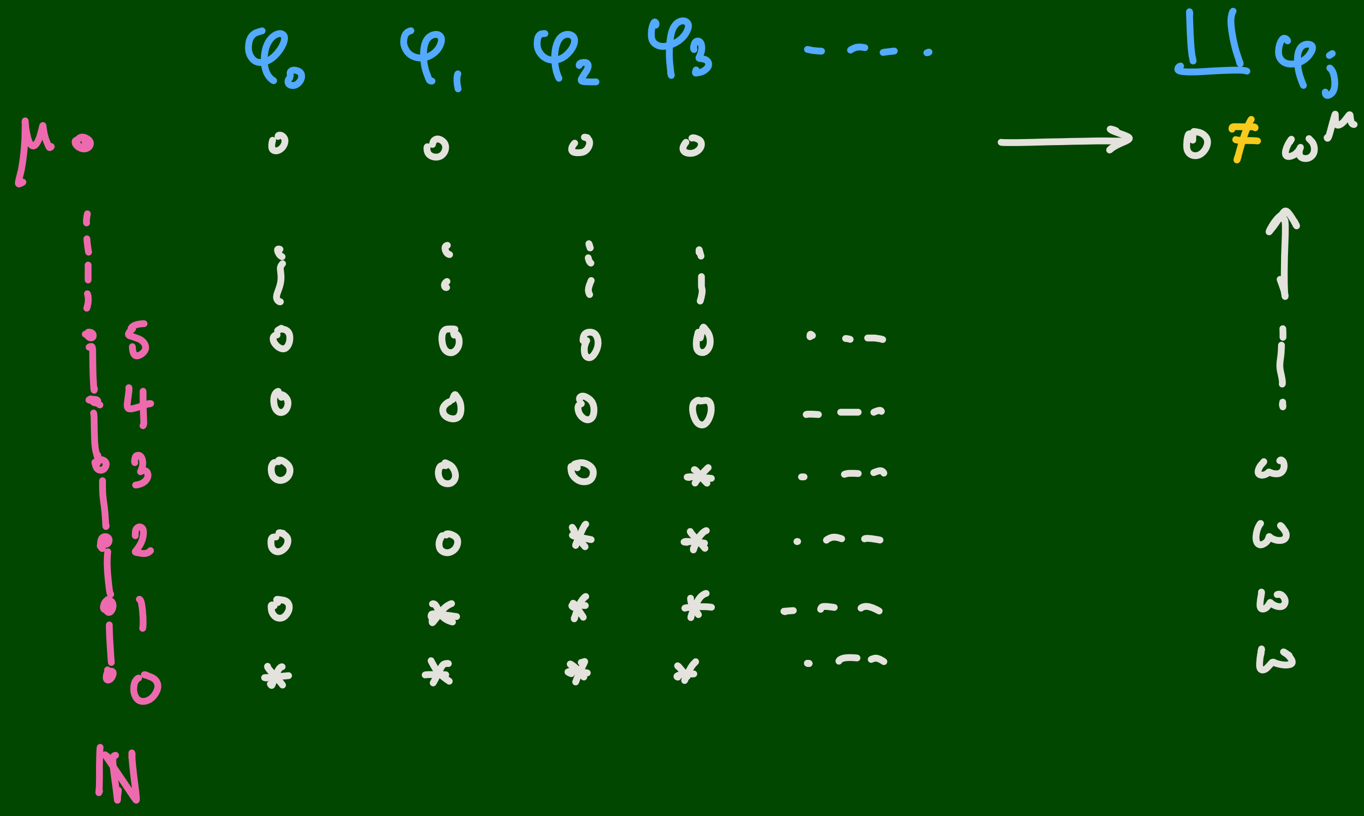
⚠ Example



$(\mathbb{N}, \leq)$ is not compact in the Alexandroff topology

A1

⚠

Example: $\varphi \longmapsto \prod_{i\in I} x_i^*(\varphi)/\mu$ may not preserve infinite coproducts.

- Consider the topos $\mathcal{E}$ of presheaves on $(\mathbb{N}, \leq)$.
- For each $i \in \mathbb{N}$, $x_i^* := ev_i : \mathcal{E} \longrightarrow \mathbf{Set}$ is a point of $\mathcal{E}$.
$$\varphi \longmapsto \varphi i$$
- Let μ be a non-principal ultrafilter on $\mathbb{N}$, $\begin{array}{l} x_\mu^* : \mathcal{E} \longrightarrow \mathbf{Set} \\ \varphi \longmapsto \prod_i x_i^* \varphi/\mu \end{array}$.
- Consider, for any $j \in \mathbb{N}$, $\varphi_j : \mathbb{N}^{op} \longrightarrow \mathbf{Set}$ with $\varphi_j(i) = \{*\}$ if $i \leq j$, $\emptyset$ else.
- For any $j \in \mathbb{N}$, $x_\mu^*(\varphi_j) = \prod_{i\in I} x_i^* \varphi_j/\mu$, and

$$x_i^* \varphi_j = \varphi_j i = \begin{cases} \{*\} & \text{if } i \leq j \\ \emptyset & \text{if } j < i \end{cases}.$$ Thus, $\{i \in I \mid x_i^* \varphi_j \neq \emptyset\}$ is finite, and hence not in μ, so, for any $j \in \mathbb{N}$, $x_\mu^*(\varphi_j) = \emptyset$, so $\coprod_j x_\mu^* \varphi_j = \emptyset$.

- However, $\varphi := \coprod_{j\in\mathbb{N}} \varphi_j$ has $\varphi(i) = \{(j,*) : i \leq j\}$ for any $i \in \mathbb{N}$.

Thus, $x_\mu^*(\varphi) = \prod_{i\in I} \varphi i/\mu \neq \emptyset$: it contains e.g. $[((i,*))_{i\in\mathbb{N}}]_\mu$.