

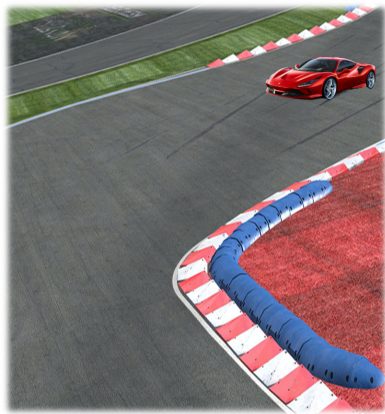
# CS159 Lecture 3: Model Predictive Control

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Caltech

Spring 2021

# Today's Class: Model Predictive Control



Compute a control policy mapping **continuous** states to **continuous** control actions

$$\pi : \mathbb{R}^n \rightarrow \mathbb{R}^d.$$

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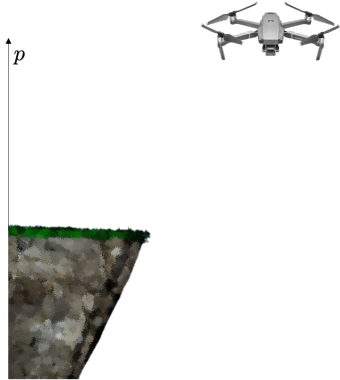
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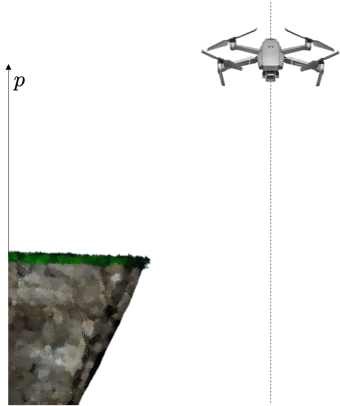
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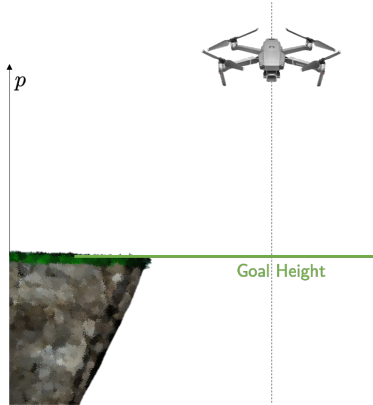
# Problem Formulation



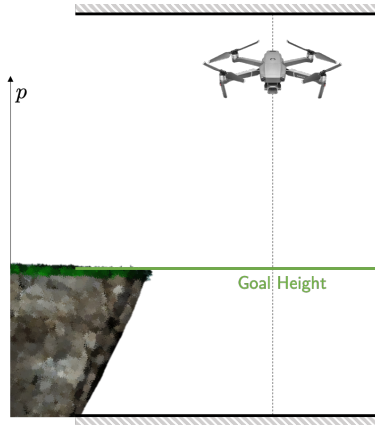
# Problem Formulation



# Problem Formulation



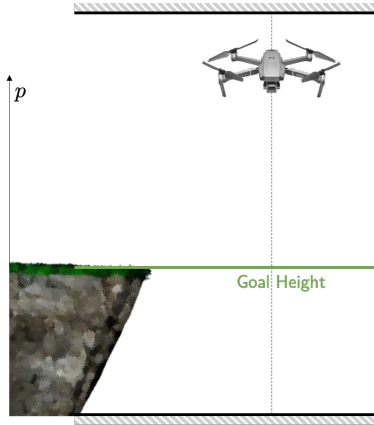
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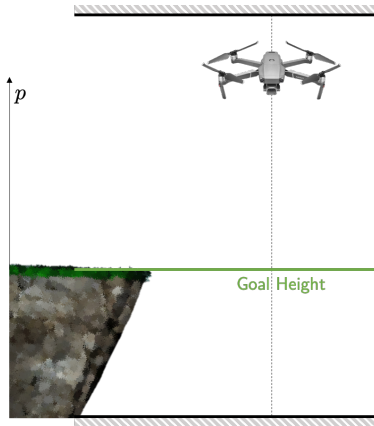
# Problem Formulation

## ► State

$$x = \begin{bmatrix} p \\ v \end{bmatrix} = \begin{bmatrix} \text{position} \\ \text{velocity} \end{bmatrix}$$



# Problem Formulation

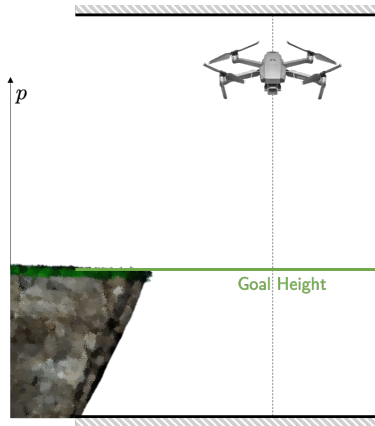


- State

$$x = \begin{bmatrix} p \\ v \end{bmatrix} = \begin{bmatrix} \text{position} \\ \text{velocity} \end{bmatrix}$$

- Input  $u = a = \text{acceleration}$

# Problem Formulation



## ► State

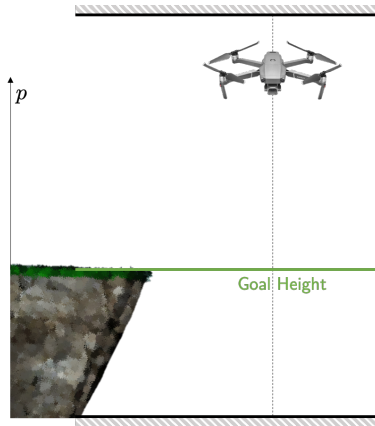
$$x = \begin{bmatrix} p \\ v \end{bmatrix} = \begin{bmatrix} \text{position} \\ \text{velocity} \end{bmatrix}$$

## ► Input $u = a =$ acceleration

## ► Dynamics

$$\begin{bmatrix} p_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & dt \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_k \\ v_k \end{bmatrix} + \begin{bmatrix} 0 \\ a_k \end{bmatrix}$$

# Problem Formulation



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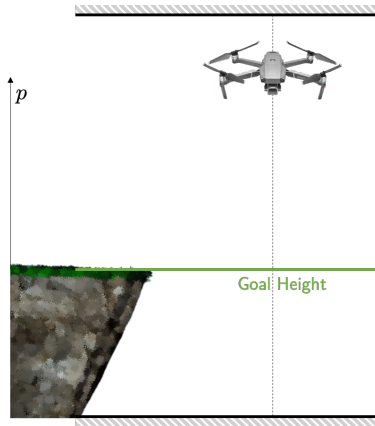
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## ► Cost $x_k^\top Q x_k + u_k^\top R u_k$

# Problem Formulation



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## ► Dynamics

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## ► Cost $x_k^\top Q x_k + u_k^\top R u_k$

## ► Constraints

$$\begin{bmatrix} -5 \\ -5 \\ -0.5 \end{bmatrix} \leq \begin{bmatrix} p_k \\ v_k \\ a_k \end{bmatrix} \leq \begin{bmatrix} 5 \\ 5 \\ 0.5 \end{bmatrix}$$

# Problem Formulation

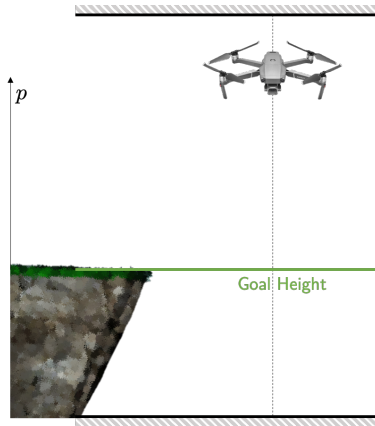
Our goal is to find a control policy which solves the following infinite time optimal control problem:

$$\begin{aligned} \pi^* = \arg \min_{\pi} \quad & \sum_{k=0}^{\infty} (x_k^{\top} Q x_k + u_k^{\top} R u_k) \\ \text{subject to} \quad & x_{k+1} = A x_k + B u_k, \quad \forall k \in \{0, 1, \dots\}, \\ & x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad \forall k \in \{0, 1, \dots\}, \\ & u_k = \pi(x_k), \quad \forall k \in \{0, 1, \dots\}, \\ & x_0 = x(0) \end{aligned} \tag{1}$$

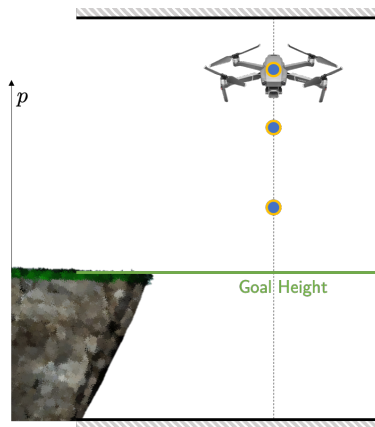
where

- ▶  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^d$  is a control policy.
- ▶  $\mathcal{X}$  and  $\mathcal{U}$  are constraint sets.

# Receding Horizon Control

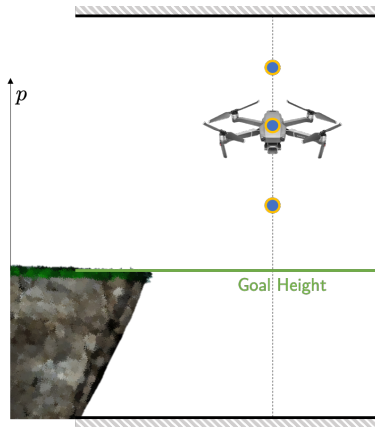


# Receding Horizon Control



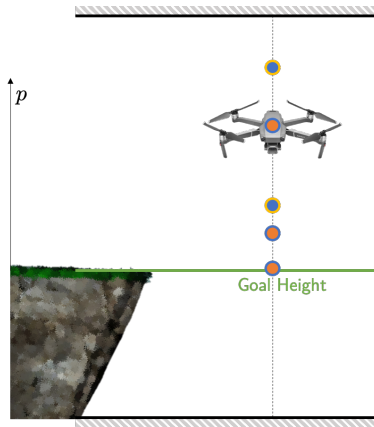
- Plan  $\{u_t^*, \dots, u_{t+N-1}^*\}$  using the batch approach.

# Receding Horizon Control



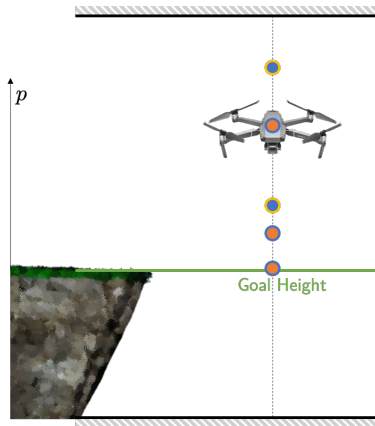
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# Receding Horizon Control



- Plan  $\{u_t^*, \dots, u_{t+N-1}^*\}$  using the batch approach.

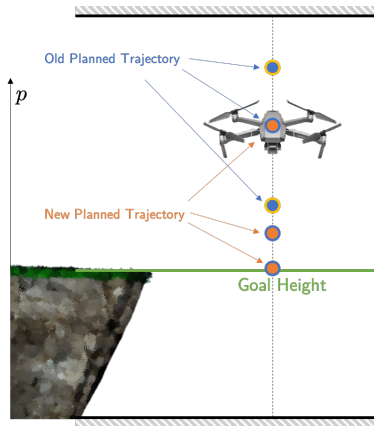
# Receding Horizon Control



- ▶ Plan  $\{u_t^*, \dots, u_{t+N-1}^*\}$  using the batch approach.
- ▶ The MPC policy

$$\pi^{\text{mpc}}(x(t)) = u_t^*.$$

# Receding Horizon Control



- ▶ Plan  $\{u_t^*, \dots, u_{t+N-1}^*\}$  using the batch approach.
- ▶ The MPC policy

$$\pi^{\text{mpc}}(x(t)) = u_t^*.$$

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Nonlinear Systems

# History of MPC\*

- ▶ **A. I. Propoi, 1963**, “Use of linear programming methods for synthesizing sampled-data automatic systems”, *Automation and Remote Control*.
- ▶ Mid 1990s: extensive theoretical effort devoted to provide conditions for guaranteeing feasibility and closed-loop stability
- ▶ 2000s: development of tractable robust MPC approaches; nonlinear and hybrid MPC;
- ▶ 2010s: stochastic MPC; distributed large-scale MPC; economic MPC
- ▶ **late 2010s**: safe integration with learning-based approaches; distributionally robust MPC; fast solvers for nonlinear systems

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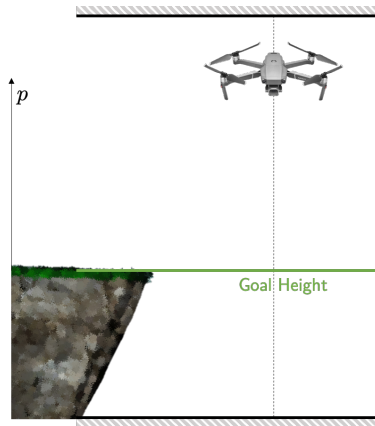
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# Numerical Example



## ► State

$$x = \begin{bmatrix} p \\ v \end{bmatrix} = \begin{bmatrix} \text{position} \\ \text{velocity} \end{bmatrix}$$

## ► Input $u = a =$ acceleration

## ► Dynamics

$$\begin{bmatrix} p_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & dt \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_k \\ v_k \end{bmatrix} + \begin{bmatrix} 0 \\ a_k \end{bmatrix}$$

## ► Cost $x_k^\top Q x_k + u_k^\top R u_k$

## ► Constraints

$$\begin{bmatrix} -5 \\ -5 \\ -0.5 \end{bmatrix} \leq \begin{bmatrix} p_k \\ v_k \\ a_k \end{bmatrix} \leq \begin{bmatrix} 5 \\ 5 \\ 0.5 \end{bmatrix}$$

# Numerical Example – MPC vs LQR

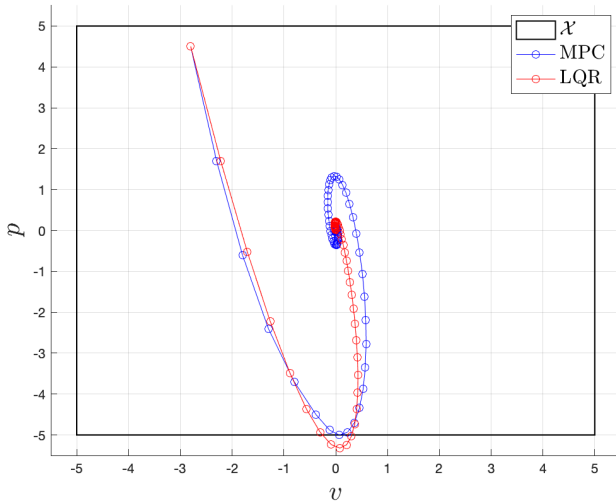
## Model Predictive Control

$$\begin{aligned} \min_{u_t, \dots, u_{t+N-1}} \quad & \sum_{k=0}^{N-1} (x_k^\top Q x_k + u_k^\top R u_k) + x_{t+N}^\top Q_F x_{t+N} \\ \text{subject to} \quad & x_{k+1} = A x_k + B u_k \\ & x_k \in \mathcal{X}, u_k \in \mathcal{U} \\ & x_t = x(0), x_N \in \mathcal{X}_F \end{aligned}$$

## Linear Quadratic Regulator

$$\begin{aligned} \min_{u_t, u_{t+1}, \dots} \quad & \sum_{k=0}^{\infty} (x_k^\top Q x_k + u_k^\top R u_k) \\ \text{subject to} \quad & x_{k+1} = A x_k + B u_k \\ & x_t = x(0) \end{aligned}$$

# Numerical Example – MPC vs LQR



Tuning  $Q = I$ ,  $Q_F = 10I$  and  $R = 10^4$

# MPC Features

## Pros

- ▶ Can handle
  - ▶ linear models
  - ▶ nonlinear models
  - ▶ any cost
  - ▶ constraints
  - ▶ delays
  - ▶ disturbances

## Cons

- ▶ Computationally demanding (Need to solve an FTOCP)

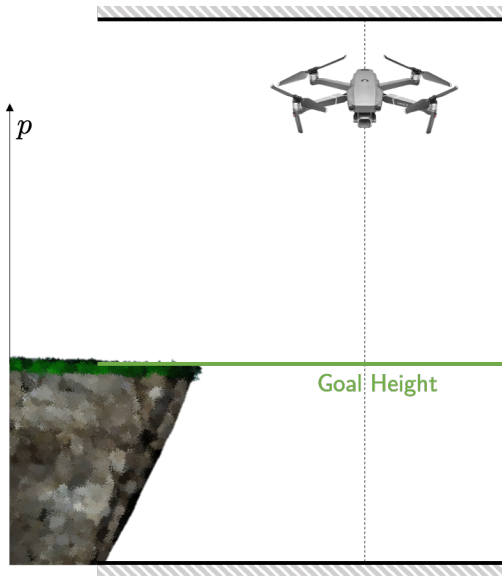
# Drone Regulation Problem

## Model Predictive Control

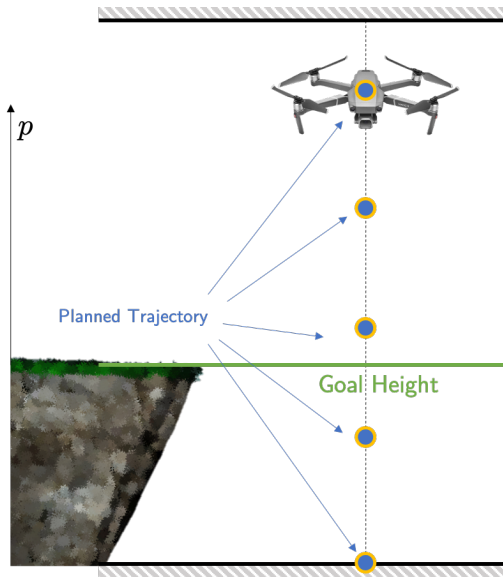
$$\begin{aligned} \min_{u_t, \dots, u_{t+N-1}} \quad & \sum_{k=0}^{N-1} (x_k^\top Q x_k + u_k^\top R u_k) + x_{t+N}^\top Q_F x_{t+N} \\ \text{subject to} \quad & x_{k+1} = A x_k + B u_k \\ & x_k \in \mathcal{X}, u_k \in \mathcal{U} \\ & x_t = x(0), x_N \in \mathcal{X}_F \end{aligned}$$

- Longer horizon  $\rightarrow$  more variables  $\rightarrow$  the computational cost increases

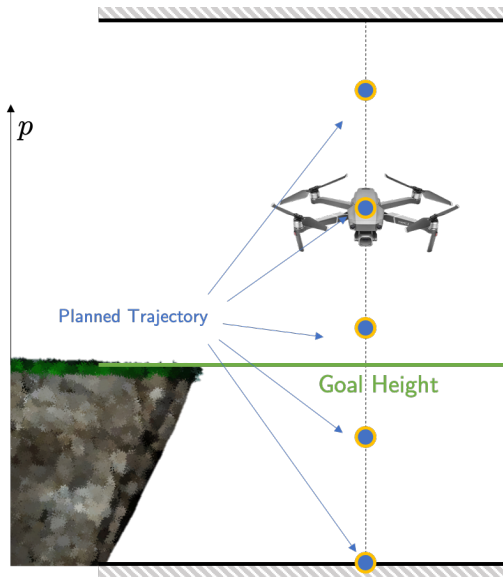
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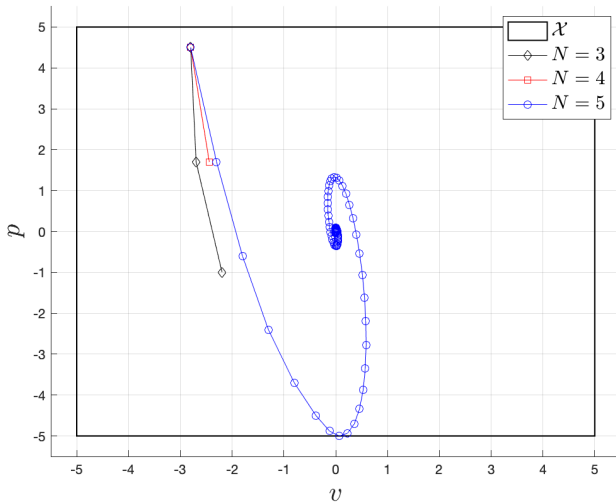
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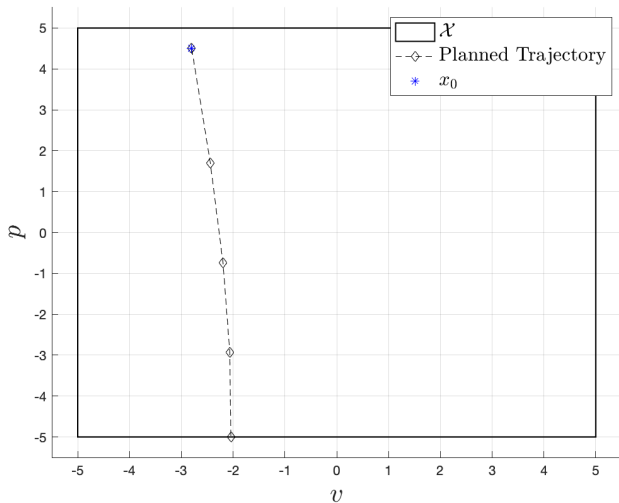


# Drone Regulation Problem



The MPC problem is not feasible at time step  $t = 3$  when  $N = 3$ .  
The MPC problem is not feasible at time step  $t = 1$  when  $N = 4$ .

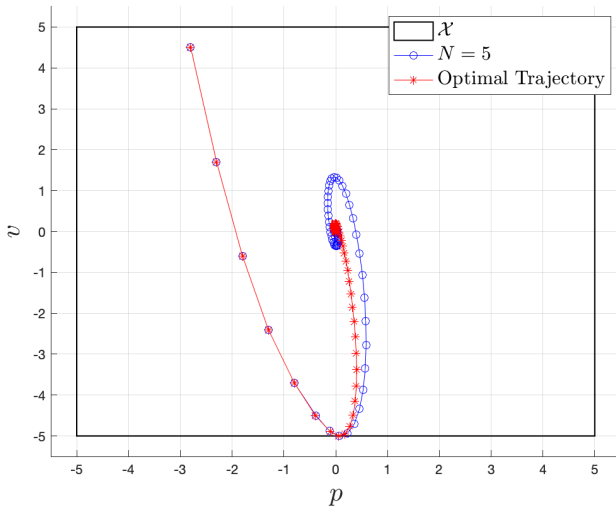
# Drone Regulation Problem



Tuning  $Q = I$ ,  $Q_F = 10I$  and  $R = 10^4$ .

Planned trajectory at time  $t = 0$  for  $N = 4$ .

# Drone Regulation Problem



The closed-loop trajectory is not guaranteed to be optimal.

# Summary

The MPC may take “short-sighted” control actions that lead to the following issues:

- ▶ **Feasibility**. After some steps the MPC problem is not feasible (Infeasibility occurs without disturbances and model mismatch!)
- ▶ **Stability**. The closed-loop system may not reach the desired goal state!
- ▶ **Optimality**. The controller may not be optimal!

Why do we care about feasibility in model-based RL?

# Why do we care about feasibility in model-based RL?

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## Deep Reinforcement Learning in a Handful of Trials using Probabilistic Dynamics Models

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Kurtland Chua

Roberto Calandra

Rowan McAllister

Sergey Levine

Berkeley Artificial Intelligence Research

University of California, Berkeley

{kchua, roberto.calandra, rmcallister, svlevine}@berkeley.edu

### Abstract

Model-based reinforcement learning (RL) algorithms can attain excellent sample efficiency, but often lag behind the best model-free algorithms in terms of asymptotic performance. This is especially true with high-capacity parametric function approximators, such as deep networks. In this paper, we study how to bridge this gap, by employing uncertainty-aware dynamics models. We propose a new algorithm called probabilistic ensembles with trajectory sampling (PETS) that combines uncertainty-aware deep network dynamics models with sampling-based uncertainty propagation. Our comparison to state-of-the-art model-based and model-free deep RL algorithms shows that our approach matches the asymptotic performance of model-free algorithms on several challenging benchmark tasks, while requiring significantly fewer samples (e.g., 8 and 125 times fewer samples than Soft Actor Critic and Proximal Policy Optimization respectively on the half-cheetah task).

# Why do we care about feasibility in model-based RL?

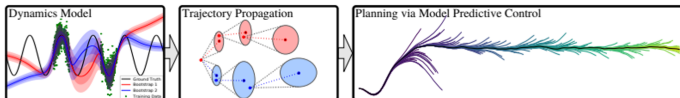


Figure 1: Our method (PE-TS): **Model:** Our probabilistic ensemble (PE) dynamics model is shown as an ensemble of two bootstraps (bootstrap disagreement far from data captures epistemic uncertainty: our subjective uncertainty due to a lack of data), each a probabilistic neural network that captures aleatoric uncertainty (inherent variance of the observed data). **Propagation:** Our trajectory sampling (TS) propagation technique uses our dynamics model to re-sample each particle (with associated bootstrap) according to its probabilistic prediction at each point in time, up until horizon  $T$ . **Planning:** At each time step, our MPC algorithm computes an optimal action sequence, applies the first action in the sequence, and repeats until the task-horizon.

**Key Idea:** Learn a dynamical model using an ensemble of DNN and solve a FTOCP via the batch approach.

# Why do we care about feasibility in model-based RL?

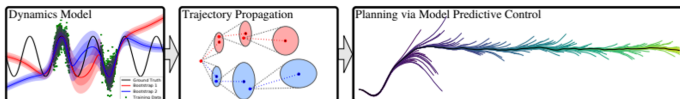


Figure 1: Our method (PE-TS): **Model:** Our probabilistic ensemble (PE) dynamics model is shown as an ensemble of two bootstraps (bootstrap disagreement far from data captures epistemic uncertainty: our subjective uncertainty due to a lack of data), each a probabilistic neural network that captures aleatoric uncertainty (inherent variance of the observed data). **Propagation:** Our trajectory sampling (TS) propagation technique uses our dynamics model to re-sample each particle (with associated bootstrap) according to its probabilistic prediction at each point in time, up until horizon  $T$ . **Planning:** At each time step, our MPC algorithm computes an optimal action sequence, applies the first action in the sequence, and repeats until the task-horizon.

**Key Idea:** Learn a dynamical model using an ensemble of DNN and solve a FTOCP via the batch approach.

**Key Message:** Recursive feasibility issues will be a problem also in this more complicated set-up.

# Why do we care about feasibility in model-based RL?

## Safety Augmented Value Estimation From Demonstrations (SAVED): Safe Deep Model-Based RL for Sparse Cost Robotic Tasks

Brijen Thananjeyan , Ashwin Balakrishna, Ugo Rosolia, Felix Li, Rowan McAllister, Joseph E. Gonzalez, Sergey Levine, Francesco Borrelli , and Ken Goldberg 

**Abstract**—Reinforcement learning (RL) for robotics is challenging due to the difficulty in hand-engineering a dense cost function, which can lead to unintended behavior, and dynamical uncertainty, which makes exploration and constraint satisfaction challenging. We address these issues with a new model-based reinforcement learning algorithm, Safety Augmented Value Estimation from Demonstrations (SAVED), which uses supervision that only identifies task completion and a modest set of suboptimal demonstrations to constrain exploration and learn efficiently while handling complex constraints. We then compare SAVED with 3 state-of-the-art model-based and model-free RL algorithms on 6 standard simulation benchmarks involving navigation and manipulation and a physical knot-tying task on the da Vinci surgical robot. Results suggest that SAVED outperforms prior methods in terms of success rate, constraint satisfaction, and sample efficiency, making it feasible to safely learn a control policy directly on a real robot in less than an hour. For tasks on the robot, baselines succeed less than 5 % of the time while SAVED has a success rate of over 75 % in the first 50 training iterations. Code and supplementary material is available at <https://tinyurl.com/saved-rl>.

**Index Terms**—Reinforcement learning, imitation learning, optimal control.

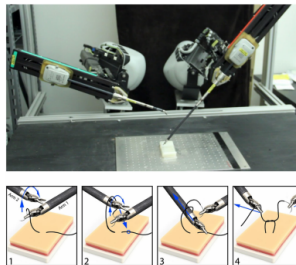
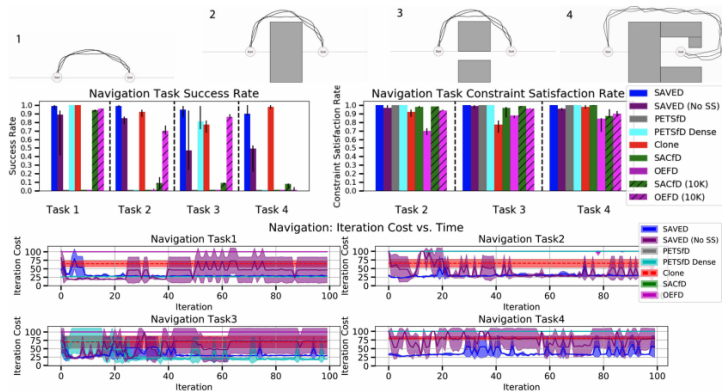


Fig. 1. SAVED is able to safely learn maneuvers on the da Vinci surgical

B. Thananjeyan\*, A. Balakrishna\*, U. Rosolia, F. Li, R. McAllister, J. E. Gonzalez, S. Levine, F. Borrelli, and K. Goldberg. "Safety augmented value estimation from demonstrations (saved): Safe deep model-based rl for sparse cost robotic tasks." *IEEE Robotics and Automation Letters* 5 (2020).

# Why do we care about feasibility in model-based RL?



**Key Message:** Receding horizon planning may lead to safety constraints violation!

B. Thananjeyan\*, A. Balakrishna\*, U. Rosolia, F. Li, R. McAllister, J. E. Gonzalez, S. Levine, F. Borrelli, and K. Goldberg. "Safety augmented value estimation from demonstrations (saved): Safe deep model-based rl for sparse cost robotic tasks." IEEE Robotics and Automation Letters 5 (2020).

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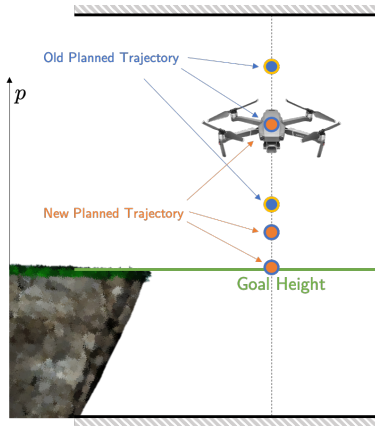
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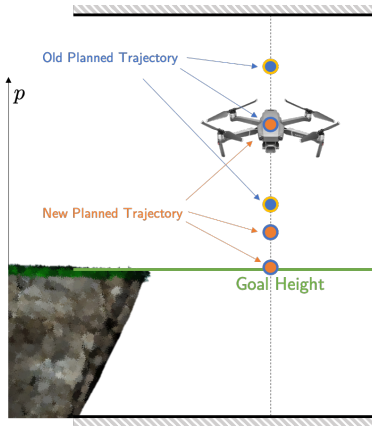
Nonlinear Systems

# Notation

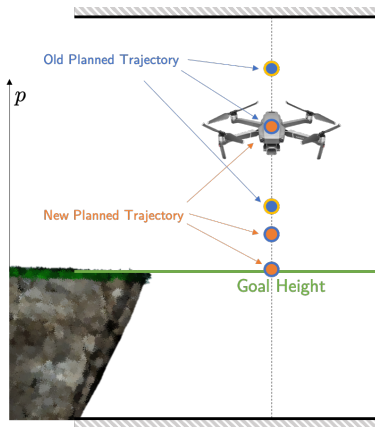


# Notation

- Need to distinguish between the optimal plan at time  $t$  and the optimal plan at time  $t + 1$ .

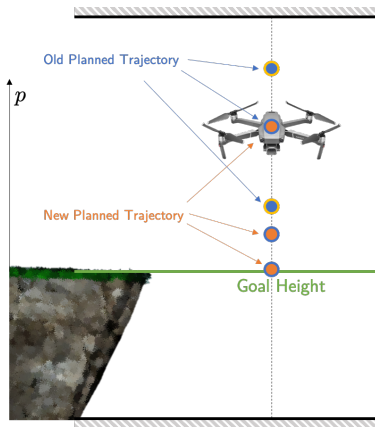


# Notation



- ▶ Need to distinguish between the optimal plan at time  $t$  and the optimal plan at time  $t + 1$ .
- ▶  $u_{k|t}^*$  is the input that we will apply at time  $k$ , if we follow the plan computed at time  $t$ .

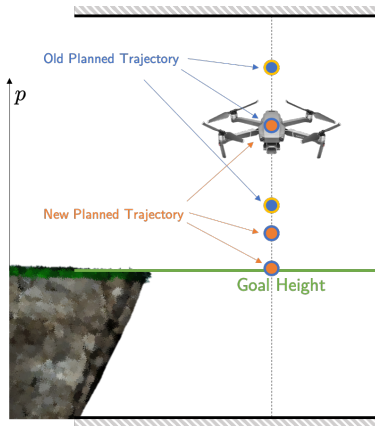
# Notation



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- ▶  $u_{k|t}^*$  is the input that we will apply at time  $k$ , if we follow the plan computed at time  $t$ .
- ▶ At time  $t$ , we plan

$$\{u_{t|t}^*, u_{t+1|t}^*, \dots, u_{t+N-1|t}^*\}$$

# Notation



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- ▶ At time  $t + 1$ , we plan

$$\{u_{t+1|t+1}^*, u_{t+2|t+1}^*, \dots, u_{t+N|t+1}^*\}$$

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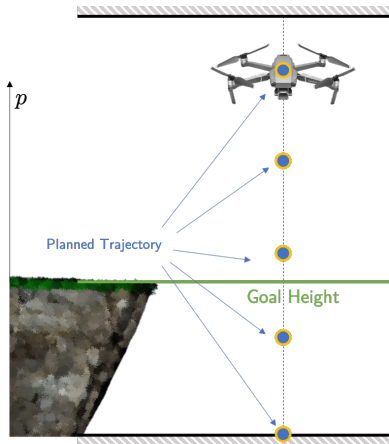
Nonlinear Systems

# Constraint Satisfaction and Stability

$$\begin{aligned} J_t^*(x(0)) = & \min_{u_{t|t}, \dots, u_{t+N-1|t}} \sum_{k=0}^{N-1} h(x_{k|t}, u_{k|t}) + V(x_{t+N|t}) \\ & \text{subject to } x_{k+1|t} = Ax_{k|t} + Bu_{k|t}, \forall k \in \{t, \dots, t+N-1\} \\ & x_{k|t} \in \mathcal{X}, u_{k|t} \in \mathcal{U}, \forall k \in \{t, \dots, t+N-1\} \\ & x_{t|t} = x(0), x_N \in \mathcal{X}_F \end{aligned}$$

**Solution:** The terminal cost  $V(x_{t+N|t})$  and terminal constraint  $\mathcal{X}_F$ , often referred to as terminal components, should approximate the tail of cost and constraints beyond the prediction horizon.

# Drone Regulation Problem



# Constraint Satisfaction and Stability – Solution

## Recursive feasibility

If at time  $t$  the MPC problem is feasible  $\rightarrow$  the MPC problem is feasible at time  $t + 1$ .

# Constraint Satisfaction and Stability – Solution

## Recursive feasibility

If at time  $t$  the MPC problem is feasible  $\rightarrow$  the MPC problem is feasible at time  $t + 1$ .

## Stability

We to guarantee that the closed-loop is stable. Informally, we want to guarantee that

$$\lim_{t \rightarrow \infty} ||x_t|| = 0.$$

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## Recursive Feasibility

Consider the terminal set  $\mathcal{X}_F = \mathbf{x}_g = \{0\}$ , which is an unforced equilibrium point for the system (i.e.,  $\mathbf{x}_g = A\mathbf{x}_g$ ).



## Recursive Feasibility

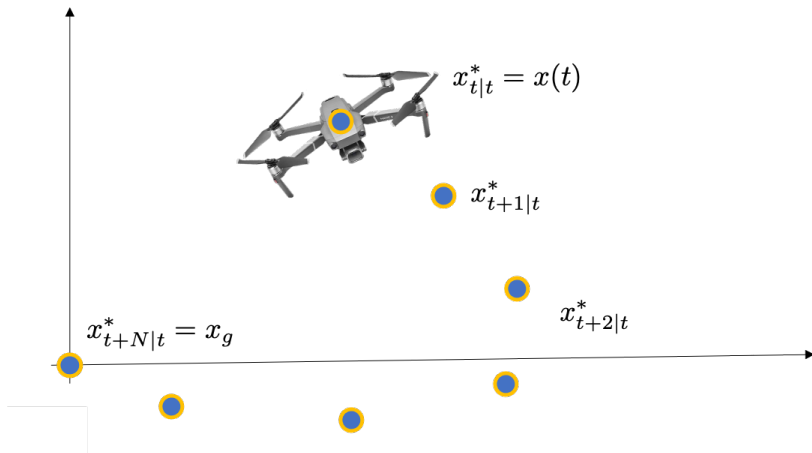
Consider the terminal set  $\mathcal{X}_F = \mathbf{x}_g = \{0\}$ , which is an unforced equilibrium point for the system (i.e.,  $\mathbf{x}_g = A\mathbf{x}_g$ ).



Assume that at time  $t = 0$  the MPC problem is feasible.

## Recursive Feasibility

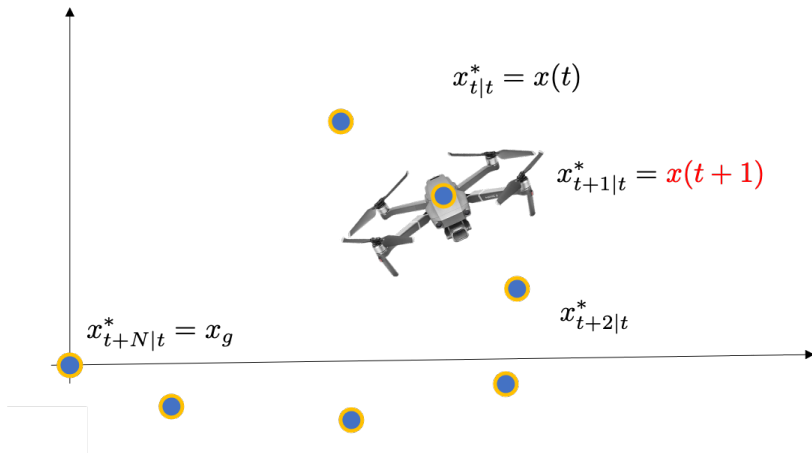
Consider the terminal set  $\mathcal{X}_F = x_g = \{0\}$ , which is an unforced equilibrium point for the system (i.e.,  $x_g = Ax_g$ ).



Let  $\{x_{t|t}^*, \dots, x_{t+N|t}^*\}$  and  $\{u_{t|t}^*, \dots, u_{t+N-1|t}^*\}$  be the optimal state-input sequences.

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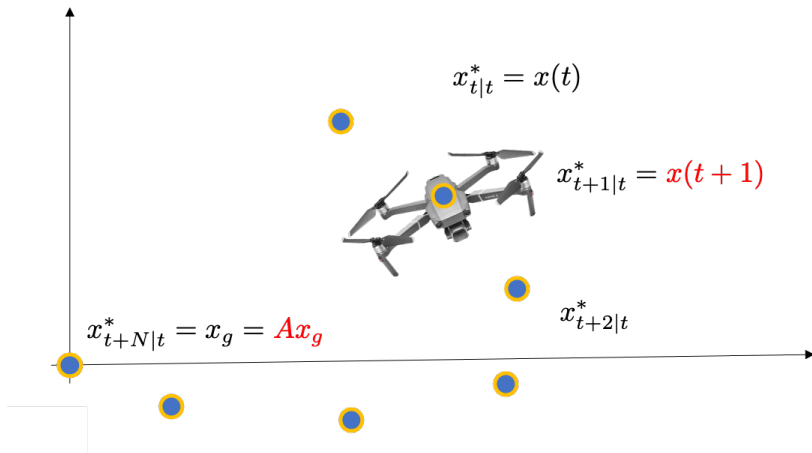
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Apply  $u_{t|t}^*$  to the system.

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Then  $\{x_{t+1|t}^*, \dots, x_{t+N|t}^*, x_g\}$  and  $\{u_{t+1|t}^*, \dots, u_{t+N-1|t}^*, 0\}$  are feasible state-input sequences for the FTOCP at  $t+1$ .

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**Stability**

Nonlinear Systems

## Stability

Consider the terminal cost  $V(x) = x^\top Q_F x$ .

Recall that  $h(x, u) = x^\top Q x + u^\top R u$ .



# Stability

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**Important:** A formal proof of stability is based on Lyapunov theory. In what follows, we only show that  $\lim_{t \rightarrow \infty} J_t^*(x_t) = 0$ .

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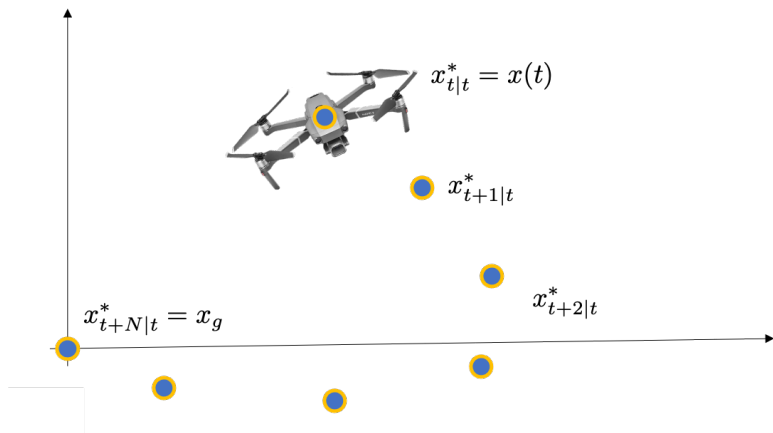


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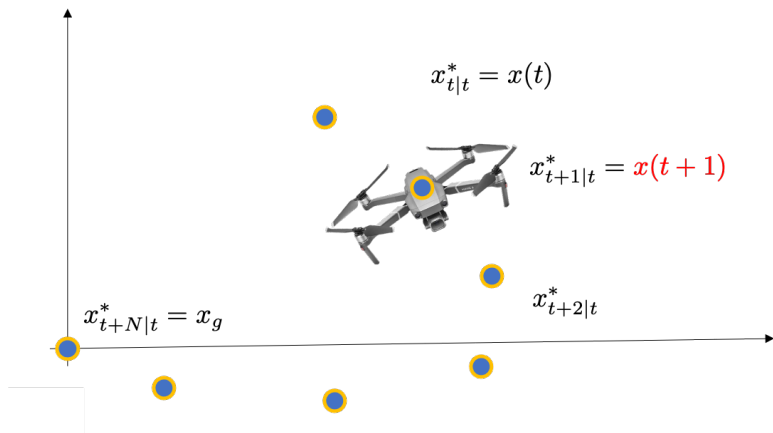


Let  $J_t^*(x(t)) = \sum_{k=t}^{t+N-1} h(x_{k|t}^*, u_{k|t}^*) + V(x_{t+N|t}^*)$  be the optimal cost.

## Stability

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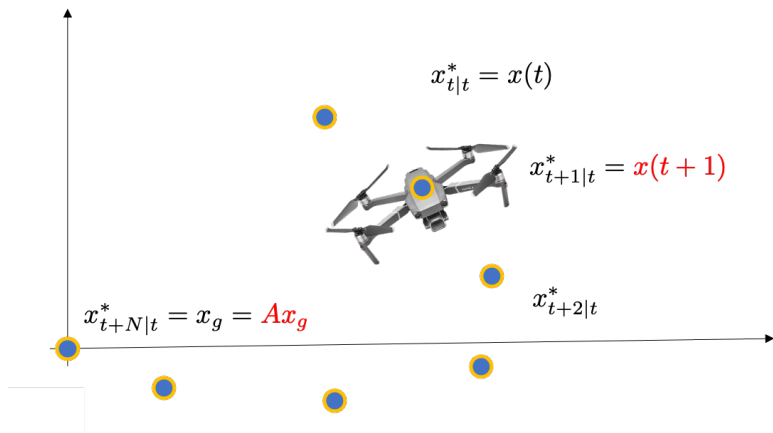


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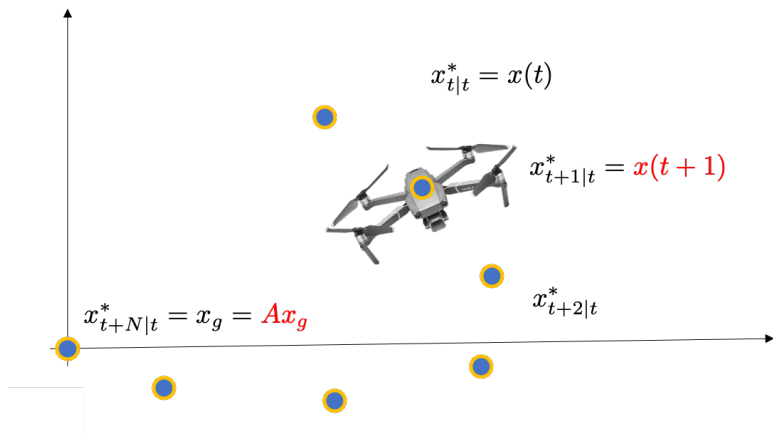


Then  $\{x_{t+1|t}^*, \dots, x_{t+N|t}^*, x_g\}$  and  $\{u_{t+1|t}^*, \dots, u_{t+N-1|t}^*, 0\}$  are feasible state-input sequences for the FTOCP at  $t+1$ .

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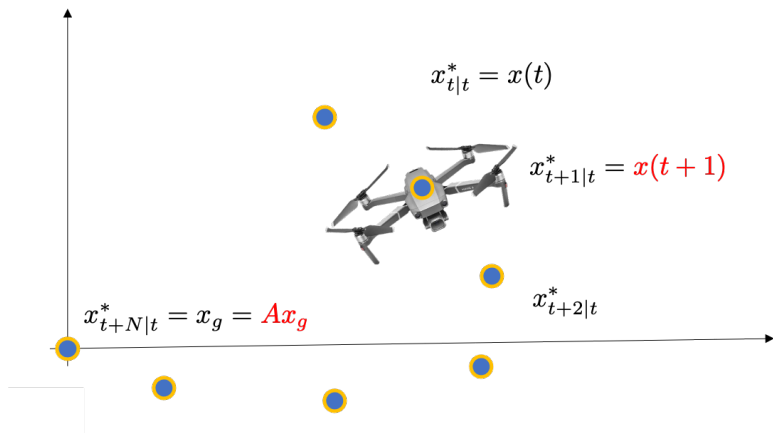


The cost is  $J_t^*(x(t)) - h(x_{t|t}^*, u_{t|t}^*)$ .

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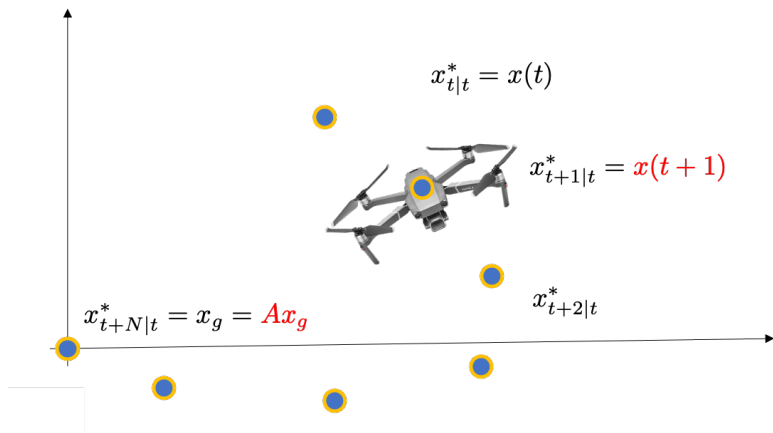


Then  $J_{t+1}^*(x(t+1)) \leq J_t^*(x(t)) - h(x_{t|t}^*, u_{t|t}^*)$ .

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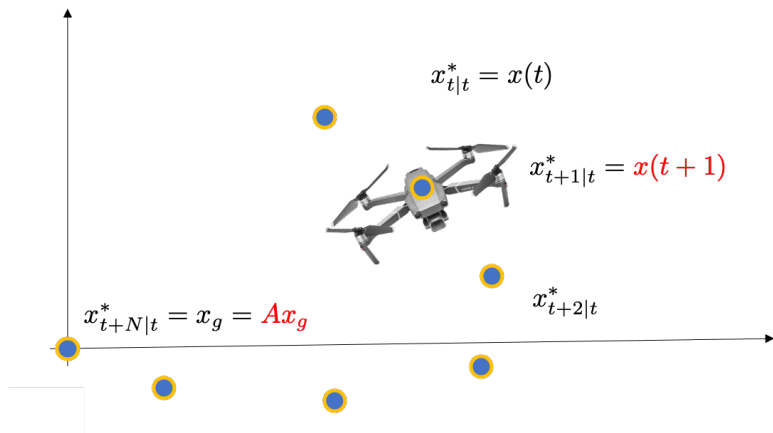


Thus  $J_{t+1}^*(x(t+1)) < J_t^*(x(t))$  for all  $x(t) \neq x_g$ .

## Stability

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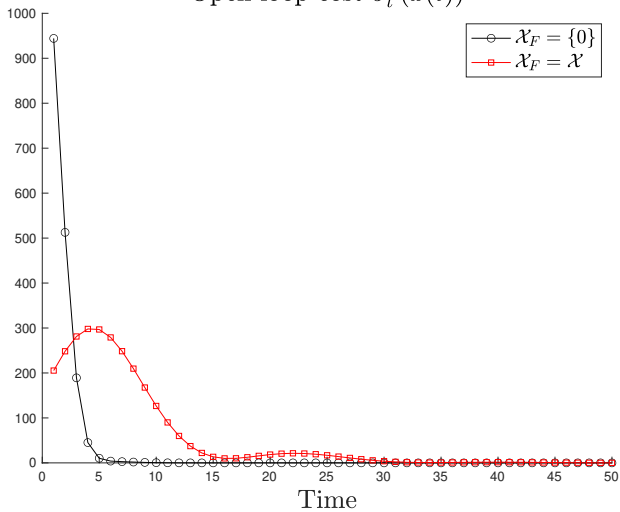
Recall that  $h(x, u) = x^\top Q x + u^\top R u$ .



Therefore,  $\lim_{t \rightarrow \infty} J_t^*(x(t)) = 0$  (Assuming  $\mathcal{X}$  and  $\mathcal{U}$  contain the origin and are compact or  $\|x\| \rightarrow \infty \implies J_t^*(x) \rightarrow \infty$ ).

# Stability

Open-loop cost  $J_t^*(x(t))$



# Summary of Safety and Stability Properties

**Key Message:** When the MPC terminal components are not designed correctly, the closed-loop system may violate safety constraints and convergence to the goal state/set is not guaranteed

**A solution:** We have shown that for the terminal set  $\mathcal{X}_F = \{0\}$  and the terminal cost  $V(x) = 0$ .

- ▶ The MPC problem is feasible at all times
- ▶ The closed-loop system is stable as for the positive definite open-loop cost we have  $J_{t+1}^*(x(t+1)) < J_t^*(x(t))$ ,  $\forall x(t) \notin \mathcal{X}_F$

**Main drawback:** The terminal constraint set affects the region of attraction of the controller.

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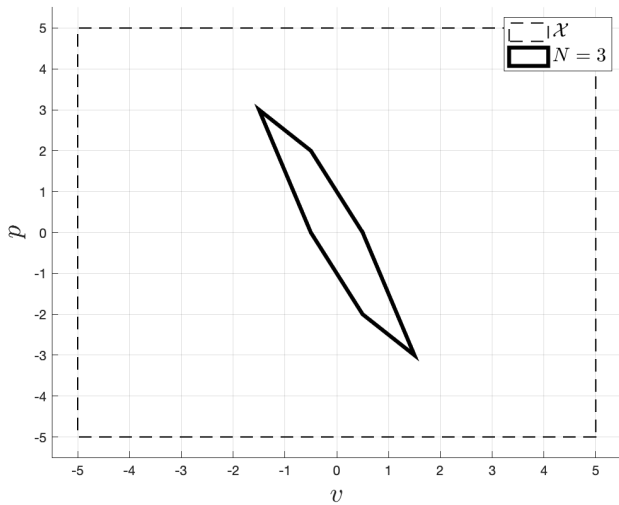
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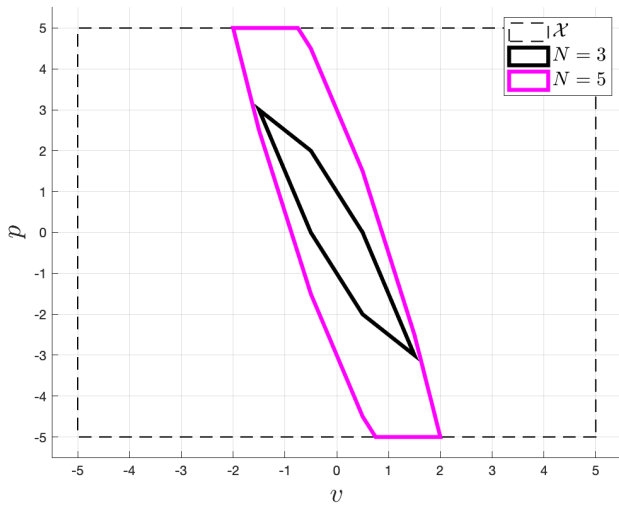
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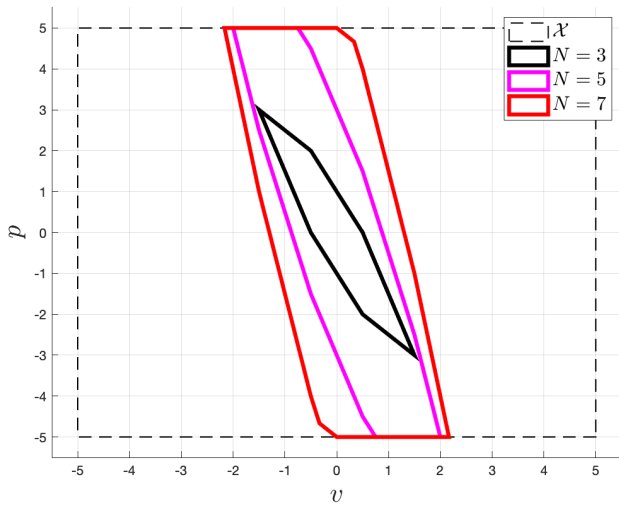
# Region of Attraction



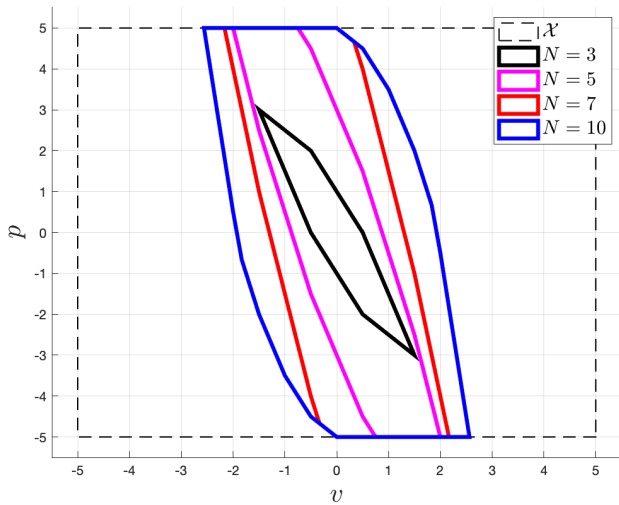
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# Nonlinear MPC



## Extension to Nonlinear MPC

Consider the nonlinear system dynamics:  $x(t+1) = f(x(t), u(t))$

$$\begin{aligned} J_t^*(x(0)) = & \min_{u_{t|t}, \dots, u_{t+N-1|t}} \sum_{k=0}^{T-1} h(x_{k|t}, u_{k|t}) + \textcolor{red}{V}(x_{t+T|t}) \\ & \text{subject to} \quad x_{k+1|t} = f(x_{k|t}, u_{k|t}), \forall k \in \{t, \dots, t+N-1\} \\ & \quad x_{k|t} \in \mathcal{X}, u_{k|t} \in \mathcal{U}, \forall k \in \{t, \dots, t+N-1\} \\ & \quad x_{t|t} = x(0), x_N \in \textcolor{red}{\mathcal{X}}_F \end{aligned}$$

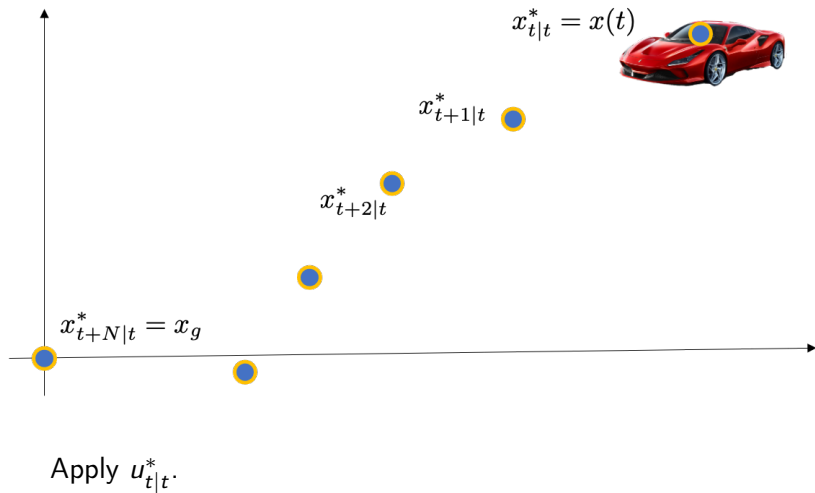
- Our derivations did not rely on linearity!!!

# Nonlinear MPC

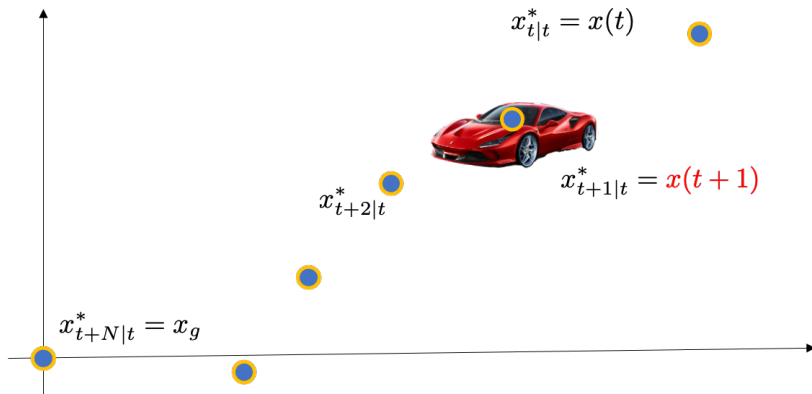


Assume that at time  $t = 0$  the MPC problem is feasible.

# Nonlinear MPC

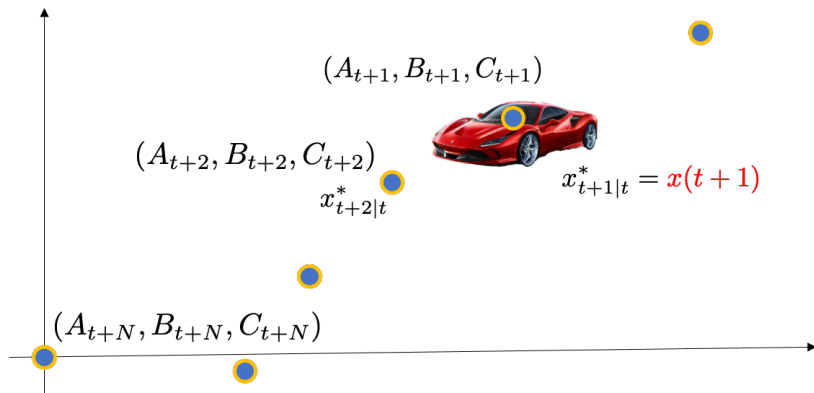


# Nonlinear MPC



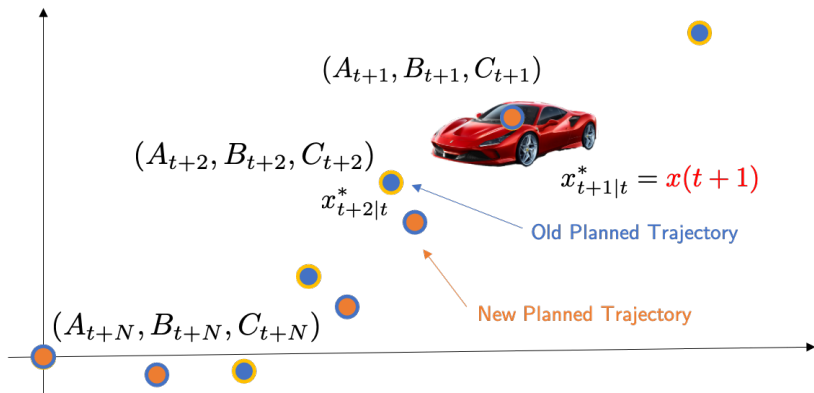
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# Nonlinear MPC



We can linearize around the feasible trajectory! (e.g.,  $A_k = A(x_{k|t}^*, u_{k|t}^*)$ ,  $B_k = B(x_{k|t}^*, u_{k|t}^*)$  and  $C_k = C(x_{k|t}^*, u_{k|t}^*)$ ).

# Nonlinear MPC



# What is next?

## Safety and Convergence:

In the next lectures we will see how to

- ▶ Compute the terminal components to guarantee safety and convergence (hard computationally)
- ▶ Learn the terminal components from data