

CS159 Lecture 5: LMPC and Model Learning

Ugo Rosolia

Caltech

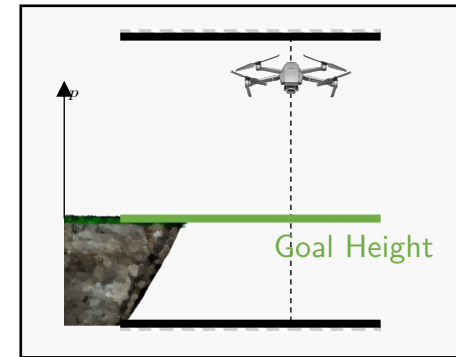
Spring 2021

Today's Class

- ▶ Recap of Lecture #4
- ▶ LMPC implementation
- ▶ Autonomous Racing Experiments
- ▶ Model learning in MPC (a brief summary)
- ▶ Planning under uncertainty

Recap of Lecture #4

Why did we focus on feasibility?

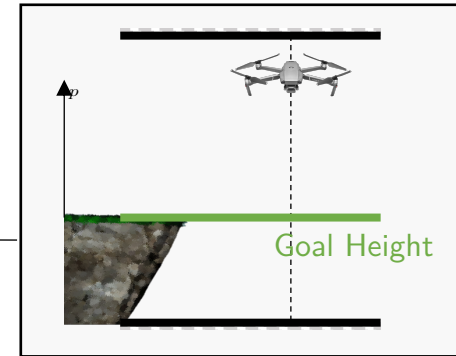


Recap of Lecture #4

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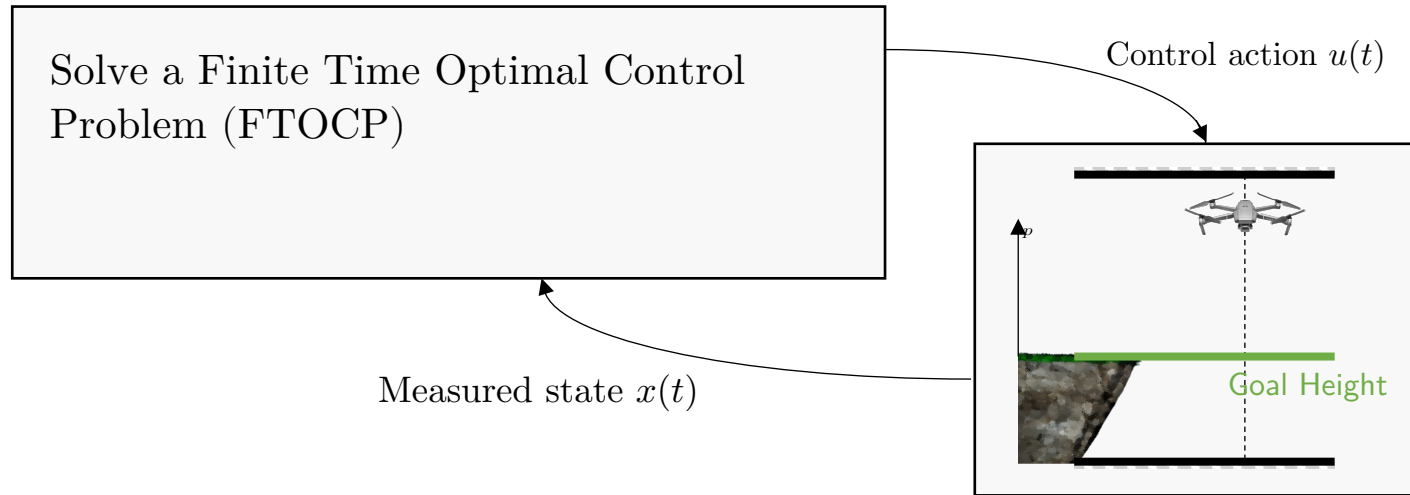
Solve a Finite Time Optimal Control Problem (FTOCP)

Measured state $x(t)$



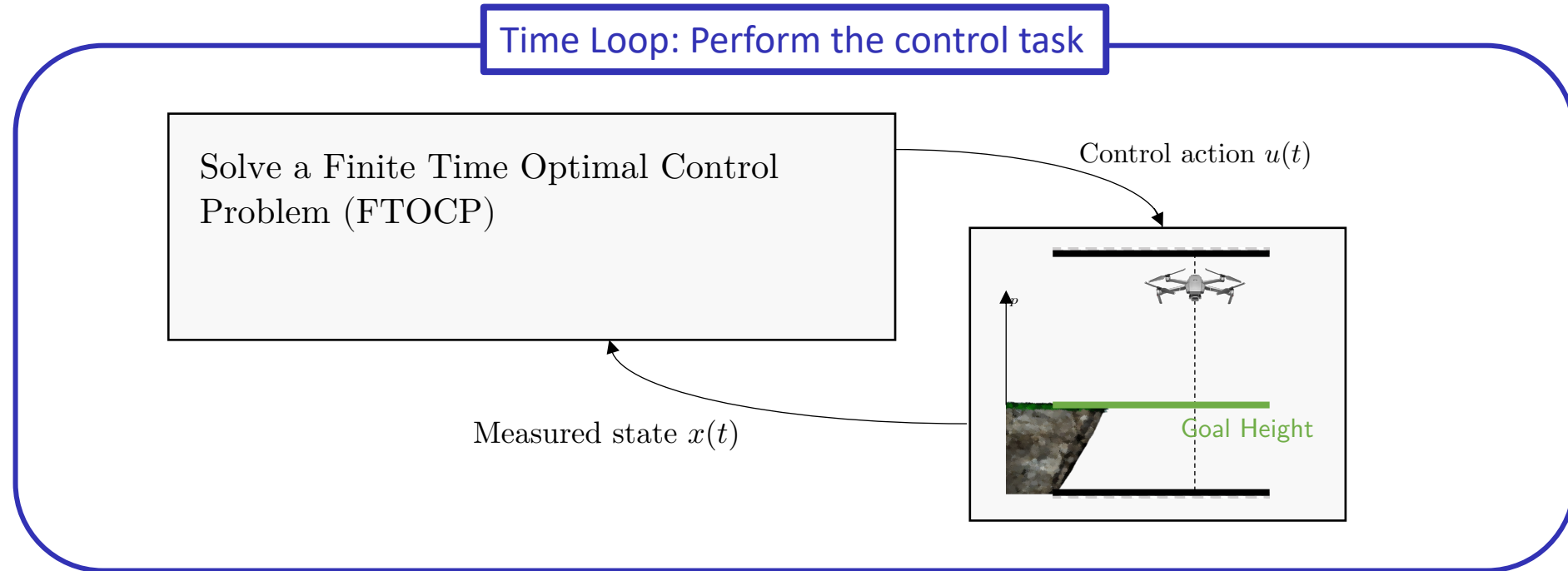
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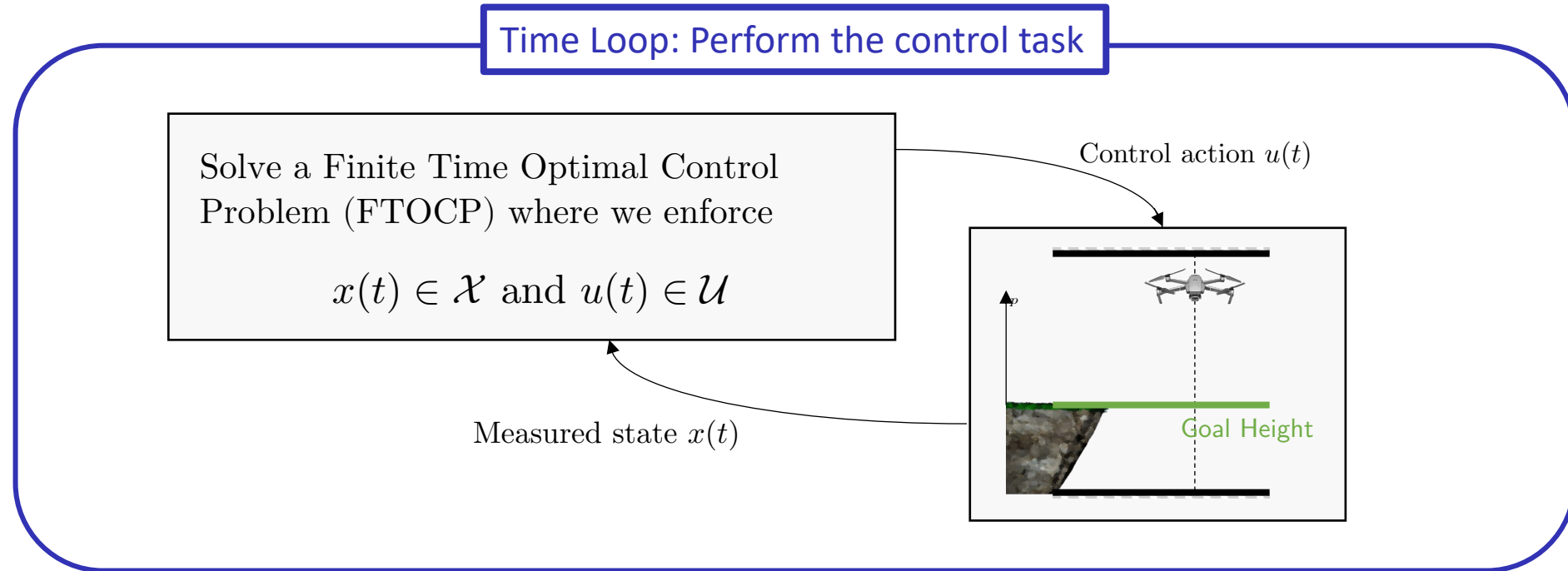
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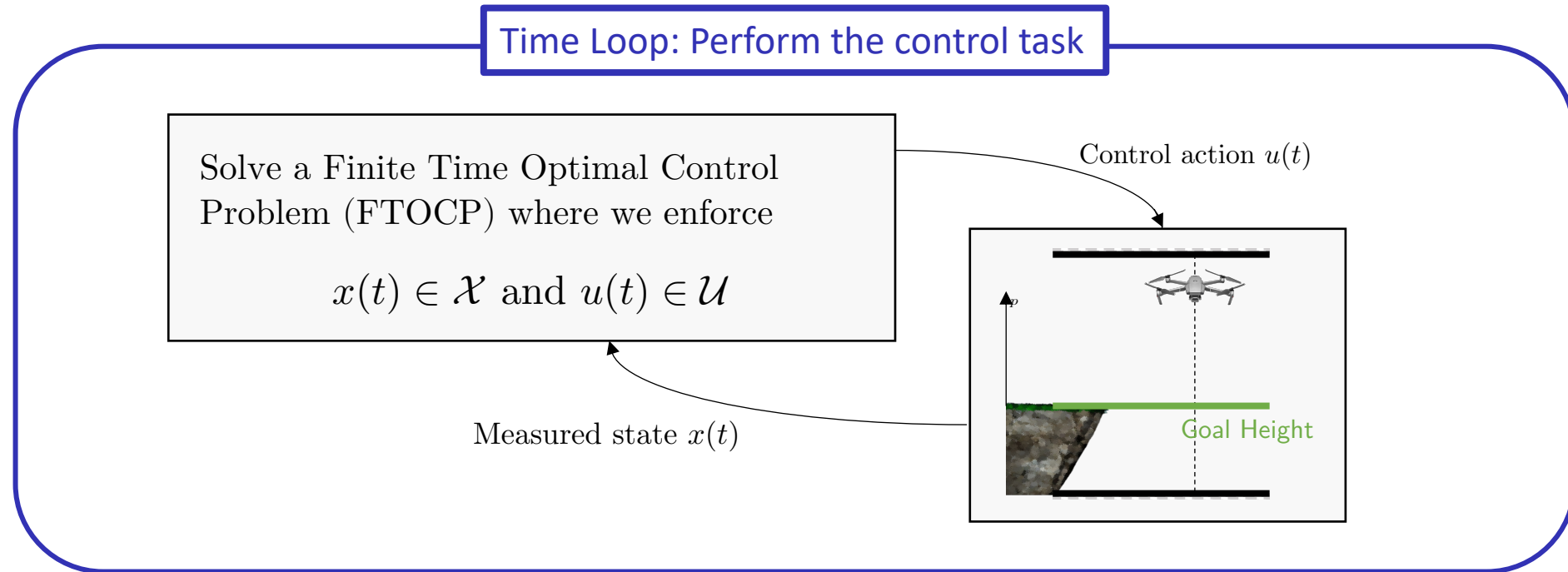
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Recap of Lecture #4

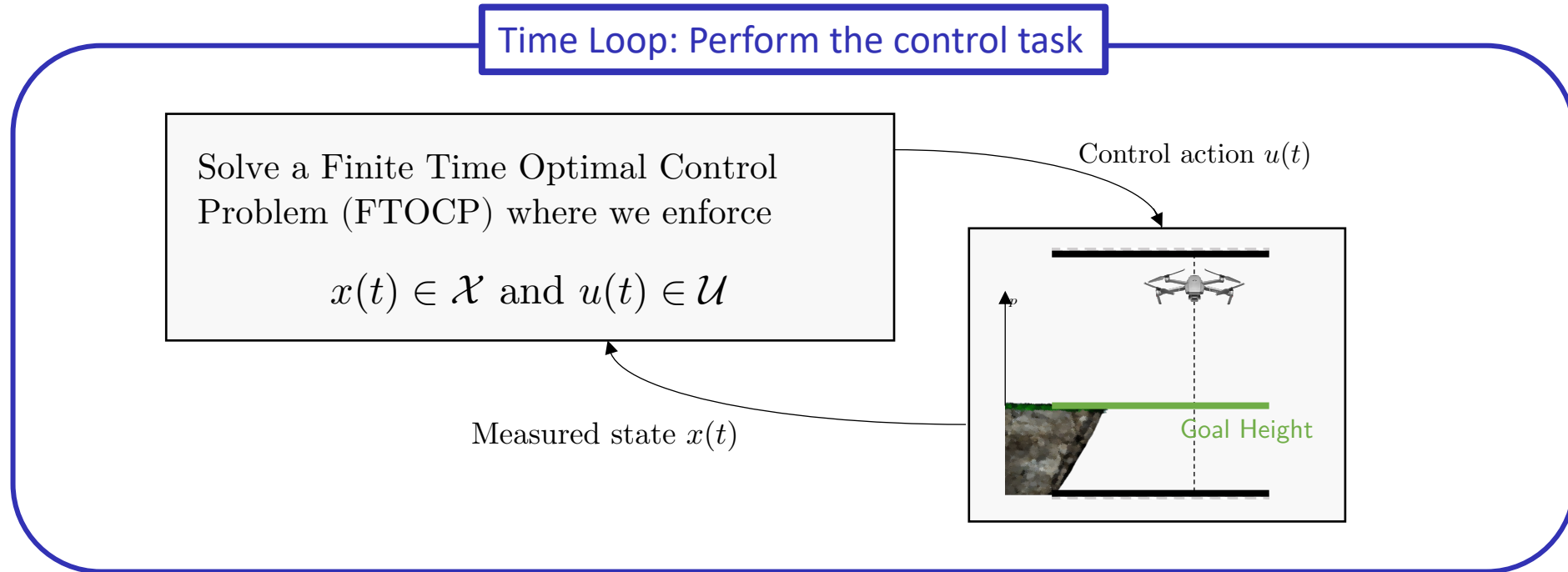
Why did we focus on feasibility?



Feasibility of the FTCOP at all times

Recap of Lecture #4

Why did we focus on feasibility?



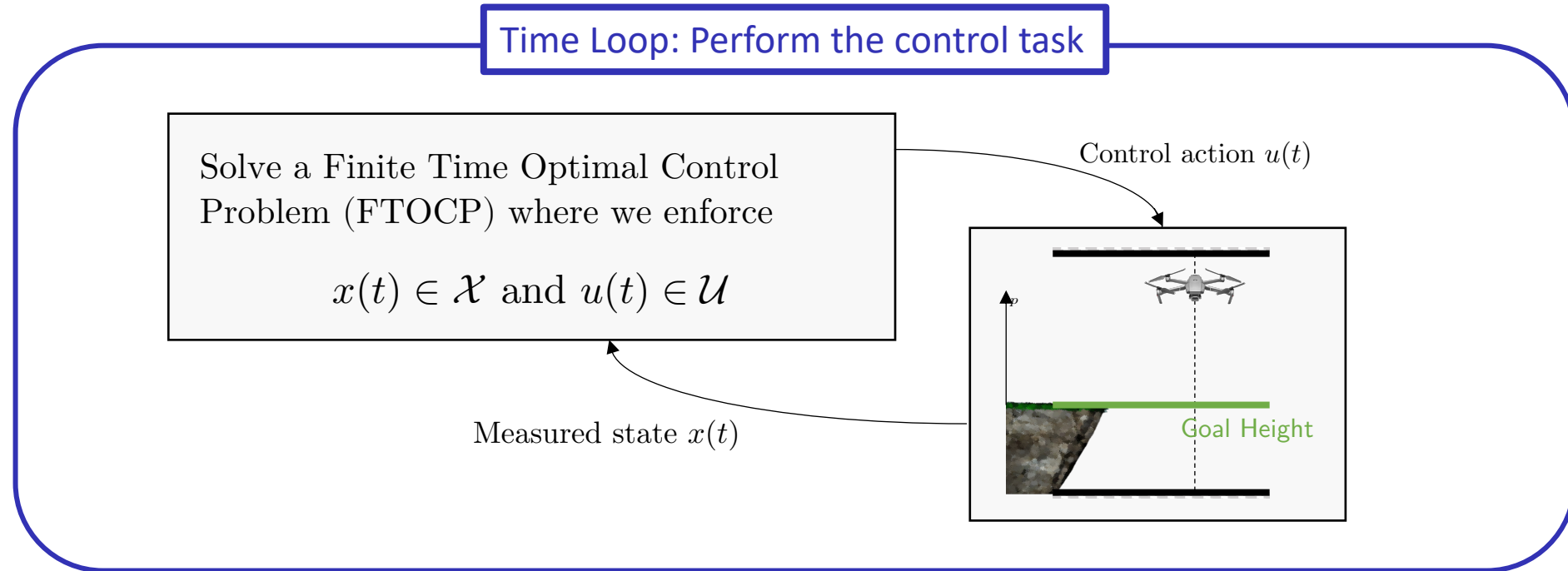
Feasibility of the FTCOP at all times



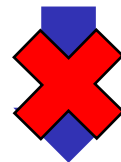
Safety

Recap of Lecture #4

Why did we focus on feasibility?



Feasibility of the FTCOP at all times



Not true when constraints are relaxed with slack variables!

Safety

Recap of Lecture #4

Stability Proof:

Recap of Lecture #4

Stability Proof:

We have shown that

Recap of Lecture #4

Stability Proof:

We have shown that

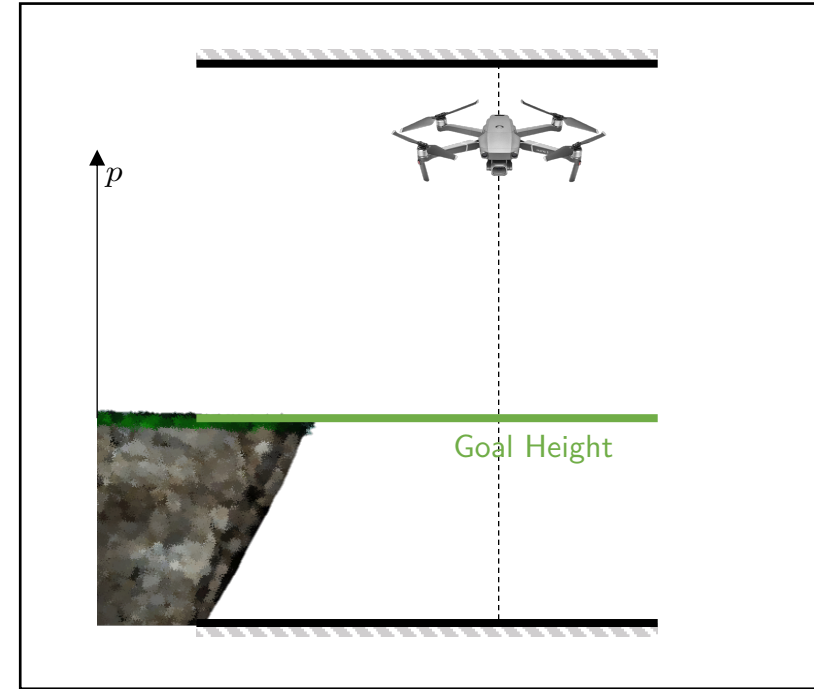
$$J_t^*(x(t)) > J_{t+1}^*(x(t+1)), \quad \forall x(t) \neq 0$$

Recap of Lecture #4

Stability Proof:

We have shown that

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Recap of Lecture #4

Stability Proof:

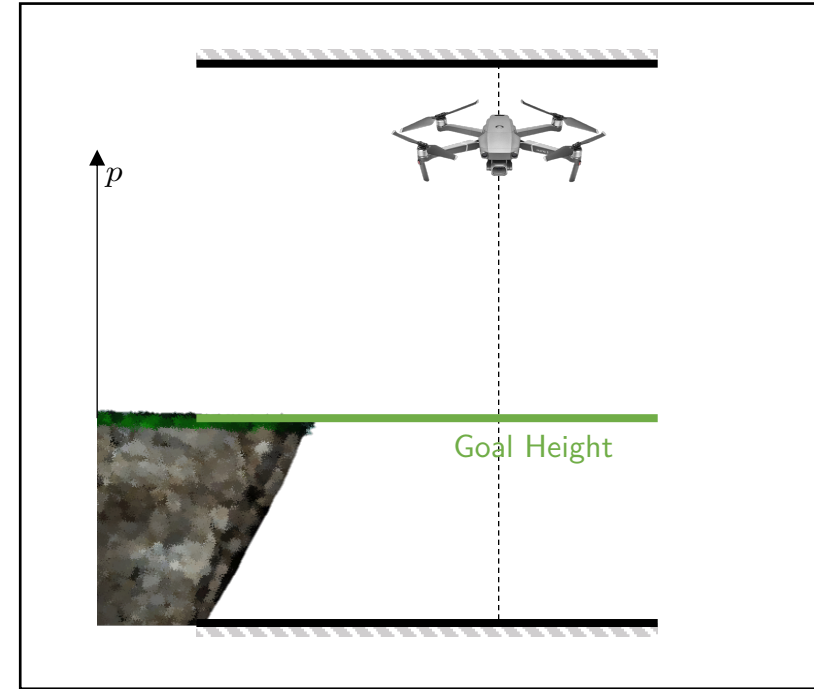
We have shown that

$$J_t^*(x(t)) > J_{t+1}^*(x(t+1)), \quad \forall x(t) \neq 0$$

$$J_t^*(x) > 0, \quad \forall x \neq 0,$$

$$\|x\|_2 \rightarrow \infty \implies J_t^*(x) \rightarrow \infty$$

$$J_t^*(0) = 0$$



Recap of Lecture #4

Stability Proof:

We have shown that

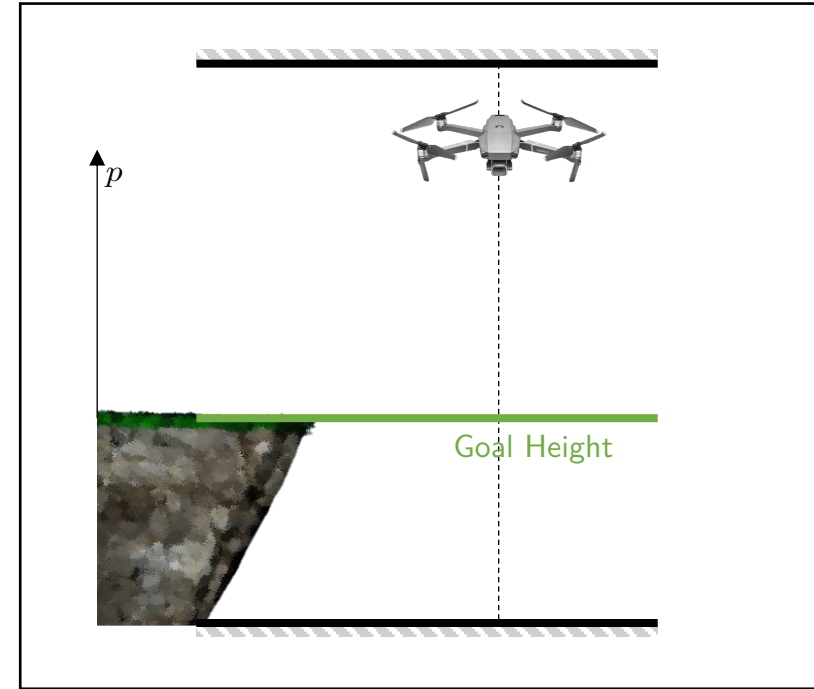
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$$\lim_{t \rightarrow \infty} J_t^*(x(t)) = 0$$

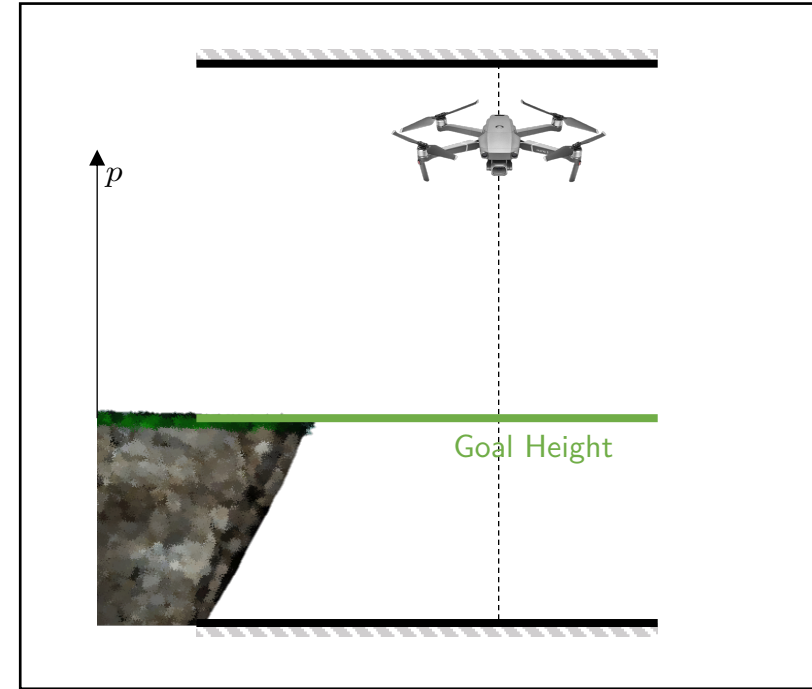


Recap of Lecture #4

Stability Proof:

We have shown that

$$\left. \begin{aligned} J_t^*(x(t)) &> J_{t+1}^*(x(t+1)), \quad \forall x(t) \neq 0 \\ J_t^*(x) &> 0, \quad \forall x \neq 0, \\ \|x\|_2 \rightarrow \infty &\implies J_t^*(x) \rightarrow \infty \\ J_t^*(0) &= 0 \end{aligned} \right\} \lim_{t \rightarrow \infty} J_t^*(x(t)) = 0$$



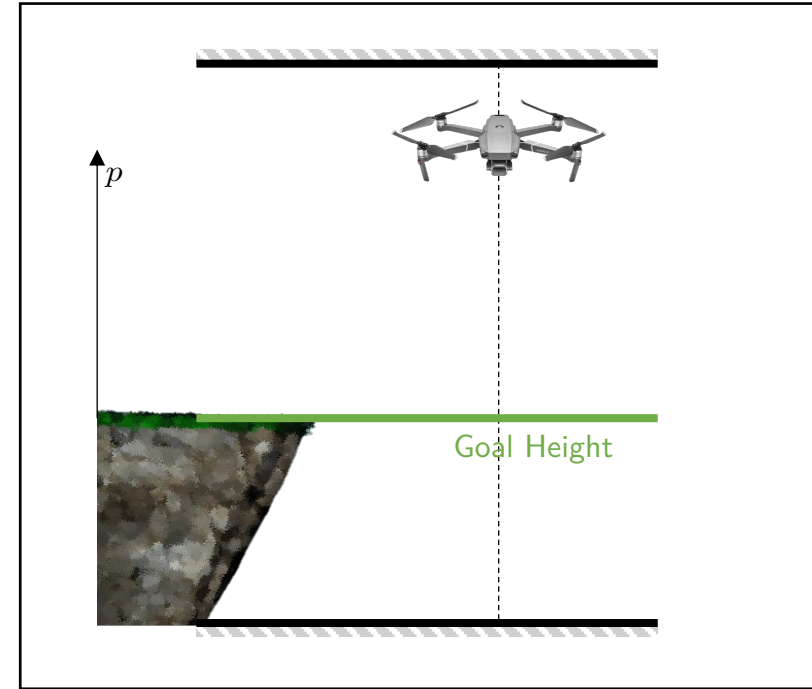
To show stability we need to that $J_t^*(x)$ is Lipchitz continuous and assume that the state and input constraint sets are compact

Recap of Lecture #4

Stability Proof:

We have shown that

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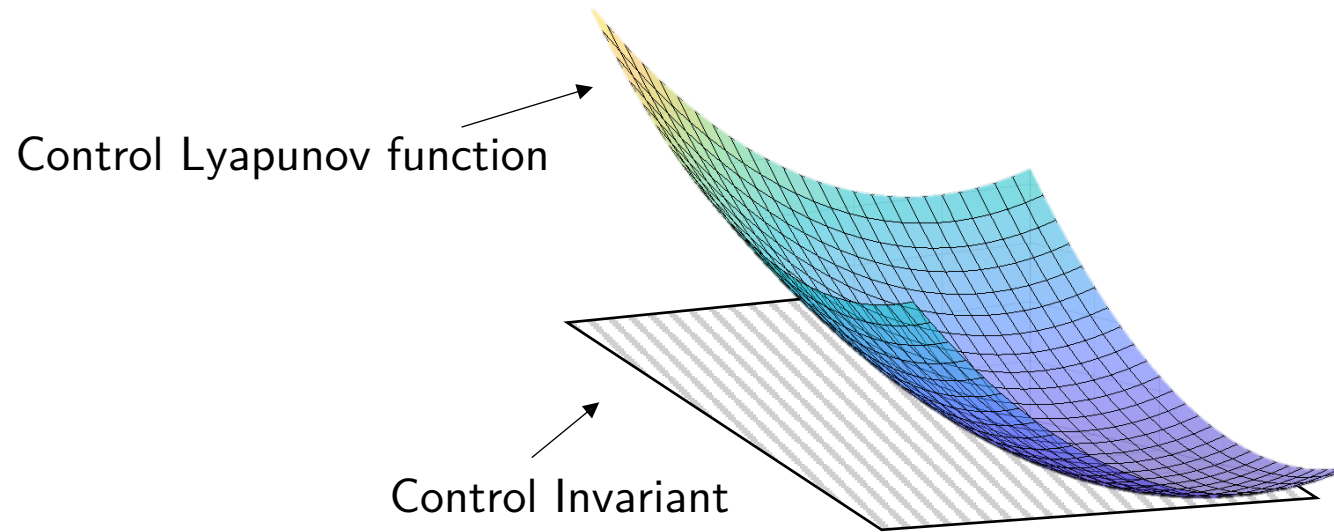
To show stability we need to that $J_t^*(x)$ is Lipschitz continuous and that

$$\alpha_1(x) \leq J_t^*(x) \leq \alpha_2(x)$$

where $\alpha_{\neq}$ and $\alpha_{\leq}$ are class K functions (can be relaxed to continuity around the origin).

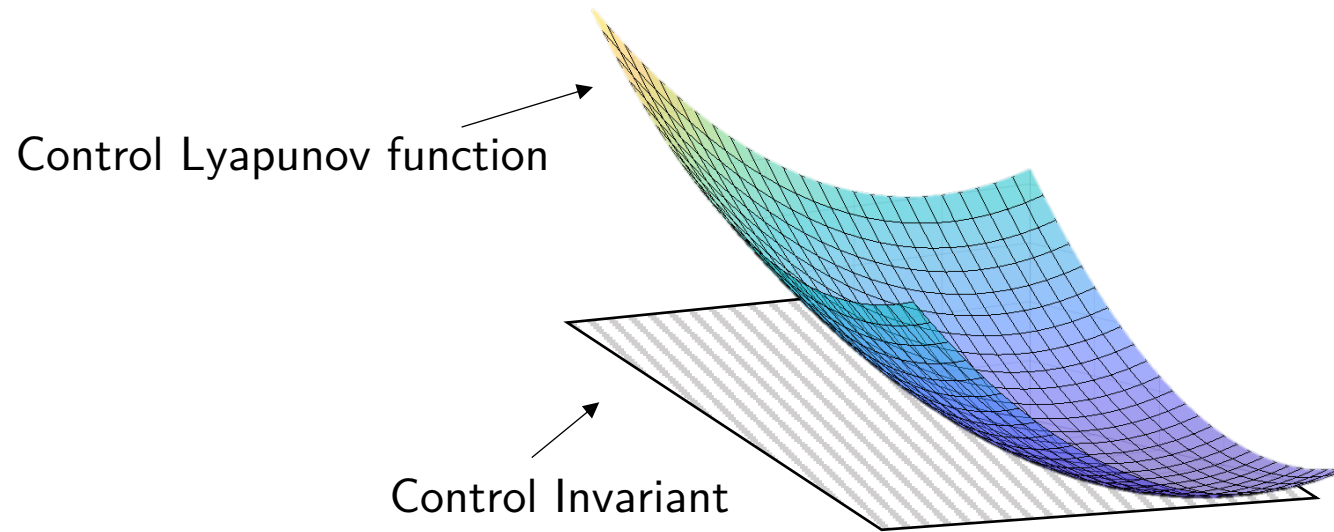
Recap of Lecture #4

Design of the terminal MPC components



Recap of Lecture #4

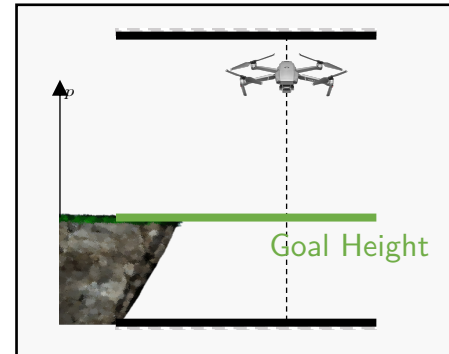
Design of the terminal MPC components



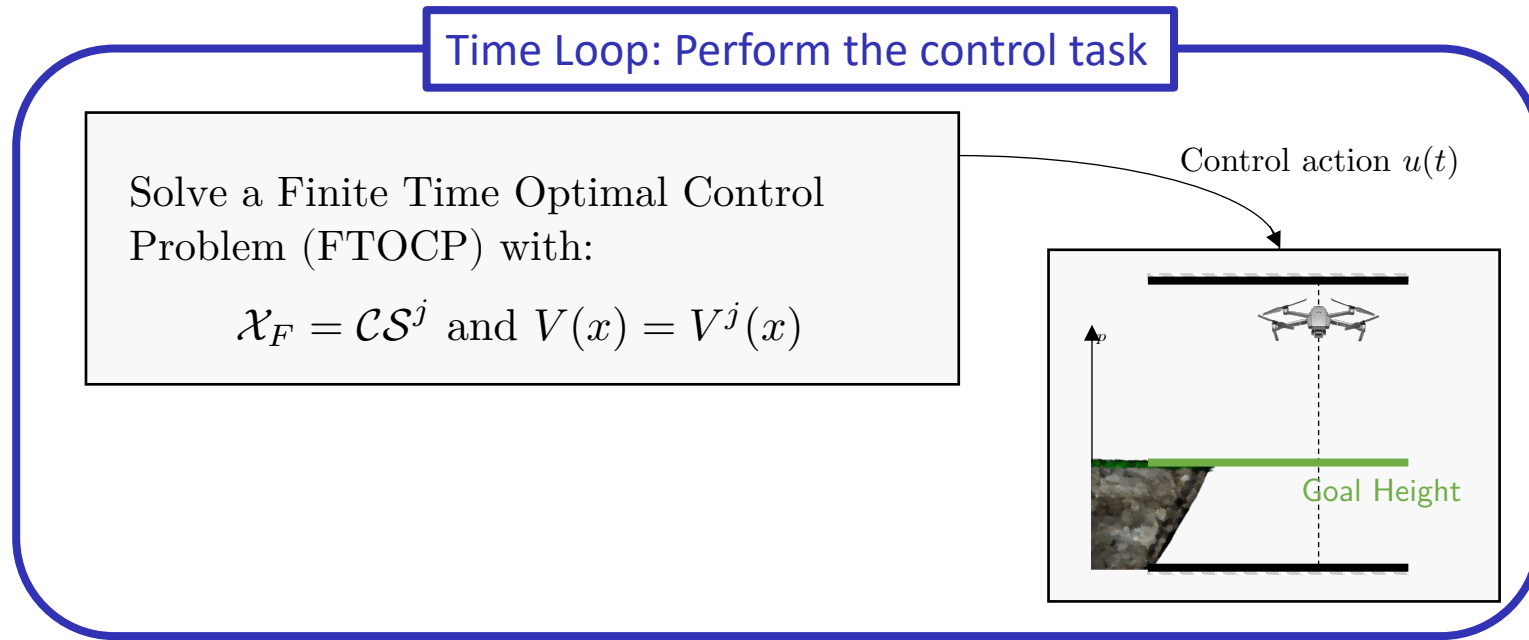
Safety and Stability

LMPC Diagram

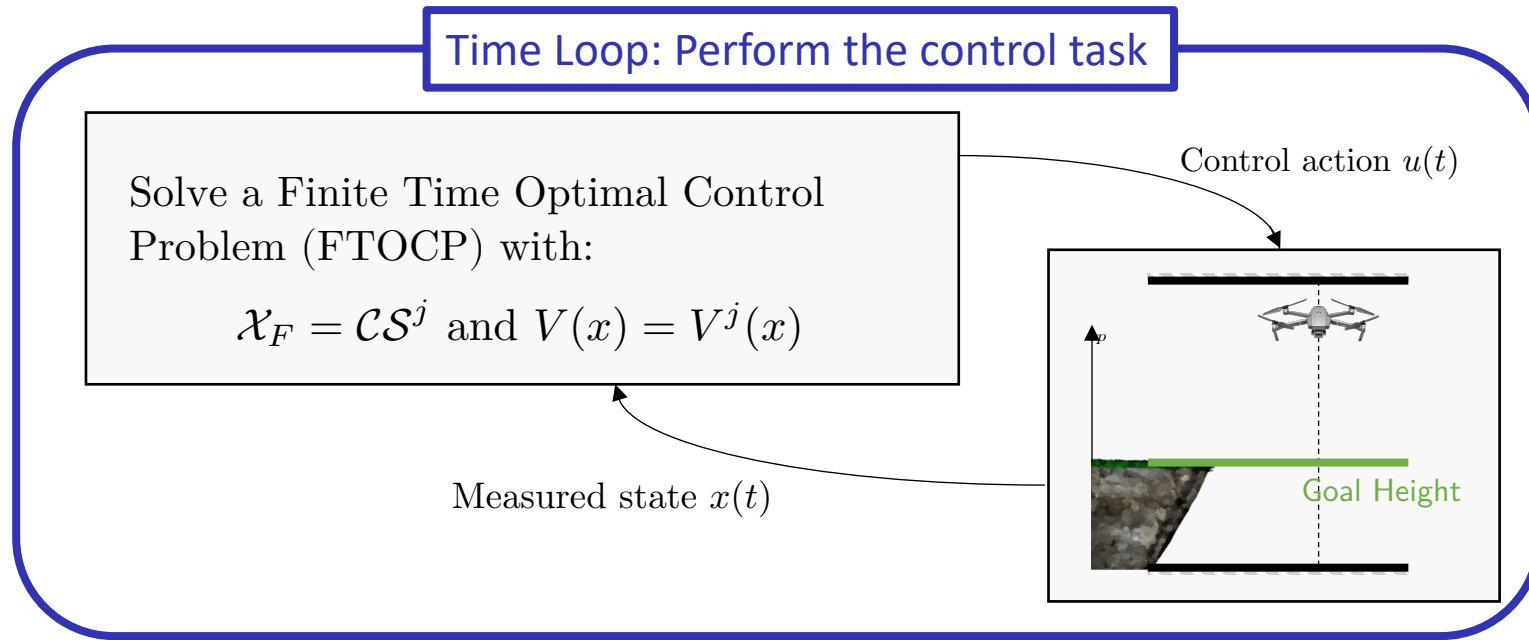
Time Loop: Perform the control task



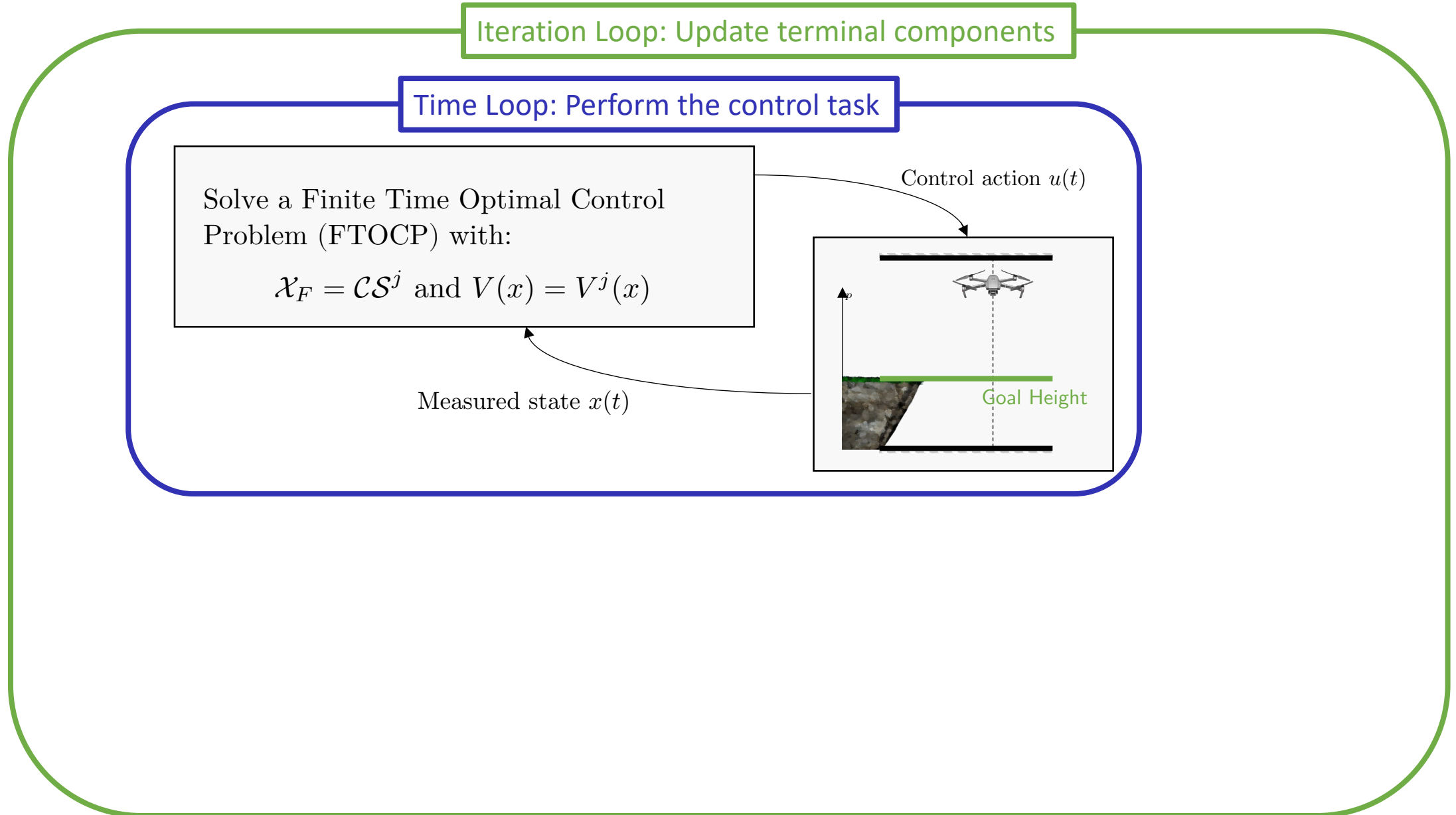
LMPC Diagram



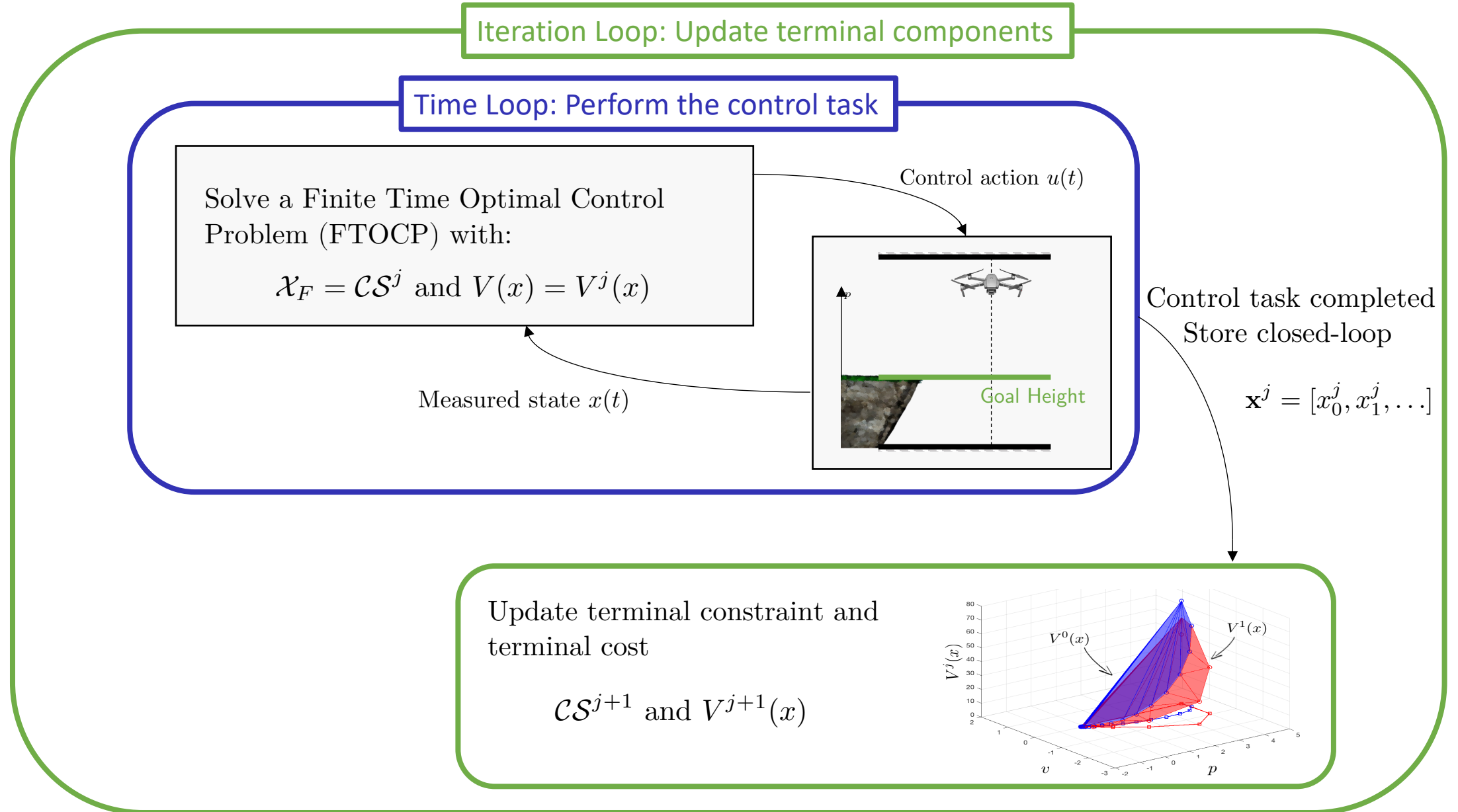
LMPC Diagram



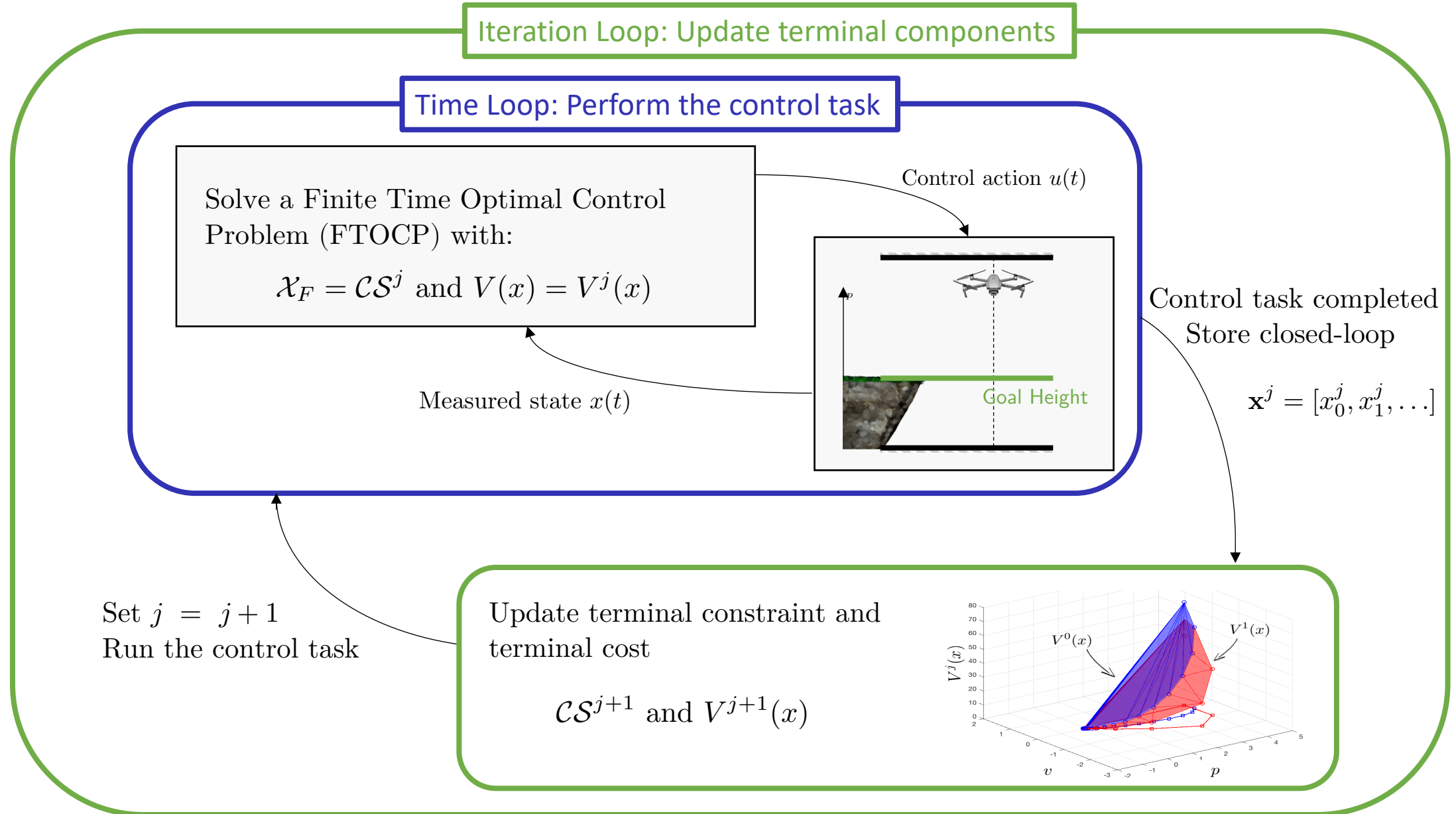
LMPC Diagram



LMPC Diagram



LMPC Diagram



LMPC Recap

Theorem

Let $\mathbf{x}^j = [x_0^j, x_1^j, \dots]$ be the closed-loop trajectory from the starting state x_S at iteration j . Consider sequence $\{\mathbf{x}^j\}$ of closed-loop trajectories and assume that for $c < \infty$ we have that

$$\mathbf{x}^c = \mathbf{x}^{c+1}$$

Then we have that

- ▶ At each iteration state and input constraints are satisfied.
- ▶ The closed-loop cost $J_{0 \rightarrow \infty}^j(x_S)$ is non-increasing, i.e.,

$$J_{0 \rightarrow \infty}^{j+1}(x_S) = \sum_{t=0}^{\infty} h(x_t^{j+1}, u_t^{j+1}) \leq \sum_{t=0}^{\infty} h(x_t^j, u_t^j) = J_{0 \rightarrow \infty}^j(x_S)$$

- ▶ $\mathbf{x}^c = \mathbf{x}^*$, under mild conditions (LICQ holds at each time t)

Non-increasing closed-loop cost: Proof

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Step 1: $J_{0 \rightarrow \infty}^{j-1}(x_S) \geq J_{0 \rightarrow N}^{LMPC,j}(x_S)$

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$$J_{0 \rightarrow \infty}^{j-1}(x_S) = \sum_{k=0}^{\infty} h(x_k^{j-1}, u_k^{j-1}) = \sum_{k=0}^{N-1} h(x_k^{j-1}, u_k^{j-1}) + \sum_{k=N}^{\infty} h(x_k^{j-1}, u_k^{j-1})$$

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$$\rightarrow J_{0 \rightarrow \infty}^{j-1}(x_S) \geq \underbrace{\sum_{k=0}^{N-1} h(x_k^{j-1}, u_k^{j-1})}_{\text{Cost associated with a feasible trajectory}} + Q^{j-1}(x_N^{j-1})$$

Cost associated with a feasible trajectory

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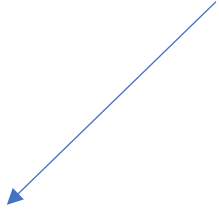
Step 2: $J_{0 \rightarrow N}^{LMPC,j}(x_S) \geq J_{0 \rightarrow \infty}^j(x_S)$

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From standard MPC arguments


$$J_{1 \rightarrow 1+N}^{LMPC,j}(x_1^j) - J_{0 \rightarrow N}^{LMPC,j}(x_0^j) \leq -h(x_0^j, u_0^j)$$

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$$\rightarrow J_{0 \rightarrow N}^{LMPC,j}(x_0^j) \geq \lim_{t \rightarrow \infty} J_{t \rightarrow t+N}^{LMPC,j}(x_t^j) + \sum_{k=0}^{\infty} h(x_k^j, u_k^j)$$

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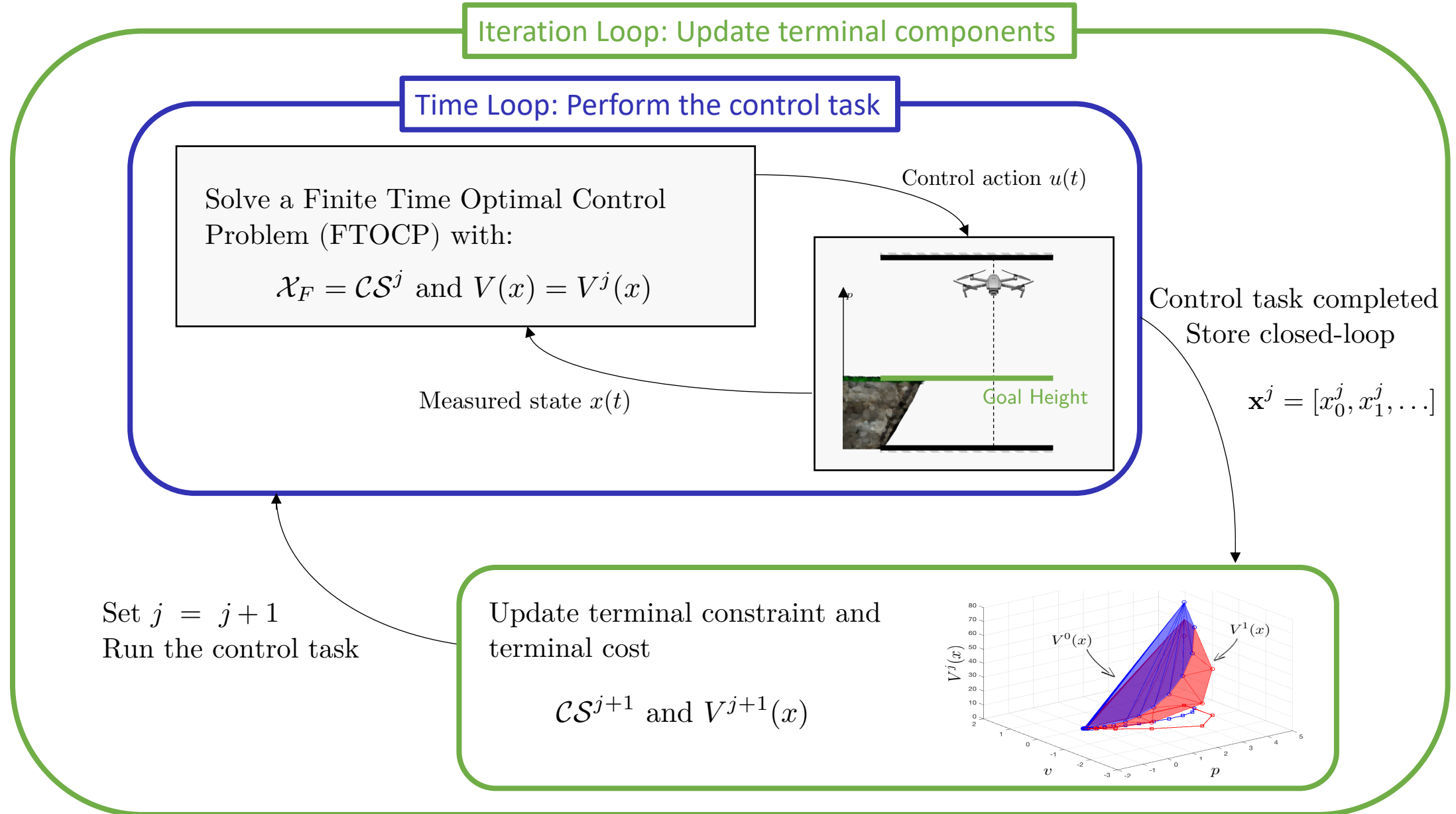
Step 2: $J_{0 \rightarrow N}^{LMPC,j}(x_S) \geq J_{0 \rightarrow \infty}^j(x_S)$

Conclusion

The iteration cost $J_{0 \rightarrow \infty}^j(x_s)$ is non-increasing at each iteration,

$$J_{0 \rightarrow \infty}^{j-1}(x_S) \geq J_{0 \rightarrow N}^{LMPC,j}(x_S) \geq J_{0 \rightarrow \infty}^j(x_S)$$

LMPC Diagram



Linear(ized) LMPC

At time t of iteration j solve the following Constrained Finite Time Optimal Control Problem (FTOCP)

$$J_{t \rightarrow t+N}^{\text{LMPC},j}(x_t^j) = \min_{u_{t|t}, \dots, u_{t+N-1|t}} \sum_{k=t}^{t+N-1} h(x_{k|t}, u_{k|t}) + V^{j-1}(x_{t+N|t})$$

s.t.

$$x_{k+1|t} = Ax_{k|t} + Bu_{k|t}, \quad \forall k \in [t, \dots, t+N-1]$$

$$x_{t|t} = x_t^j,$$

$$x_{k|t} \in \mathcal{X}, \quad u_{k|t} \in \mathcal{U}, \quad \forall k \in [t, \dots, t+N-1]$$

$$x_{t+N|t} \in \mathcal{CS}^{j-1}$$

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$$\boxed{x_{t+N|t} = \sum_{i=0}^{j-1} \sum_k x_k^i \lambda_k^i, \quad \sum_{i=0}^{j-1} \sum_k \lambda_k^i = 1, \quad \lambda_k^i \geq 0.} \iff \boxed{x_{t+N|t} \in \mathcal{CS}^{j-1}}$$

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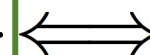
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$$V^{j-1}(x_{t+N|t})$$

$$x_{t+N|t} = \sum_{i=0}^{j-1} \sum_k x_k^i \lambda_k^i, \quad \sum_{i=0}^{j-1} \sum_k \lambda_k^i = 1, \quad \lambda_k^i \geq 0.$$



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s.t.

$$x_{k+1|t} = Ax_{k|t} + Bu_{k|t}, \quad \forall k \in [t, \dots, t+N-1]$$

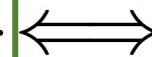
$$x_{t|t} = x_t^j,$$

$$x_{k|t} \in \mathcal{X}, \quad u_{k|t} \in \mathcal{U}, \quad \forall k \in [t, \dots, t+N-1]$$



$$V^{j-1}(x_{t+N|t})$$

$$x_{t+N|t} = \sum_{i=0}^{j-1} \sum_k x_k^i \lambda_k^i, \quad \sum_{i=0}^{j-1} \sum_k \lambda_k^i = 1, \quad \lambda_k^i \geq 0.$$



$$x_{t+N|t} \in \mathcal{CS}^{j-1}$$

- **Convex optimization** problem over inputs and lambdas
- Safety and performance improvement **guarantees still hold** (simple proofs as before)
- Converges to global optimal solution (LICQ required)

Learning MPC for Autonomous Racing

Real-time implementation on the Berkeley Autonomous Race Car (BARC)

Problem Formulation

Minimum Time Control Problem

$$\min_{T, \mathbf{u}} \boxed{T} \quad \text{Control objective}$$

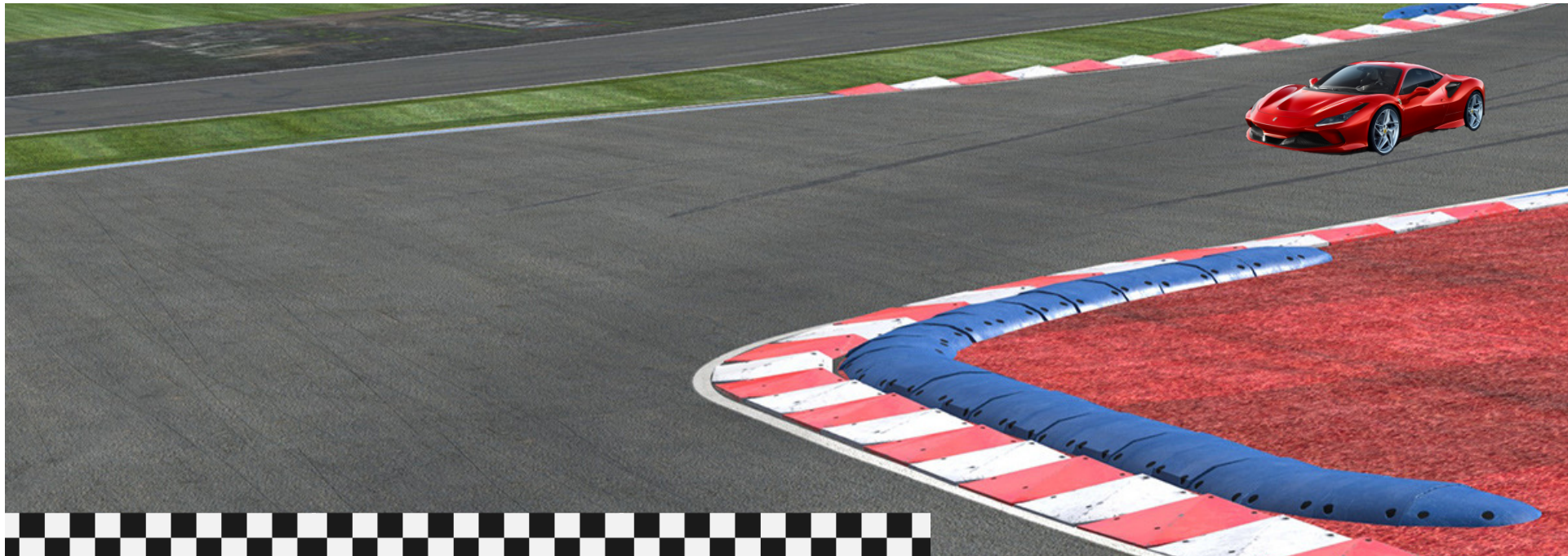
$$\boxed{x_0 = x_s, \quad x_T = \mathcal{X}_F} \quad \text{Start \& end position}$$

System dynamics
System constraints

$$\boxed{x_{k+1} = f(x_k, u_k), \quad \forall k \in \{0, \dots, T-1\}}$$

Safety constraints

$$\boxed{x_k \in \mathcal{X}, \quad u_k \in \mathcal{U}, \quad \forall k \in \{0, \dots, T-1\}}$$





Learning Model Predictive Controller full-size vehicle experiments

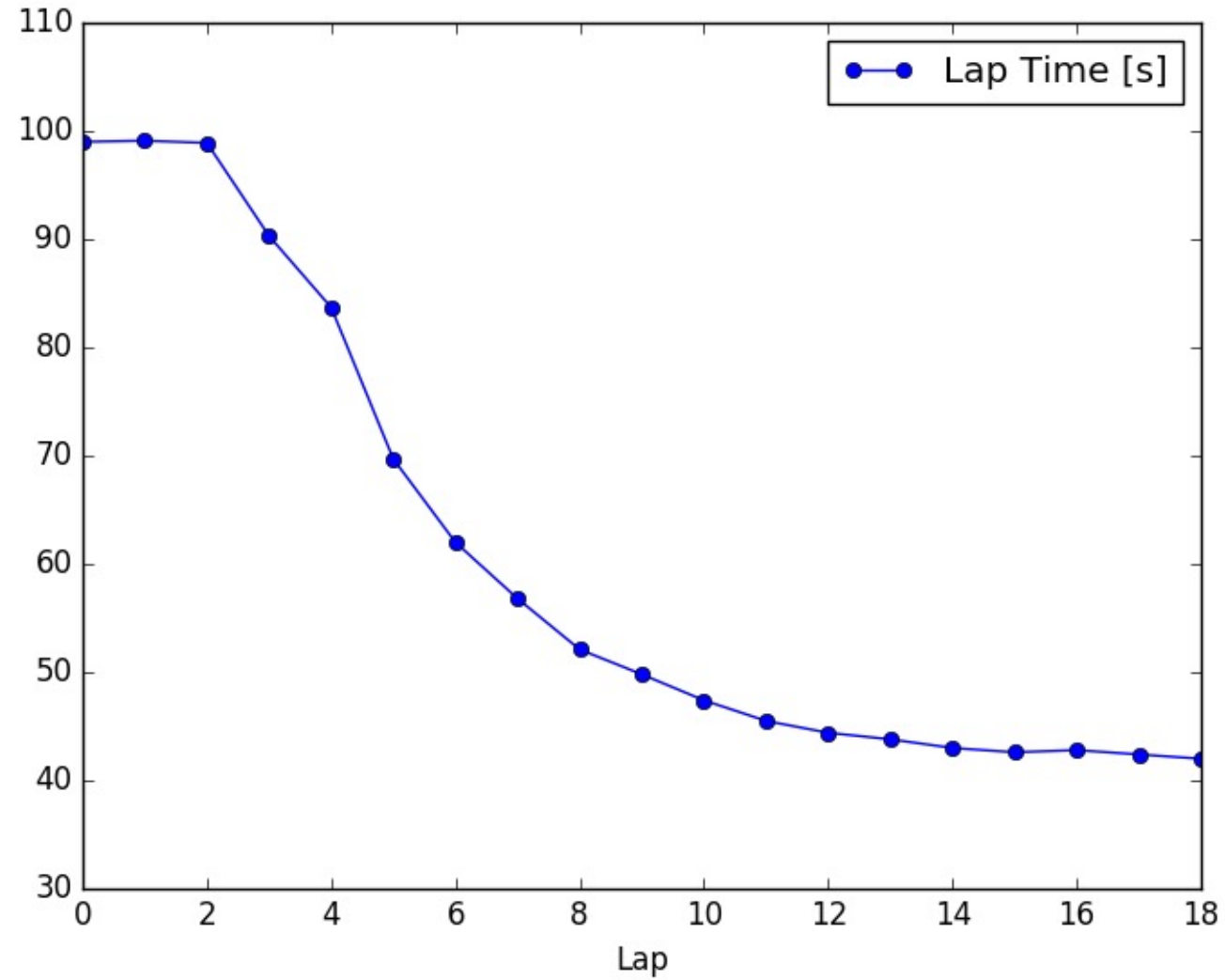
Credits: Siddharth Nair, Nitin Kapania and Ugo Rosolia



Learning Model Predictive Controller full-size vehicle experiments

Credits: Siddharth Nair, Nitin Kapania and Ugo Rosolia

Lap Time

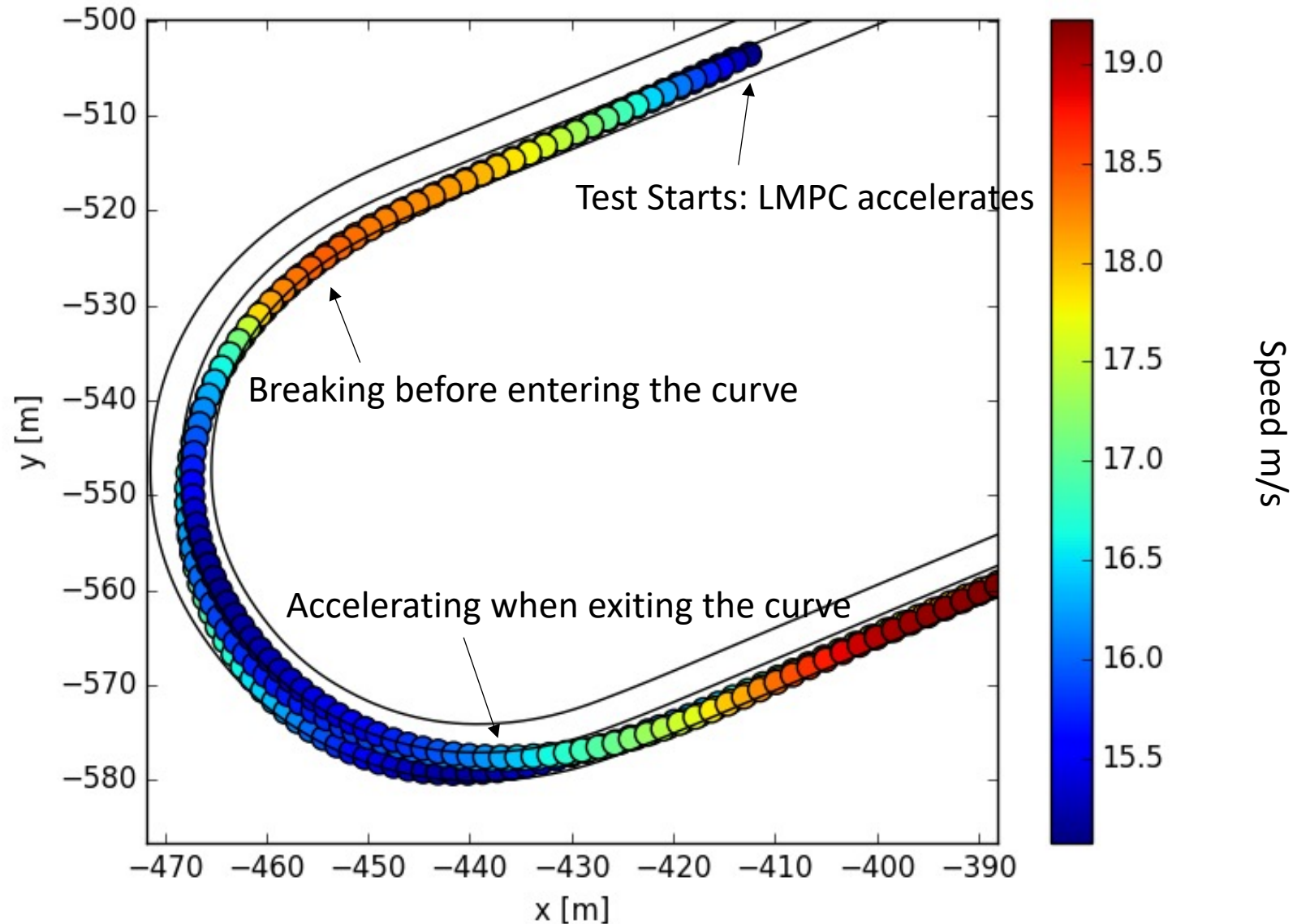




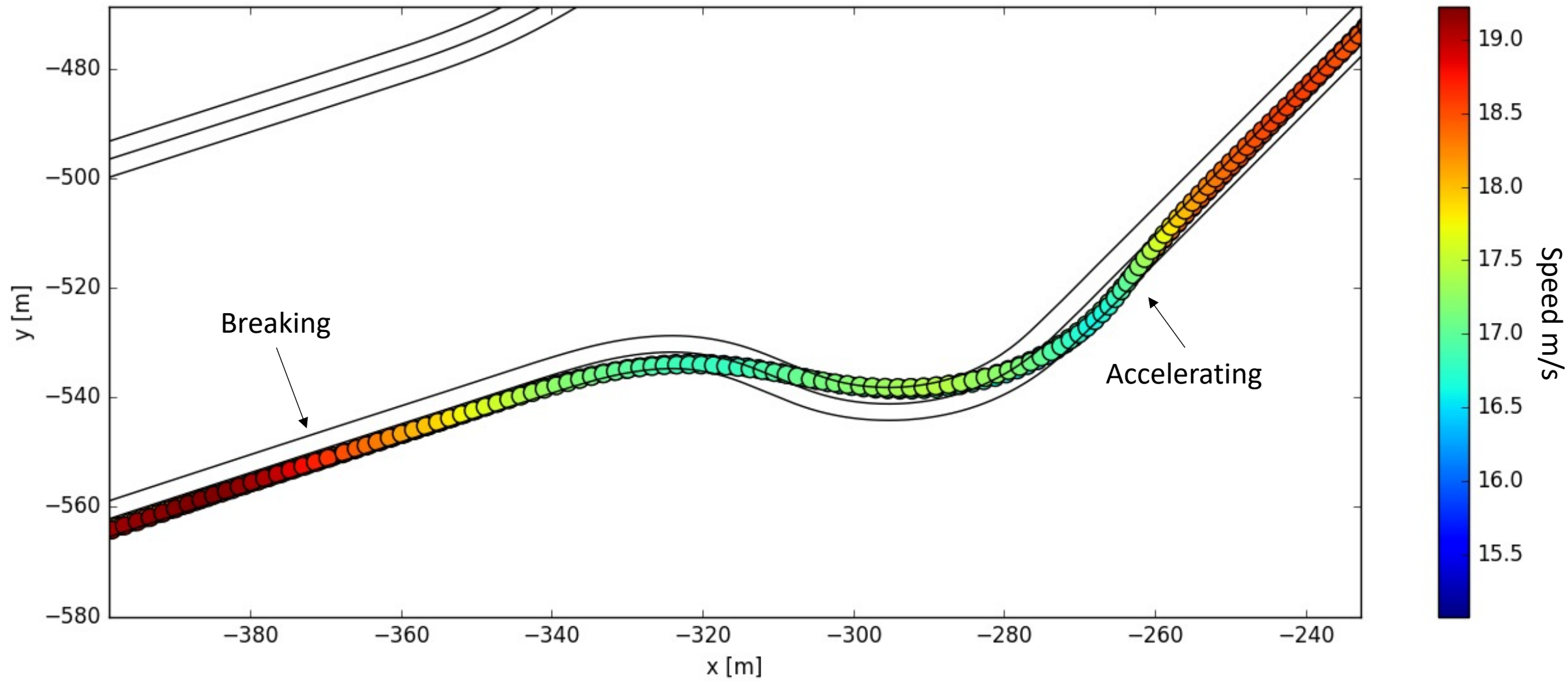
Learning Model Predictive Controller full-size vehicle experiments

Credits: Siddharth Nair, Nitin Kapania and Ugo Rosolia

Velocity Profile at Convergence (Curve 1)



Velocity Profile at Convergence (Chicane)



Problem Formulation

Minimum Time Control Problem

$$\min_{T, \mathbf{u}} \boxed{T} \quad \text{Control objective}$$

$$\boxed{x_0 = x_s, \quad x_T = x_F} \quad \text{Start \& end position}$$

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Problem Formulation

Minimum Time Control Problem

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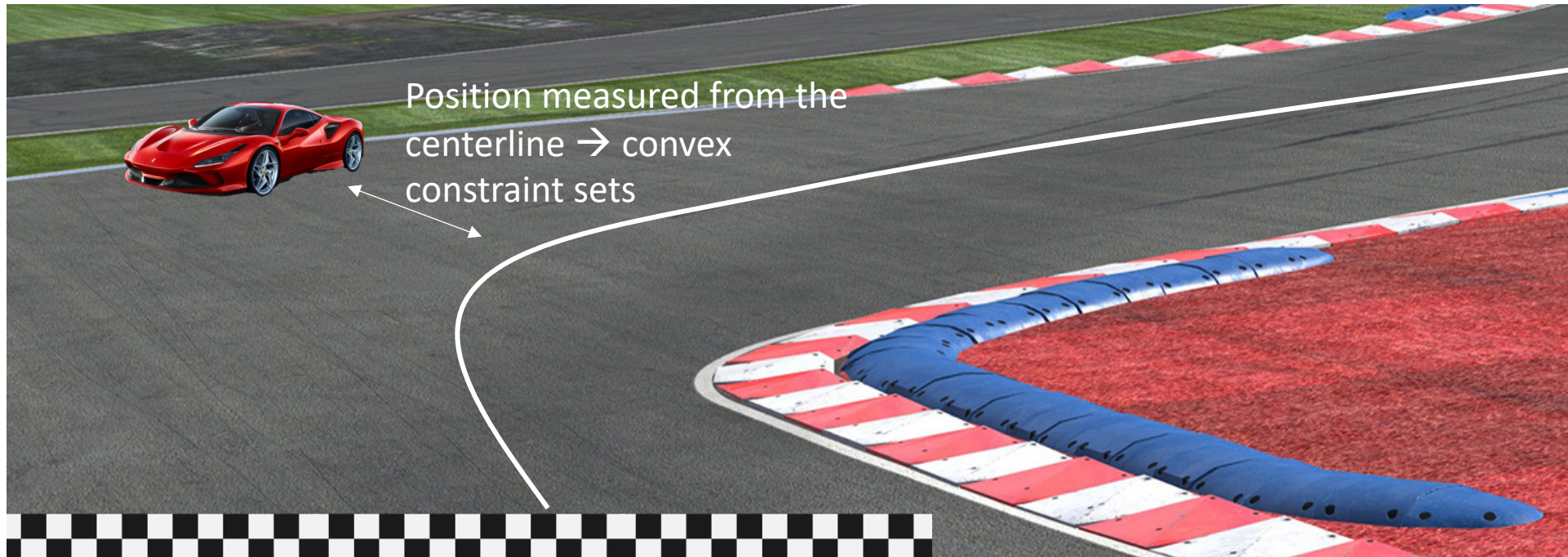
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System dynamics
System constraints

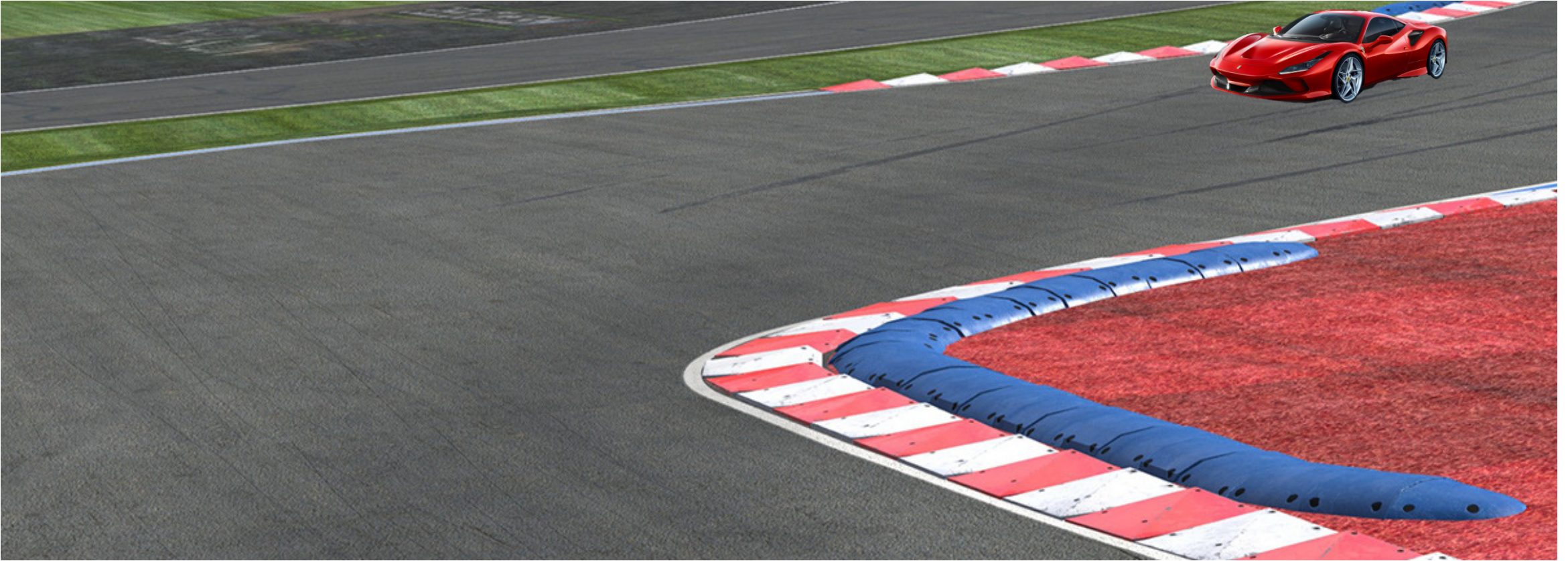
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Safety constraints

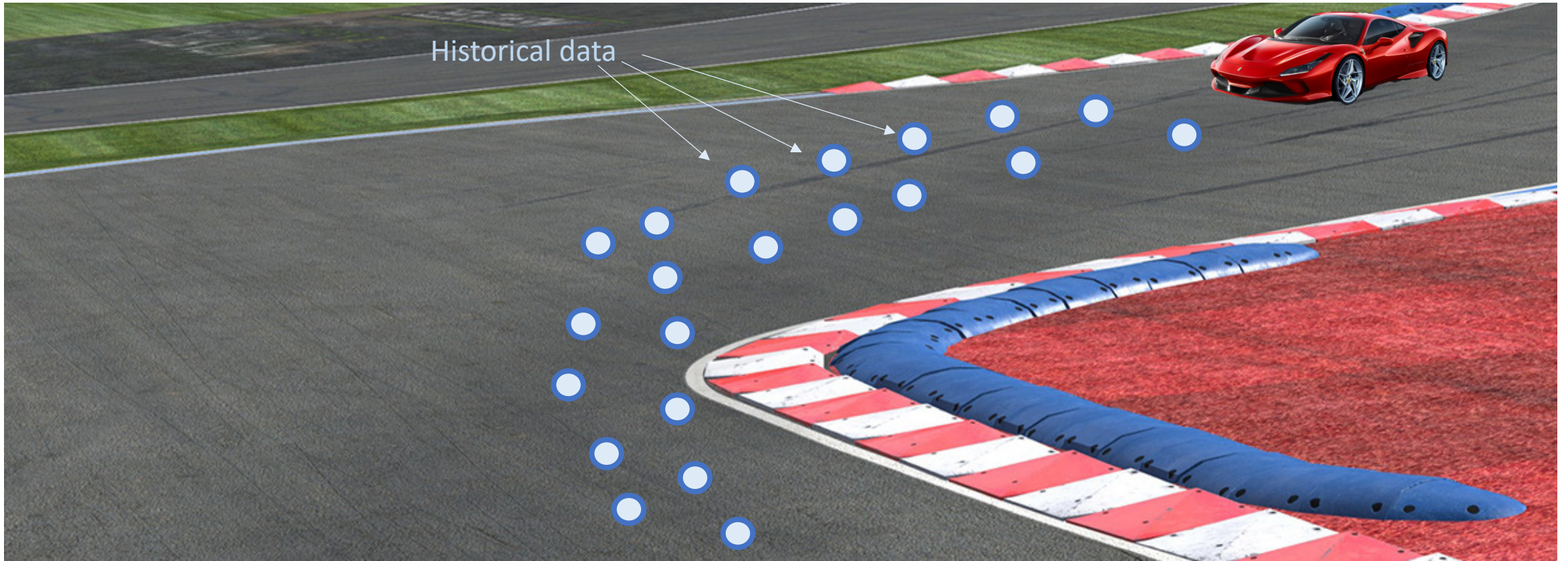
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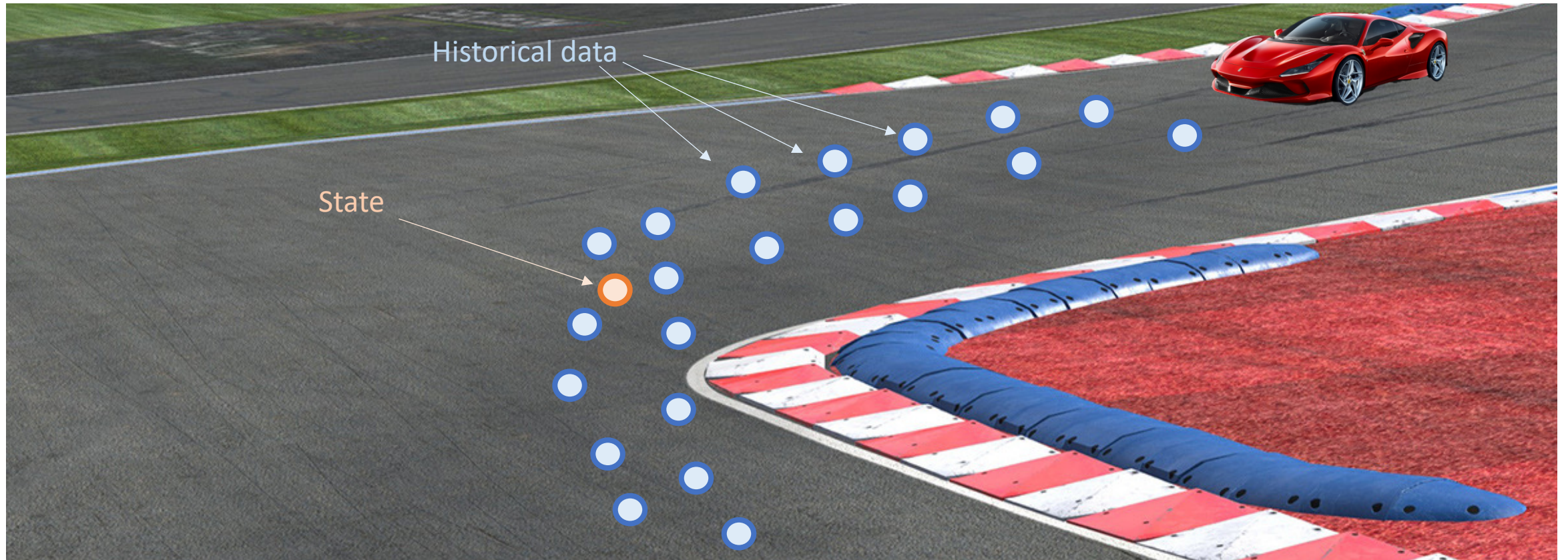
Local Approximations



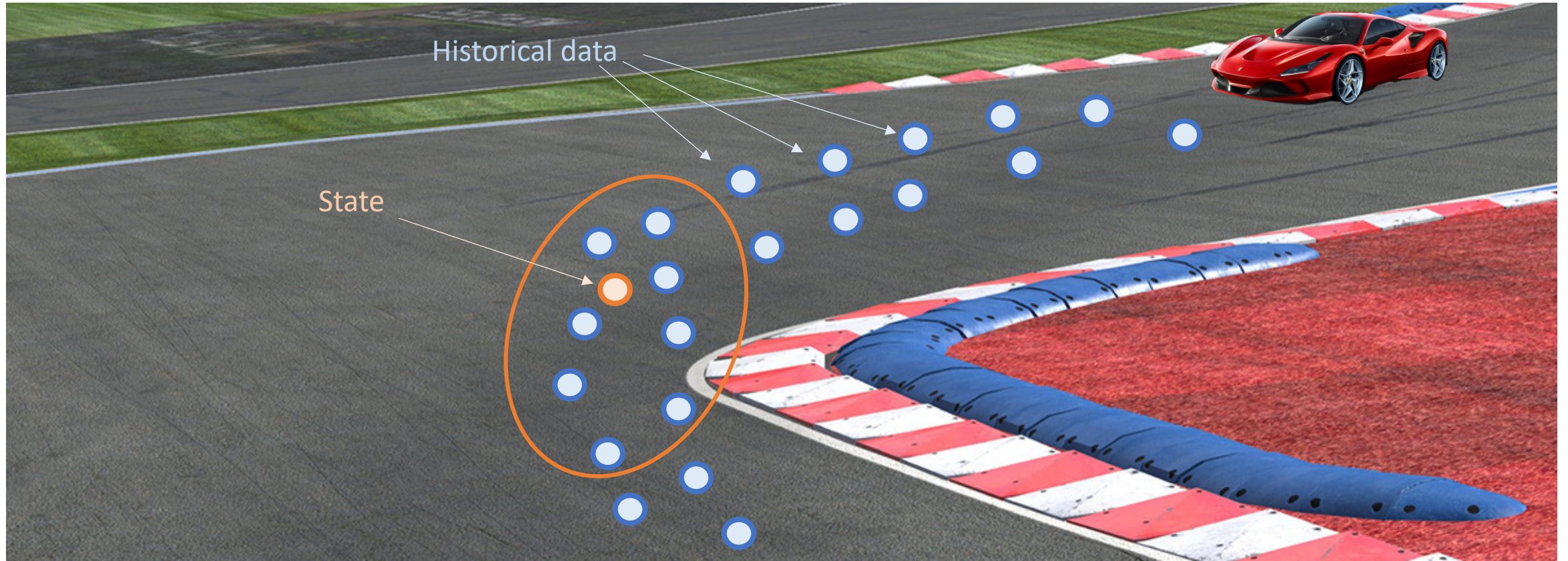
Local Approximations



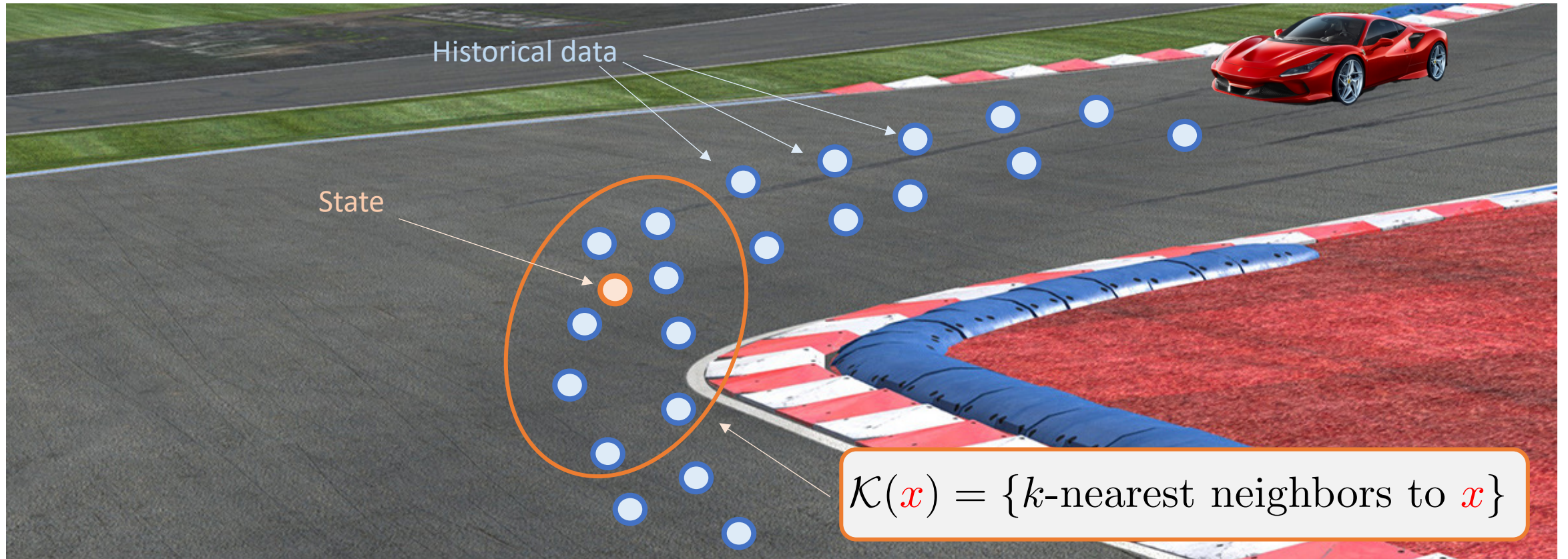
Local Approximations



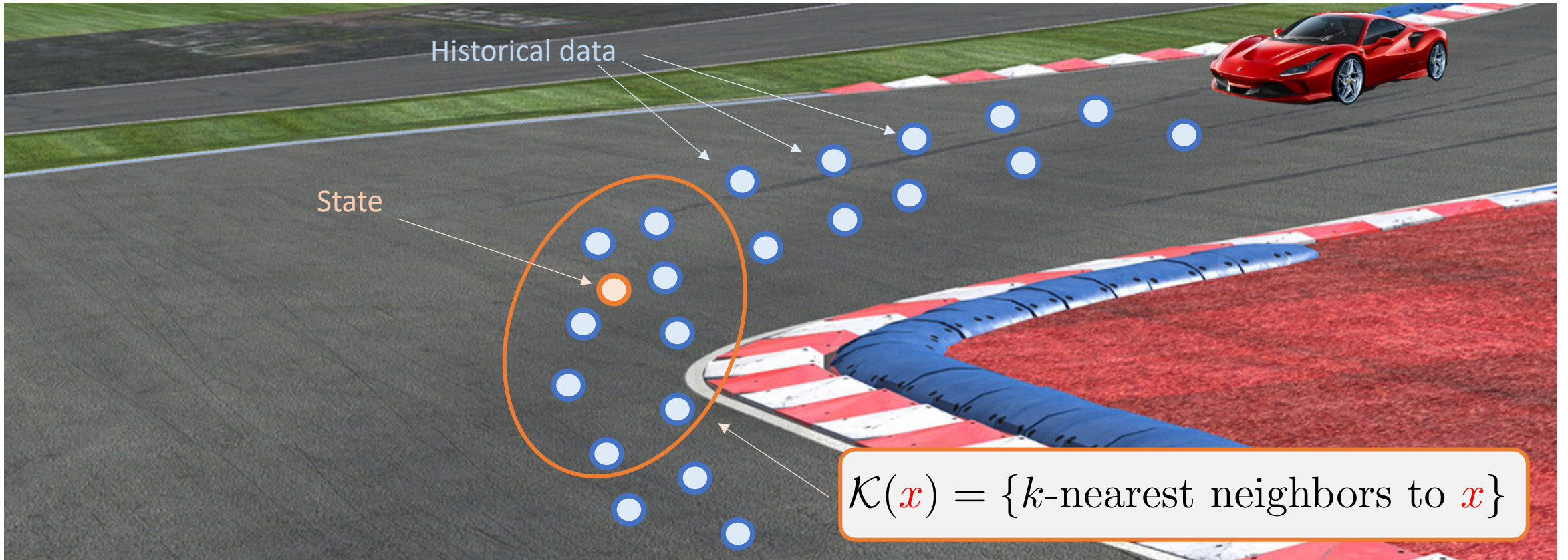
Local Approximations



Local Approximations



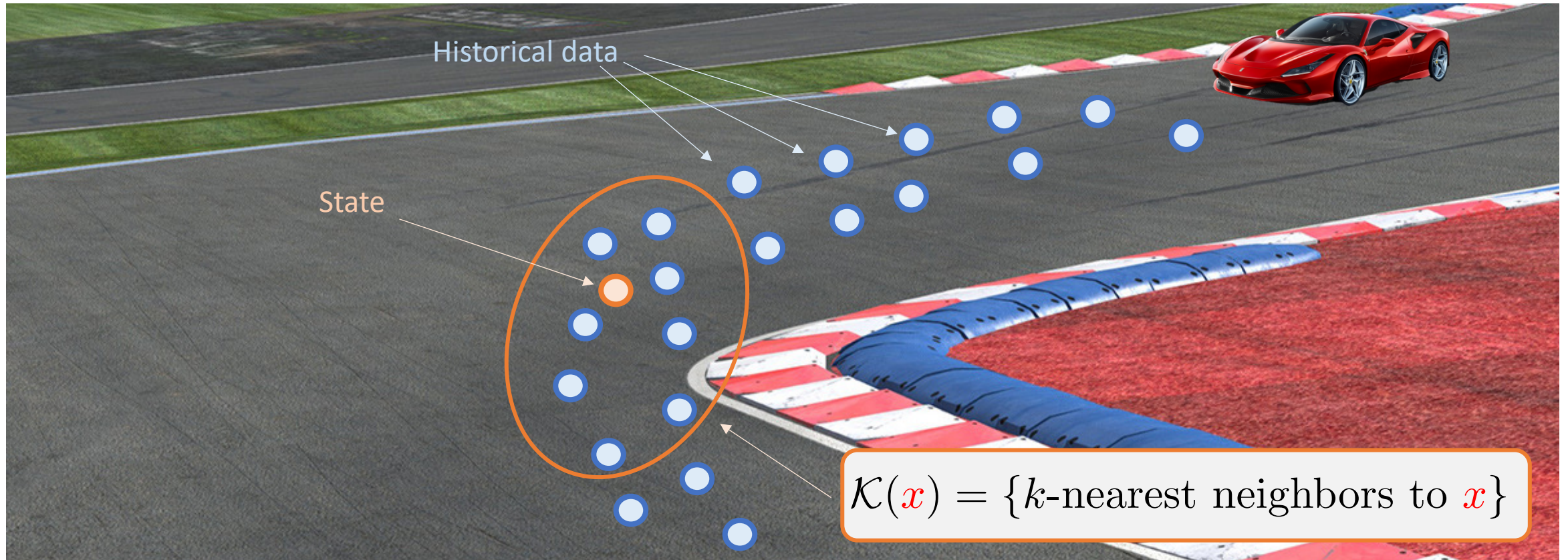
Local Approximations



Local convex safe set approximation:

$$\mathcal{CS}^j(\boldsymbol{x}) = \text{conv} \left(\bigcup_{\boldsymbol{x}_t^j \in \mathcal{K}(\boldsymbol{x})} \boldsymbol{x}_t^j \right)$$

Local Approximations



Local value function approximation:

$$V^j(\bar{x}, x) = \min_{\lambda_t^i \geq 0} \sum_{x_t^i \in \mathcal{K}^j(x)} J_t^i(x_t^i) \lambda_t^i$$

$$\text{subject to } \sum_{x_t^i \in \mathcal{K}^j(x)} x_t^i \lambda_t^i = \bar{x}, \sum_i \sum_t \lambda_t^i = 1$$

Learning Model Predictive Controller

At time t of iteration j solve the following Constrained Finite Time Optimal Control Problem (CFTOCP)

$$J_{t \rightarrow t+N}^{\text{LMPC},j}(x_t^j) = \min_{u_{t|t}^j, \dots, u_{t+N-1|t}^j} \sum_{k=t}^{t+N-1} h(x_{k|t}^j, u_{k|t}^j) + V^{j-1}(x_{t+N|t}^j, x_{t+N}^+)$$

s.t.

$$x_{k+1|t}^j = A_{k|t}^j x_{k|t}^j + B_{k|t}^j u_{k|t}^j + C_{k|t}^j$$

$$x_{t|t}^j = x_t^j,$$

$$x_{k|t}^j \in \mathcal{X}, \quad u_{k|t}^j \in \mathcal{U}, \quad \forall k \in [t, \dots, t+N-1]$$

$$x_{t+N|t}^j \in \mathcal{CS}^{j-1}(x_{t+N}^+),$$

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Computed online
using LMPC strategy

where $x_{t+N}^+ = x_{t+N-1|t-1}^*$

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where $x_{t+N}^+ = x_{t+N-1|t-1}^*$ or $x_{t+N}^+ = \hat{f}(x_{t+N-1|t-1}^*, u_f)$

Learning Model Predictive Controller

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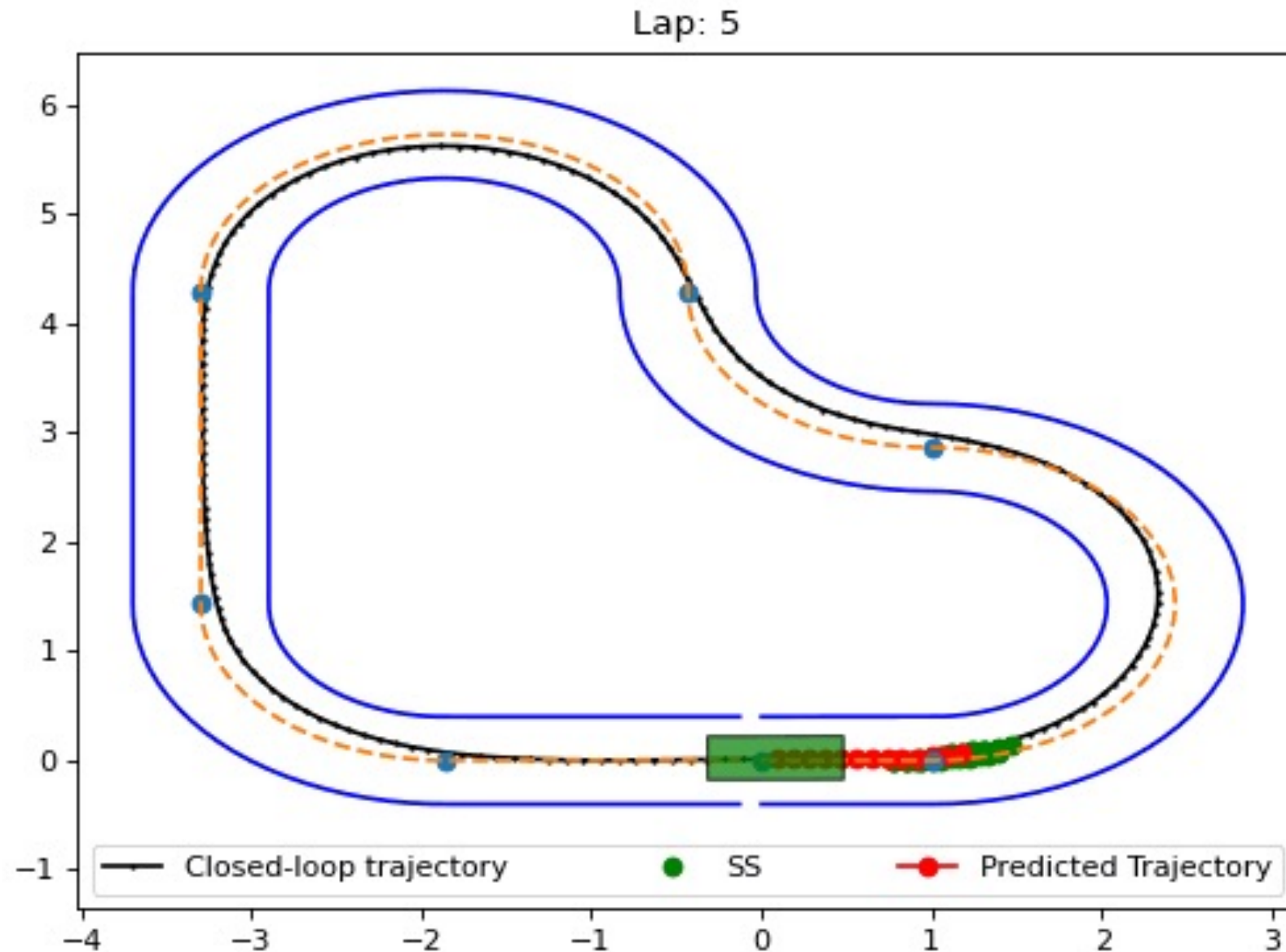
$$x_{t+N|t}^j \in \mathcal{CS}^{j-1}(x_{t+N}^+),$$

Computed
online

Computed online
using LMPC strategy

Then apply to the system the control input $u_t^j = u_{t|t}^{*,j}$

Safe Set and Value Function Approximations



System ID in Autonomous Racing

- Nonlinear Dynamical System,

$$\ddot{x} = \dot{y}\dot{\psi} + \frac{1}{m} \sum_i F_{x_i}$$

$$\ddot{y} = -\dot{x}\dot{\psi} + \frac{1}{m} \sum_i F_{y_i}$$

$$\ddot{\psi} = \frac{1}{I_z} (a(F_{y_{1,2}}) - b(F_{y_{2,3}}) + c(-F_{x_{1,3}} + F_{x_{2,4}}))$$

$$\dot{X} = \dot{x} \cos \psi - \dot{y} \sin \psi, \quad \dot{Y} = \dot{x} \sin \psi + \dot{y} \cos \psi$$

System ID in Autonomous Racing

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Kinematic Equations

System ID in Autonomous Racing

- Nonlinear Dynamical System,

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Kinematic Equations

- Identifying the Dynamical System

$$x_{k+1|t}^j = \begin{bmatrix} \dot{x}_{k+1|t} \\ \dot{y}_{k+1|t} \\ \ddot{\psi} \\ \psi_{k+1|t} \\ X_{k+1|t} \\ Y_{k+1|t} \end{bmatrix} = \begin{bmatrix} \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{bmatrix} x_{k|t}^j + \begin{bmatrix} \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{bmatrix} \begin{bmatrix} u_{k|t}^j \\ 1 \end{bmatrix}$$

Linearization around predicted trajectory

System ID in Autonomous Racing

► Nonlinear Dynamical System,

$$\begin{aligned}
 \ddot{x} &= \dot{y}\dot{\psi} + \frac{1}{m} \sum_i F_{x_i} \\
 \ddot{y} &= -\dot{x}\dot{\psi} + \frac{1}{m} \sum_i F_{y_i} \\
 \ddot{\psi} &= \frac{1}{I_z} (a(F_{y_{1,2}}) - b(F_{y_{2,3}}) + c(-F_{x_{1,3}} + F_{x_{2,4}})) \\
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 \end{aligned}$$

Dynamic Equations

Kinematic Equations

► Identifying the Dynamical System

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Dynamic Equations

Kinematic Equations

► Identifying the Dynamical System

Local Linear Regression

$$x_{k+1|t}^j = \begin{bmatrix} \dot{x}_{k+1|t} \\ \dot{y}_{k+1|t} \\ \ddot{\psi} \\ \psi_{k+1|t} \\ X_{k+1|t} \\ Y_{k+1|t} \end{bmatrix} = \begin{bmatrix} \arg \min_{i,s} \sum K(x_{k|t}^j - x_s^i) || \Lambda_y \begin{bmatrix} x_{k|t}^j \\ u_{k|t}^j \\ 1 \end{bmatrix} - y_{s+1}^i ||, \forall y \in \{\dot{x}, \dot{y}, \ddot{\psi}\} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{bmatrix} x_{k|t}^j + \begin{bmatrix} \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{bmatrix} \begin{bmatrix} u_{k|t}^j \\ 1 \end{bmatrix}$$

Linearization around predicted trajectory

System Identification – Design Steps

Identifying the Dynamical System

Local Linear Regression

$$x_{k+1|t}^j = \begin{bmatrix} \dot{x}_{k+1|t} \\ \dot{y}_{k+1|t} \\ \ddot{\psi} \\ \psi_{k+1|t} \\ X_{k+1|t} \\ Y_{k+1|t} \end{bmatrix} = \begin{bmatrix} \arg \min_{i,s} \sum K(x_{k|t}^j - x_s^i) || \Lambda_y \begin{bmatrix} x_{k|t}^j \\ u_{k|t}^j \\ 1 \end{bmatrix} - y_{s+1}^i ||, \forall y \in \{\dot{x}, \dot{y}, \ddot{\psi}\} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{bmatrix} x_{k|t}^j + \begin{bmatrix} \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{bmatrix} \begin{bmatrix} u_{k|t}^j \\ 1 \end{bmatrix}$$

Linearization around predicted trajectory

Implementation Details

System Identification – Design Steps

Identifying the Dynamical System

Local Linear Regression

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Linearization around predicted trajectory

Implementation Details

- Compute trajectory to linearize around from previous optimal inputs and stored data

System Identification – Design Steps

Identifying the Dynamical System

Local Linear Regression

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Linearization around predicted trajectory

Implementation Details

- ▶ Compute trajectory to linearize around from previous optimal inputs and stored data
- ▶ Enforce model-based sparsity in local linear regression

System Identification – Design Steps

Identifying the Dynamical System

Local Linear Regression

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Linearization around predicted trajectory

Implementation Details

- ▶ Compute trajectory to linearize around from previous optimal inputs and stored data
- ▶ Enforce model-based sparsity in local linear regression
- ▶ Perform local linear regression at the linearization points using a subset of the stored data

System Identification – Design Steps

Identifying the Dynamical System

Local Linear Regression

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Linearization around predicted trajectory

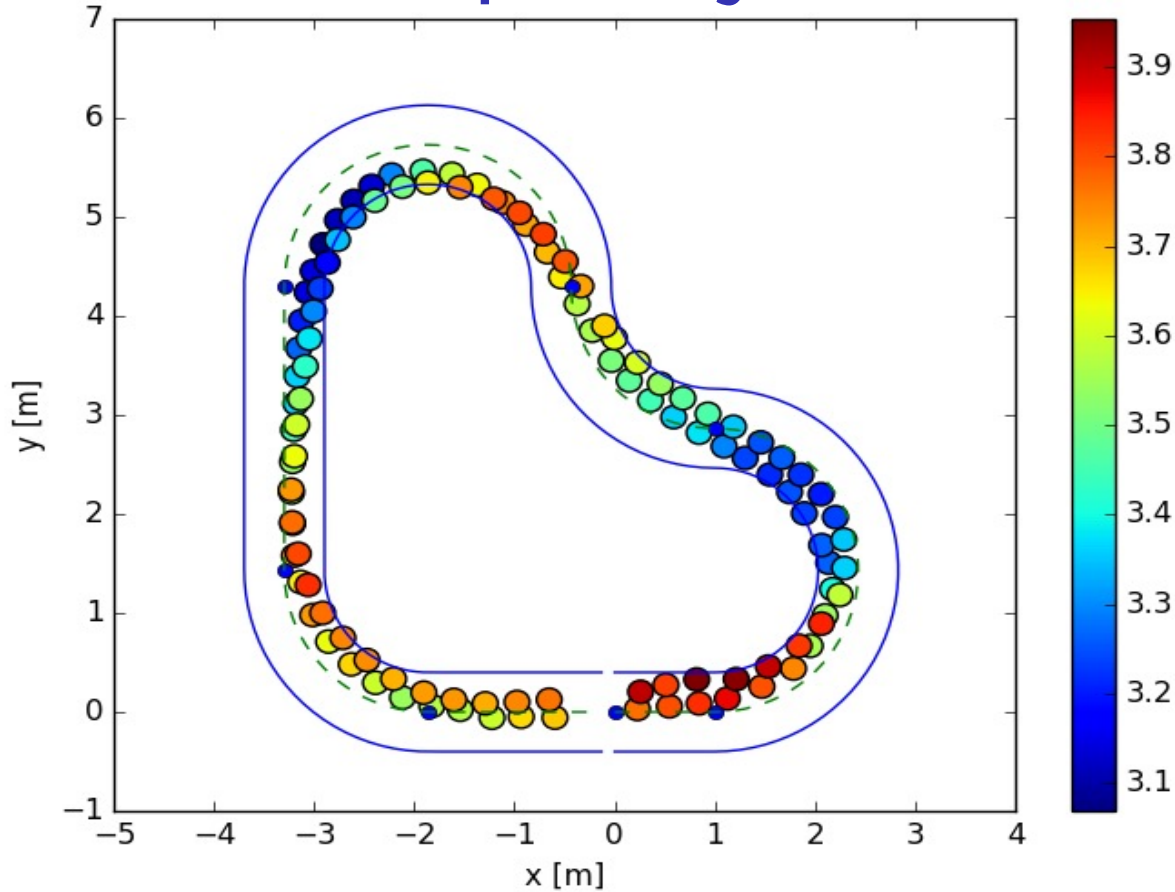
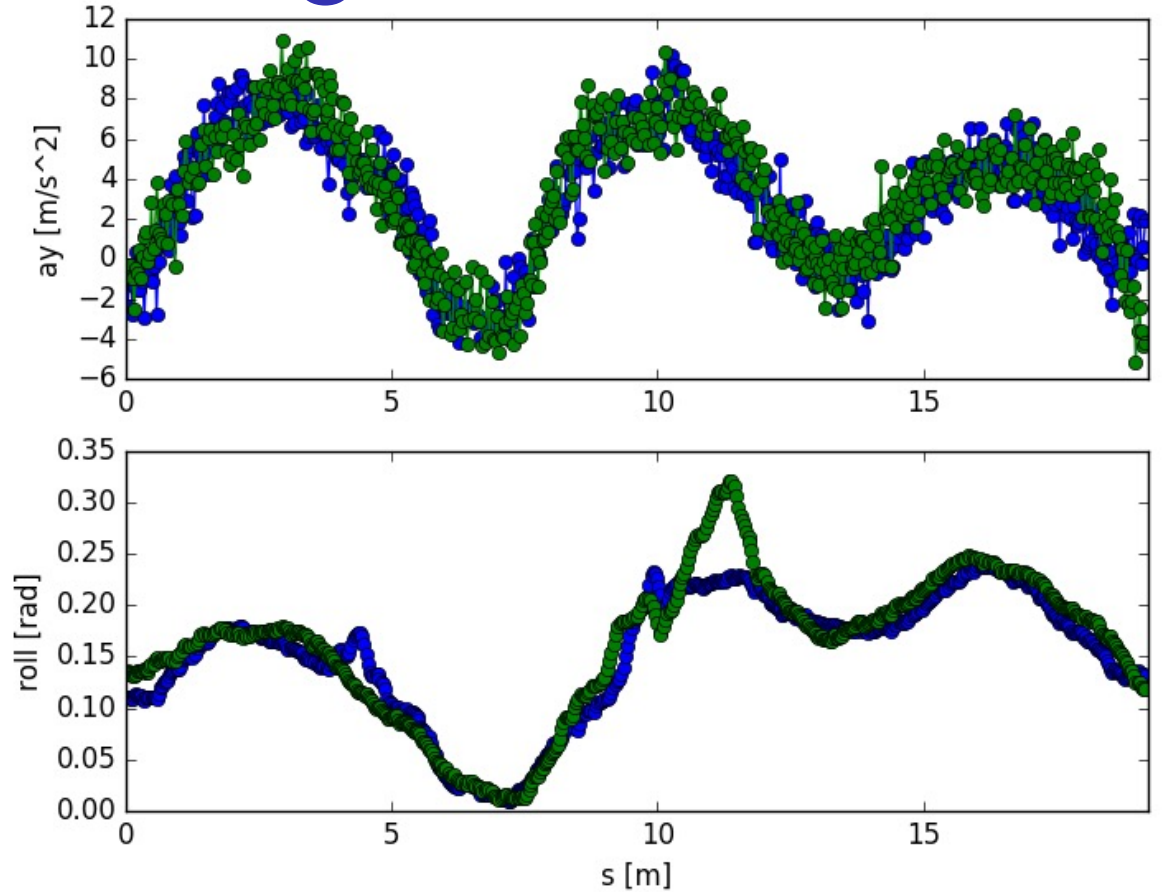
Implementation Details

- ▶ Compute trajectory to linearize around from previous optimal inputs and stored data
- ▶ Enforce model-based sparsity in local linear regression
- ▶ Perform local linear regression at the linearization points using a subset of the stored data
- ▶ Use kernel $K()$ to weight differently data as a function of distance to the linearization trajectory



Learning Model Predictive Control for Autonomous Racing

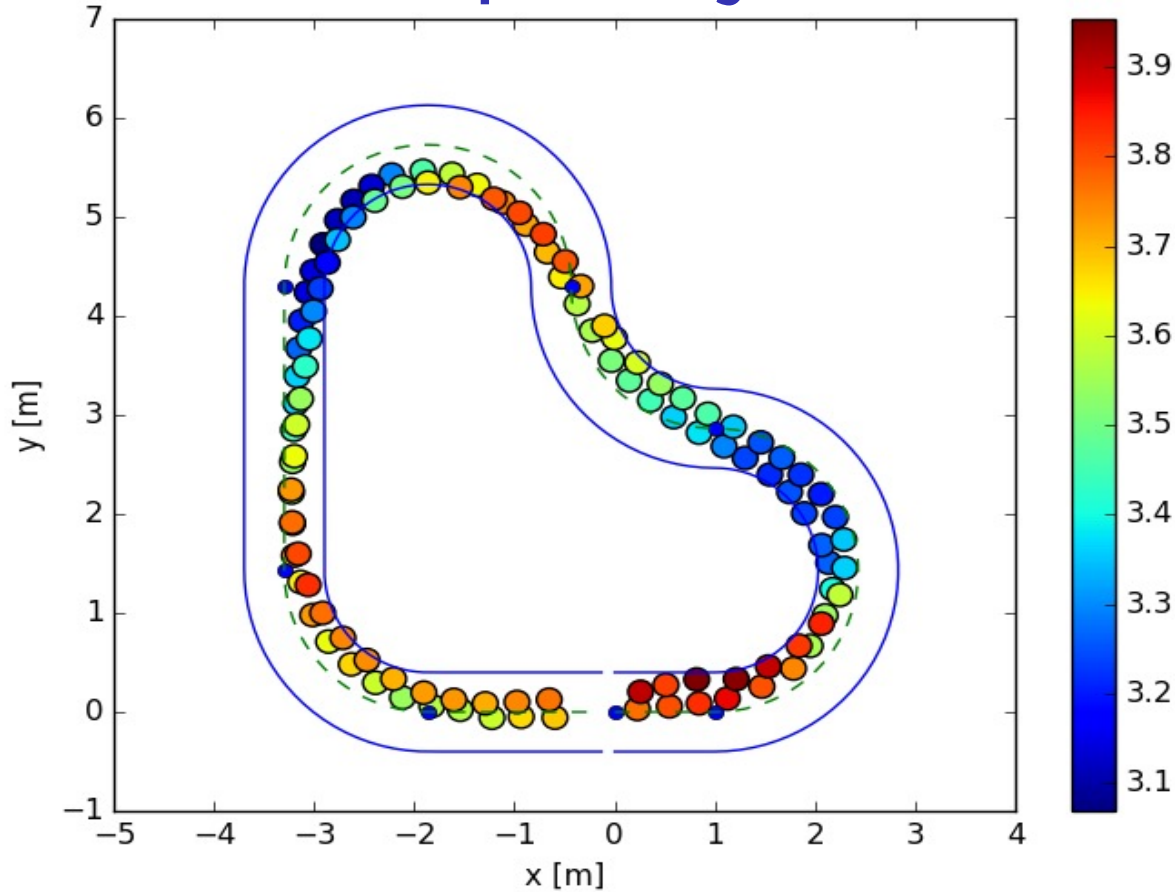
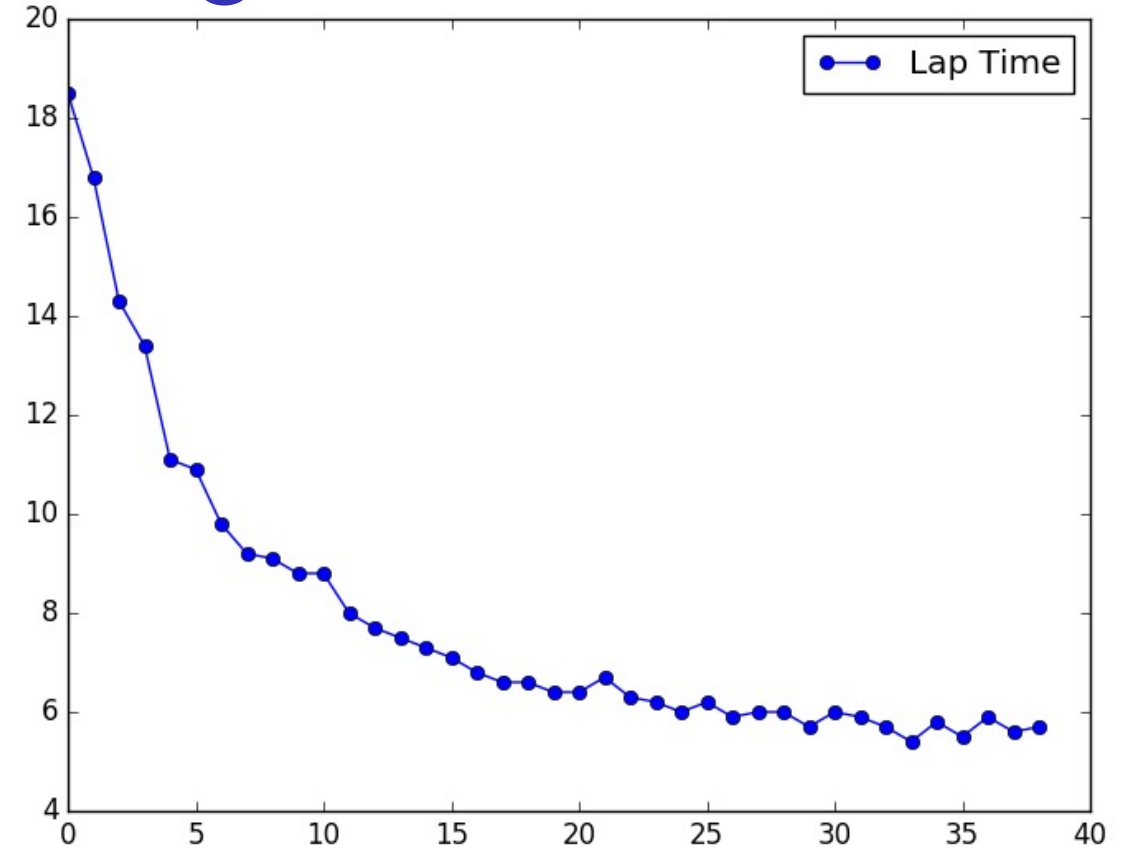
Closed-loop Trajectories at Convergence



Remarks

- ▶ A **reference trajectory is not needed** for the controller implementation
- ▶ The controller converges after few laps: the learning process is data efficient
- ▶ The **controller safely explores the state space** iteratively improving the lap time

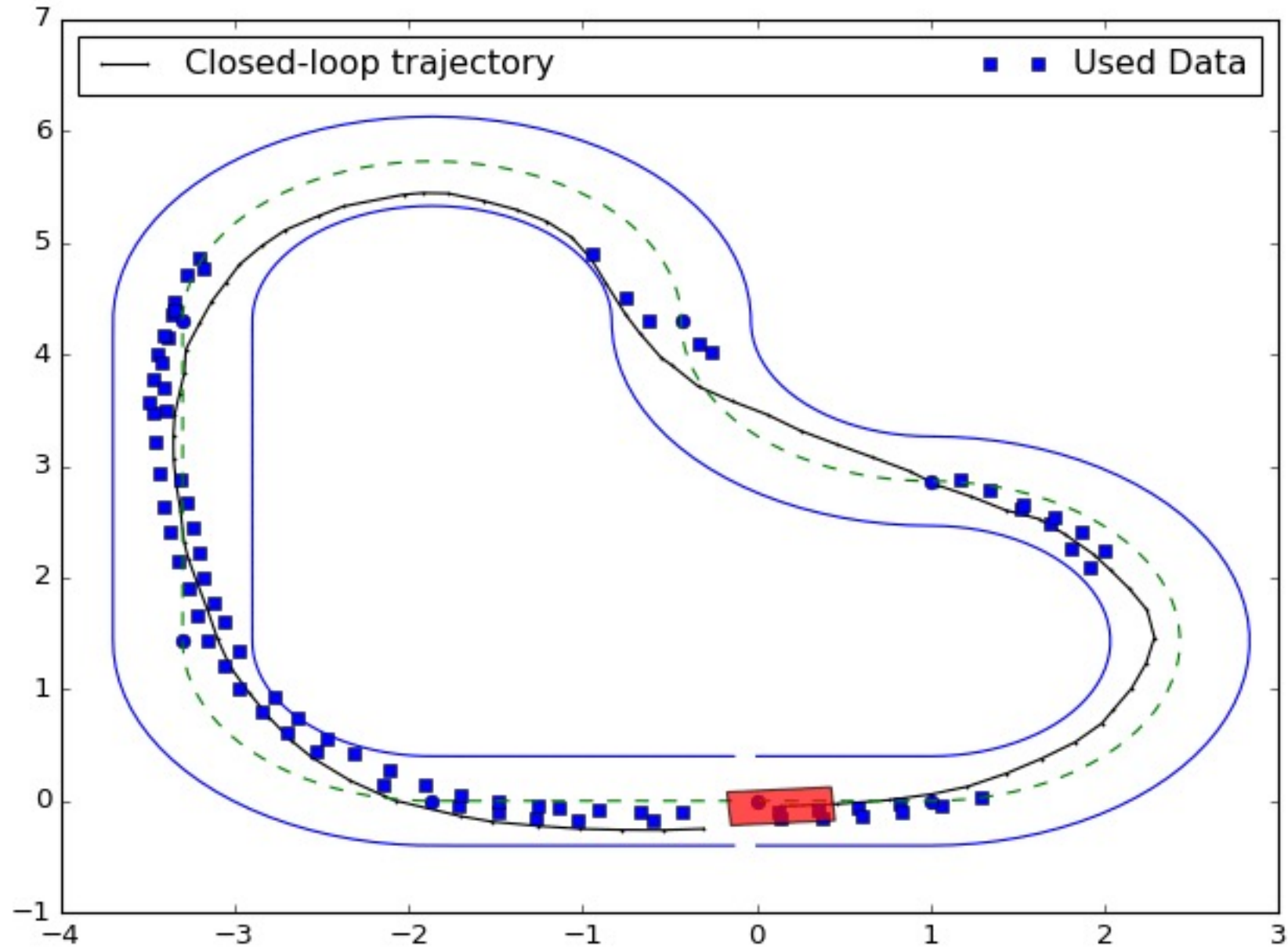
Closed-loop Trajectories at Convergence



Remarks

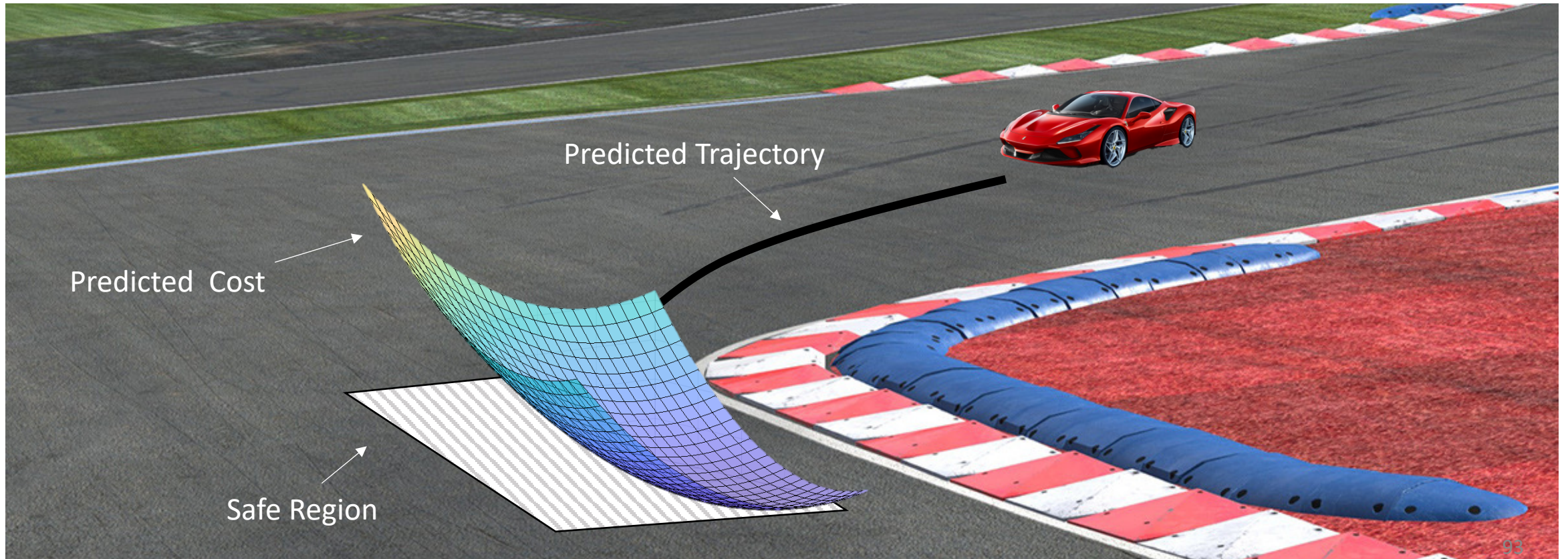
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- ▶ The controller converges after few laps: the learning process is data efficient
- ▶ The **controller safely explores the state space** iteratively improving the lap time

Data Point Selection



The key components

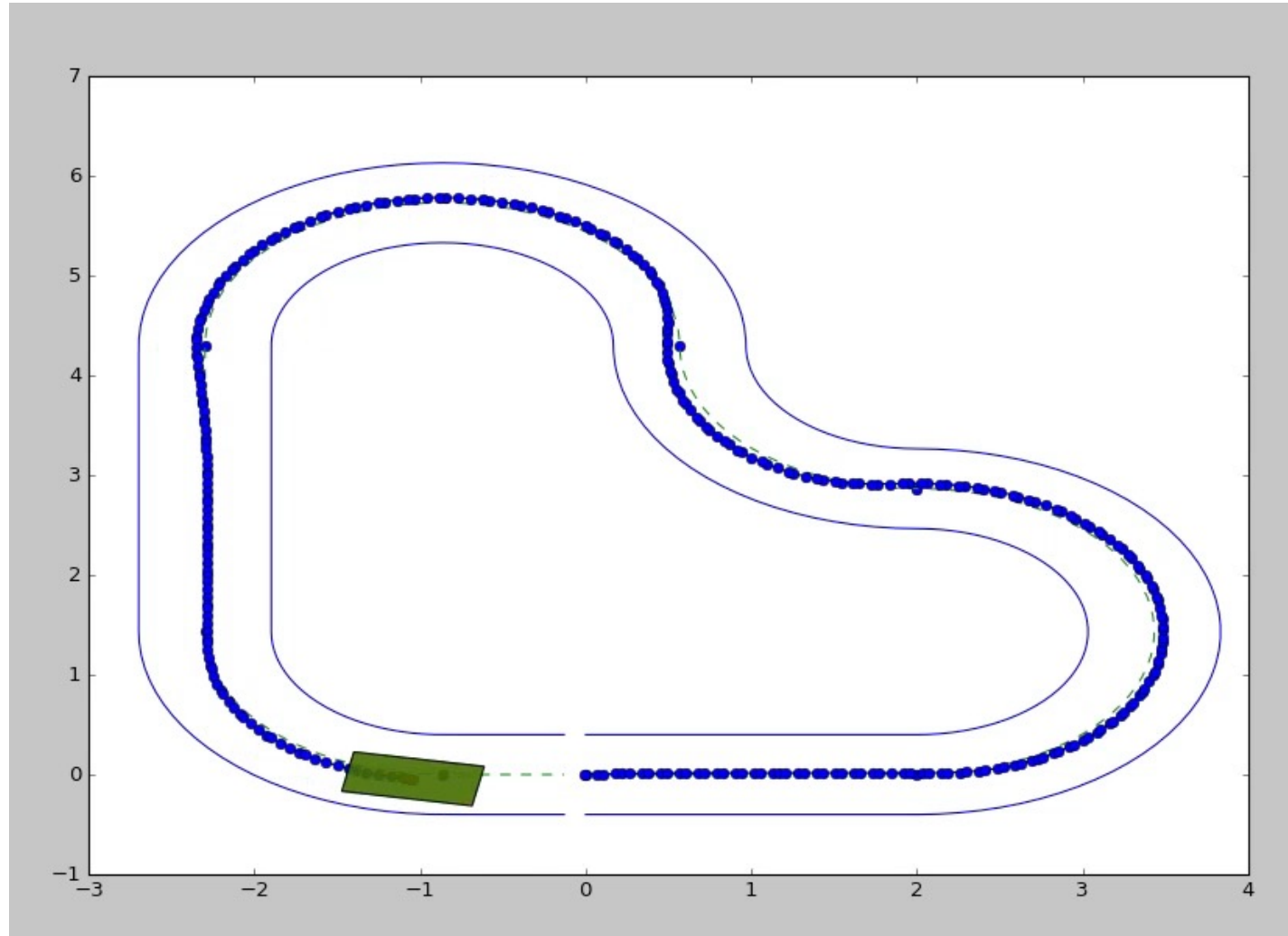
- ▶ Predicted trajectory given by **prediction model**
- ▶ Predicted cost estimated by **value function approximation**
- ▶ Safe region estimated by the **safe set**



Do you need the safe set? – Yes

LMPC without Invariant Set

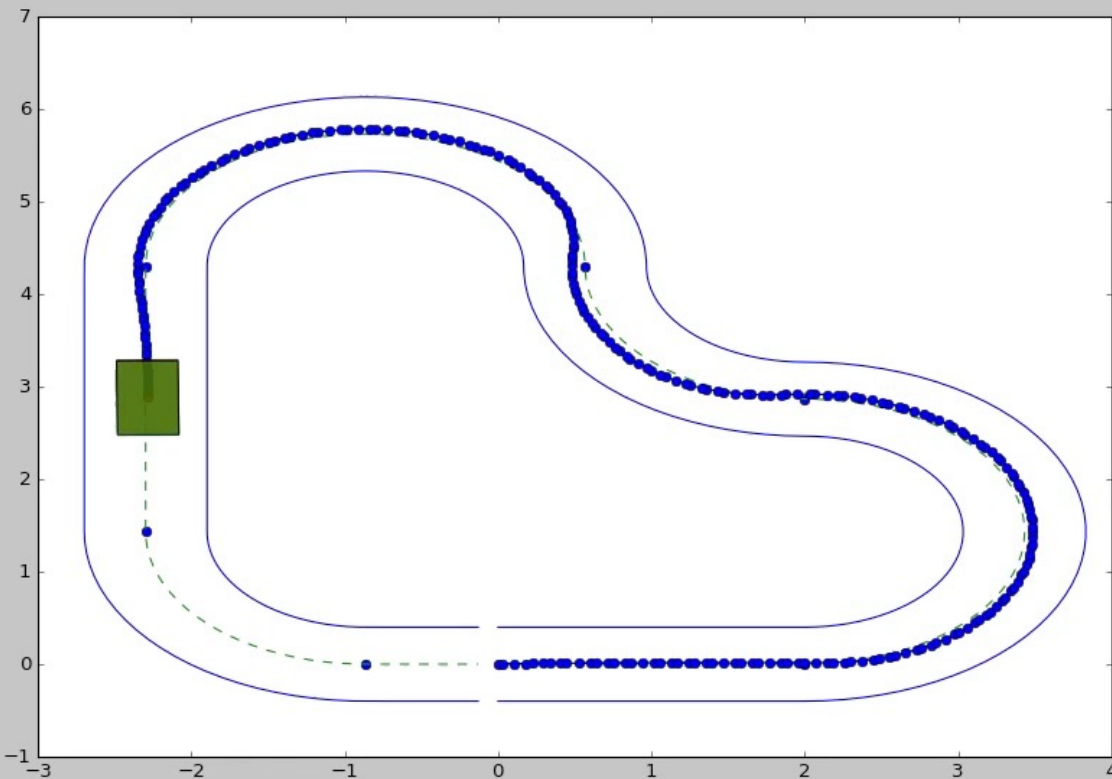
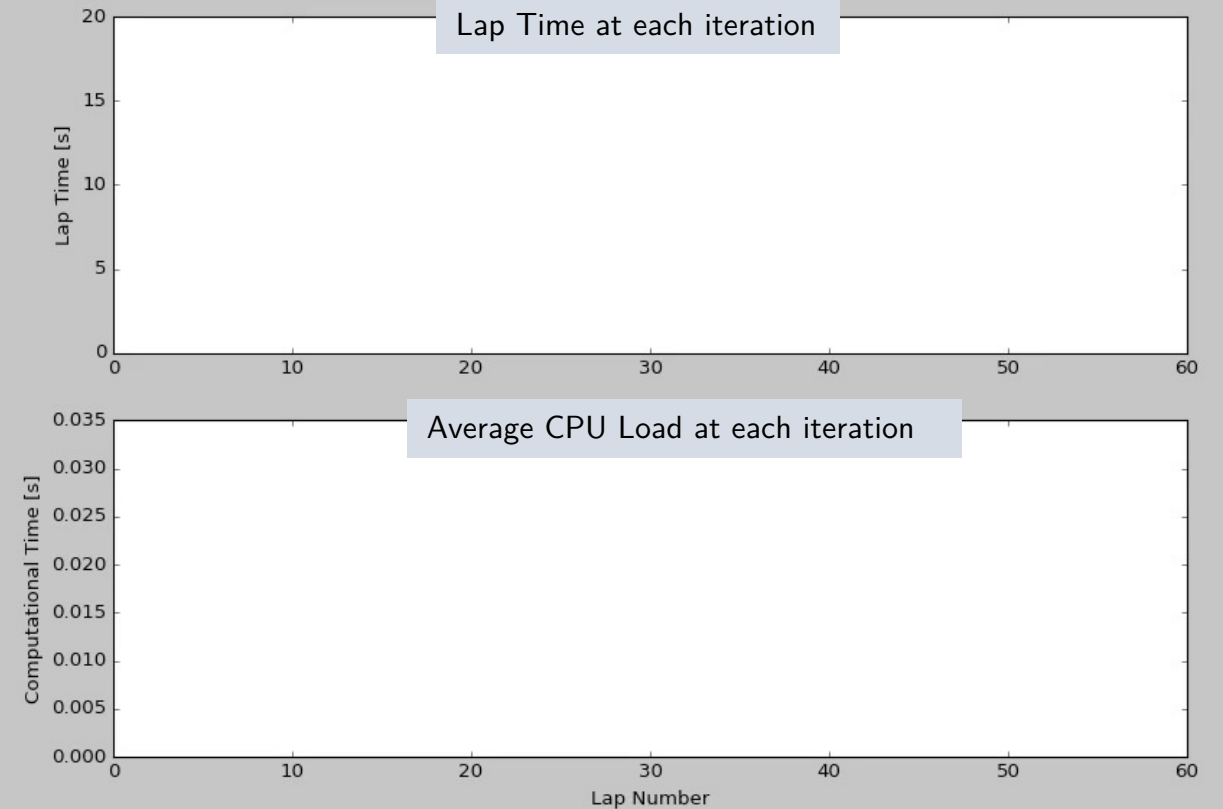
The controller extrapolates the value function on the V_x dimension



Do you need to Predict to Learn? Yes

When the LMPC horizon is $N = 1$ the controller

- ▶ solves the Bellman equation using the value function approximation
- ▶ does not explore the state space as it cannot plan outside the safe set



Comparison with Approximate DP (aka RL)

- ▶ Some references:
 - ❖ Bertsekas paper connecting MPC and ADP [1], books on RL and OC [2,3]
 - ❖ Lewis and Vrabie survey [4]
 - ❖ Recht survey [5]
- ▶ LMPC highlights
 - ❖ **Continuous** state and action formulation
 - ❖ Constraints satisfaction and **Sampled Safe Sets**
 - ❖ **Terminal constructed locally** based on cost/model driven exploration
 - ❖ Terminal at stored state is “exact” and **upperbounds** at intermediate points

[1] D. Bertsekas, “Dynamic programming and suboptimal control: A survey from ADP to MPC.” European Journal of Control 11.4-5 (2005)

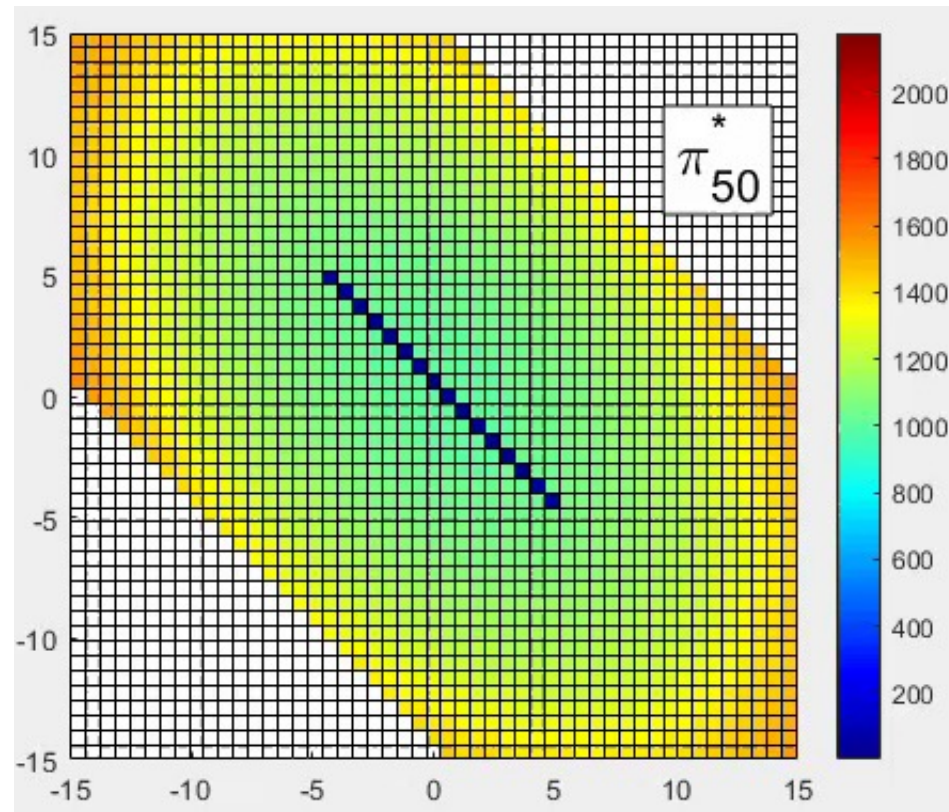
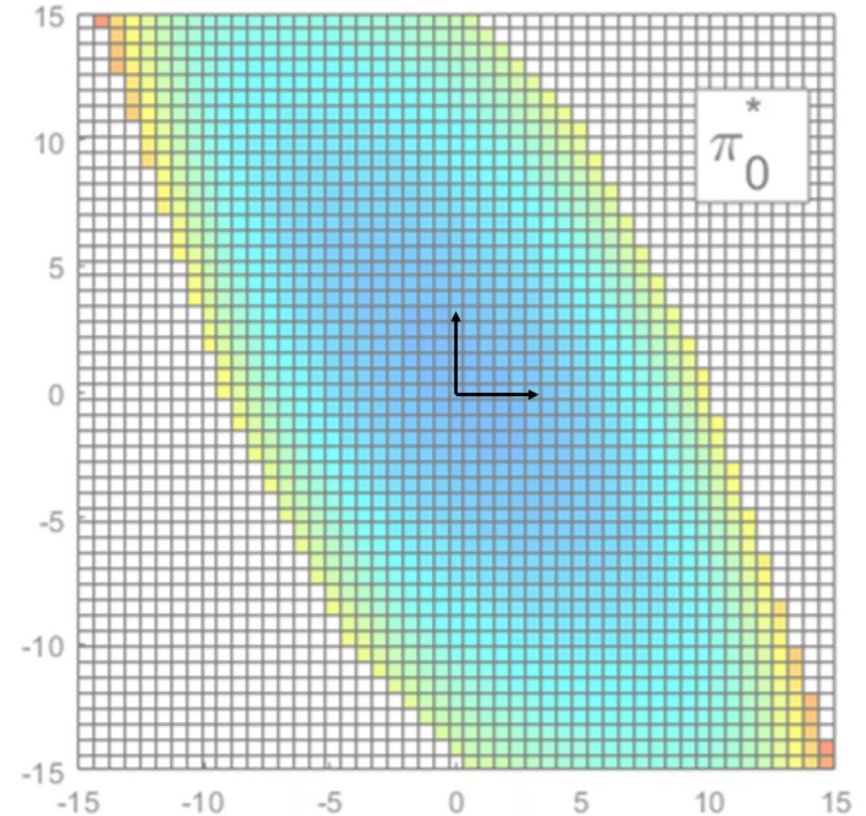
[2] D. Bertsekas, “Reinforcement learning and optimal control.” Athena Scientific, 2019.

[3] D. Bertsekas, “Distributed Reinforcement Learning” http://web.mit.edu/dimitrib/www/RL_2_Rollout_&_PI.pdf

[4] F. Lewis, Frank, and D. Vrabie. "Reinforcement learning and adaptive dynamic programming for feedback control." IEEE circuits and systems magazine 9.3 (2009)

[5] R. Benjamin. "A tour of reinforcement learning: The view from continuous control." Annual Review of Control, Robotics, and Autonomous Systems 2 (2019)

Forward Value Iteration



Dynamic Programming:

- ▶ Gridding, global properties
- ▶ Backward, one-step iteration

LMPC:

- ▶ No Gridding, local properties
- ▶ Forward, multi-step prediction
- ▶ LICQ required for optimality

Model Learning in MPC

A short and non-comprehensive summary

The complexity of the prediction model

The complexity of the prediction model

Linear

$$x_{k+1} = Ax_k + Bu_k + w_k$$

The complexity of the prediction model

Linear

$$x_{k+1} = Ax_k + Bu_k + w_k$$

Gray Box

known  unknown 

$$x_{k+1} = f(x_k, u_k) + g(x_k, u_k) + w_k$$

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Black Box

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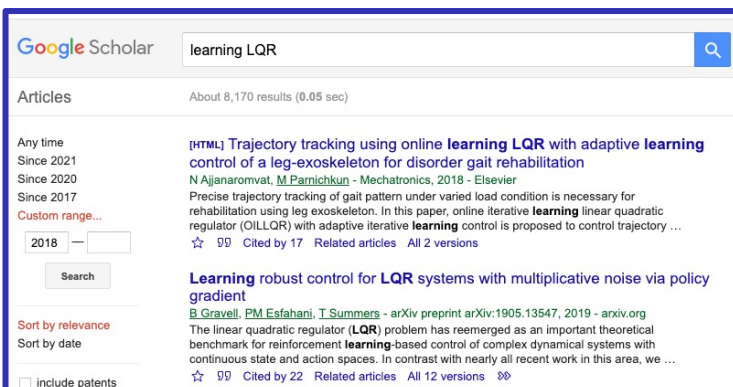
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The image shows two screenshots of Google Scholar search results for the query "learning LQR". The top screenshot shows "About 8,170 results (0.05 sec)" and a list of articles, with the first article highlighted. The bottom screenshot shows "About 1,440 results (0.05 sec)" and a list of articles, with the first article highlighted. Both screenshots have red circles around the search bar, the result count, and the first article.

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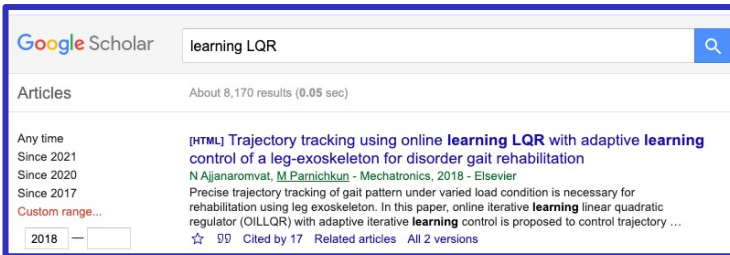
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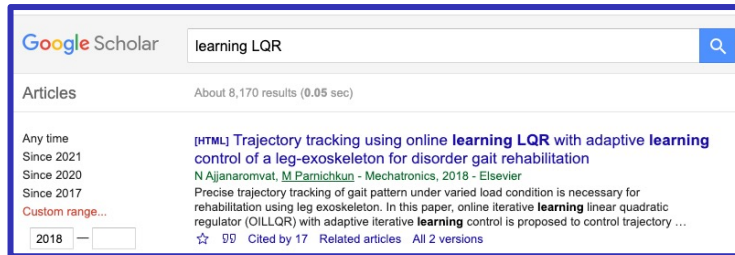
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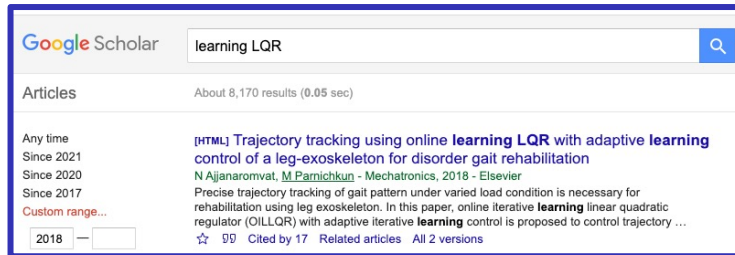
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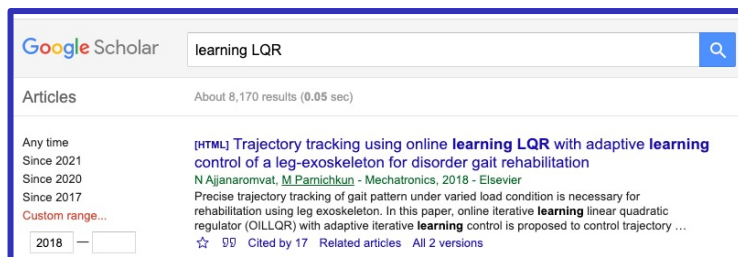
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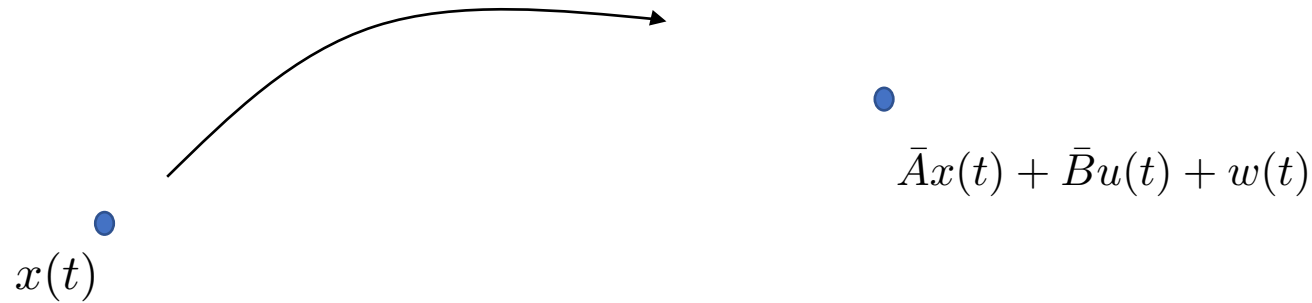

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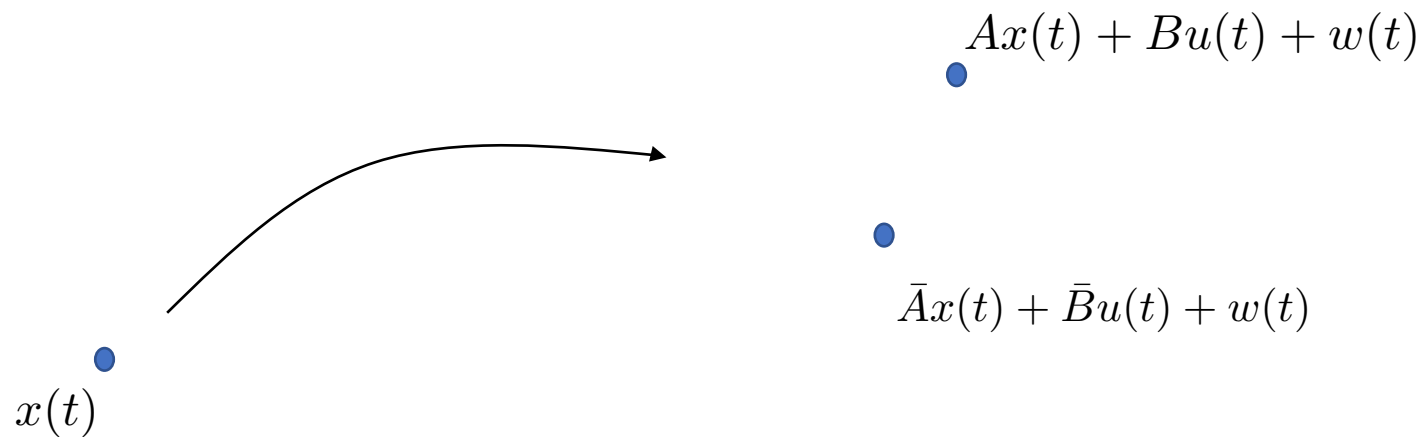


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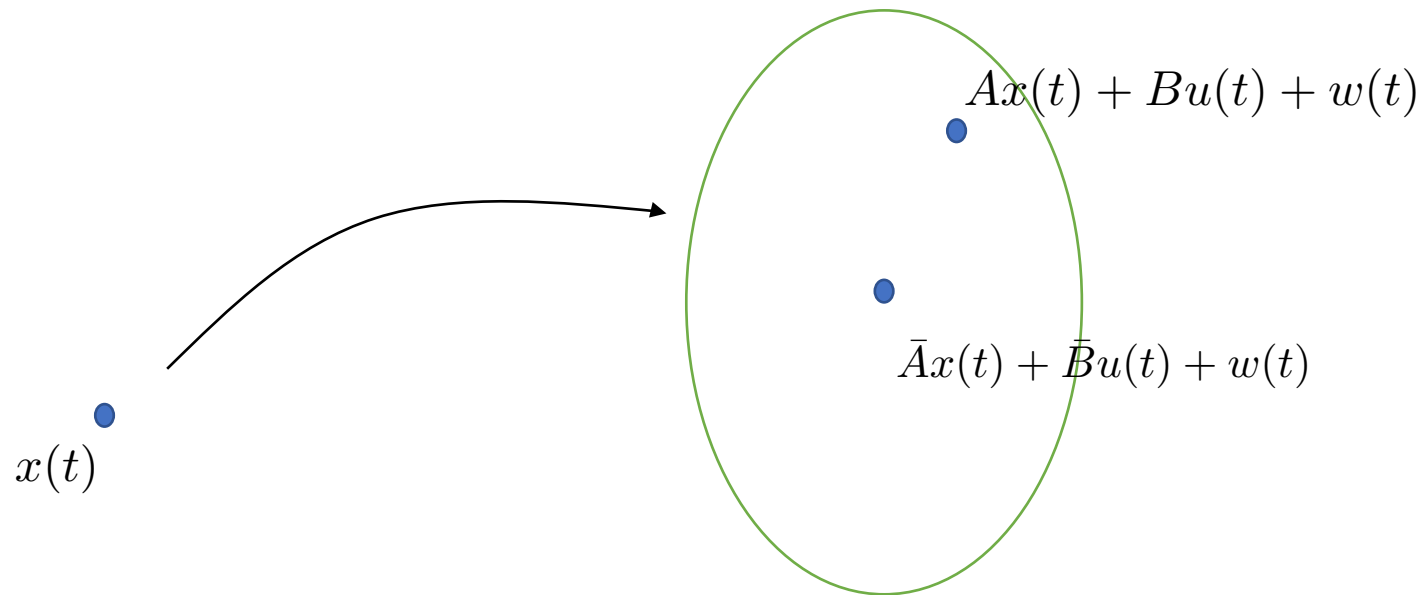


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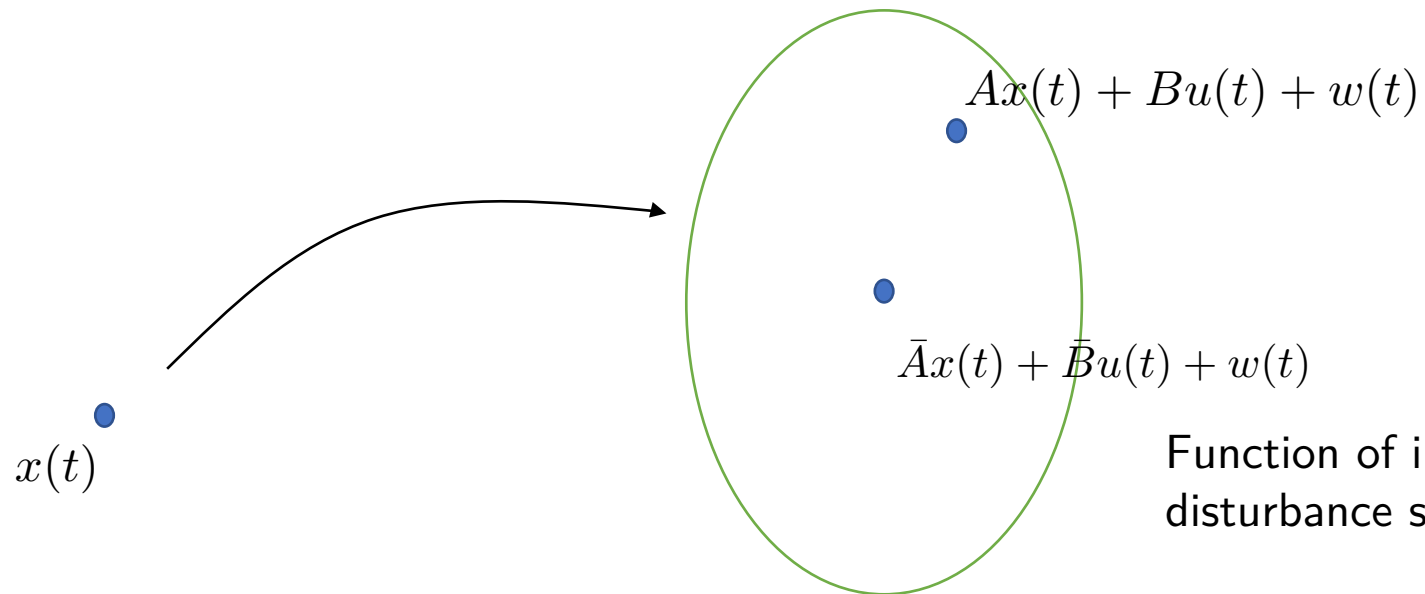


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Function of initial condition, applied input, disturbance support and the identification error:

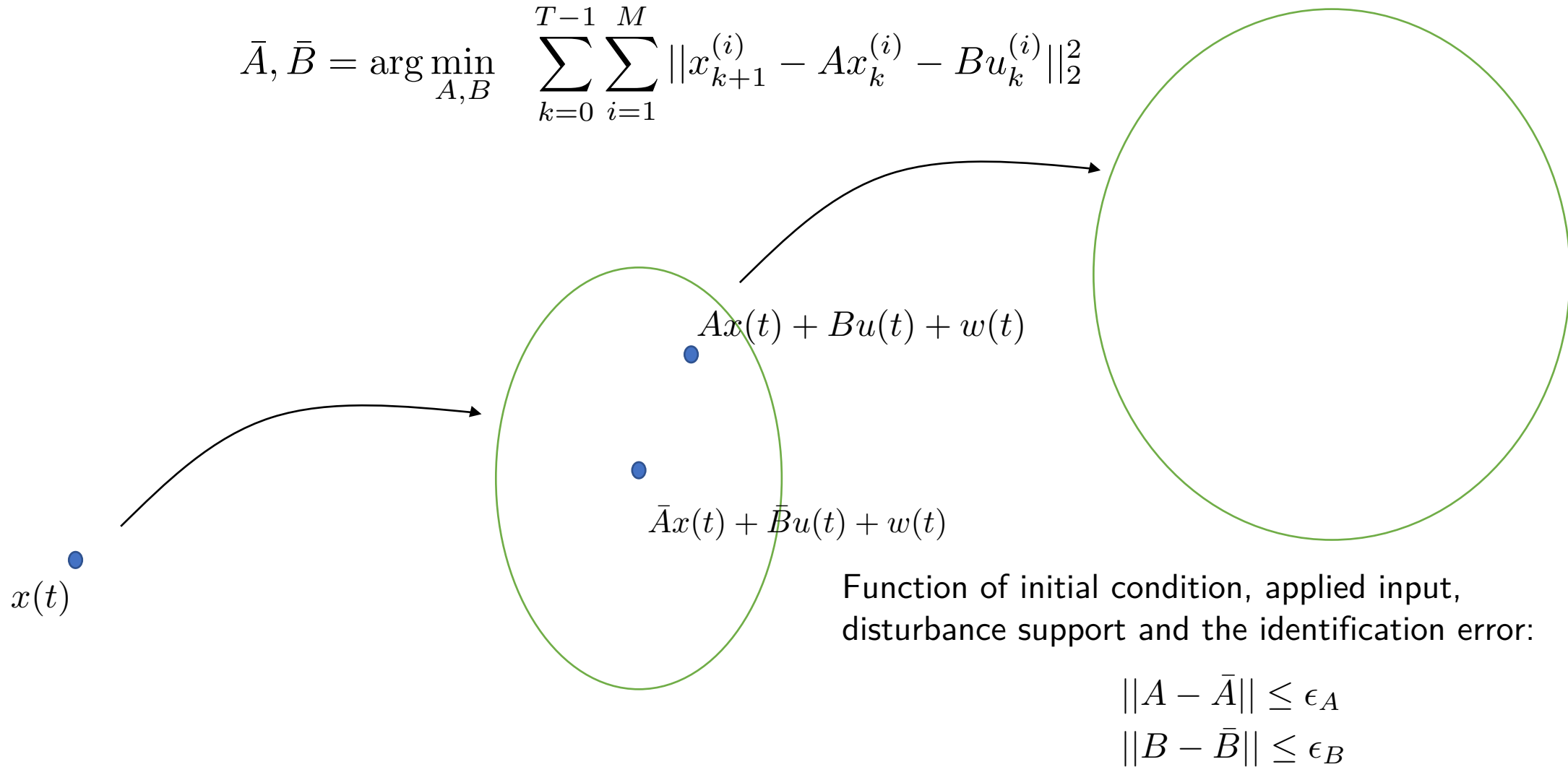
$$\|A - \bar{A}\| \leq \epsilon_A$$

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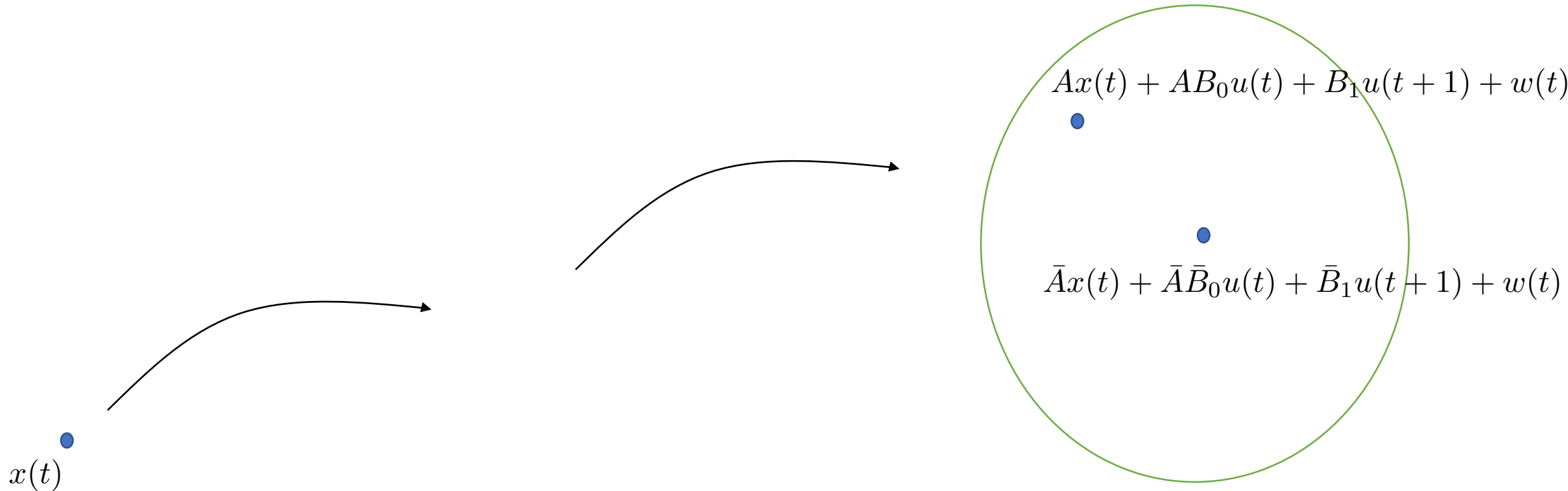
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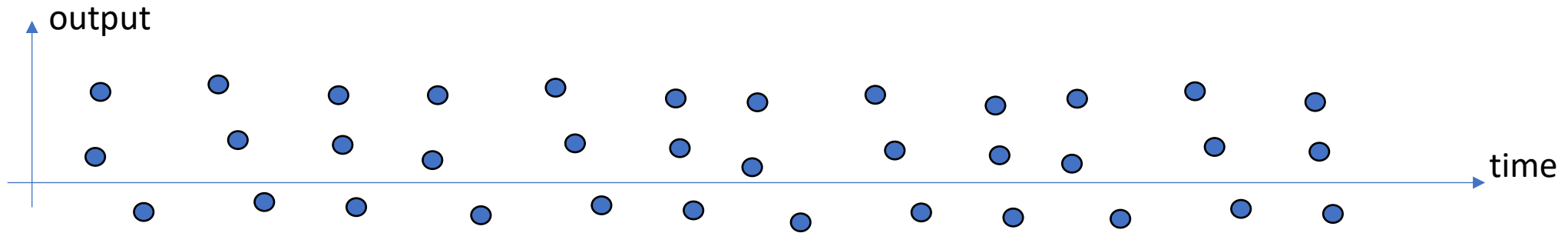
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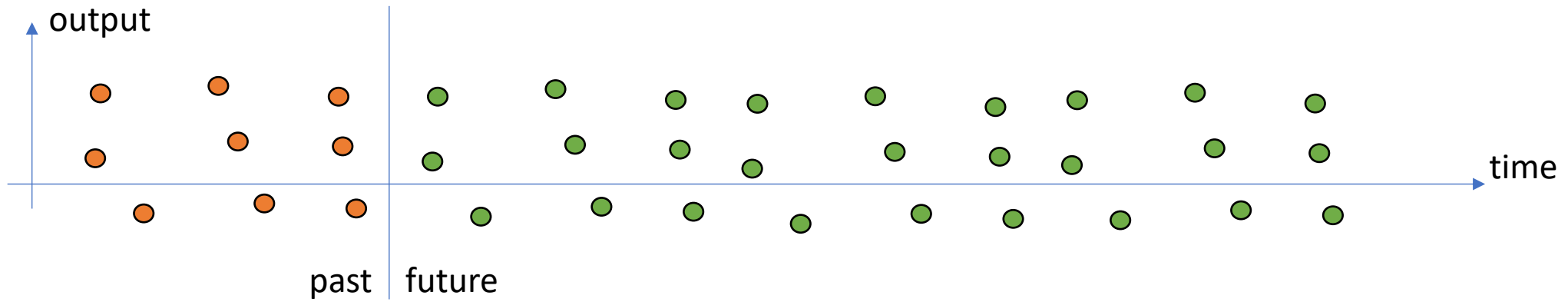


Learning Linear Models: Data-Based Prediction

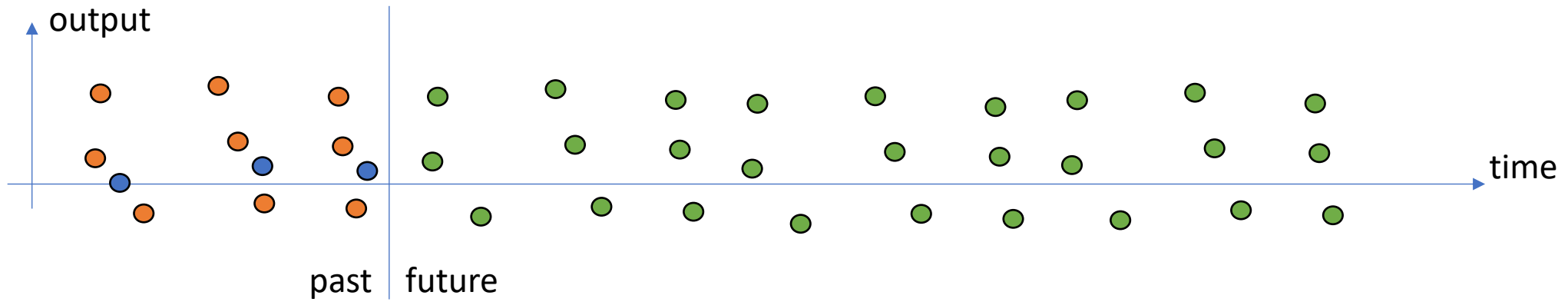
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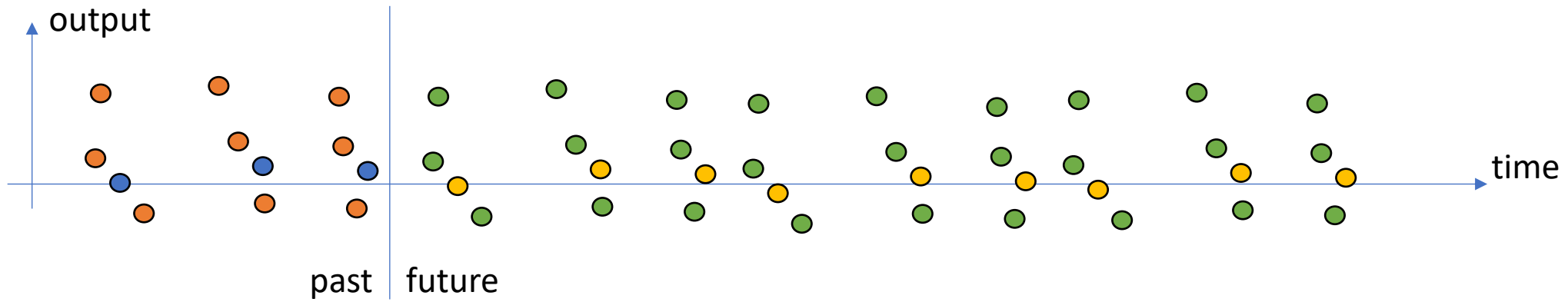
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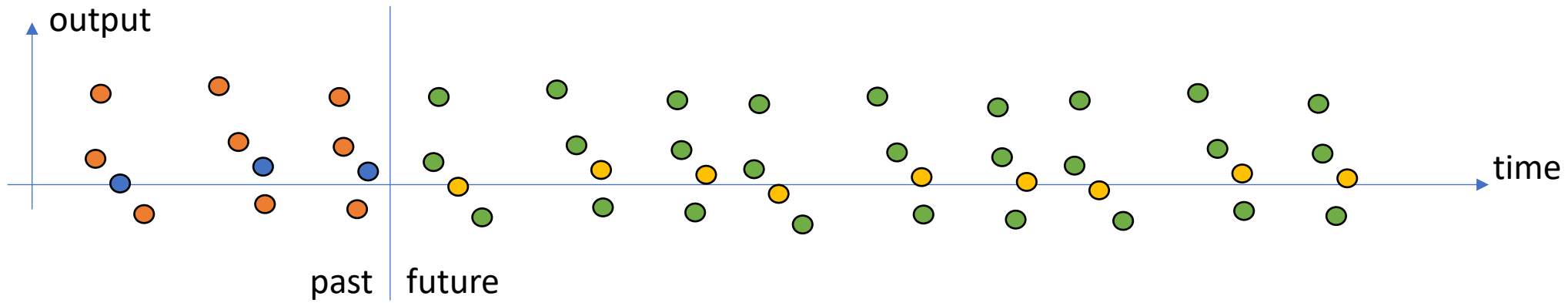
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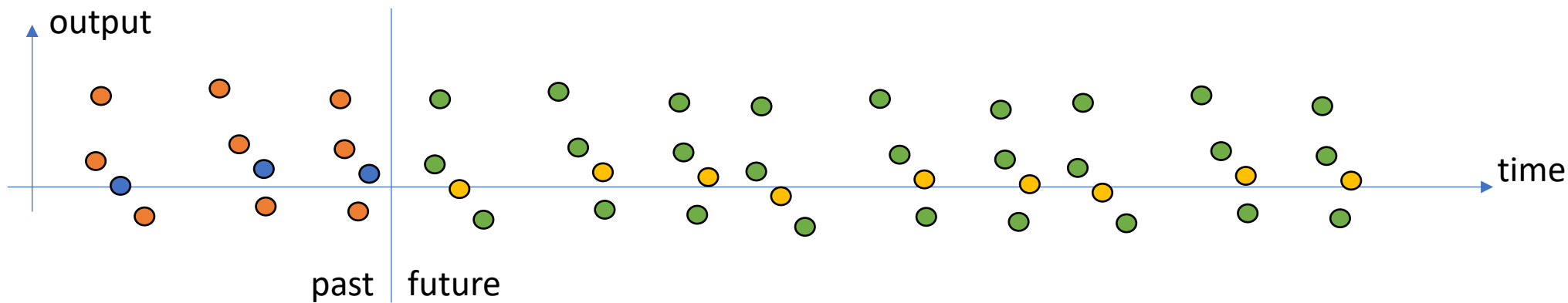


Construct the Hankel Matrix using Data

$$\mathcal{H}_L(u) := \begin{pmatrix} u_1 & \cdots & u_{T-L+1} \\ \vdots & \ddots & \vdots \\ u_L & \cdots & u_T \end{pmatrix}$$

$$\begin{pmatrix} U_p \\ U_f \end{pmatrix} := \mathcal{H}_{T_{\text{ini}}+N}(u^d), \quad \begin{pmatrix} Y_p \\ Y_f \end{pmatrix} := \mathcal{H}_{T_{\text{ini}}+N}(y^d),$$

Learning Linear Models: Data-Based Prediction



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Replace the model

$$\begin{aligned} & \underset{g, u, y}{\text{minimize}} && \sum_{k=0}^{N-1} \left(\|y_k - r_{t+k}\|_Q^2 + \|u_k\|_R^2 \right) \\ & \text{subject to} && \begin{pmatrix} U_p \\ Y_p \\ U_f \\ Y_f \end{pmatrix} g = \begin{pmatrix} u_{\text{ini}} \\ y_{\text{ini}} \\ u \\ y \end{pmatrix}, \\ & && u_k \in \mathcal{U}, \forall k \in \{0, \dots, N-1\}, \\ & && y_k \in \mathcal{Y}, \forall k \in \{0, \dots, N-1\}. \end{aligned}$$

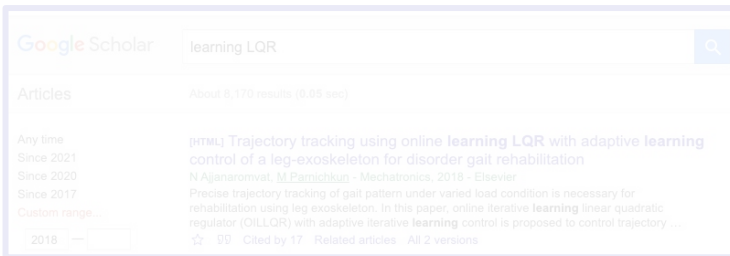
The complexity of the prediction model

Linear

$$x_{k+1} = Ax_k + Bu_k + w_k$$

Marco C., Campi, and Erik Weyer. "Finite sample properties of system identification methods." *IEEE Transactions on Automatic Control* 47.8 (2002): 1329-1334.

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Gray Box

$$x_{k+1} = f(x_k, u_k) + g(x_k, u_k) + w_k$$

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Kendall Lowrey, Aravind Rajeswaran, Sham Kakade, Emanuel Todorov, and Igor Mordatch. "Plan online, learn offline: Efficient learning and exploration via model-based control." in *Proc. Int. Conf. Mach. Learn.*, 2019.

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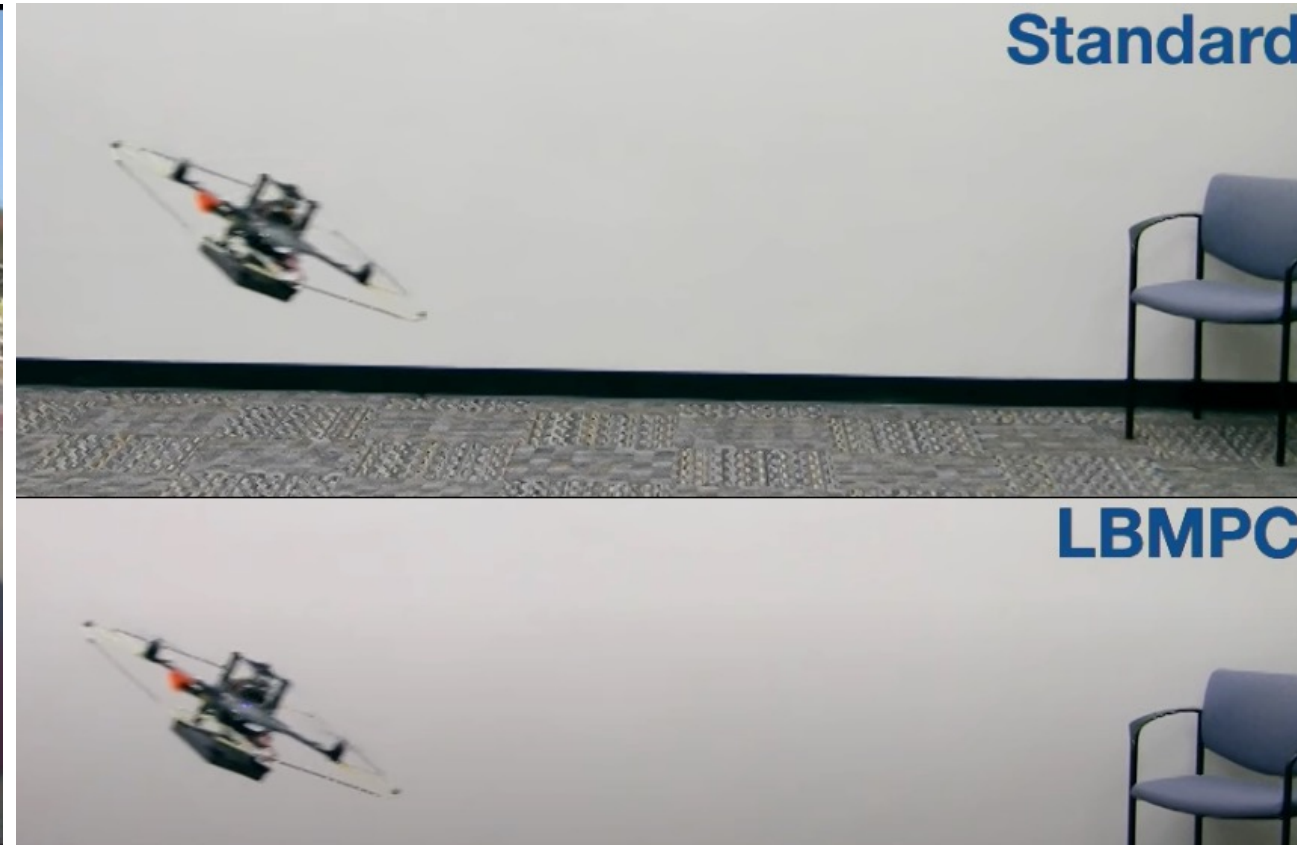
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Bouffard, Patrick, Anil Aswani, and Claire Tomlin. "Learning-based model predictive control on a quadrotor: Onboard implementation and experimental results." In *2012 IEEE International Conference on Robotics and Automation*, pp. 279-284. IEEE, 2012.

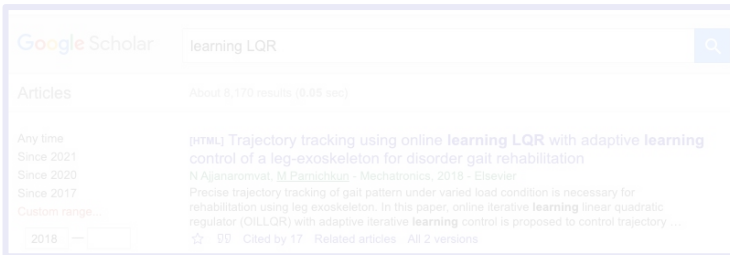
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Deisenroth, Marc, and Carl E. Rasmussen. "PILCO: A model-based and data-efficient approach to policy search." In *Proceedings of the 28th International Conference on machine learning (ICML-11)*, pp. 465-472. 2011.

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Black Box Model

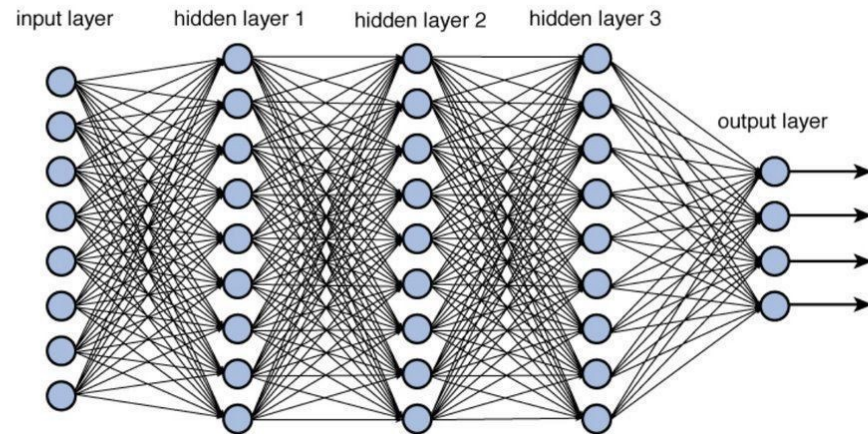
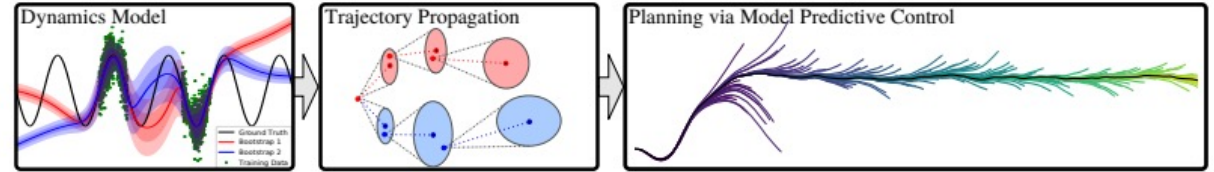


Figure 1: Our method (PE-TS): **Model:** Our probabilistic ensemble (PE) dynamics model is shown as an ensemble of two bootstraps (bootstrap disagreement far from data captures epistemic uncertainty: our subjective uncertainty due to a lack of data), each a probabilistic neural network that captures aleatoric uncertainty (inherent variance of the observed data). **Propagation:** Our trajectory sampling (TS) propagation technique uses our dynamics model to re-sample each particle (with associated bootstrap) according to its probabilistic prediction at each point in time, up until horizon T . **Planning:** At each time step, our MPC algorithm computes an optimal action sequence, applies the first action in the sequence, and repeats until the task-horizon.

Black Box Model

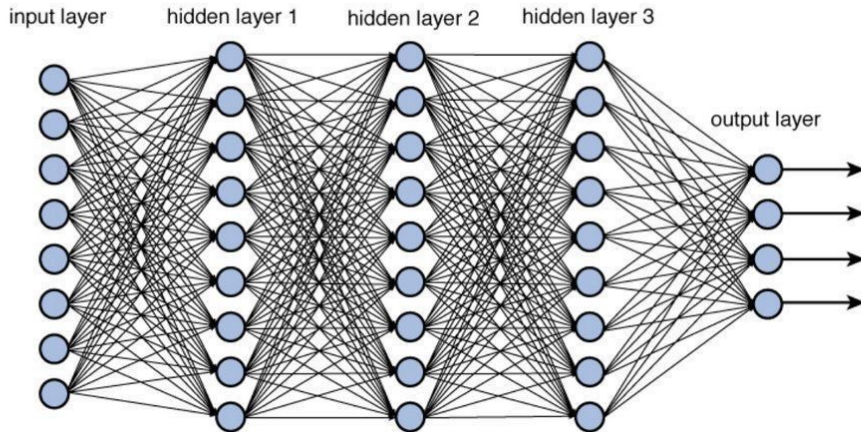
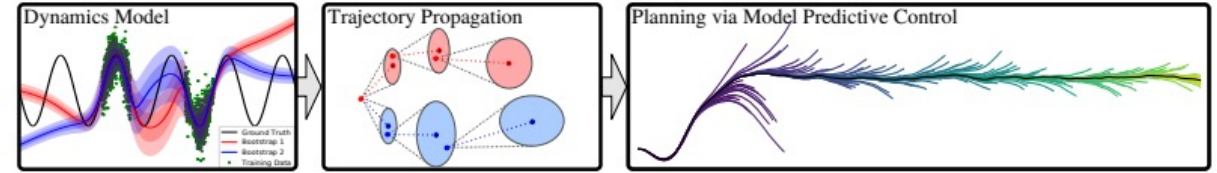
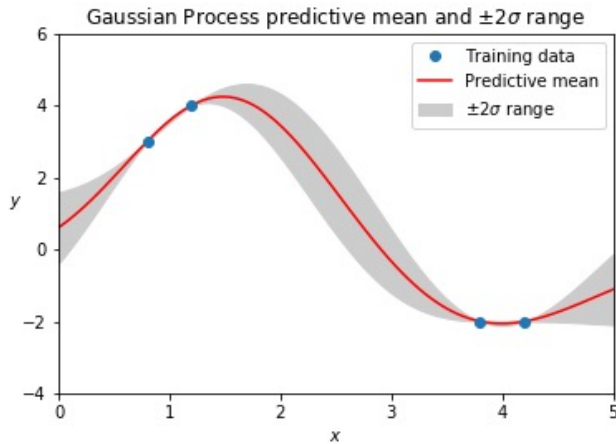
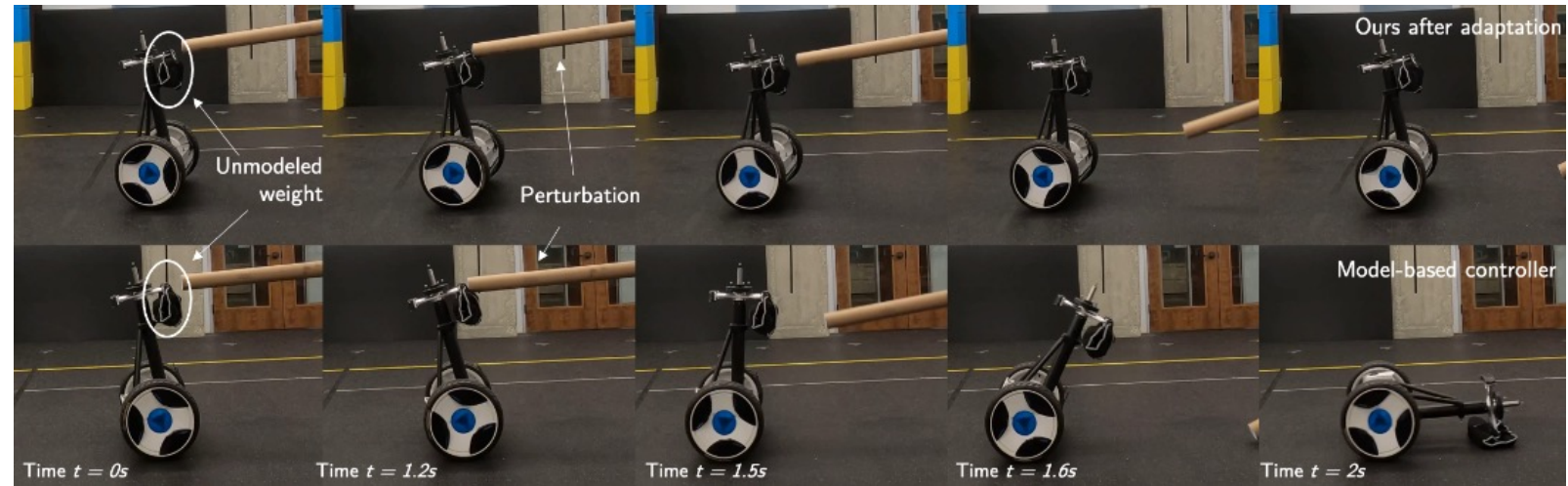


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Ivan D. Jimenez, Rodriguez, Ugo Rosolia, Aaron D. Ames, and Yisong Yue. "Learning Unstable Dynamics with One Minute of Data: A Differentiation-based Gaussian Process Approach." *arXiv preprint arXiv:2103.04548* (2021).

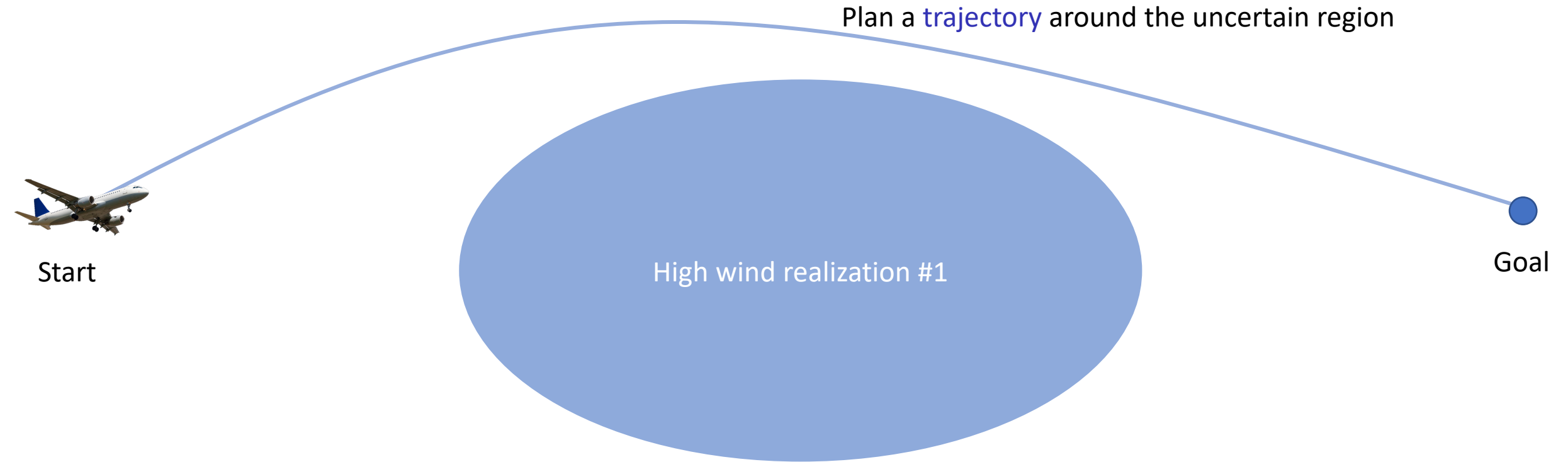
Planning under uncertainty

Optimizing over control policies

Why is planning in uncertain environments harder?



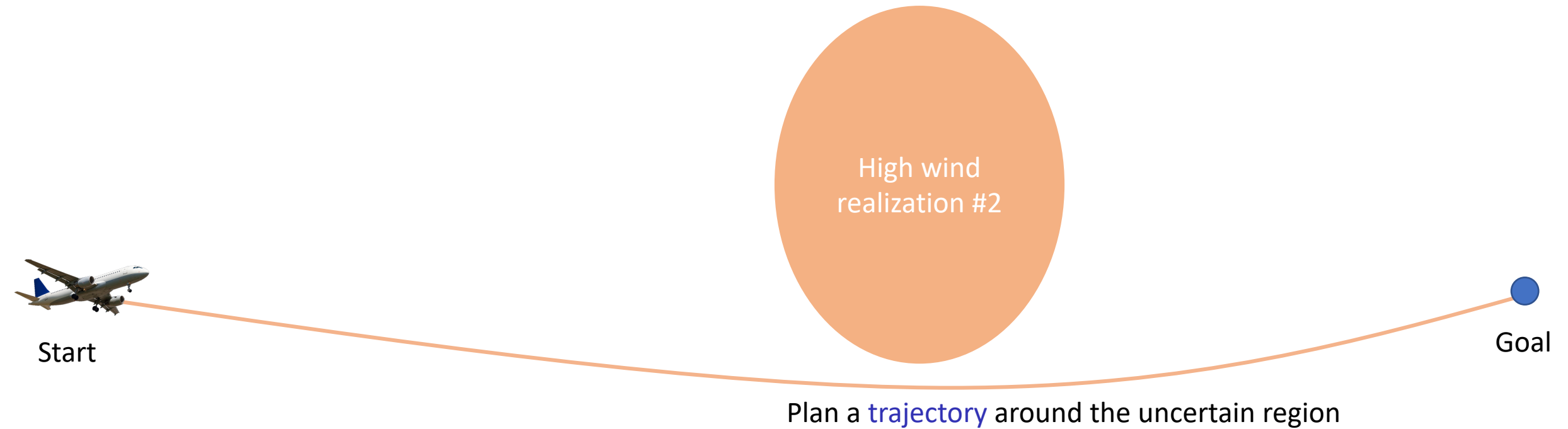
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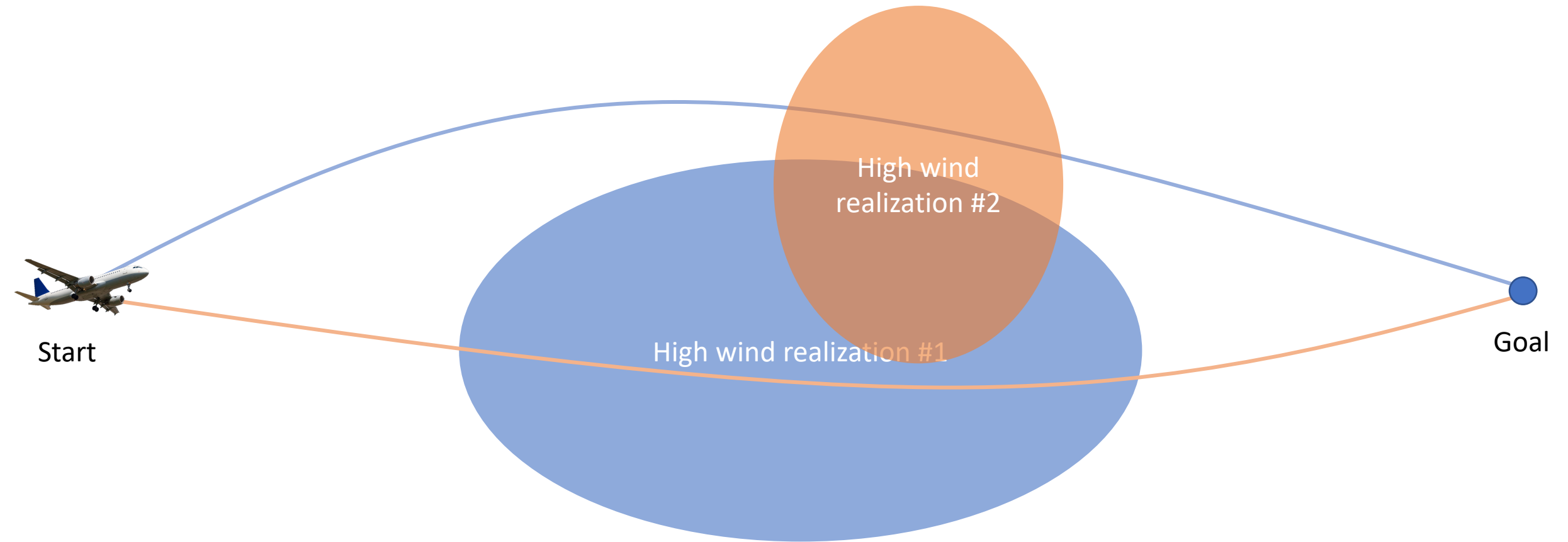
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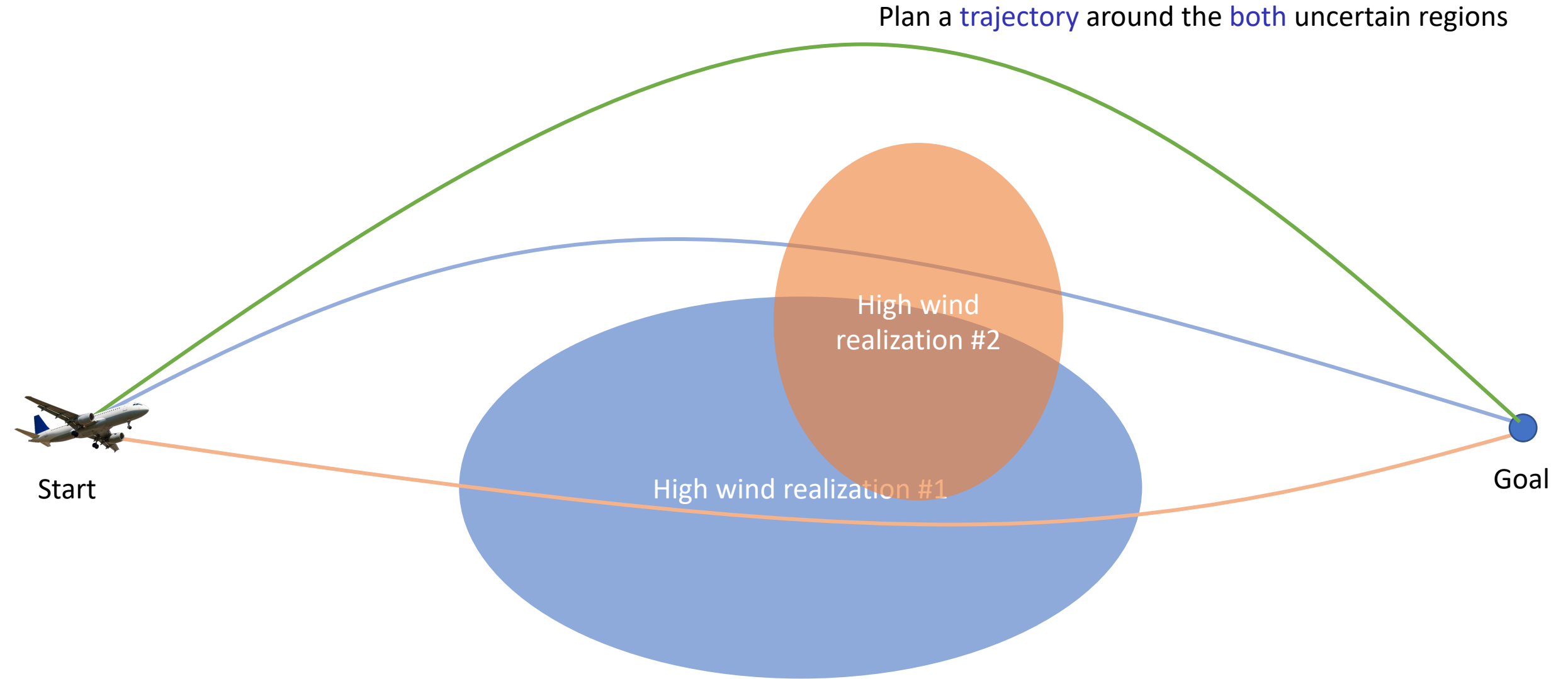
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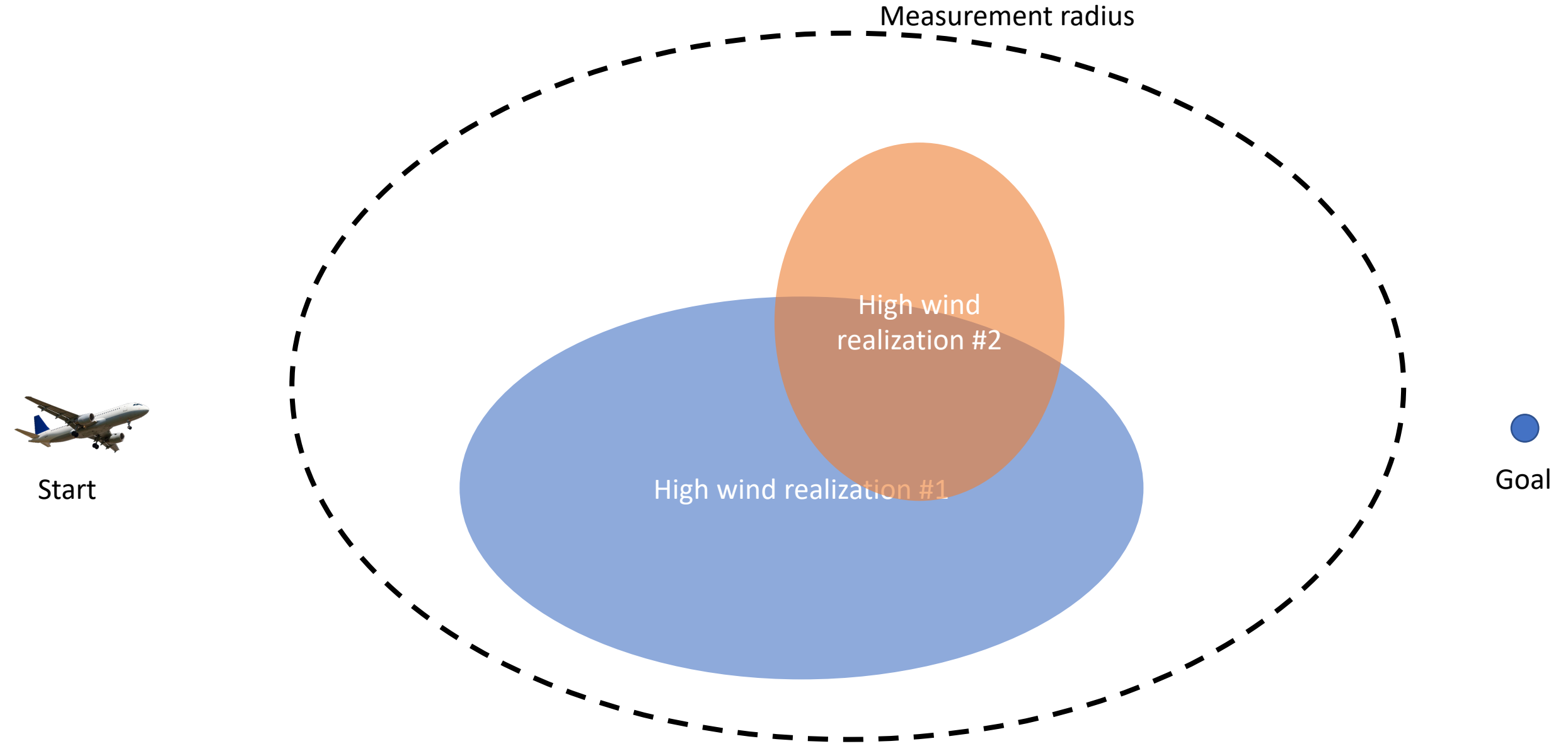
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Why is planning in uncertain environments harder?

We compute a **policy** and the trajectory depends on the measurements

Bifurcation point

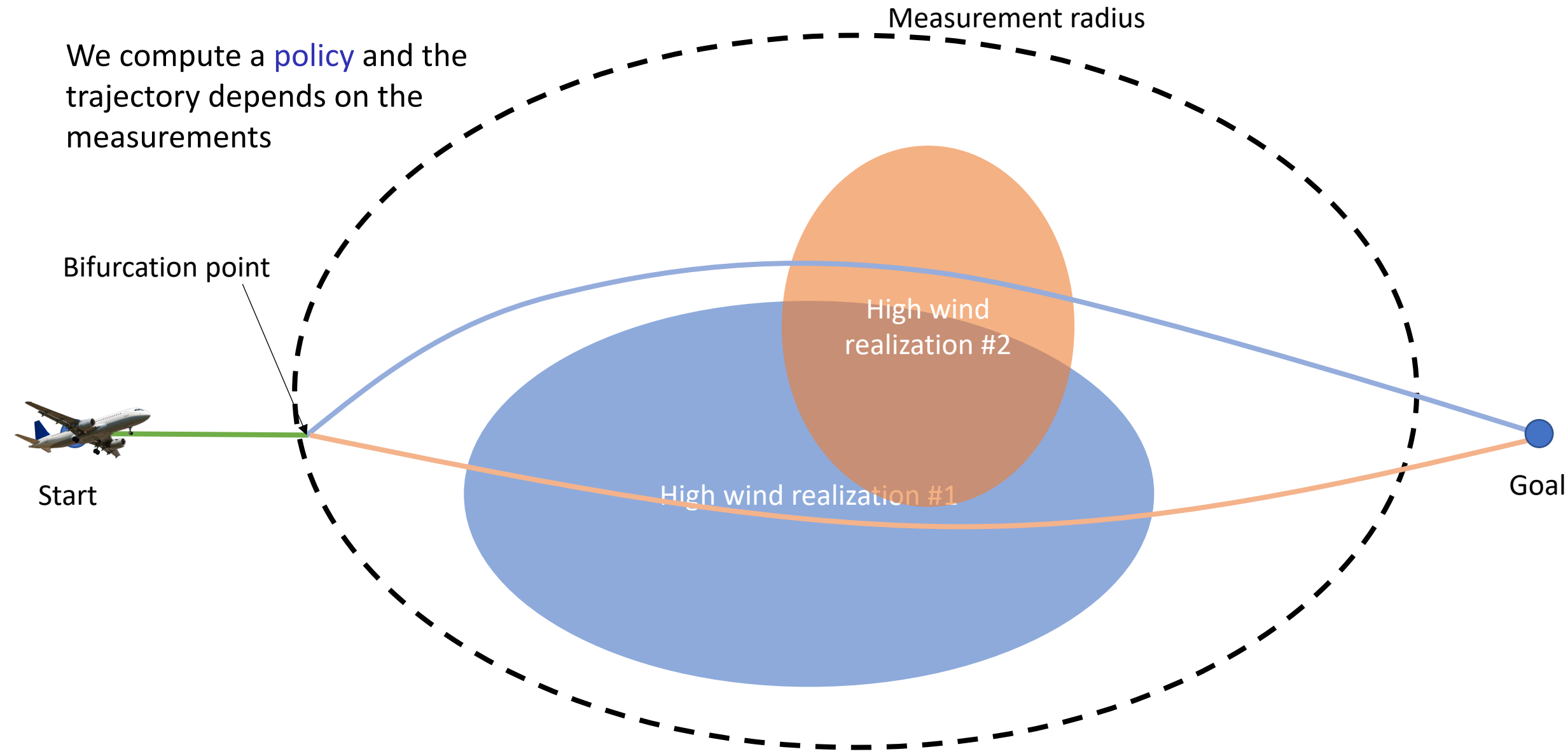
Measurement radius

High wind realization #2

High wind realization #1

Start

Goal



Problem Formulation

True system dynamics: $x_{k+1} = Ax_k + Bu_k + w_k$

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Robust constraints:

Given a control policy $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^d$ designed using the nominal model, we want to guarantee that the closed-loop system

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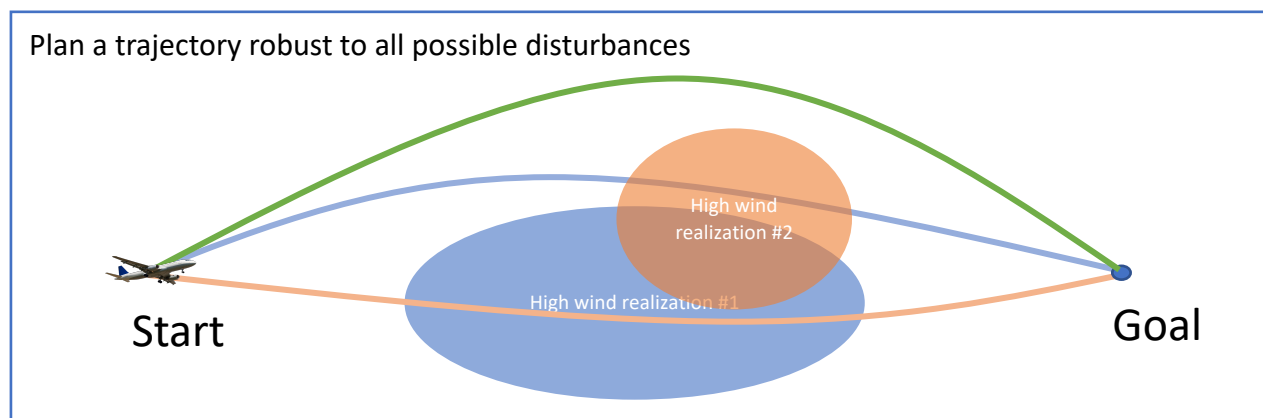
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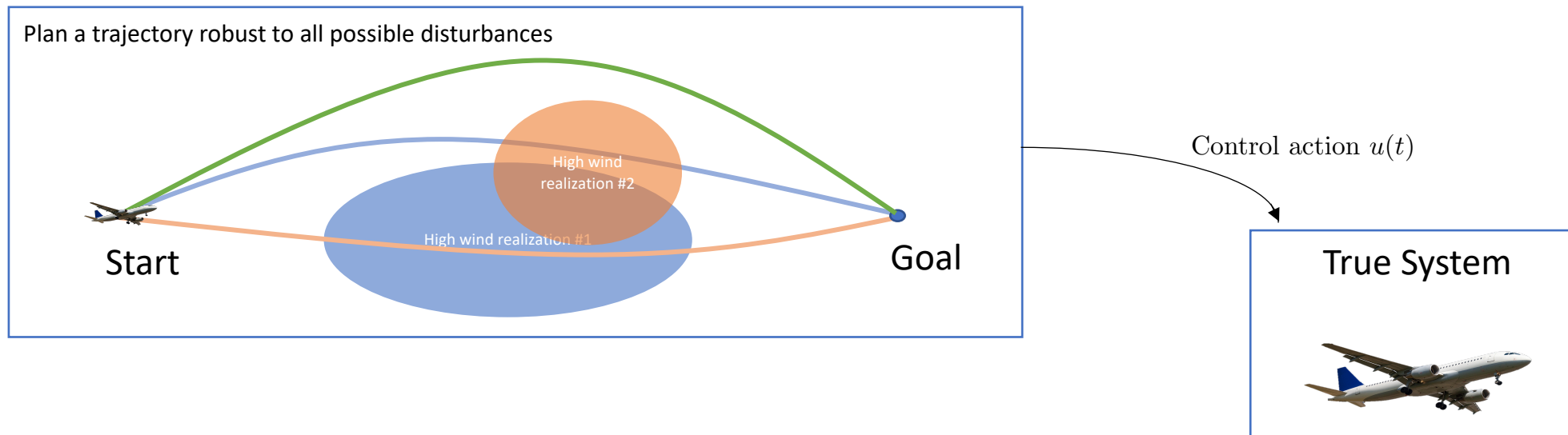
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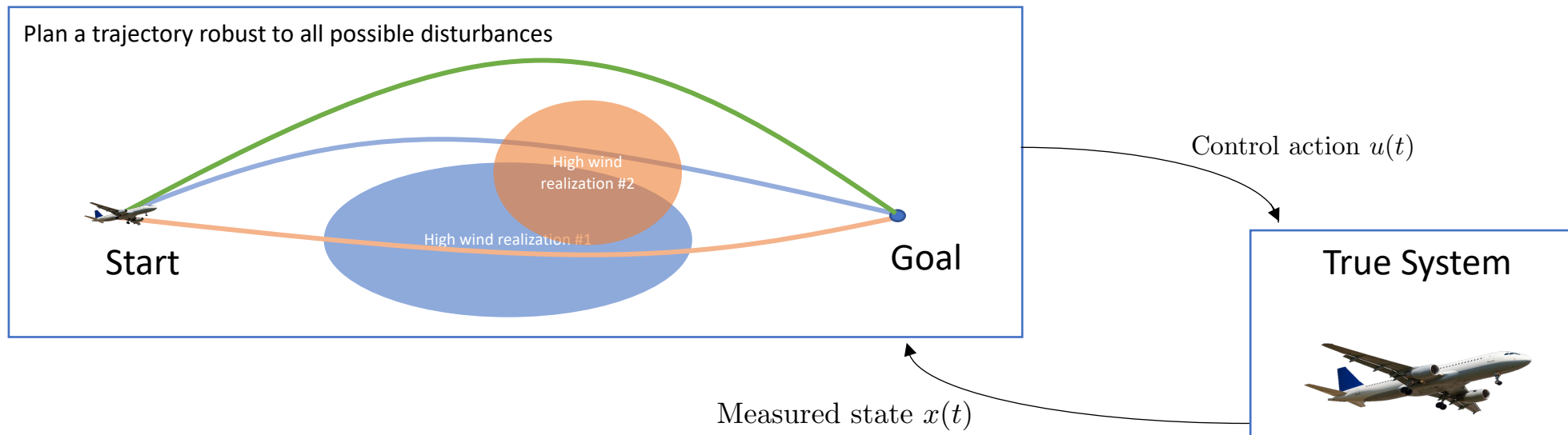
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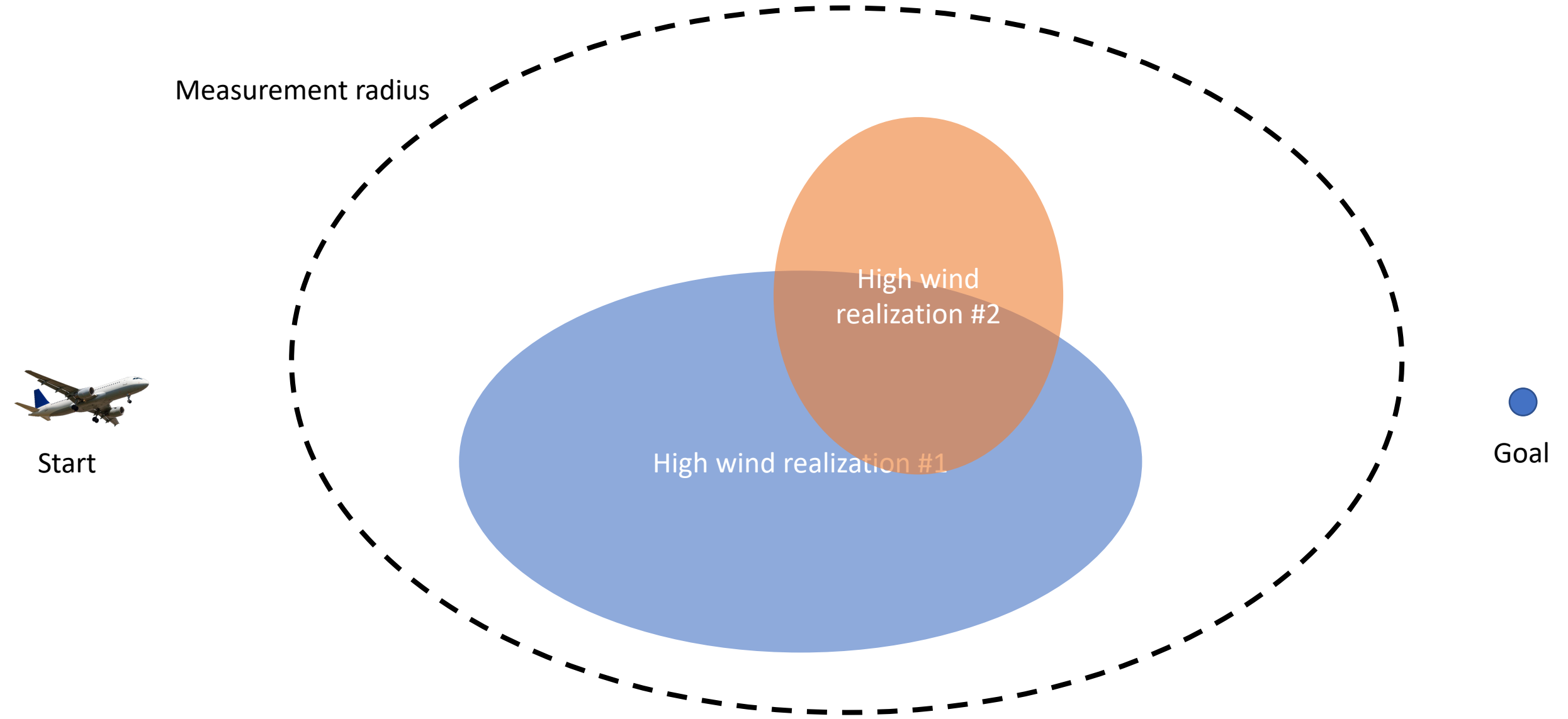
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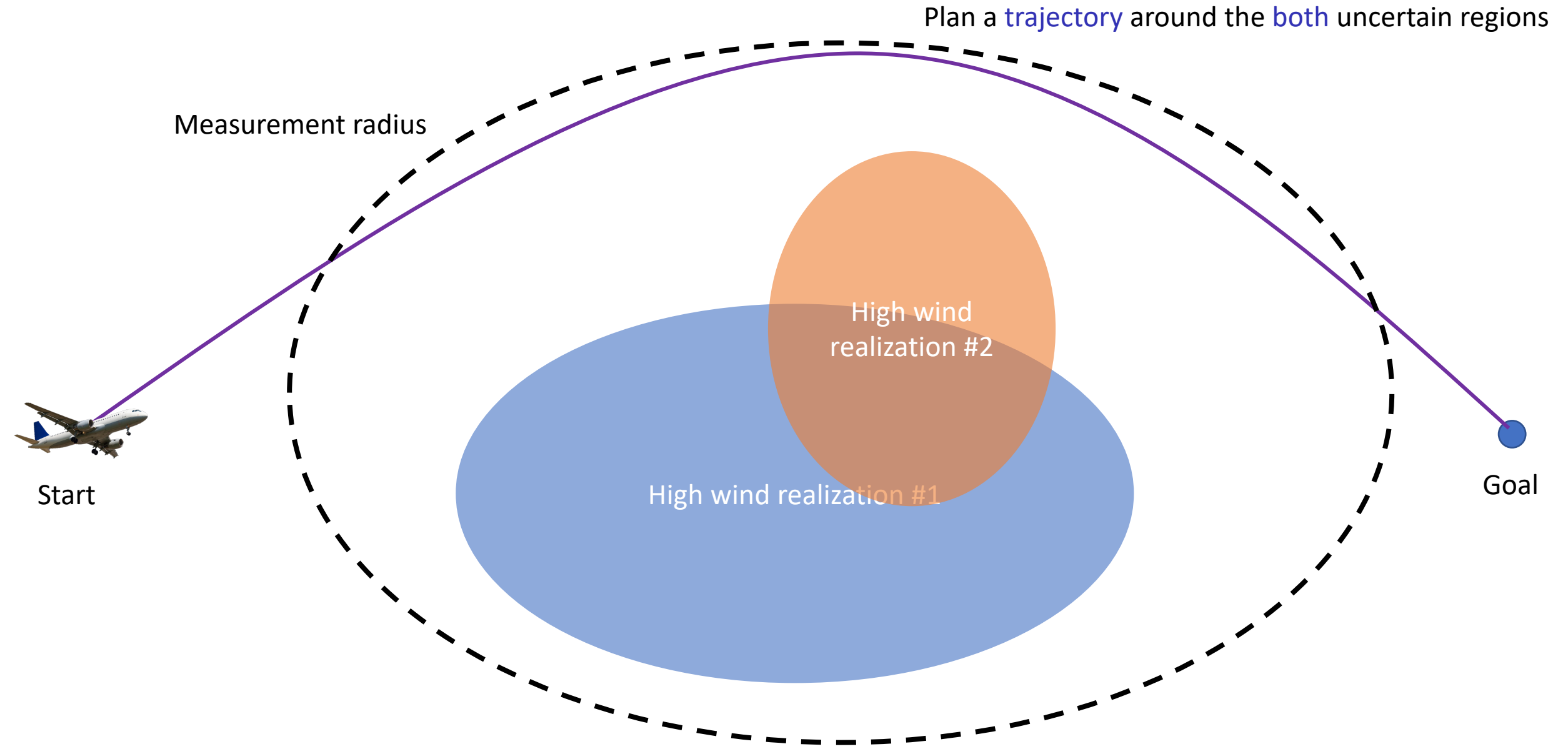
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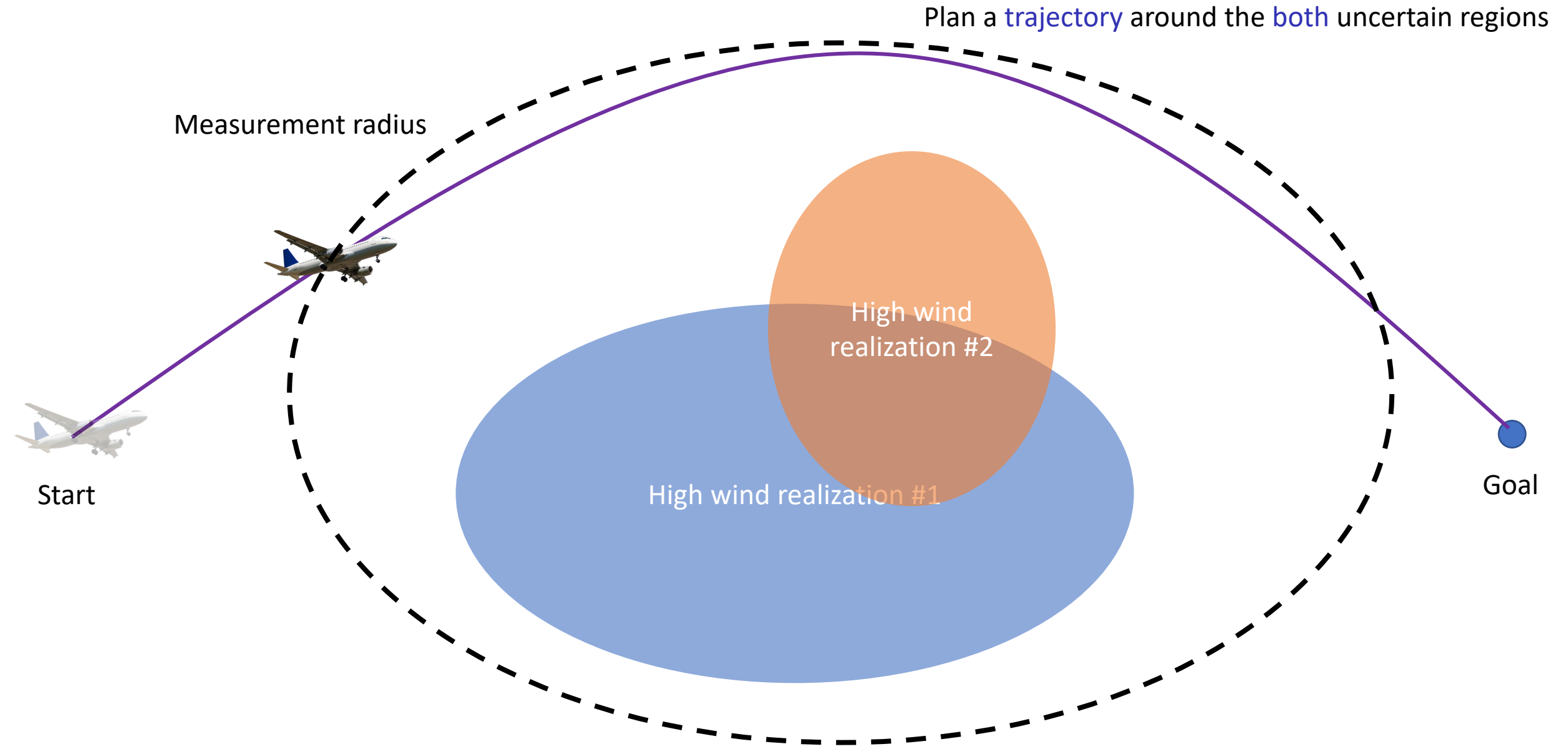
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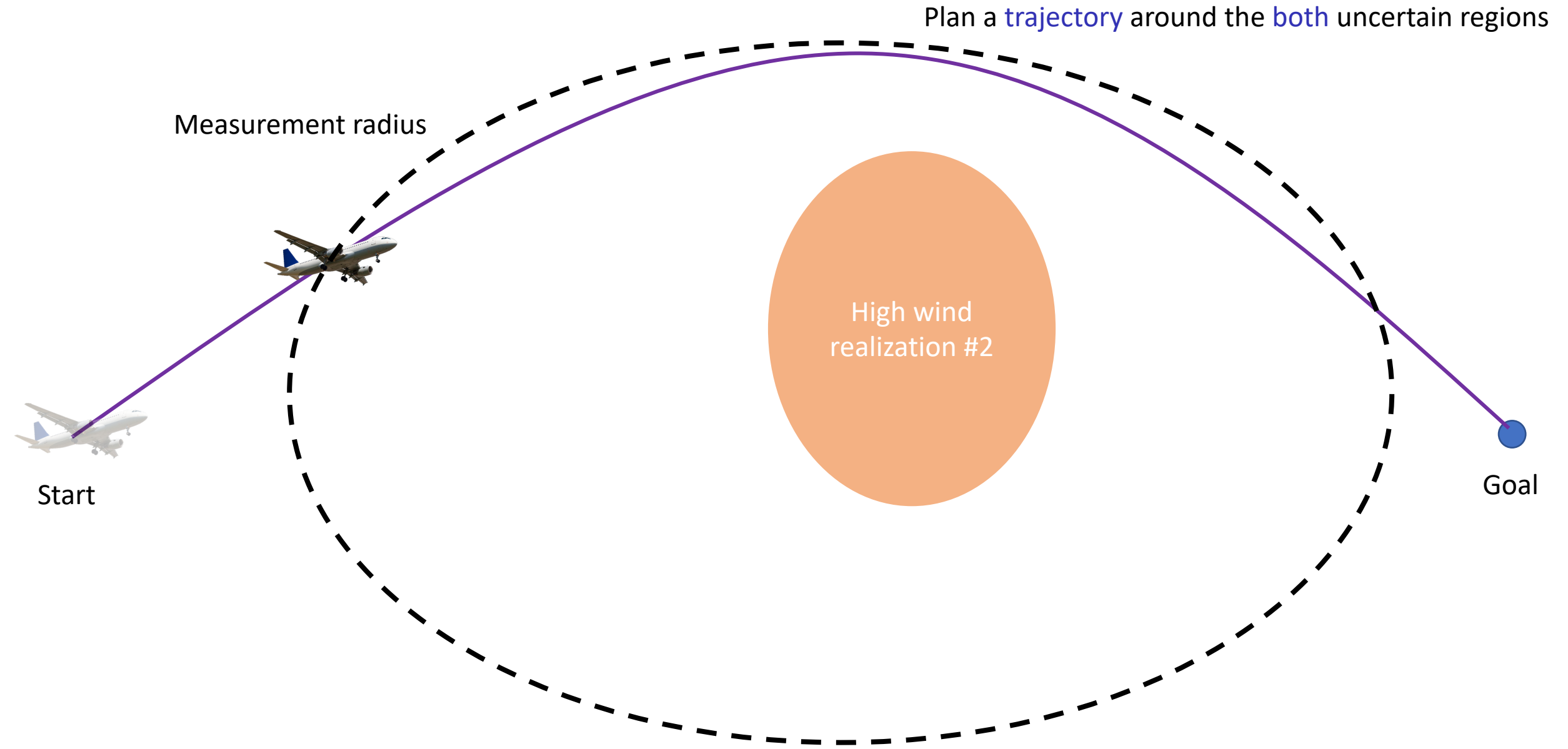
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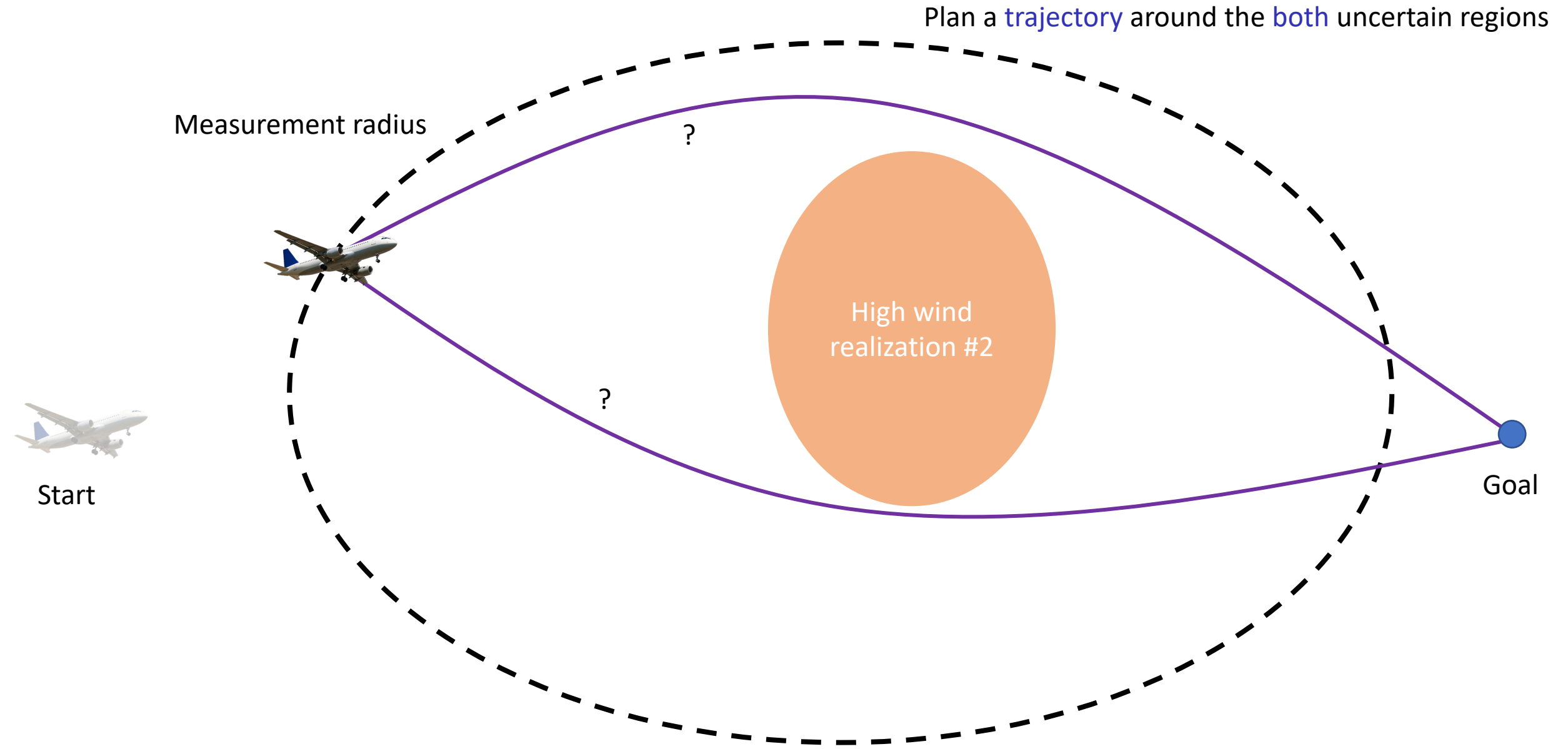
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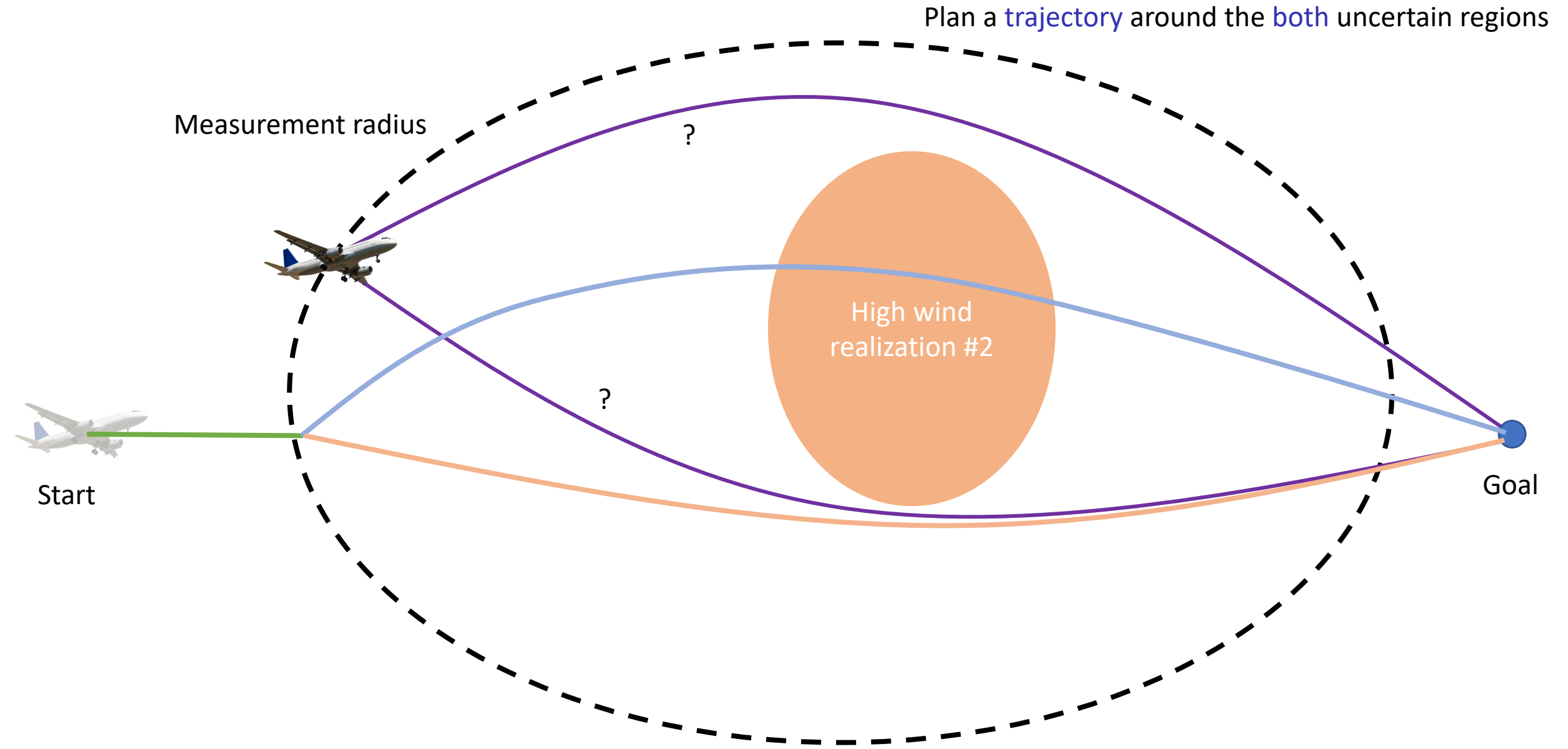
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Why should I care?

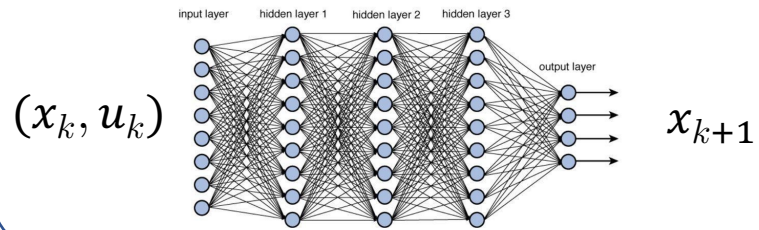
Why should I care?

Model-based RL

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Model-based RL

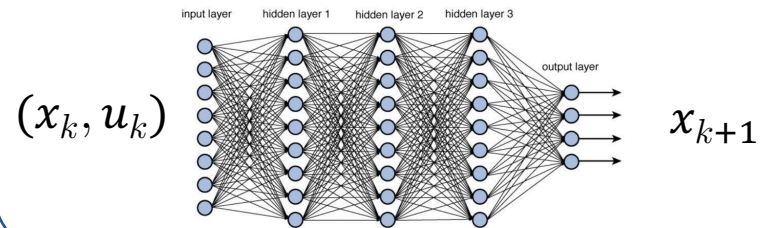
Step 1: Learn a Model



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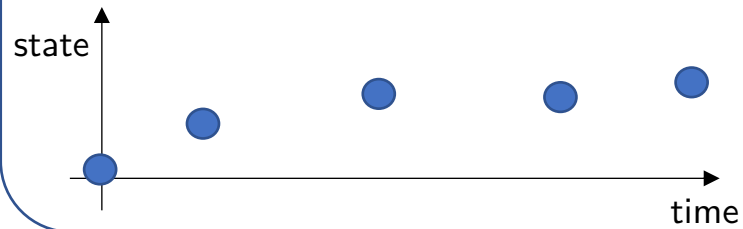
Model-based RL

Step 1: Learn a Model



Step 2: Receding horizon planning

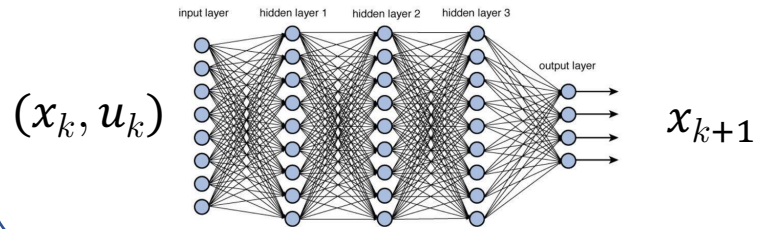
$$\min_{\pi} \mathbb{E}_{\substack{a_t \sim \pi \\ s_t \sim p}} \left[\sum_{t=0}^H \gamma^t c(a_t, s_t) \right]$$



Why should I care?

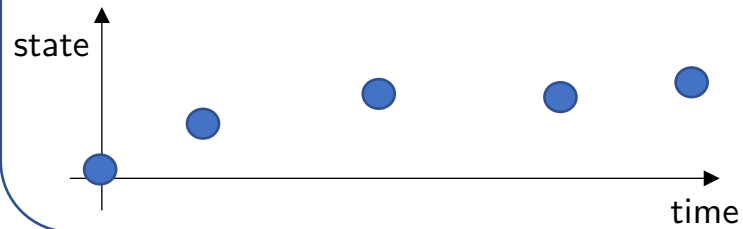
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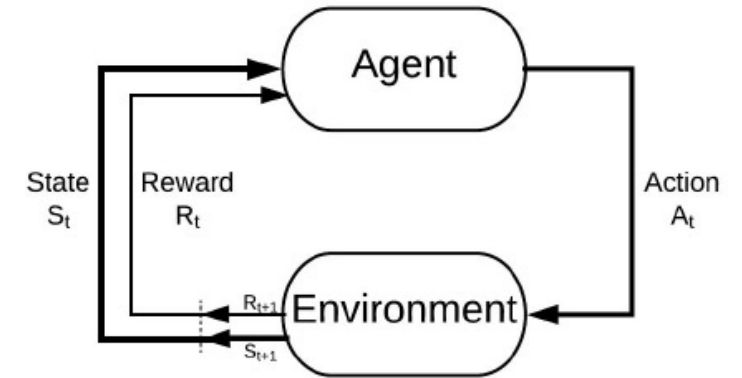
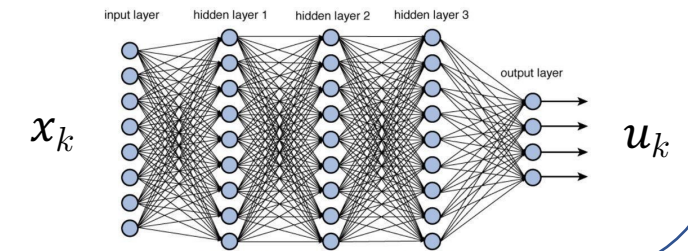
Step 2: Receding horizon planning

$$\min_{a_t \sim \pi, s_t \sim p} \mathbb{E} \left[\sum_{t=0}^H \gamma^t c(a_t, s_t) \right]$$



Model-free RL

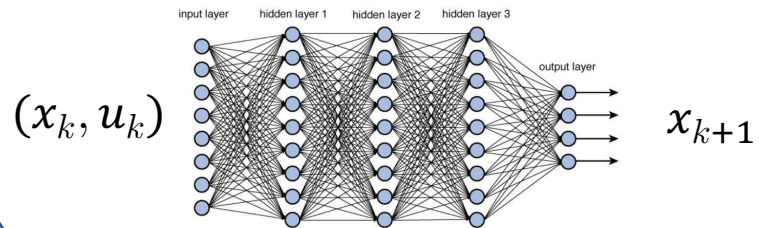
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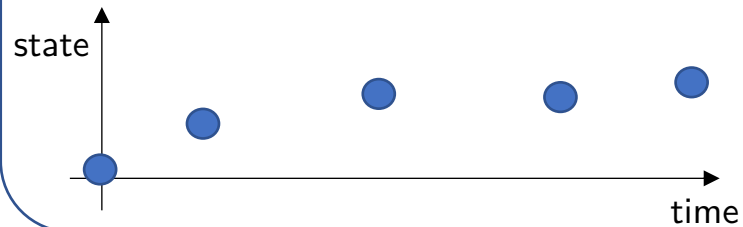
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Step 2: Receding horizon planning

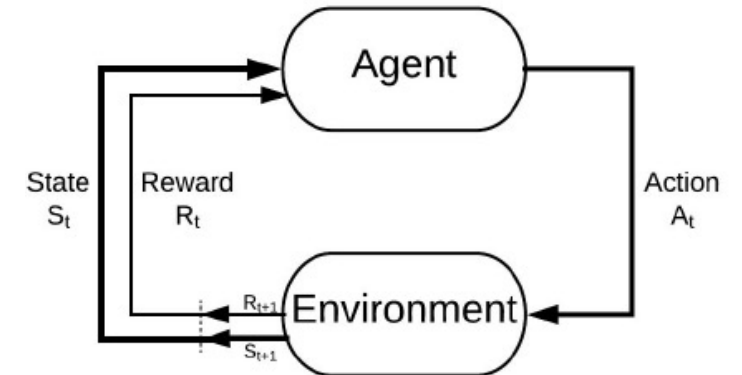
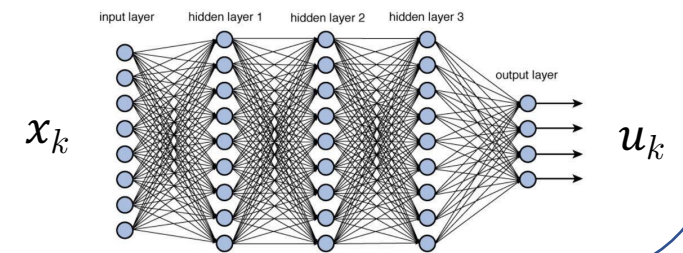
$$\min_{a_t \sim \pi, s_t \sim p} \mathbb{E} \left[\sum_{t=0}^H \gamma^t c(a_t, s_t) \right]$$



Which one has better asymptotic performance?

Model-free RL

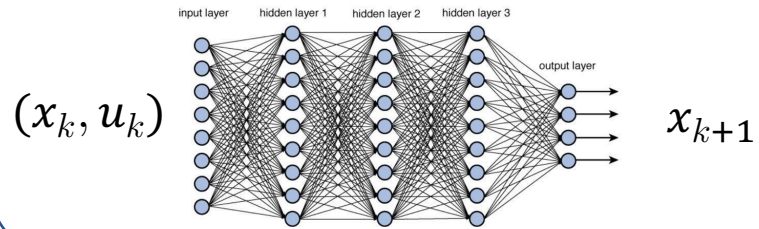
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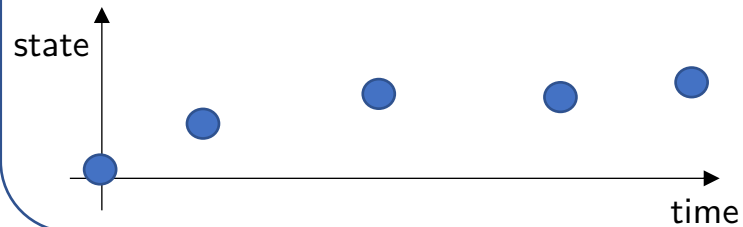
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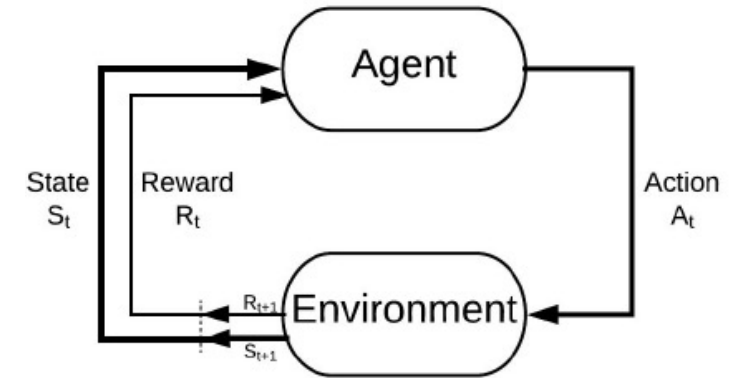
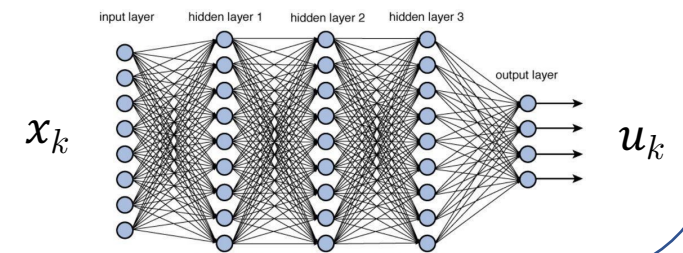
$$\min_{a_t \sim \pi, s_t \sim p} \mathbb{E} [\sum_{t=0}^H \gamma^t c(a_t, s_t)]$$



Which one has better asymptotic performance?

Model-free RL

Step 1: Learn a policy

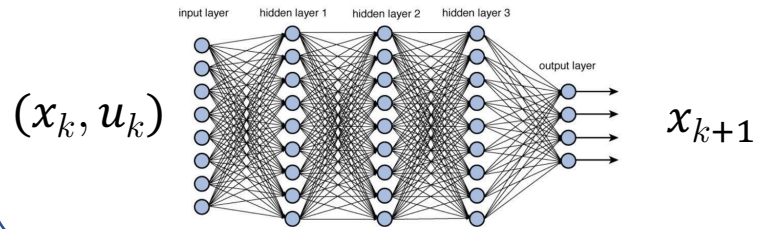


What can go wrong with model-based RL?

Why should I care?

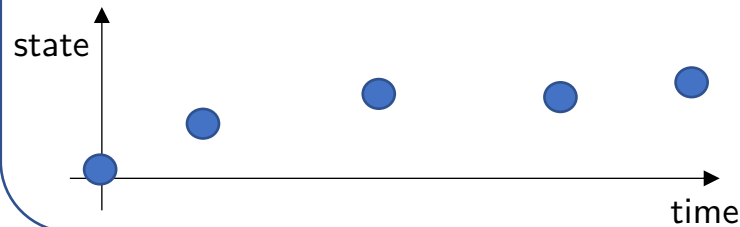
Model-based RL

Step 1: Learn a Model



Step 2: Receding horizon planning

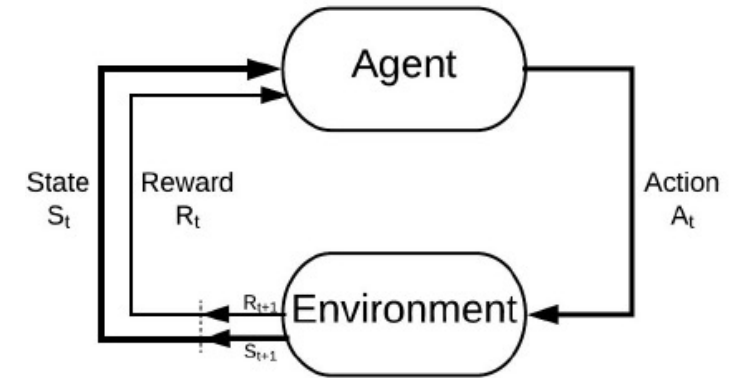
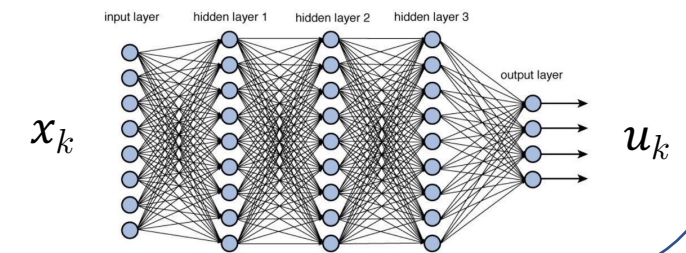
$$\min_{a_t \sim \pi, s_t \sim p} \mathbb{E} \left[\sum_{t=0}^H \gamma^t c(a_t, s_t) \right]$$



Which one has better asymptotic performance?

Model-free RL

Step 1: Learn a policy



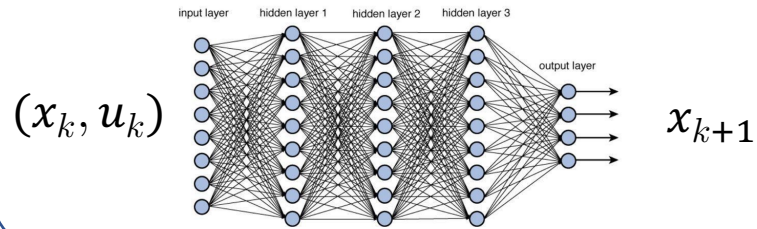
What can go wrong with model-based RL?

- The prediction horizon is short → need a value function and safe set

Why should I care?

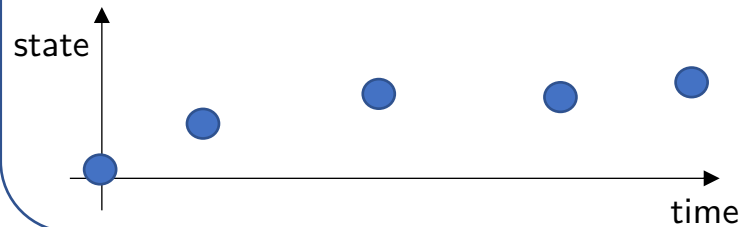
Model-based RL

Step 1: Learn a Model



Step 2: Receding horizon planning

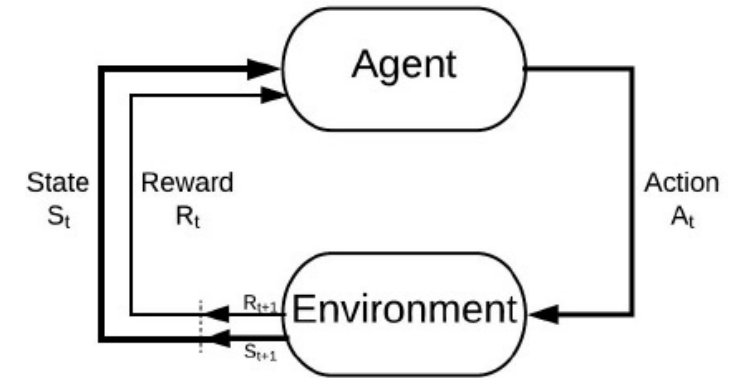
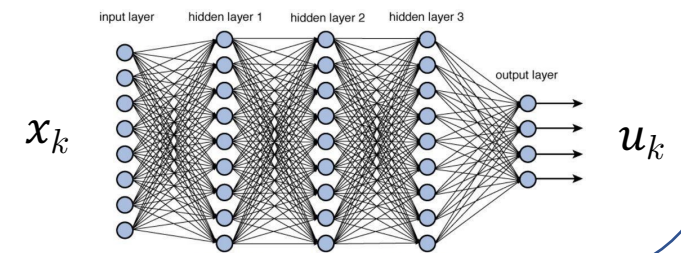
$$\min_{a_t \sim \pi, s_t \sim p} \mathbb{E} [\sum_{t=0}^H \gamma^t c(a_t, s_t)]$$



Which one has better asymptotic performance?

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- ▶ The prediction horizon is short $\rightarrow$ need a value function and safe set
- ▶ Planning trajectories is suboptimal! We should plan over policies

Why should I care?

Model-based RL

Benchmarking Model-Based Reinforcement Learning

Tingwu Wang^{1,2}, Xuchan Bao^{1,2}, Ignasi Clavera³, Jerrick Hoang^{1,2}, Yeming Wen^{1,2},
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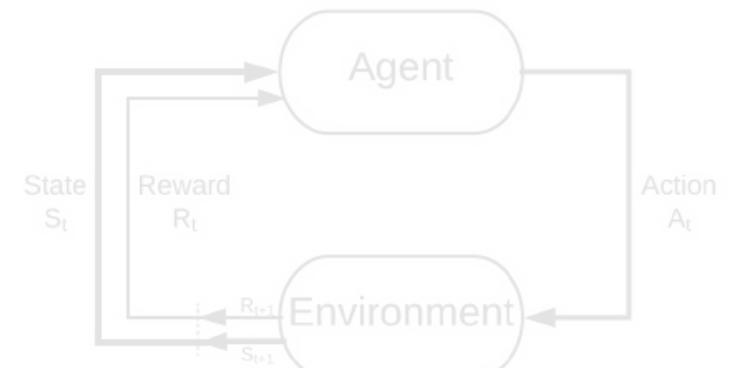
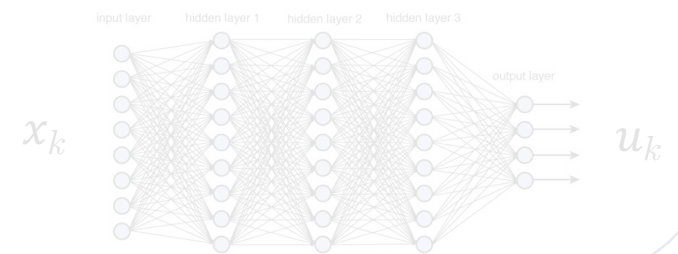
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Abstract

Model-based reinforcement learning (MBRL) is widely seen as having the potential to be significantly more sample efficient than model-free RL. However, research in model-based RL has not been very standardized. It is fairly common for authors to experiment with self-designed environments, and there are several separate lines of research, which are sometimes closed-sourced or not reproducible. Accordingly, it is an open question how these various existing MBRL algorithms perform relative to each other. To facilitate research in MBRL, in this paper we gather a wide collection of MBRL algorithms and propose over 18 benchmarking environments specially designed for MBRL. We benchmark these algorithms with unified problem settings, including noisy environments. Beyond cataloguing performance, we explore and unify the underlying algorithmic differences across MBRL algorithms. We characterize three key research challenges for future MBRL research: the dynamics bottleneck, the planning horizon dilemma, and the early-termination dilemma. Finally, to maximally facilitate future research on MBRL, we open-source our benchmark in <http://www.cs.toronto.edu/~tingwuwang/mbrl.html>.

Model-free RL

Step 1: Learn a policy



has better
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Model-free RL

Model-Based Reinforcement Learning via Meta-Policy Optimization

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Pieter Abbeel

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Abstract: Model-based reinforcement learning approaches carry the promise of being data efficient. However, due to challenges in learning dynamics models that sufficiently match the real-world dynamics, they struggle to achieve the same asymptotic performance as model-free methods. We propose Model-Based Meta-Policy-Optimization (MB-MPO), an approach that foregoes the strong reliance on accurate learned dynamics models. Using an ensemble of learned dynamic models, MB-MPO meta-learns a policy that can quickly adapt to any model in the ensemble with one policy gradient step. This steers the meta-policy towards internalizing consistent dynamics predictions among the ensemble while shifting the burden of behaving optimally w.r.t. the model discrepancies towards the adaptation step. Our experiments show that MB-MPO is more robust to model imperfections than previous model-based approaches. Finally, we demonstrate that our approach is able to match the asymptotic performance of model-free methods while requiring significantly less experience.

Keywords: Reinforcement Learning, Meta-Learning, Model-Based, Model-Free

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How do we fix this problem?

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Statement 2: Under the assumptions of this theorem for any $0 \leq t_0 \leq t$

$$\begin{aligned}\Phi_{\tilde{z}}(t) - \Phi_{\tilde{z}}(t_0) &\geq \Phi_{\tilde{z}^*}(t) - \Phi_{\tilde{z}^*}(t_0) \\ &= -\frac{1}{h} \Delta \bar{\vartheta}^T(t) L^{-1} \Delta \bar{\vartheta}(t) + o_{\omega}(h)\end{aligned}$$

where

$$\begin{aligned}\Phi_{\tilde{z}}(t) &:= \int_{t_0}^t (2\bar{u}^T(\tau) d\bar{\vartheta}(\tau) + \bar{u}^T(\tau) L\bar{u}(\tau) d\tau) \\ \Delta \bar{\vartheta}(t) &:= B_{\tilde{u}}^T P[B_{\tilde{u}}(u_{\text{comp}}(t) - x^*(t))h \\ &\quad + (K + A_0 C_0^T) \Delta y(t)], \quad \Delta y(t) = y(t) - y(t - h)\end{aligned} \quad (20)$$

and this minimum is reachable for

$$\begin{aligned}\bar{u}(t) dt &= \bar{u}^*(t) dt := -L^{-1} B_{\tilde{u}}^T P[B_{\tilde{u}}(u_{\text{comp}}(t) + x^*(t))dt \\ &\quad + (K + A_0 C_0^T) dy(t)].\end{aligned}$$

As a result, we have: $d\Phi_{\tilde{z}^*}(t) \leq 0$ (in symbolic form).

Proof of Statement 2: Using the Euler-Maruyama's formula [3], [9] we obtain the following relation: $(h := t - t_0 \rightarrow 0)$: $\Phi_{\tilde{z}}(t) - \Phi_{\tilde{z}}(t_0) = 2\bar{u}^T(t) \Delta \bar{\vartheta}(t) + \bar{u}^T(t) L\bar{u}(t)h + o_{\omega}(h)$.

Minimizing then the right side for each fixed t , we derive

$$\begin{aligned}\min_{\bar{u}(t)} [\Phi_{\tilde{z}}(t) - \Phi_{\tilde{z}}(t_0)] &= \Phi_{\tilde{z}^*}(t) - \Phi_{\tilde{z}^*}(t_0) \\ &= -\frac{1}{h} \Delta \bar{\vartheta}^T(t) L^{-1} \Delta \bar{\vartheta}(t) + o_{\omega}(h) \leq o_{\omega}(h).\end{aligned} \quad (21)$$

Hence, taking into account the definition (20), we have $d\Phi_{\tilde{z}^*}(t) \leq 0$. $\square$

Rewriting (18) in differential form and taking into account that P is the solution of the Riccati equation (see Assumption 4 of this theorem) we finally derive

$$dV(e(t)) \stackrel{\text{a.s.}}{\leq} [\varphi(t) + I(t)] dt + S^T(t) dw(t) - e^T(t) Q_e e(t) dt.$$

According to the Kronecker lemma, the Large Number law for martingale, and Skorohod lemma (see [2] and [4]), we obtain the result of the theorem. Details of this proof can be found in [13].

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Min-Max Feedback Model Predictive Control for Constrained Linear Systems

P. O. M. Scaekaert and D. Q. Mayne

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Index Terms—Feedback, min-max optimization, model predictive control.

I. INTRODUCTION

Model predictive control (MPC) is a control methodology that is becoming mature. The basic aspects of the method are well understood, and stabilizing formulations of the control law are documented in the literature, both for linear and nonlinear processes [1], [2].

The MPC strategy optimizes an open-loop control sequence each sample, to minimize a nominal cost function, subject to state and input constraints. The optimization is usually based on the assumption that the process model is exact and that disturbances are constant. Because the control law ignores the possibility of future changes in disturbance and model mismatch, closed-loop performance can be poor [3], with likely violations of the constraints, when disturbances or model mismatch are present.

Some formulations of MPC have been proposed that address these issues [4], [5]. These methods rely on a min-max optimization to optimize performance. However, these formulations optimize a control profile over all possible disturbance (or model mismatch) sequences, which is often computationally intractable.



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Optimization over state feedback policies for robust control with constraints

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^bDepartment of Aeronautics and Department of Electrical and Electronic Engineering, Imperial College London, Exhibition Road, London SW7 2BX, UK

Received 15 March 2005; received in revised form 8 June 2005; accepted 25 August 2005

Abstract

This paper is concerned with the optimal control of linear discrete-time systems subject to unknown but bounded state disturbance and mixed polytopic constraints on the state and input. It is shown that the class of admissible affine state feedback control policies with prior states is equivalent to the class of admissible feedback policies that are affine functions of the past disturbance sequence. This leads to a broad class of constrained finite horizon robust and optimal control problems, where the optimization is over affine state feedback policies, can be solved in a computationally efficient fashion using convex optimization methods. This equivalence result is used to design a robust receding horizon control (RHC) state feedback policy such that the closed-loop system is input-to-state stable (ISS) and the cost is minimized for all time and all allowable disturbance sequences. The cost to be minimized in the associated finite horizon optimization problem is quadratic in the disturbance-free state and input sequences. The value of the receding horizon control law can be calculated sample instant using a single, tractable and convex quadratic program (QP) if the disturbance set is polytopic, or a tractable second-order cone program (SOCP) if the disturbance set is given by a 2-norm bound. © 2006 Elsevier Ltd. All rights reserved.

Keywords: Robust control; Constraint satisfaction; Robust optimization; Predictive control; Optimal control

1. Introduction

This paper is concerned with the control of constrained discrete-time linear systems that are subject to additive, but bounded disturbances on the state. The main aim is to provide results that allow for the efficient computation of an optimal and stabilizing state feedback control policy that ensures a given set of state and input constraints are satisfied for all time, despite the presence of the disturbances. This is a problem that has been studied for some time now in the optimal control literature (Bertsekas & Rhodes, 1973) and a number of different

& Diaz-Bobillo, 1995) or predictive control (Munoz & Morari, 1993; Mayne, Rawlings, Rao, & Scaekaert, 2000).

It is generally accepted that if disturbances are accounted for in the formulation of a constrained control problem, then the optimization has to be over admissible state feedback policies, rather than open-loop sequences, otherwise infeasibility and instability may occur (Mayne et al., 2000). However, optimization over (nonlinear) feedback policies is particularly challenging. The simplest is to ignore the disturbance and rely on the inherent robustness of deterministic model predictive control applied to the nominal system (Scaekaert & Rawlings, 1995; Marrufo, Álamo, & Camacho, 2002). Open-loop model predictive control that determines the



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Brief paper

Robust model predictive control of constrained linear systems with bounded disturbances[☆]

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Received 9 February 2004; received in revised form 14 July 2004; accepted 23 August 2004

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Abstract

This paper provides a novel solution to the problem of robust model predictive control of constrained, linear, discrete-time systems in the presence of bounded disturbances. The optimal control problem that is solved online includes, uniquely, the initial state of the model employed in the problem as a decision variable. The associated value function is zero in a disturbance invariant set that serves as the 'origin' when bounded disturbances are present, and permits a strong stability result, namely robust exponential stability of the disturbance invariant set for the controlled system with bounded disturbances, to be obtained. The resultant online algorithm is a quadratic program of similar complexity to that required in conventional model predictive control. © 2004 Elsevier Ltd. All rights reserved.

Keywords: Robust model predictive control; Robustness; Bounded disturbances

1. Introduction

Model predictive control is widely employed for the control of constrained systems and an extensive literature on the subject exists some of which is reviewed in Bemporad and Morari (1999); Allgöwer, Badgwell, Qin, Rawlings, and Wright (1999); Mayne, Rawlings, Rao, and Scaekaert (2000). Several methods for achieving robustness have been considered. The simplest is to ignore the disturbance and rely on the inherent robustness of deterministic model predictive control applied to the nominal system (Scaekaert & Rawlings, 1995; Marrufo, Álamo, & Camacho, 2002). Open-loop model predictive control that determines the

$\{u_0, u_1, \dots, u_{N-1}\}$ of control actions was proposed in Zheng and Morari (1993); this method cannot contain the 'spread' of predicted trajectories resulting from disturbances. Hence feedback model predictive control in which the decision variable is a policy π , which is a sequence $\{\mu_0(\cdot), \mu_1(\cdot), \dots, \mu_{N-1}(\cdot)\}$ of control laws, was advocated in, for example, Mayne (1995); Kothare, Balakrishnan, and Morari (1996); Lee and Yu (1997); Scaekaert and Mayne (1998); De Nicolao, Magni, and Scattolini (2000); Magni, Nijmeijer, and van der Schaft (2001); Magni, De Nicolao, Scattolini, and Allgöwer (2003). Determination of a control policy is usually prohibitively difficult so research has concentrated on various simplifying approximations. (Mayne

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Optimization over state feedback policies for robust control with constraints

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Abstract

This paper is concerned with the optimal control of linear discrete-time systems subject to unknown but bounded state disturbance and mixed polytopic constraints on the state and input. It is shown that the class of admissible affine state feedback control policies with prior states is equivalent to the class of admissible feedback policies that are affine functions of the past disturbance sequence. This leads to a broad class of constrained finite horizon robust and optimal control problems, where the optimization is over affine state feedback policies, can be solved in a computationally efficient fashion using convex optimization methods. This equivalence result is used to design a robust receding horizon control (RHC) state feedback policy such that the closed-loop system is input-to-state stable (ISS) and the cost is minimized in the associated finite horizon optimal control problem. The value of the receding horizon control law can be calculated sample instant using a single, tractable and convex quadratic program (QP) if the disturbance set is polytopic, or a tractable second-order cone program (SOCP) if the disturbance set is given by a 2-norm bound. © 2006 Elsevier Ltd. All rights reserved.

Keywords: Robust control; Constraint satisfaction; Robust optimization; Predictive control; Optimal control

1. Introduction

This paper is concerned with the control of constrained discrete-time linear systems that are subject to additive, but bounded disturbances on the state. The main aim is to provide results that allow for the efficient computation of an optimal and stabilizing state feedback control policy that ensures a given set of state and input constraints are satisfied for all time, despite the presence of the disturbances. This is a problem that has been studied for some time now in the optimal control literature (Bertsekas & Rhodes, 1973) and a number of different

& Diaz-Bobillo, 1995) or predictive control (Mangas & Morari, 1999; Allgöwer, Badgwell, Qin, Rawlings, and Wright (1999); Mayne, Rawlings, Rao, and Scaekaert (2000)).

It is generally accepted that if disturbances are accounted for in the formulation of a constrained control problem, then the optimization has to be over admissible state feedback policies, rather than open-loop sequences, otherwise infeasibility and instability may occur (Mayne et al., 2000). However, optimization over (nonlinear) feedback policies is particularly challenging. Current proposals for this problem include: (i) using finite dimensional optimization, such as linear programming (LP) (Bertsekas & Rhodes, 1973) or quadratic programming (QP) (Mangas & Morari, 1999); (ii) using



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Brief paper

Robust model predictive control of constrained linear systems with bounded disturbances[☆]

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Abstract

This paper provides a novel solution to the problem of robust model predictive control of constrained, linear, discrete-time systems in the presence of bounded disturbances. The optimal control problem that is solved online includes, uniquely, the initial state of the model employed in the problem as a decision variable. The associated value function is zero in a disturbance invariant set that serves as the 'origin' when bounded disturbances are present, and permits a strong stability result, namely robust exponential stability of the disturbance invariant set for the controlled system with bounded disturbances, to be obtained. The resultant online algorithm is a quadratic program of similar complexity to that required in conventional model predictive control. © 2004 Elsevier Ltd. All rights reserved.

Keywords: Robust model predictive control; Robustness; Bounded disturbances

1. Introduction

Model predictive control is widely employed for the control of constrained systems and an extensive literature on the subject exists some of which is reviewed in Bemporad and Morari (1999); Allgöwer, Badgwell, Qin, Rawlings, and Wright (1999); Mayne, Rawlings, Rao, and Scaekaert (2000). Several methods for achieving robustness have been considered. The simplest is to ignore the disturbance and rely on the inherent robustness of deterministic model predictive control applied to the nominal system (Scaekaert & Rawlings, 1995; Marruado, Álamo, & Camacho, 2002). Open-loop model predictive control that determines the

$\{u_0, u_1, \dots, u_{N-1}\}$ of control actions was proposed in Zheng and Morari (1993); this method cannot contain the 'spread' of predicted trajectories resulting from disturbances. Hence feedback model predictive control in which the decision variable is a policy π , which is a sequence $\{\mu_0(\cdot), \mu_1(\cdot), \dots, \mu_{N-1}(\cdot)\}$ of control laws, was advocated in, for example, Mayne (1995); Kothare, Balakrishnan, and Morari (1996); Lee and Yu (1997); Scaekaert and Mayne (1998); De Nicolao, Magni, and Scattolini (2000); Magni, Nijmeijer, and van der Schaft (2001); Magni, De Nicolao, Scattolini, and Allgöwer (2003). Determination of a control policy is usually prohibitively difficult so research has concentrated on various simplifying approximations. (Mayne

The problem is NP-hard!

How do we fix this problem?

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Statement 2: Under the assumptions of this theorem for any $0 \leq t_0 \leq t$

$$\begin{aligned}\Phi_{\tilde{z}}(t) - \Phi_{\tilde{z}}(t_0) &\geq \Phi_{\tilde{z}^*}(t) - \Phi_{\tilde{z}^*}(t_0) \\ &= -\frac{1}{h} \Delta \bar{v}^T(t) L^{-1} \Delta \bar{v}(t) + o_{\omega}(h)\end{aligned}$$

where

$$\begin{aligned}\Phi_{\tilde{z}}(t) &:= \int_{t_0}^t (2\bar{u}^T(\tau) \Delta \bar{v}(\tau) + \bar{u}^T(\tau) L \bar{u}(\tau) d\tau) \\ \Delta \bar{v}(t) &:= B_{\tilde{u}}^T P [B_{\tilde{u}}(u_{\text{comp}}(t) - x^*(t))h \\ &\quad + (K + A_0 C_0^T) \Delta y(t)], \quad \Delta y(t) = y(t) - y(t - h)\end{aligned} \quad (20)$$

and this minimum is reachable for

$$\begin{aligned}\bar{u}(t) dt &= \bar{u}^*(t) dt := -L^{-1} B_{\tilde{u}}^T P [B_{\tilde{u}}(u_{\text{comp}}(t) + x^*(t)) dt \\ &\quad + (K + A_0 C_0^T) dy(t)].\end{aligned}$$

As a result, we have: $d\Phi_{\tilde{z}^*}(t) \leq 0$ (in symbolic form).

Proof of Statement 2: Using the Euler-Maruyama's formula [3], [9] we obtain the following relation: $(h := t - t_0 \rightarrow 0): \Phi_{\tilde{z}}(t) - \Phi_{\tilde{z}}(t_0) = 2\bar{u}^T(t) \Delta \bar{v}(t) + \bar{u}^T(t) L \bar{u}(t)h + o_{\omega}(h)$.

Minimizing then the right side for each fixed t , we derive

$$\begin{aligned}\min_{\bar{u}(t)} [\Phi_{\tilde{z}}(t) - \Phi_{\tilde{z}}(t_0)] &= \Phi_{\tilde{z}^*}(t) - \Phi_{\tilde{z}^*}(t_0) \\ &= -\frac{1}{h} \Delta \bar{v}^T(t) L^{-1} \Delta \bar{v}(t) + o_{\omega}(h) \leq o_{\omega}(h).\end{aligned} \quad (21)$$

Hence, taking into account the definition (20), we have $d\Phi_{\tilde{z}^*}(t) \leq 0$. $\square$

Rewriting (18) in differential form and taking into account that P is the solution of the Riccati equation (see Assumption 4 of this theorem) we finally derive

$$dV(e(t)) \stackrel{\text{a.s.}}{\leq} [\varphi(t) + I(t)] dt + S^T(t) dw(t) - e^T(t) Q e(t) dt.$$

According to the Kronecker lemma, the Large Number law for martingale, and Skorohod lemma (see [2] and [4]), we obtain the result of the theorem. Details of this proof can be found in [13].

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Min-Max Feedback Model Predictive Control for Constrained Linear Systems

P. O. M. Scaekaert and D. Q. Mayne

Abstract—Min-max feedback formulations of model predictive control are discussed, both in the fixed and variable horizon contexts. The schemes the authors discuss introduce, in the control optimization, the notion that feedback is present in the receding-horizon implementation of the control. This leads to improved performance, compared to standard model predictive control, and resolves the feasibility difficulties that arise with the min-max techniques that are documented in the literature. The stabilizing properties of the methods are discussed as well as practical implementation details.

Index Terms—Feedback, min-max optimization, model predictive control.

I. INTRODUCTION

Model predictive control (MPC) is a control methodology that is becoming mature. The basic aspects of the method are well understood, and stabilizing formulations of the control law are documented in the literature, both for linear and nonlinear processes [1], [2].

The MPC strategy optimizes an open-loop control sequence each sample, to minimize a nominal cost function, subject to state and input constraints. The optimization is usually based on the assumption that the process model is exact and that disturbances are constant. Because the control law ignores the possibility of future changes in disturbance and model mismatch, closed-loop performance can be poor [3], with likely violations of the constraints, when disturbances or model mismatch are present.

Some formulations of MPC have been proposed that address these issues [4], [5]. These methods rely on a min-max optimization to optimize performance. However, these formulations optimize a control profile over all possible disturbance (or model mismatch) sequences, which is computationally intractable for most practical systems.



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Optimization over state feedback policies for robust control with constraints

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Abstract

This paper is concerned with the optimal control of linear discrete-time systems subject to unknown but bounded state disturbances and mixed polytopic constraints on the state and input. It is shown that the class of admissible affine state feedback control policies with prior states is equivalent to the class of admissible feedback policies that are affine functions of the past disturbance sequence. This equivalence result is used to solve a broad class of constrained finite horizon robust and optimal control problems, where the optimization is over affine state feedback policies, can be solved in a computationally efficient fashion using convex optimization methods. This equivalence result is used to solve robust receding horizon control (RHC) state feedback policy such that the closed-loop system is input-to-state stable (ISS) and the cost is minimized for all time and all allowable disturbance sequences. The cost to be minimized in the associated finite horizon optimal control problem is quadratic in the disturbance-free state and input sequences. The value of the receding horizon control law can be calculated sample instant using a single, tractable and convex quadratic program (QP) if the disturbance set is polytopic, or a tractable second-order cone program (SOCP) if the disturbance set is given by a 2-norm bound. © 2006 Elsevier Ltd. All rights reserved.

Keywords: Robust control; Constraint satisfaction; Robust optimization; Predictive control; Optimal control

1. Introduction

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It is generally accepted that if disturbances are accounted for in the formulation of a constrained control problem, then the optimization has to be over admissible state feedback policies, rather than open-loop sequences, otherwise infeasibility and instability may occur (Mayne et al., 2000). However, optimization over (nonlinear) feedback policies is particularly challenging. Current proposals for this problem include: (i) using finite dimensional optimization, such as linear programming (LP) or quadratic programming (QP) (Bertsekas & Rhodes, 1973); (ii) using receding horizon control (RHC) (Bertsekas & Rhodes, 1973); (iii) using model predictive control (MPC) (Bertsekas & Rhodes, 1973); (iv) using robust control (RC) (Bertsekas & Rhodes, 1973); (v) using stochastic control (SC) (Bertsekas & Rhodes, 1973); (vi) using adaptive control (AC) (Bertsekas & Rhodes, 1973); (vii) using hybrid control (HC) (Bertsekas & Rhodes, 1973); (viii) using fuzzy control (FC) (Bertsekas & Rhodes, 1973); (ix) using neural network control (NNC) (Bertsekas & Rhodes, 1973); (x) using genetic algorithm control (GAC) (Bertsekas & Rhodes, 1973); (xi) using evolutionary control (EC) (Bertsekas & Rhodes, 1973); (xii) using swarm intelligence control (SIC) (Bertsekas & Rhodes, 1973); (xiii) using particle swarm optimization (PSO) (Bertsekas & Rhodes, 1973); (xiv) using differential evolution (DE) (Bertsekas & Rhodes, 1973); 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